The Generalized De Rham-Witt Complex and Congruence Differential Equations

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1 Introduction.

The De Rham-Witt complex is a powerful instrument for studying the crystalline cohomology of a smooth projective variety over a perfect field of positive characteristic. In [9] the De Rham-Witt complex is constructed for schemes on which some prime number p is zero. Here in section 2 we construct on every scheme X on which 2 is invertible the *generalized De Rham-Witt complex* $\underline{\mathcal{W}\Omega}^{\cdot}_X$; this is a Zariski sheaf of anti-commutative differential graded algebras with the additional structures and properties described in (2.1)–(2.6). Section 3 gives the (obvious) definition of the *relative generalized De Rham-Witt complex* $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$ for $f: X \to S$ a morphism of schemes over $\mathbf{Z}[\frac{1}{2}]$.

Now consider a proper smooth morphism $f: X \to S$ of smooth schemes over some open part of Spec $\mathbf{Z}[\frac{1}{2}]$. For simplicity we assume that S is affine. Let s be a closed point of S with residue field $k(s)$ of characteristic $p > 2$. Let X_s be the fiber of f over s. Using the functoriality of the constructions and the projection onto the p-typical part (see (2.4)) one obtains for all $m \geq 0$ a *specialization map*

$$\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S}) \to \mathbf{H}^m(X_s, \mathcal{W}\Omega^{\cdot}_{X_s})$$

compatible with the Frobenius endomorphisms $\underline{F}_p$ on its source and target. Here $\mathcal{W}\Omega^{\cdot}_{X_s}$ is the classical De Rham-Witt complex on X_s. Since X_s is a smooth proper scheme over the perfect field $k(s)$ one knows from ([9] II(1.4),(2.8)) that $\mathbf{H}^m(X_s, \mathcal{W}\Omega^{\cdot}_{X_s})$ is isomorphic to the crystalline cohomology $\mathrm{H}^m_{\mathrm{crys}}(X_s)$ of X_s.

On the other hand from (2.2) one gets a homomorphism

$$\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S}) \to \mathbf{H}^m(X, \Omega^{\cdot}_{X/S})$$

for every $m \geq 0$. In order to turn this effectively into a result on the interaction of Frobenius on the crystaline cohomology of the fibers X_s and the Gauss-Manin

*this paper was published in *Arithmetic Algebraic Geometry*, Van der Geer, Oort, Peters (eds), Progress in Mathematics 89, Boston: Birkhuser (1991) connection on the De Rham cohomology of X/S we must however replace our complexes by the projective systems of complexes

$$\begin{array}{ll} \{\underline{\mathcal{W}\Omega}^{\cdot}_X \bmod N\}_{N\in\mathbf{N}}\ , & \{\underline{\mathcal{W}\Omega}^{\cdot}_{X/S} \bmod N\}_{N\in\mathbf{N}}\ , \\ \{\Omega^{\cdot}_{X/S} \bmod N\}_{N\in\mathbf{N}}\ , & \{\mathcal{W}\Omega^{\cdot}_{X_s} \bmod N\}_{N\in\mathbf{N}}\ , \end{array}$$

indexed by the multiplicative monoid of the positive integers. This respects the relation with crystalline cohomology: see (5.9). The reason for looking at the complexes modulo N is that now the Frobenius homomorphism F_N induces a homomorphism

$$F_N^* : \mathrm{H}^m(X, \underline{\mathcal{W}\mathcal{O}}_X) \to \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_X \bmod N)$$

for every $m \geq 0$ (see (4.5)). This simple observation is the key to theorem (4.6) and its corollary (4.7) which states that matrices which come from the action of the Frobenius operator F_N on the generalized Witt vector cohomology group $\mathrm{H}^m(X, \underline{\mathcal{W}\mathcal{O}}_X)$ provide solutions to the differential equations taken modulo N associated with the Gauss-Manin connection acting on $\mathrm{H}^{r-m}(X, \Omega^r_{X/S})$. In a number of interesting examples these matrices can explicitly be calculated via Čech cocycles for generalized Witt vector cohomology (see [18] (5.6)).

A result of this type was first observed by Igusa for the Legendre family of elliptic curves $y^2 = x(x-1)(x-\lambda)$ over $S = \mathrm{Spec}\,\mathbf{Z}[\lambda, (2\lambda(1-\lambda))^{-1}]$ with $N = p$ an odd prime number. Manin [15] proved it for general smooth families of curves in characteristic $p > 0$. Katz ([11] prop. (2.3.6.3)) generalized it to higher dimensions, but still in characteristic $p > 0$. In [12] Katz reinterpreted Igusa's observation and greatly generalized it to congruence differential equations modulo arbitrary N for the coefficients of formal expansions of differential forms.

In [13] Katz used the expansion coefficients of differential forms to describe the *top slope quotient crystal* of $\mathbf{H}^m_{\mathrm{DR}}(X/S)$ for certain families of varieties X/S. In section 5 we prove a similar result for the *unit root sub-crystal*. One can imagine a kind of Hodge symmetry relating (5.6) and the main theorem of [13].

Our work on $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$ was in part motivated by the remark in [13] p.246 that the result of op. cit. might perhaps be considered as evidence for the existence of a theory of De Rham-Witt with parameters. A second motivation comes from the comparison isomorphism between crystalline cohomology in characteristic p and De Rham cohomology in characteristic 0 (see [1] (7.26.3)). There the De Rham side has no natural Frobenius and the crystalline side has no Hodge filtration. $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$ is an object with Frobenius and good filtrations of its own, which sees both $\mathrm{H}^*_{\mathrm{crys}}$ and $\mathbf{H}^*_{\mathrm{DR}}$; see (5.9) for a more explicit example of how this works out.

2 Construction of the generalized De Rham-Witt complex.

In this section we give the construction of the generalized De Rham-Witt complex $\underline{\mathcal{W}\Omega}^{\cdot}_X$ on a scheme X on which 2 is invertible. This is a Zariski sheaf of anti-commutative differential graded algebras with the additional structures and properties described in (2.1)–(2.6).

2.1 Let $\underline{\mathcal{W}\Omega}^i_X$ be the homogeneous component of degree i of $\underline{\mathcal{W}\Omega}^{\cdot}_X$. Then $\underline{\mathcal{W}\Omega}^i_X = 0$ for $i < 0$ and $\underline{\mathcal{W}\Omega}^0_X$ is the sheaf of generalized Witt vectors on X (cf. [3, 8, 2]). We shall usually write $\underline{\mathcal{W}\mathcal{O}}_X$ instead of $\underline{\mathcal{W}\Omega}^0_X$.

2.2 Let $\Omega^{\cdot}_X$ be the absolute De Rham complex on X i.e. the complex of differential forms relative to $\mathbf{Z}$. Put $\widetilde{\Omega}^i_X := \Omega^i_X/(i!\text{-torsion in } \Omega^i_X)$ and $\widetilde{\Omega}^{\cdot}_X := \bigoplus_{i\geq 0} \widetilde{\Omega}^i_X$. Then there is a homomorphism of sheaves of differential graded algebras

$$\pi : \underline{\mathcal{W}\Omega}^{\cdot}_X \to \widetilde{\Omega}^{\cdot}_X;$$

in degree 0 this is the projection onto the first Witt vector coordinate $\underline{\mathcal{W}\mathcal{O}}_X \to \mathcal{O}_X$.

2.3 For every integer $m \geq 1$ and every $i \geq 0$ there are homomorphisms of additive groups

$$F_m, V_m : \underline{\mathcal{W}\Omega}^i_X \to \underline{\mathcal{W}\Omega}^i_X$$

satisfying the following relations

$$\begin{array}{lll} F_m V_m = m, & F_m F_n = F_{mn}, & V_m V_n = V_{mn}, \\ V_m d = m d V_m, & dF_m = m F_m d, & F_m d V_m = d, \\ F_m(ab) = (F_m a)(F_m b), & V_m(a(F_m b)) = (V_m a)b, & \end{array}$$

for all m, n and for all sections a, b of $\underline{\mathcal{W}\Omega}^{\cdot}_X$, and

$$V_n F_m = F_m V_n \qquad \text{if} \qquad (n, m) = 1;$$

here $d : \underline{\mathcal{W}\Omega}^i_X \to \underline{\mathcal{W}\Omega}^{i+1}_X$ is the differential of the differential graded algebra $\underline{\mathcal{W}\Omega}^{\cdot}_X$. On the sheaf of generalized Witt vectors $\underline{\mathcal{W}\mathcal{O}}_X$ the operators F_m and V_m coincide with the usual Frobenius and Verschiebung operators (cf.[3, 8, 2]).

