Crystals, quivers and dessins d'enfants

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Recently, physicists (in particular A. Hanany and co-authors) working on the AdS/CFT correspondence found a way to associate quivers with certain planar lattice polygons [2]. At closer inspection, the method, which they describe through examples, turns out to start like N.G. de Bruijn's method of constructing Penrose tilings and quasi-crystals [4]. While the latter uses grids 'parallel' to the sides of a regular pentagon, the former uses grids 'parallel' to the sides of the given lattice polygon. The method yields crystals (periodic rhombus tilings) if the slopes of the polygon's sides are rational, and quasi-crystals (non-periodic rhombus tilings) if the slopes are irrational. Our present interest is in lattice polygons.

So, let $\mathcal{P}$ be a polygon in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_N$ be a collection of (distinct) lattice points on the boundary $\partial\mathcal{P}$ of $\mathcal{P}$, including all vertices, ordered according to their appearance as one walks along $\partial\mathcal{P}$ counterclockwise. Let $\mathbf{b}_i = \mathbf{v}_{i+1} - \mathbf{v}_i$ for $i = 1, \ldots, N$, $\mathbf{v}_{N+1} = \mathbf{v}_1$. Think of these as column vectors. Let $\mu = \frac{1}{6}N(N-1)(N-2)$ and index the coordinates of $\mathbb{R}^\mu$ by strictly increasing triples $i < j < k \in \{1, \ldots, N\}$. Associated with these data is the linear map

$$P : \mathbb{R}^N \to \mathbb{R}^\mu\,, \qquad P(\gamma_1, \ldots, \gamma_N) = \left(\det\begin{pmatrix} \mathbf{b}_i & \mathbf{b}_j & \mathbf{b}_k \\ \gamma_i & \gamma_j & \gamma_k \end{pmatrix}\right)_{ijk}\,.$$

The kernel of P is the linear 2-plane in $\mathbb{R}^N$ spanned by the rows of the matrix $(\mathbf{b}_1, \ldots, \mathbf{b}_N)$. The matrix for P is just made from the Plücker coordinates of this linear 2-plane in $\mathbb{R}^N$. For every $\mathbf{c}$ in the image of P the set $P^{-1}(\mathbf{c})$ is an affine 2-plane in $\mathbb{R}^N$. In $\mathbb{R}^N$ one also has the standard N-grid formed by the hyperplanes $H_{i,m} = \{x_i = m\}$ for $1 \le i \le N$, $m \in \mathbb{Z}$. Intersecting this standard N-grid with $P^{-1}(\mathbf{c})$ gives an N-grid in the plane. Next consider the map

$$F : \mathbb{R}^N \to \mathbb{Z}^N\,, \qquad F(\gamma_1, \ldots, \gamma_N) = (\lfloor\gamma_1\rfloor \ldots, \lfloor\gamma_1\rfloor)\,,$$

where $\lfloor x \rfloor$ for a real number x denotes the largest integer $\le x$. The map F is constant on the connected components of the grid complement in $P^{-1}(\mathbf{c})$ and takes different values on different components. For $\mathbf{c}$ in the image of P we now let $S_\mathbf{c}$ denote the piecewise linear surface in $\mathbb{R}^N$ which is the union of the 2-dimensional squares with sides of length 1 and vertices in the set $F(P^{-1}(\mathbf{c}))$. An appropriate projection map $W : \mathbb{R}^N \to \mathbb{R}^2$ makes $S_\mathbf{c}$ appear as a rhombus tiling of the plane $\mathbb{R}^2$; for this the columns of W must be vectors of length 1 and for the angles between the columns there are some restrictions. The choice of W corresponds to the 'isoradial embeddings and R-charges' in [2] §3. The piecewise linear surface $S_\mathbf{c}$ is invariant under translations by vectors from the lattice $\mathbb{L} = \mathbb{Z}^N \cap \ker P$. Let $\phi : \mathbb{R}^N \to \mathbb{R}$ denote the map which assigns to a vector the sum of its coordinates. It is invariant under translations by vectors from $\mathbb{L}$, because $\sum_i \mathbf{b}_i = 0$. Summarizing: *for every* $\mathbf{c}$ *in the image of* P *we have*

$$\text{the torus} \quad S_\mathbf{c}/\mathbb{L}\,, \qquad \text{the map} \quad \phi : S_\mathbf{c}/\mathbb{L} \to \mathbb{R}\,;$$

the torus is equipped with a piecewise linear structure given by a tiling with squares and the map is piecewise linear.