Obviously F_m does not commute with d. However one obtains an endomorphism $\underline{F}_m$ of the sheaf of differential graded algebras $\underline{\mathcal{W}\Omega}^{\cdot}_X$ by taking $\underline{F}_m = m^i F_m$ on $\underline{\mathcal{W}\Omega}^i_X$.

2.4 Let p be an odd prime number and let X be a scheme of characteristic p. Then there is an idempotent endomorphism E_p of the differential graded algebra $\underline{\mathcal{W}\Omega}^{\cdot}_X$ which projects $\underline{\mathcal{W}\Omega}^{\cdot}_X$ onto its p-typical part: $E_p \underline{\mathcal{W}\Omega}^{\cdot}_X = \cap\{\ker F_m \mid m \text{ prime} \neq p\}$. So $E_p \underline{\mathcal{W}\Omega}^{\cdot}_X$ is a sheaf of anti-commutative differential graded algebras. Its component in degree 0 is the sheaf $\mathcal{W}\mathcal{O}_X$ of p-typical Witt vectors on X. Since E_p commutes with F_p and V_p, the operators F_p and V_p act on $E_p \underline{\mathcal{W}\Omega}^{\cdot}_X$ and here in characteristic p they commute: $V_p F_p = F_p V_p = p$. There is an isomorphism of sheaves of differential graded algebras

$$\underline{\mathcal{W}\Omega}^{\cdot}_X \simeq (E_p \underline{\mathcal{W}\Omega}^{\cdot}_X)^{\mathbf{N} \setminus p\mathbf{N}},$$

where the right hand side is the product of copies of $E_p \underline{\mathcal{W}\Omega}^{\cdot}_X$ indexed by the set of positive integers prime to p. Let $\mathcal{W}\Omega^{\cdot}_X$ be the De Rham-Witt complex on

X constructed by Deligne and Illusie [9]. There is a surjective homomorphism of sheaves of differential graded algebras from $\mathcal{W}\Omega^{\cdot}_X$ onto $E_p\underline{\mathcal{W}\Omega}^{\cdot}_X$ compatible with the operators F_p and V_p on both sides. If X is a smooth scheme over a perfect field of characteristic p this is an isomorphism:

$$\mathcal{W}\Omega^{\cdot}_X \simeq E_p\underline{\mathcal{W}\Omega}^{\cdot}_X.$$

2.5 Let X be a scheme over $\mathbf{Q}$. Then there is an idempotent endomorphism E_0 of the differential graded algebra $\underline{\mathcal{W}\Omega}^{\cdot}_X$ with image $E_0\underline{\mathcal{W}\Omega}^{\cdot}_X = \bigcap_{m>1} \ker F_m$. There are isomorphisms of sheaves of differential graded algebras

$$E_0\underline{\mathcal{W}\Omega}^{\cdot}_X \simeq \Omega^{\cdot}_X, \qquad \underline{\mathcal{W}\Omega}^{\cdot}_X \simeq (\Omega^{\cdot}_X)^{\mathbf{N}} \;.$$

2.6 The constructions are functorial: let $f : Y \to X$ be a morphism of schemes over $\mathbf{Z}[\frac{1}{2}]$. Then there is a homomorphism $\underline{\mathcal{W}\Omega}^{\cdot}_X \to f_*\underline{\mathcal{W}\Omega}^{\cdot}_Y$ of sheaves of differential graded algebras on X compatible with the operators F_m and V_m on both sides. On (hyper-) cohomology groups this induces homomorphisms like

$$\mathrm{H}^n(X, \underline{\mathcal{W}\Omega}^i_X) \to \mathrm{H}^n(Y, \underline{\mathcal{W}\Omega}^i_Y), \qquad \mathbf{H}^n(X, \underline{\mathcal{W}\Omega}^{\cdot}_X) \to \mathbf{H}^n(Y, \underline{\mathcal{W}\Omega}^{\cdot}_Y).$$

The construction of the generalized De Rham-Witt complex with its additional structures is essentially given in [19]. However in op. cit. it is specialized to characteristic p situations too early for our present purpose. Therefore we shall briefly recall the constructions in in such a way that the general statements in (2.1) –(2.6) become completely justified.

2.7 Let A be a commutative ring with 1. In [19] K-theory is used to construct an anti-commutative graded ring with 1

$$\widetilde{K}_*(\mathrm{End}(A)) = \bigoplus_{i\geq 0} \widetilde{K}_i(\mathrm{End}(A))$$

equipped with homomorphisms for the additive structure

$$F_m, V_m : \widetilde{K}_i(\mathrm{End}(A)) \to \widetilde{K}_i(\mathrm{End}(A)), \qquad i \geq 0,$$

for every positive integer m, and with a derivation

$$d : \widetilde{K}_i(\mathrm{End}(A)) \to \widetilde{K}_{i+1}(\mathrm{End}(A)), \qquad i \geq 0.$$

The relations listed in (2.3) hold also for F_m, V_m and d on $\widetilde{K}_*(\mathrm{End}(A))$ except for $F_m dV_m = d$ which here only holds for odd m, and for $d^2 = 0$ which here is weakened to $2d^2 = 0$ (see [19] theorem (1.8)).

2.8 In [20] the graded ring $\widetilde{K}_*(\mathrm{End}(A))$ is equipped with a decreasing filtration by homogeneous ideals $\{\mathrm{Fil}_n\widetilde{K}_*(\mathrm{End}(A))\}_{n\geq 1}$ with $\mathrm{Fil}_1\widetilde{K}_*(\mathrm{End}(A)) = \widetilde{K}_*(\mathrm{End}(A))$. Define

$$\widetilde{K}_*(\mathrm{End}(A))^c := \lim_{\leftarrow n}[\widetilde{K}_*(\mathrm{End}(A))/\mathrm{Fil}_n\widetilde{K}_*(\mathrm{End}(A))]$$

with the topology of a projective limit of discrete spaces. From the proof of the proposition in [20] one gets

$$F_m(\mathrm{Fil}_{mn}) \subset \mathrm{Fil}_n, \qquad V_m(\mathrm{Fil}_n) \subset \mathrm{Fil}_{mn}, \qquad d(\mathrm{Fil}_{qn}) \subset \mathrm{Fil}_n$$

for all m, $n \geq 1$ and for some positive integer q independent of n and A. Thus F_m, V_m and d extend to continuous operators on $\widetilde{K}_*(\mathrm{End}(A))^c$ satisfying the same relations as in (2.7). Moreover, $\widetilde{K}_0(\mathrm{End}(A))^c$ is canonically isomorphic with the ring of generalized Witt vectors over A ([20] p.220).

2.9 If 2 is invertible in A it is also invertible in $\widetilde{K}_0(\mathrm{End}(A))^c$. Therefore all relations in (2.3)(see also (2.7)) are valid for the operators acting on $\widetilde{K}_*(\mathrm{End}(A))^c$.

2.10 From now on we assume that 2 is invertible in A.

Definition $\underline{\mathcal{W}\Omega}^{\cdot}_A$:= closure of the graded subring of $\widetilde{K}_*(\mathrm{End}(A))^c$ generated by $\widetilde{K}_0(\mathrm{End}(A))^c$ and $d\widetilde{K}_0(\mathrm{End}(A))^c$.

2.11 In particular, $\underline{\mathcal{W}\Omega}^0_A$ is the ring of generalized Witt vectors over A. Its additive group is isomorphic with the multiplicative group $(1+tA[[t]])^*$. Every element of the latter group can be written uniquely in the form $\prod_{n\geq 1}(1-a_nt^n)^{-1}$ with all $a_n \in A$. The elements of $\underline{\mathcal{W}\Omega}^0_A$ can therefore be written uniquely as $\sum_{n\geq 1} V_n\underline{\underline{a_n}}$, where $\underline{\underline{a}}$ is the Witt vector which corresponds to the power series $(1-at)^{-1}$.

2.12 Proposition *For all* $i \geq 0$ *and* $m \geq 1$ *one has*

$$d\underline{\mathcal{W}\Omega}^i_A \subset \underline{\mathcal{W}\Omega}^{i+1}_A, \qquad V_m\underline{\mathcal{W}\Omega}^i_A \subset \underline{\mathcal{W}\Omega}^i_A, \qquad F_m\underline{\mathcal{W}\Omega}^i_A \subset \underline{\mathcal{W}\Omega}^i_A.$$

Proof The results for d and V_m follow immediately from the relations in (2.3). The problem for F_m is easily reduced to showing $F_mdV_n\underline{\underline{a}} \in \underline{\mathcal{W}\Omega}^1_A$ for all m,$n \geq 1$ and $a \in A$; the Witt vector $\underline{\underline{a}}$ is defined in (2.11). In view of the relations in (2.3) we may even assume $(m,n) = 1$. Choose integers q and r such that $qn+rm = 1$. Then

$$F_mdV_n\underline{\underline{a}} = qV_nF_md\underline{\underline{a}} + rdF_mV_n\underline{\underline{a}}.$$

Formula (8.3.3) in [19] shows

$$F_md\underline{\underline{a}} = \underline{\underline{a}}^{m-1}d\underline{\underline{a}}$$

Thus we find

$$F_mdV_n\underline{\underline{a}} = q(V_n\underline{\underline{a}}^{m-1})d(V_n\underline{\underline{a}}) + rdF_mV_n\underline{\underline{a}} \in \underline{\mathcal{W}\Omega}^1_A \tag{1}$$

□

2.13 The preceding construction depends functorially on A: if $g : A \to B$ is a homomorphism of $\mathbf{Z}[\frac{1}{2}]$-algebras, then there is a continuous homomorphism of graded topological rings

$$g_* : \underline{\mathcal{W}\Omega}^{\cdot}_A \to \underline{\mathcal{W}\Omega}^{\cdot}_B$$

which commutes with the operators d, F_m and V_m $(m \geq 1)$. It sends the Witt vector $\underline{\underline{a}} \in \underline{\mathcal{W}\Omega}^0_A$ to $\underline{\underline{g(a)}} \in \underline{\mathcal{W}\Omega}^0_B$.

2.14 We get on every scheme X over $\mathbf{Z}[\frac{1}{2}]$ a pre-sheaf for the Zariski topology

$$(\text{Zariski open } U \subset X) \mapsto \underline{\mathcal{W}\Omega}^{\cdot}_{\Gamma(U,\mathcal{O}_X)}.$$

We define

$$\underline{\mathcal{W}\Omega}^{\cdot}_X := \text{ the sheaf associated with the above pre-sheaf.}$$

and call this the *generalized De Rham-Witt complex* of X. We shall usually write $\underline{\mathcal{W}\mathcal{O}}_X$ instead of $\underline{\mathcal{W}\Omega}^0_X$.

This completes the construction of $\underline{\mathcal{W}\Omega}^{\cdot}_X$ and of the operators d, F_m and V_m $(m \geq 1)$ acting on it. It follows from (2.7) and (2.9) that $\underline{\mathcal{W}\Omega}^{\cdot}_X$ with d is a sheaf of anti-commutative differential graded algebras and that the relations in (2.3) hold. Moreover (2.8) proves (2.1); in particular, $\underline{\mathcal{W}\mathcal{O}}_X$ is the sheaf of generalized Witt vectors on X. The functoriality property in (2.6) is a consequence of (2.13).

We now turn to the construction of the homomorphism π in (2.2).

2.15 Let A be a commutative ring with 1 and 2 invertible in A. By [19](3.4) (see also [20]) we have a bilinear pairing

$$\langle , \rangle : \widetilde{K}_i(\text{End}(A))^c \times \widetilde{K}_0(\text{Nil}(\mathbf{Z}[t]/(t^2))) \to K_{i+1}(A[t]/(t^2)).$$

From this we get in particular a homomorphism

$$\langle , \underline{\underline{t}} \rangle : \widetilde{K}_i(\text{End}(A))^c \to K_{i+1}(A[t]/(t^2))$$

where $\underline{\underline{t}}$ is the element of $\widetilde{K}_0(\text{Nil}(\mathbf{Z}[t]/(t^2)))$ defined in [19](5.2). On the other hand one has Gersten's map (see [2] p.206 (3.2),(3.3))

$$\text{dlog } : K_{i+1}(A[t]/(t^2)) \to \Omega^{i+1}_{A[t]/(t^2)};$$

here we work with differential forms relative to $\mathbf{Z}$. Let

$$\rho : \underline{\mathcal{W}\Omega}^i_A \to \Omega^{i+1}_{A[t]/(t^2)}$$

be the composite $\text{dlog}\langle , \underline{\underline{t}} \rangle$ restricted to $\underline{\mathcal{W}\Omega}^i_A$.

2.16 The group $\underline{\mathcal{W}\Omega}^i_A$ is topologically generated by the elements

$$(V_{n_0}\underline{\underline{a_0}})d(V_{n_1}\underline{\underline{a_1}}) \cdots d(V_{n_i}\underline{\underline{a_i}})$$

with $n_0, \ldots, n_i \in \mathbf{N}, a_0, \ldots, a_i \in A$ (cf.(2.11)).

Lemma *In the above situation let* $\alpha = (V_{n_0}\underline{\underline{a_0}})d(V_{n_1}\underline{\underline{a_1}})\cdots d(V_{n_i}\underline{\underline{a_i}})$. *If all* $n_j = 1$ *then*

$$\rho(\alpha) = (-1)^i i! d(-ta_0 da_1 \wedge \cdots \wedge da_i).$$

Otherwise $\rho(\alpha) = 0$.

Proof Let $n = \max(n_0, \ldots, n_i)$. Assume first $n \geq 2$. Using (2.3) one easily rewrites α in the form $\alpha = V_n\beta + dV_n\gamma$ with $\beta \in \underline{\mathcal{W}\Omega}^i_A, \gamma \in \underline{\mathcal{W}\Omega}^{i-1}_A$. From [19](3.2) one gets $\langle \alpha, \underline{\underline{t}} \rangle = \langle \beta, F_n\underline{\underline{t}} \rangle + (-1)^i \langle \gamma, F_n d\underline{\underline{t}} \rangle$. Loc.cit. (1.6) and (5.2) show $F_n\underline{\underline{t}} = \underline{\underline{t}}^n = 0$. Loc.cit. (8.3.3) yields $F_n d\underline{\underline{t}} = \underline{\underline{t}}^{n-1} d\underline{\underline{t}} = 0$ for $n \geq 3$. For $n = 2$ we compute $F_2 d\underline{\underline{t}} = 2^{-1} dF_2\underline{\underline{t}} = 0$. This proves $\rho(\alpha) = 0$ if $n \geq 2$.
The formula for $\rho(\alpha)$ in case $n = 1$ follows from [19](7.6) and [2] p.206 (3.3); more precisely the argument is as follows. By functoriality it suffices to prove the formula for the case that $a_0, \ldots, a_i$ are the indeterminates in the polynomial ring $P := \mathbf{Z}[\frac{1}{2}][a_0, \ldots, a_i]$. Set $Q := P[a_0^{-1}, \ldots, a_i^{-1}]$. Using the injectivity of the natural homomorphism

$$\Omega^{i+1}_{P[t]/(t^2)} \to \Omega^{i+1}_{Q[t]/(t^2)}$$

and functoriality we see that it suffices to prove the formula with $a_0, \ldots, a_i$ in Q. Then [19](3.1) and the proof of [19](7.6) give

$$\langle \alpha, \underline{\underline{t}} \rangle = \{1 - ta_0 a_1 \cdots a_i, a_1, \ldots, a_i\}.$$

The right hand side is a Steinberg symbol in $K_{i+1}(Q[t]/(t^2))$. Applying [2] p.206 (3.3) to compute the dlog of this Steinberg symbol we find

$$\rho(\alpha) = (-1)^i i! (1 - ta_0 a_1 \cdots a_i)^{-1} d(-ta_0) \wedge da_1 \wedge \cdots \wedge da_i$$

This is equal to $(-1)^i i! d(-ta_0 da_1 \wedge \cdots \wedge da_i)$ because $t^2 = 0$ and $\frac{1}{2} \in Q$. □

2.17 Lemma *Let* A *be as in* (2.15). *Define*

$$\psi : \Omega^i_A \to \Omega^{i+1}_{A[t]/(t^2)}, \qquad \psi(\eta) = (-1)^i i! d(-t\eta).$$

Then

$$\ker \psi = (i! - \text{torsion in } \Omega^i_A) := \ker(i! : \Omega^i_A \to \Omega^i_A).$$

Proof Consider the map $\Omega^1_{A[t]/(t^2)} \to \Omega^1_A \oplus \Omega^1_{A[t]/(t^2)/A}$ which is the direct sum of the map induced by $t \mapsto 0$ and the map taking differentials relative to A. Its $(i+1)$-fold exterior power over $A[t]/(t^2)$ is a map $\Omega^{i+1}_{A[t]/(t^2)} \to \Omega^{i+1}_A \oplus \Omega^i_A dt$ which sends $d(t\eta)$ to $(-1)^i \eta dt$. The lemma is now clear. □

2.18 Define $\widetilde{\Omega}^i_A := \Omega^i_A/(i!\text{-torsion in } \Omega^i_A)$. Then the map ψ from (2.17) induces an isomorphism $\overline{\psi} : \widetilde{\Omega}^i_A \xrightarrow{\sim} \text{image } \psi$. From (2.16) one sees that the image of ρ is contained in the image of ψ. So we can compose ρ with $\overline{\psi}^{-1}$. Define

$$\pi_i := \overline{\psi}^{-1}\rho : \underline{\mathcal{W}\Omega}^i_A \to \widetilde{\Omega}^i_A.$$

2.19 From (2.16) one obtains explicit formulas for π_i:

$$\pi_i((V_{n_0}\underline{\underline{a_0}})d(V_{n_1}\underline{\underline{a_1}})\cdots d(V_{n_i}\underline{\underline{a_i}})) = 0 \text{ if some } n_j \neq 1$$

$$\pi_i(\underline{\underline{a_0}}d\underline{\underline{a_1}}\cdots d\underline{\underline{a_i}}) = a_0 da_1 \wedge \cdots \wedge da_i.$$

These formulas show that π_i is surjective. They also show that the direct sum π of the maps π_i is a homomorphism of differential graded algebras $\pi : \underline{\mathcal{W}\Omega}^\cdot_A \to \widetilde{\Omega}^\cdot_A$.

2.20 Let X be a scheme over $\mathbf{Z}[\frac{1}{2}]$. Then sheafification of the above construction provides the homomorphism of sheaves of differential graded algebras on X

$$\pi : \underline{\mathcal{W}\Omega}^\cdot_X \to \widetilde{\Omega}^\cdot_X$$

for (2.2).

2.21 Let P be a set of prime numbers and let X be a scheme such that every prime number in P is invertible in $\mathcal{O}_X$. Then every $l \in P$ is also invertible in $\underline{\mathcal{W}\mathcal{O}}_X$. Moreover with notations as in (2.8) we have $F_l\mathrm{Fil}_1 \subset \mathrm{Fil}_1$ and $V_l\mathrm{Fil}_1 \subset \mathrm{Fil}_l$ for every l. Therefore the expression

$$E^P := \prod_{l\in P}(1 - l^{-1}V_lF_l)$$

defines an operator on $\underline{\mathcal{W}\Omega}^\cdot_X$. One easily checks that it is an idempotent operator, that it commutes with d, V_p and F_p for all primes $p \notin P$ and that $E^P(ab) = (E^Pa)(E^Pb)$ for all sections a,b of $\underline{\mathcal{W}\Omega}^\cdot_X$. Furthermore it is clear that for every $l \in P$ the image of E^P is contained in $\ker F_l$ and that E^P is the identity on $\ker F_l$. Consequently

$$E^P\underline{\mathcal{W}\Omega}^\cdot_X = \bigcap_{l\in P} \ker F_l.$$

Let $\overline{P} \subset \mathbf{N}$ be the multiplicatively closed subset with 1 generated by P. Then there is an isomorphism of sheaves of differential graded algebras

$$\underline{\mathcal{W}\Omega}^\cdot_X \simeq (E^P\underline{\mathcal{W}\Omega}^\cdot_X)^{\overline{P}};$$

on homogeneous sections of degree i the map $\to$ sends a to $(m^iE^PF_ma)_{m\in\overline{P}}$ and the map $\leftarrow$ sends $(b_m)_{m\in\overline{P}}$ to $\sum_{m\in\overline{P}} m^{-i-1}V_mb_m$. All this is an easy consequence of the relations in (2.3). We apply it in the situation of (2.5) (resp. (2.4)) with P the set of all primes (resp. all primes $\neq p$) and write E_0 (resp. E_p) for E^P. The results relating in (2.4) $E_p\underline{\mathcal{W}\Omega}^\cdot_X$ to the De Rham-Witt complex of Deligne and Illusie are proved in [19] section 8. The isomorphism

$$E_0\underline{\mathcal{W}\Omega}^\cdot_X \simeq \Omega^\cdot_X$$

in (2.5) is proved as follows. Note that $E_0 V_m = 0$ for all $m \geq 2$. Combining this with the definition of the ring structure on generalized Witt vectors one sees that there is a ring homomorphism

$$\lambda : \mathcal{O}_X \to E_0 \underline{\mathcal{W}\mathcal{O}}_X$$

which on sections is defined by $\lambda(a) = E_0 \underline{\underline{a}}$ (see (2.11) for $\underline{\underline{a}}$). Because of the universal property of $\Omega^{\cdot}_X$ this homomorphism from $\mathcal{O}_X$ into the degree 0 component of the differential graded algebra $E_0 \underline{\mathcal{W}\Omega}^{\cdot}_X$ extends uniquely to a homomorphism of differential graded algebras

$$\lambda : \Omega^{\cdot}_X \to E_0 \underline{\mathcal{W}\Omega}^{\cdot}_X.$$

This homomorphism is surjective because $\underline{\mathcal{W}\Omega}^{\cdot}_X$ is topologically generated by the sections described in (2.16) and because $E_0 V_m = 0$ for $m \geq 2$. A simple computation shows that $\pi\lambda$ is the identity map on $\Omega^{\cdot}_X$, where π is the homomorphism π from (2.2) restricted to the image of E_0. This proves that π induces an isomorphism $E_0 \underline{\mathcal{W}\Omega}^{\cdot}_X \simeq \Omega^{\cdot}_X$.

3 The relative generalized De Rham-Witt complex.

3.1 Let $f : X \to S$ be a morphism of schemes over $\mathbf{Z}[\frac{1}{2}]$. We define the relative generalized De Rham-Witt complex $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$ on X to be the quotient of $\underline{\mathcal{W}\Omega}^{\cdot}_X$ by the closure of the ideal generated by $d(f^{-1}\underline{\mathcal{W}\mathcal{O}}_S)$. It is clear that $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$ is a sheaf of anti-commutative differential graded algebras with $\underline{\mathcal{W}\Omega}^0_{X/S} = \underline{\mathcal{W}\mathcal{O}}_X$. The homomorphism $\pi : \underline{\mathcal{W}\Omega}^{\cdot}_X \to \widetilde{\Omega}^{\cdot}_X$ induces a homomorphism

$$\pi : \underline{\mathcal{W}\Omega}^{\cdot}_{X/S} \to \widetilde{\Omega}^{\cdot}_{X/S},$$

where $\Omega^{\cdot}_{X/S}$ is the usual relative De Rham complex of X/S and $\widetilde{\Omega}^i_{X/S} := \Omega^i_{X/S}/(i!-\text{torsion})$. Using the relations in (2.3) and formula (1) in (2.12) one easily checks that the operators F_m and V_m on $\underline{\mathcal{W}\Omega}^{\cdot}_X$ map the ideal $(d(f^{-1}\underline{\mathcal{W}\mathcal{O}}_S)) \cdot \underline{\mathcal{W}\Omega}^{\cdot}_X$ into itself and thus induce operators F_m and V_m on $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$. The relations in (2.3) pass without change to $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S}$. Notice also the analogue of the functoriality property (2.6): a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

leads naturally to a homomorphism $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S} \to g_* \underline{\mathcal{W}\Omega}^{\cdot}_{Y/T}$.

3.2 Suppose S is the spectrum of a perfect field of characteristic $p > 2$. Then F_p is surjective on $\underline{\mathcal{W}\mathcal{O}}_S$. Because of $d\, F_p^r = p^r\, F_p^r\, d = V_p^r\, F_p^{2r}\, d$ in characteristic p and $V_p^r \,\mathrm{Fil}_1 \subset \mathrm{Fil}_{p^r}$ (see (2.8)), the subsheaf $d(f^{-1}\underline{\mathcal{W}\mathcal{O}}_S)$ of $\underline{\mathcal{W}\Omega}^{\cdot}_X$ is zero. So $\underline{\mathcal{W}\Omega}^{\cdot}_{X/S} = \underline{\mathcal{W}\Omega}^{\cdot}_X$ if S is the spectrum of a perfect field of characteristic $p > 2$.

4 Congruence differential equations.

4.1 Let $S = \operatorname{Spec} A$ be an affine scheme which is smooth over an open part of $\operatorname{Spec} \mathbf{Z}[\frac{1}{2}]$. Let $f : X \to S$ be a projective smooth morphism of relative dimension r. We assume that all Hodge cohomology groups $\mathrm{H}^j(X, \Omega^i_{X/S})$ are free A-modules and $\mathrm{H}^r(X, \Omega^r_{X/S}) \simeq A$.

4.2 These hypotheses imply that the Hodge-De Rham spectral sequence $\mathrm{E}_1^{ij} := \mathrm{H}^j(X, \Omega^i_{X/S}) \Rightarrow \mathbf{H}^{i+j}(X, \Omega^{\cdot}_{X/S})$ degenerates at E_1 (note that A is flat over $\mathbf{Z}$ and use [4] th.(5.1)). So in particular all De Rham cohomology groups $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S})$ are also free A-modules and $\mathbf{H}^{2r}(X, \Omega^{\cdot}_{X/S}) \simeq A$. Moreover the homomorphism

$$\beta : \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \to \mathrm{H}^m(X, \mathcal{O}_X),$$

induced by the projection of the complex $\Omega^{\cdot}_{X/S}$ onto its degree 0 component $\mathcal{O}_X$, is surjective and the homomorphism

$$\mathrm{H}^{r-m}(X, \Omega^r_{X/S}) \to \mathbf{H}^{2r-m}(X, \Omega^{\cdot}_{X/S}),$$

induced by the inclusion of $\Omega^r_{X/S}$ as degree r component into $\Omega^{\cdot}_{X/S}$, is injective for every $m \geq 0$. One has a perfect pairing

$$\langle , \rangle : \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \times \mathbf{H}^{2r-m}(X, \Omega^{\cdot}_{X/S}) \to \mathbf{H}^{2r}(X, \Omega^{\cdot}_{X/S}) \simeq A$$

which induces the duality

$$\mathrm{H}^{r-m}(X, \Omega^r_{X/S}) = \mathrm{H}^m(X, \mathcal{O}_X)^{\vee}.$$

4.3 Recall the Katz-Oda construction of the Gauss-Manin connection [14, 11]. The Koszul filtration $\{K^{i\cdot}\}_{i \geq 0}$ on the absolute De Rham complex $\Omega^{\cdot}_X$ is defined by

$$K^{i\cdot} := \mathrm{image}(f^*\Omega^i_S \otimes \Omega^{\cdot - i}_X \to \Omega^{\cdot}_X).$$

It satisfies

$$K^{0\cdot}/K^{1\cdot} \simeq \Omega^{\cdot}_{X/S}, \qquad K^{1\cdot}/K^{2\cdot} \simeq f^*\Omega^1_S \otimes \Omega^{\cdot -1}_{X/S}.$$

The boundary map in the long exact hypercohomology sequence associated with the exact sequence of complexes $0 \to K^{1\cdot}/K^{2\cdot} \to K^{0\cdot}/K^{2\cdot} \to K^{0\cdot}/K^{1\cdot} \to 0$ yields the Gauss-Manin connection

$$\nabla : \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \to \Omega^1_S \otimes \mathbf{H}^m(X, \Omega^{\cdot}_{X/S})$$

for $m \geq 0$.

These constructions work equally well if we take the complexes modulo a positive integer N. They then provide the Gauss-Manin connection ∇ on $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S} \bmod N)$ and show that the image of $\mathbf{H}^m(X, \Omega^{\cdot}_X \bmod N)$ lies in the kernel of ∇.

4.4 Let Diff_S be the algebra of differential operators on A relative to $\mathbf{Z}$ and let Diff'_S be the subalgebra of Diff_S generated by the derivations of A (cf. [7] (16.11)). The Gauss-Manin connection defines a Lie algebra homomorphism $\nabla : \mathrm{Der}A \to \mathrm{End}_{\mathbf{Z}}\,(\mathbf{H}^*(X, \Omega^{\cdot}_{X/S}))$ so that $\nabla(D)$ is the composite of ∇ with $D \otimes 1$. This Lie algebra homomorphism extends to an algebra homomorphism

$$\nabla : \mathrm{Diff}'_S \to \mathrm{End}_{\mathbf{Z}}\,(\mathbf{H}^*(X, \Omega^{\cdot}_{X/S})).$$

4.5 Fix a positive integer N. Because of the relation $dF_N = NF_N d$ one can extend the homomorphism $F_N : \underline{\mathcal{WO}}_X \to \underline{\mathcal{WO}}_X \bmod N$ to a homomorphism of complexes

$$F'_N : \underline{\mathcal{WO}}_X \to \underline{\mathcal{W}\Omega}^{\cdot}_X \bmod N$$

where $\underline{\mathcal{WO}}_X$ is viewed as a complex concentrated in degree 0. This leads to a homomorphism

$$F^*_N : \mathrm{H}^m(X, \underline{\mathcal{WO}}_X) \to \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_X \bmod N)$$

for every $m \geq 0$. One has the following commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^m(X, \underline{\mathcal{WO}}_X) & \xrightarrow{F^*_N} & \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_X \bmod N) \\
F_N \downarrow & & \downarrow \tau_N \\
\mathrm{H}^m(X, \underline{\mathcal{WO}}_X) & & \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \bmod N \\
\pi \downarrow & & \downarrow \beta \\
\mathrm{H}^m(X, \mathcal{O}_X) & \longrightarrow & \mathrm{H}^m(X, \mathcal{O}_X) \bmod N
\end{array}$$

where τ_N is induced by $\underline{\mathcal{W}\Omega}^{\cdot}_X \bmod N \to \Omega^{\cdot}_X \bmod N \to \Omega^{\cdot}_{X/S} \bmod N$; notice that there is no $i!$-torsion in Ω^i_X and $\Omega^i_{X/S}$ because X is smooth over a subring of $\mathbf{Q}$.

4.6 Theorem *Let $f : X \to S$ be as in (4.1). Fix an integer $m \geq 0$. Take a basis $\{\omega_1, \ldots, \omega_h\}$ of $\mathrm{H}^m(X, \mathcal{O}_X)$. Let $\{\breve{\omega}_1, \ldots, \breve{\omega}_h\}$ be the dual basis of $\mathrm{H}^{r-m}(X, \Omega^r_{X/S})$. Take $\xi \in \mathrm{H}^m(X, \underline{\mathcal{WO}}_X)$ and define for every positive integer N $B_{N,1}, \ldots, B_{N,h} \in A$ by*

$$\pi F_N\, \xi = \sum_{j=1}^{h} B_{N,j}\, \omega_j$$

Suppose $P_1, \ldots, P_h \in \mathrm{Diff}'_S$ are such that

$$\nabla(P_1)\, \breve{\omega}_1 + \cdots + \nabla(P_h)\, \breve{\omega}_h = 0 \qquad \textit{in} \quad \mathbf{H}^{2r-m}(X, \Omega^{\cdot}_{X/S}), \qquad (2)$$

then

$$P_1\, B_{N,1} + \cdots + P_h\, B_{N,h} \equiv 0 \bmod N.$$

for all $N \in \mathbf{N}$.

Proof From (4.5) one deduces for all j

$$\langle \tau_N F_N^* \xi, \breve{\omega}_j \rangle \equiv B_{N,j} \bmod N.$$

The map τ_N factors via $\mathbf{H}^m(X, \Omega^{\cdot}_X \bmod N)$. Therefore the image of τ_N in $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S} \bmod N)$ is contained in the kernel of the Gauss-Manin connection. So for every derivation D of A we have

$$\nabla(D)(\tau_N F_N^* \xi) = 0 \qquad \text{in} \quad \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \bmod N.$$

In view of the compatibility of $\langle , \rangle$ and ∇ ([14] th.1) we find for all $D \in \mathrm{Der}A$ and all j

$$DB_{N,j} \equiv \langle \tau_N F_N^* \xi, \nabla(D) \breve{\omega}_j \rangle \bmod N.$$

The theorem is now obvious. □

4.7 In [18](2.6) it is shown that the hypotheses in (4.1) imply that the map $\pi : \mathrm{H}^m(X, \underline{\mathcal{WO}}_X) \to \mathrm{H}^m(X, \mathcal{O}_X)$ is surjective. So there are elements $\tilde{\omega}_1, \ldots, \tilde{\omega}_h \in \mathrm{H}^m(X, \underline{\mathcal{WO}}_X)$ such that $\pi\, \tilde{\omega}_i = \omega_i$ for $i = 1, \ldots, h$. Define for $N \in \mathbf{N}$ the $h \times h$-matrix B_N with entries in A by

$$\pi F_N \, \underline{\tilde{\omega}} = B_N \, \underline{\omega}$$

where $\underline{\omega}$ resp. $\underline{\tilde{\omega}}$ is the column vector with components $\omega_1, \ldots, \omega_h$ resp. $\tilde{\omega}_1, \ldots, \tilde{\omega}_h$. For a prime number p the matrix $B_p \bmod p$ is known as the *Hasse-Witt matrix* of $X \otimes \mathbf{F}_p$ in degree m (cf. [11] p.27).

Corollary *The congruence differential equations in theorem* (4.6) *are valid for the rows of the matrices* B_N.

4.8 *Example* The Gauss-Manin connection makes $\mathbf{H}^{2r-m}(X, \Omega^{\cdot}_{X/S})$ a module over the algebra Diff'_S (a so-called $\mathcal{D}$-module [17]). The full set of differential equations (2) (or a generating subset thereof) gives a presentation for the sub-Diff'_S-module generated by $\mathrm{H}^{r-m}(X, \Omega^r_{X/S})$. In practice in explicit examples one finds these differential equations as Picard-Fuchs equations for the periods of regular differential forms.

For explicit examples based on families of curves of the form

$$y^n = x^a(x-1)^b(x-\lambda)^c$$

with $n, a, b, c \in \mathbf{N}$, $(n, a, b, c) = 1$, $a, b, c < n$ and connected with hypergeometric differential equations we refer to section 5 of [18]; there one also finds a full detail example illustrating (4.7).

Further explicit examples of Picard-Fuchs equations can be found in ([6] p. 73-76) for the 1-parameter family of elliptic curves

$$X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0$$

and for the 1-parameter family of K3 surfaces

$$W^4 + X^4 + Y^4 + Z^4 - 4\lambda WXYZ = 0$$

and in [16] for the 2-parameter family of K3 surfaces

$$w^2 = xy(1-x)(1-y)(1-\lambda x - \mu y).$$

For these examples the matrices B_N (see (4.7)) can be calculated with the method of ([18] (5.6)).

5 Reconstruction of the unit root crystal.

In this section we prove theorem (5.6). This theorem shows great similarities with the main theorem of [13]. The two theorems seem related by a kind of Hodge symmetry. The actual congruences in our theorem look however weaker than the congruences in Katz's theorem. I do not yet understand this phenomenon.

5.1 We keep the situation and assumptions of (4.1). Fix an integer $m \geq 0$ and a prime number $p > 2$. We assume condition HW(m) of [13]:

hypothesis: For every point $\mathrm{Spec}\, k \to S$ with k a perfect field of characteristic p the Frobenius endomorphism F_p on $\mathrm{H}^m(X \otimes k, \mathcal{O}_{X\otimes k})$ is bijective.

Now fix a basis $\omega_1, \ldots, \omega_h$ of $\mathrm{H}^m(X, \mathcal{O}_X)$. Take elements $\tilde{\omega}_1, \ldots, \tilde{\omega}_h$ in $\mathrm{H}^m(X, \underline{\mathcal{WO}}_X)$ such that $\pi\tilde{\omega}_i = \omega_i$ for $i = 1, \ldots, h$ and define the matrices B_N as in (4.7). Then by [18] (4.2) the above hypothesis is equivalent with

hypothesis: The Hasse-Witt matrix B_p mod p is invertible over the ring A/pA.

5.2 Set

$$\begin{array}{lll} A_n = A/p^nA, & S_n = \mathrm{Spec}\, A_n, & X_n = X \otimes A_n, \\ A_\infty = \lim\limits_{\leftarrow n} A_n, & S_\infty = \mathrm{Spec}\, A_\infty, & X_\infty = X \otimes A_\infty. \end{array}$$

Since the ring A_∞ is formally smooth over $\mathbf{Z}_p$ it carries an endomorphism σ such that for all $a \in A_\infty$

$$\sigma(a) \equiv a^p \bmod pA_\infty.$$

In general there are many endomorphisms with this property. Given one choice for σ there is a unique homomorphism of rings

$$\lambda : A_\infty \to W(A_\infty)$$

into the ring $W(A_\infty)$ of p-typical Witt vectors over A_∞, such that $\pi\, F_p^n\, \lambda = \sigma^n$ for all $n \in \mathbf{N}$; in particular $\pi\lambda = id$ [8](17.6.9).

In the sequel we will often write a^σ instead of $\sigma(a)$. For a matrix $M = (m_{ij})$ with entries in A_∞ we set $M^{\sigma^n} = (m_{ij}^{\sigma^n}),\ \lambda(M) = (\lambda(m_{ij}))$.

5.3 In [18] theorem (3.4) it is shown that under the hypotheses of (5.1) there exist an invertible $h \times h$-matrix H with entries in A_∞ and elements $\hat{\omega}_1, \ldots, \hat{\omega}_h$ in $\mathrm{H}^m(X_\infty, \underline{\mathcal{WO}}_{X_\infty})$ such that

$$\begin{array}{l} B_{p^{n+1}} \equiv B_{p^n}^\sigma\, H \bmod p^{n+1} \quad \text{for all} \quad n \geq 0, \\ \pi\hat{\omega}_i = \omega_i \quad \text{in} \quad \mathrm{H}^m(X_\infty, \mathcal{O}_{X_\infty}) = \mathrm{H}^m(X, \mathcal{O}_X) \otimes A_\infty, \\ F_p\, \underline{\hat{\omega}} = \lambda(H)\, \underline{\hat{\omega}}, \end{array}$$

where $\underline{\hat{\omega}}$ is the column vector with components $\hat{\omega}_1, \ldots, \hat{\omega}_h$.

5.4 We apply (4.5) with X_n instead of X and with $N = p^n$. This provides homomorphims

$$\begin{array}{lcl} \tau_N\, F_N^* & : & \mathrm{H}^m(X_n, \underline{\mathcal{WO}}_{X_n}) \;\to\; \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_n \\ \psi_N\, F_N^* & : & \mathrm{H}^m(X_n, \underline{\mathcal{WO}}_{X_n}) \;\to\; \mathbf{H}^m(X_n, \underline{\mathcal{W}\Omega}^{\cdot}_{X_n/S_n} \bmod N) \end{array}$$

where ψ_N is induced by $\underline{\mathcal{W}\Omega}^{\cdot}_{X_n} \bmod N \to \underline{\mathcal{W}\Omega}^{\cdot}_{X_n/S_n} \bmod N$. Writing $\hat{\omega}_i$ also for the image of $\hat{\omega}_i$ in $\mathrm{H}^m(X_n, \underline{\mathcal{WO}}_{X_n})$ we get

$$\begin{array}{lcl} \tau_N\, F_N^*\, \hat{\omega}_1, \ldots, \tau_N\, F_N^*\, \hat{\omega}_h & \in & \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_n, \\ \psi_N\, F_N^*\, \hat{\omega}_1, \ldots, \psi_N\, F_N^*\, \hat{\omega}_h & \in & \mathbf{H}^m(X_n, \underline{\mathcal{W}\Omega}^{\cdot}_{X_n/S_n} \bmod N). \end{array}$$

Using (4.5) and (5.3) we compute

$$\tau_{p^{n+1}}\, F_{p^{n+1}}^*\, \underline{\hat{\omega}} \bmod p^n = \tau_{p^n}\, F_{p^n}^*\, F_p\, \underline{\hat{\omega}} = \tau_{p^n}\, F_{p^n}^*\, \lambda(H)\, \underline{\hat{\omega}} = H^{\sigma^n}\, \tau_{p^n}\, F_{p^n}^*\, \underline{\hat{\omega}}$$

in $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes A_n$. From this computation one obtains

$$\begin{array}{l} (H^{\sigma^n}\, H^{\sigma^{n-1}} \cdots H^{\sigma}\, H)^{-1}\, \tau_{p^{n+1}}\, F_{p^{n+1}}^*\, \underline{\hat{\omega}} \equiv \\ (H^{\sigma^{n-1}}\, H^{\sigma^{n-2}} \cdots H^{\sigma}\, H)^{-1}\, \tau_{p^n}\, F_{p^n}^*\, \underline{\hat{\omega}} \bmod p^n \end{array}$$

for all $n \geq 1$. So in $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes A_\infty = \lim\limits_{\leftarrow n} \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes A_n$ there exist elements $\varpi_1, \ldots, \varpi_h$ such that

$$\underline{\varpi} \bmod p^n = (H^{\sigma^{n-1}}\, H^{\sigma^{n-2}} \cdots H^{\sigma}\, H)^{-1}\, \tau_{p^n}\, F_{p^n}^*\, \underline{\hat{\omega}}$$

in $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes A_n$.

Define

$$\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty = \lim_{\leftarrow n} \mathbf{H}^m(X_n, \underline{\mathcal{W}\Omega}^{\cdot}_{X_n/S_n} \bmod p^n)$$

With a simple computation as above one sees that there exist elements $\lambda(\varpi_1), \ldots, \lambda(\varpi_h)$ in $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty$ such that

$$\underline{\lambda(\varpi)} \mapsto \lambda((H^{\sigma^{n-1}}\, H^{\sigma^{n-2}} \cdots H^{\sigma}\, H)^{-1})\, \psi_{p^n}\, F_{p^n}^*\, \underline{\hat{\omega}} \qquad (3)$$

in $\mathbf{H}^m(X_n, \underline{\mathcal{W}\Omega}^{\cdot}_{X_n/S_n} \bmod p^n)$.

5.5 $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty$ is a module over the ring $W(A_\infty)$. Via the homomorphism $\lambda : A_\infty \to W(A_\infty)$ it becomes also an A_∞-module. We set:

$\mathcal{U}$	:=	the sub-A_∞-module of $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty$ generated by $\varpi_1, \ldots, \varpi_h$
$\lambda(\mathcal{U})$	:=	the sub-A_∞-module of $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty$ generated by $\lambda(\varpi_1), \ldots, \lambda(\varpi_h)$

5.6 Theorem
(a) *The homomorphism* $\pi : \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty \to \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty$ *restricts to an isomorphism of* A_∞*-modules*

$$\lambda(\mathcal{U}) \simeq \mathcal{U} \qquad \text{with}$$

$$\pi\lambda(\varpi_i) = \varpi_i \quad \text{for } i = 1, \ldots, h.$$

(b) *The homomorphism* $\beta : \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty \to \mathrm{H}^m(X, \mathcal{O}_X) \otimes_A A_\infty$ *(see* 4.2)) *restricts to an isomorphism of* A_∞*-modules*

$$\mathcal{U} \simeq \mathrm{H}^m(X, \mathcal{O}_X) \otimes_A A_\infty \qquad \text{with}$$

$$\beta\, \varpi_i = \omega_i \quad \text{for } i = 1, \ldots, h.$$

Consequently

$$\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty = \mathcal{U} \oplus \mathrm{Fil}^1_{\mathrm{Hodge}} \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty.$$

(c) *The Frobenius endomorphism* F_p *on* $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty$ *stabilizes* $\lambda(\mathcal{U})$*. The matrix of* F_p^k *on* $\lambda(\mathcal{U})$ *with respect to the basis* $\lambda(\varpi_1), \ldots, \lambda(\varpi_h)$ *satisfies the congruences*

$$\mathrm{matrix}[F_p^k] \equiv \lambda[B_{p^n}^{-\sigma^k}\, B_{p^{n+k}}] \bmod p^{n+1}$$

for every $n \geq 0$*.*
(d) *The Gauss-Manin connection on* $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty$ *stabilizes* $\mathcal{U}$ *i.e.* $\nabla\mathcal{U} \subset \Omega^1_S \otimes \mathcal{U}$*. If* D *is a derivation of* A *the matrix of* $\nabla(D)$ *on* $\mathcal{U}$ *with respect to the basis* $\varpi_1, \ldots, \varpi_h$ *satisfies the congruences*

$$\mathrm{matrix}[\nabla(D)] \equiv -B_{p^{2n}}^{-1}\, D(B_{p^{2n}}) \bmod p^n$$

for every $n \geq 0$*.*

Proof The results in (a) and (b) are immediate consequences of the constructions in (5.4) and (4.5) and of the formula

$$\pi F_{p^n}\, \hat{\underline{\omega}} = (H^{\sigma^{n-1}}\, H^{\sigma^{n-2}} \cdots H^\sigma\, H)\, \underline{\omega}$$

For (c) and (d) one should first observe

$$H^{\sigma^{k-1}}\, H^{\sigma^{k-2}} \cdots H^\sigma\, H \equiv B_{p^n}^{-\sigma^k}\, B_{p^{n+k}} \bmod p^{n+1}$$

This together with formula (3) proves the result in (c).
One checks by induction that for every $x \in A_\infty$ there exist $x_i \in A_\infty (i \geq 0)$ such that for every $n \geq 0$

$$x^{\sigma^n} = \sum_{i=0}^{n} p^i\, x_i^{p^{n-i}}$$

Consequently $D(x^{\sigma^n}) \equiv 0 \mod p^n$. Using that $\tau_{p^n} F^*_{p^n}\underline{\hat{\omega}}$ lies in the kernel of Gauss-Manin one now computes

$$\begin{aligned} \nabla(D)(\underline{\varpi}) &\equiv [D((B_{p^n}^{-\sigma^n} B_{p^{2n}})^{-1})][B_{p^n}^{-\sigma^n} B_{p^{2n}}]\,\underline{\varpi} \\ &\equiv [D(B_{p^{2n}}^{-1})]\, B_{p^{2n}}\,\underline{\varpi} \qquad\qquad \mathrm{mod} p^n \end{aligned}$$

This proves (d). □

5.7 *Remark* From (c) one sees that $\underline{F}_p$ is an automorphism of $\lambda(\mathcal{U})$. Via the isomorphism in (a) it gives a Frobenius automorphism on $\mathcal{U}$. Thus $\mathcal{U}$ becomes a unit root crystal. Since the rank of $\mathcal{U}$ is $h =$ rank $\mathrm{H}^m(X, \mathcal{O}_X)$ which is the maximal rank for a unit root sub-crystal of $\mathbf{H}^m(X, \Omega^\cdot_{X/S}) \otimes A_\infty$, $\mathcal{U}$ is *the unit root sub-crystal* of $\mathbf{H}^m(X, \Omega^\cdot_{X/S}) \otimes A_\infty$ (cf. [13] p.249).

5.8 The canonical filtration on a sheaf complex $C^\cdot$ consists of the subcomplexes $t_{\leq i}C^\cdot$ for $i \in \mathbf{Z}$ defined by $(t_{\leq i}C^\cdot)^j = C^j$ for $j < i$, $= 0$ for $j > i$, $= \ker(d : C^j \to C^{j+1})$ for $j = i$. It induces on the hypercohomlogy $\mathbf{H}^m(C^\cdot)$ the increasing filtration

$$\mathrm{Fil}^i_{\mathrm{con}}\mathbf{H}^m(C^\cdot) := \mathrm{image}(\mathbf{H}^m(t_{\leq i}C^\cdot) \to \mathbf{H}^m(C^\cdot)), \qquad i \in \mathbf{Z}.$$

This gives in particular the *conjugate filtrations* on $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^\cdot_X \bmod N)$, $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^\cdot_{X/S} \bmod N)$, $\mathbf{H}^m(X_s, \mathcal{W}\Omega^\cdot_{X_s} \bmod p^n)$, $\mathbf{H}^m(X, \Omega^\cdot_{X/S} \bmod N)$ (the terminology conjugate filtration is adopted from [11, 10]). The homomorphisms

$$\begin{aligned} \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^\cdot_X \bmod N) &\to \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^\cdot_{X/S} \bmod N) \\ \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^\cdot_{X/S} \bmod N) &\to \mathbf{H}^m(X, \Omega^\cdot_{X/S} \bmod N) \\ \mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^\cdot_{X/S} \bmod N) &\to \mathbf{H}^m(X_s, \mathcal{W}\Omega^\cdot_{X_s} \bmod N) \end{aligned}$$

are compatible with the conjugate filtrations on their source and target.

5.9 Let s be a closed point of S with perfect residue field $k(s)$ of characteristic p. By ([9] p.577 (3.17.3)) the canonical homomorphism

$$\mathcal{W}\Omega^\cdot_{X_s} \bmod p^n \to \mathcal{W}_n\Omega^\cdot_{X_s}$$

onto the De Rham-Witt complex of level n is a quasi-isomorphism. Thus we get an isomorphism

$$\mathbf{H}^m(X_s, \mathcal{W}\Omega^\cdot_{X_s} \bmod p^n) \simeq \mathbf{H}^m(X_s, \mathcal{W}_n\Omega^\cdot_{X_s})$$

and a specialization homomorphism

$$\mathbf{H}^m(X_n, \underline{\mathcal{W}\Omega}^\cdot_{X_n/S_n} \bmod p^n) \to \mathbf{H}^m(X_s, \mathcal{W}_n\Omega^\cdot_{X_s}). \tag{4}$$

Moreover from ([9] II(1.4), (2.8)) one knows

$$\mathrm{H}^m_{\mathrm{crys}}(X_s) \simeq \mathbf{H}^m(X_s, \mathcal{W}\Omega^\cdot_{X_s}) \simeq \lim_{\leftarrow n} \mathbf{H}^m(X_s, \mathcal{W}_n\Omega^\cdot_{X_s}).$$

Thus the homomorphisms in (4) give in the limit

$$\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty \to \mathrm{H}^m_{\mathrm{crys}}(X_s). \tag{5}$$

The homomorphism $A \to k(s)$ corresponding with the point $s \in S$ and the homomorphism $\lambda : A_\infty \to W(A_\infty)$ lead to the composite map $A \to A_\infty \to W(A_\infty) \to W(k(s))$. From the basic comparison theorem of crystalline and De Rham cohomology ([1] (7.26.3)) and from the hypotheses in (4.1) one gets isomorphisms

$$\begin{aligned} \mathrm{H}^m_{\mathrm{crys}}(X_s) &\simeq \mathbf{H}^m(X \otimes W(k(s)), \Omega^{\cdot}_{X\otimes W(k(s))/W(k(s))}) \\ &\simeq \mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A W(k(s)) \end{aligned}$$

So $\mathrm{H}^m_{\mathrm{crys}}(X_s)$ is a free $W(k(s))$-module.

The conjugate filtration on the finite levels $\mathbf{H}^m(X_s, \mathcal{W}_n\Omega^{\cdot}_{X_s})$ induces on the limit $\mathrm{H}^m_{\mathrm{crys}}(X_s)$ the conjugate filtration $\{\mathrm{Fil}^i_{\mathrm{con}}\mathrm{H}^m_{\mathrm{crys}}(X_s)\}_{i\geq 0}$ (see [10]). One of the main results in [10] describes $\mathrm{Fil}^i_{\mathrm{con}}\mathrm{H}^m_{\mathrm{crys}}(X_s) \otimes \mathbf{Q}$ as precisely that part of $\mathrm{H}^m_{\mathrm{crys}}(X_s) \otimes \mathbf{Q}$ where Frobenius $\underline{F}_p$ acts with slopes $\leq i$. In particular $\mathrm{Fil}^0_{\mathrm{con}}\mathrm{H}^m_{\mathrm{crys}}(X_s)$ is the *unit root part* of $\mathrm{H}^m_{\mathrm{crys}}(X_s)$ ([10] III(6.8)). Since $\mathrm{H}^m_{\mathrm{crys}}(X_s)$ is a free $W(k(s))$-module, $\mathrm{Fil}^0_{\mathrm{con}}\mathrm{H}^m_{\mathrm{crys}}(X_s)$ is also a free $W(k(s))$-module. By the theory of the *conjugate spectral sequence* [10] its rank is at most $h = \dim_{k(s)} \mathrm{H}^m(X_s, \mathcal{O}_{X_s})$.

The conjugate filtration on the finite levels induces the conjugate filtration on the limit $\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty$. The specialization homomorphism (5) is compatible with the conjugate filtrations. It is clear from the construction in (5.4) and (5.5) that $\lambda(\mathcal{U})$ is contained in $\mathrm{Fil}^0_{\mathrm{con}}\mathbf{H}^m(X, \underline{\mathcal{W}\Omega}^{\cdot}_{X/S})_\infty$. So by (5) it is mapped into $\mathrm{Fil}^0_{\mathrm{con}}\mathrm{H}^m_{\mathrm{crys}}(X_s)$.

We compose (5) with the projection $\mathrm{H}^m_{\mathrm{crys}}(X_s) \to \mathrm{H}^m(X_s, \mathcal{O}_{X_s})$. One easily checks that the composite map $\lambda(\mathcal{U}) \to \mathrm{H}^m(X_s, \mathcal{O}_{X_s})$ sends $\lambda(\varpi_i)$ to $\omega_i(s)$,= the image of ω_i under the map $\mathrm{H}^m(X, \mathcal{O}_X) \to \mathrm{H}^m(X_s, \mathcal{O}_{X_s})$. Because the $W(k(s))$-rank of $\mathrm{Fil}^0_{\mathrm{con}}\mathrm{H}^m_{\mathrm{crys}}(X_s)$ is at most h and because $\{\omega_1(s), \ldots, \omega_h(s)\}$ is a $k(s)$-basis of $\mathrm{H}^m(X_s, \mathcal{O}_{X_s})$ we conclude:

5.10 Theorem *Let s be a closed point of S with perfect residue field of characteristic p. Then the specialization homomorphism* (5) *restricts to a surjection*

$$\lambda(\mathcal{U}) \to \mathrm{Fil}^0_{con}\mathrm{H}^m_{crys}(X_s). \qquad \square$$

5.11 *Remark* The conjugate filtration on $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_n$ for $n \geq 0$ induces the conjugate filtration on $\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty$. Clearly $\mathcal{U}$ is contained in $\mathrm{Fil}^0_{\mathrm{con}}\mathbf{H}^m(X, \Omega^{\cdot}_{X/S}) \otimes_A A_\infty$. One may hope that this inclusion is in fact an equality (cf. [5] p.97).

References

[1] Berthelot, P., Ogus, A. *Notes on crystalline cohomology.* Princeton: Princeton University Press 1978

[2] Bloch, S. *Algebraic K-theory and crystalline cohomology.* Publ. Math. IHES 47 (1977) 187-268

[3] Cartier, P. *Groupes formels associés aux anneaux de Witt généralisés.* resp. *Modules associés à un groupe formel commutatif. Courbes typiques.* C.R. Acad. Sc. Paris 265 (1967) 49-52 resp. 129-132

[4] Deligne, P. *Théorème de Lefschetz et critères de dégénérescence de suites spectrales.* Publ. Math. IHES 35 (1968) 259-278

[5] Deligne, P. *Cristaux ordinaires et coordonnées canoniques.* in: Surfaces algébriques. Lectures Notes in Math 868, Springer Verlag 1981

[6] Dwork, B. *p-Adic cycles.* Publ. Math. IHES 37 (1969) 27-116

[7] Grothendieck, A., Dieudonné, J. *Éléments de géométrie algébrique IV, quatrième partie.* Publ. Math. IHES 32 (1967)

[8] Hazewinkel, M. *Formal groups and applications.* New York: Academic Press 1978

[9] Illusie, L. *Complexe de De Rham-Witt et cohomologie cristalline.* Ann. scient. Éc. Norm. Sup. 12 (1979) 501-661

[10] Illusie, L., Raynaud, M. *Les suites spectrales associées au complexe de De Rham-Witt .* Publ. Math. IHES 57 (1983) 73-212

[11] Katz, N. *Algebraic solutions of differential equations (p-curvature and the Hodge filtration).* Inv. Math. 18 (1972) 1-118

[12] Katz, N. *Expansion-coefficients as approximate solution of differential equations.* Astérisque 119-120 (1984) 183-189

[13] Katz, N. *Internal reconstruction of unit-root F-crystals via expansion-coefficients.* Ann. scient. Éc. Norm. Sup. 18 (1985) 245-268

[14] Katz, N., Oda,T. *On the differentiation of De Rham cohomology classes with respect to parameters.* J. Math. Kyoto Univ. 8 (1968) 199-213

[15] Manin, Ju. *The Hasse-Witt matrix of an algebraic curve.* AMS Translations (2) 45 (1965) 245-264

[16] Nishiyama, S. *Appell's hypergeometric function* F_2 *and periods of certain elliptic K3 surfaces.* Tokyo J. Math. 10 (1987) 33-67

[17] Pham, F. *Singularités des systèmes différentiels de Gauss-Manin.* Progress in Math. 2, Boston: Birkhauser 1979

[18] Stienstra, J., van der Put, M., van der Marel, L. *On the variation of p-adic periods* , Mathematische Zeitschrift 208 (1991), p.309-325.

[19] Stienstra, J. *Cartier-Dieudonné theory for Chow groups.* J. f. d. reine u. angew. Math. 355 (1985) 1-66

[20] correction to [19] same journal vol. 362 (1985) 218-220

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