A very special and beautiful situation arises when ϕ takes only three values on the set $F(P^{-1}(\mathbf{c}))$. These values are then three consecutive integers, but since the numerical values are irrelevant for what follows, we call them R, W, B, with R being the maximum and B the minimum. Every square in the tiling on $S_{\mathbf{c}}/\mathbb{L}$ then has one R, two W and one B vertex and its diagonals are R-B and W-W. We give the W-W diagonal an orientation so that the R vertex is on its right hand side.

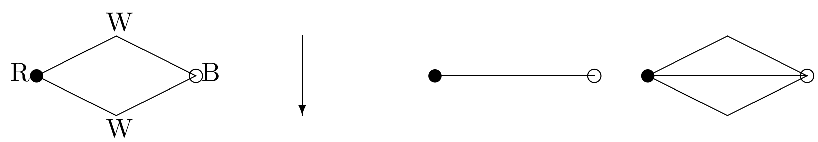

FIGURE 1. *Rhombus and its diagonals.*

The oriented W-W diagonals from the tiling on the torus $S_{\mathbf{c}}/\mathbb{L}$ form the quiver, mentioned in the title of this report. The R-B diagonals in the tiling of the plane, on the other hand, form a periodic bipartite graph. Let me emphasize here that this story tells only what I think I read in the physics literature.

A new aspect I want to add are the *dessins d'enfants*: these are the triangulations of the torus $S_{\mathbf{c}}/\mathbb{L}$ one obtains from the squares-tiling by dividing each square into two triangles by cutting it along the R-B diagonal. Each triangle has one R, one W and one B vertex. According to the general theory of dessins d'enfants [5, 1] this observation implies that $S_{\mathbf{c}}/\mathbb{L}$ can be given the structure of an elliptic curve together with a morphism $\psi : S_{\mathbf{c}}/\mathbb{L} \to \mathbb{P}^1$, everything defined over some number field, such that ψ is unramified outside the R,W,B points and sends all R points to ∞, all W points to 1, all B points to 0. In the triangulation on $S_{\mathbf{c}}/\mathbb{L}$ the cells are given as the ψ-inverse image of the upper- or lower hemisphere, the R-W edges are in $\psi^{-1}([1,\infty))$, the B-W edges are in $\psi^{-1}([0,1])$, the R-B edges are in $\psi^{-1}((\infty,0])$.

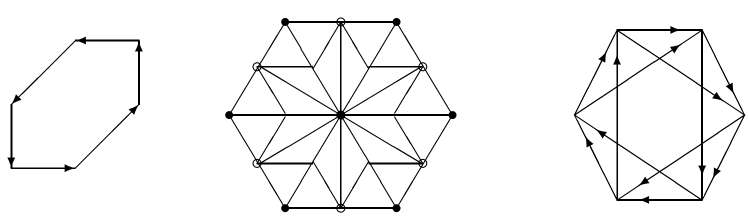

FIGURE 2. *Polygon (left), Dessin (middle) and Quiver (right).*

Figure 2 shows, as an example, the polygon, dessin and quiver for what in the physics literature is known as Model I for the Del Pezzo surface dP_3 (i.e. $\mathbb{P}^2$ with three points blown up). This dessin can be realized on the elliptic curve $\mathcal{E}$ with equation $y^2 = x^3 - 1$ by the composite of the following three maps:

$$\begin{array}{rccccccc} \text{curve:} & \mathcal{E} & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ \text{affine coordinate:} & x, y & & x & & t & & s \\ \text{branched covering:} & & y^2 = x^3 - 1 & & x^3 = \frac{t+1}{t-1} & & t^2 = \frac{s}{s-1} & \end{array}$$

The elliptic curve $\mathcal{E}$ is in fact the Fermat cubic.

Remark. I am still working on a program to construct all periodic rhombus tilings with periodicity and collection of tiles specified by the initial lattice polygon. From the thus constructed data set one can then easily select the quivers and dessins. I expect that this, for instance, will also yield Models II, III, IV of dP_3.

Remark. In the evenings of the workshop Alastair Craw, Lutz Hille, Markus Perling, Duco van Straten and I discussed possible relations between the quivers in my talk and those in the talks of Perling and Bondal. A few days later the paper [3] by Hanany, Herzog and Vegh appeared, which deals with the same issues and (partly) answers our questions.

References

[1] The Grothendieck Theory of Dessins d'Enfants, L. Schneps (ed.) London Math. Soc. Lecture Note Series 200, Cambridge University Press, 1994.

[2] Hanany, A., D. Vegh, *Quivers, Tilings, Branes and Rhombi*, arXiv:hep-th/0511063

[3] Hanany, A., C. Herzog, D. Vegh, *Brane Tilings and Exceptional Collections*, arXiv:hep-th/0602041

[4] Senechal, M., *Quasicrystals and geometry*, Cambridge University Press, 1995.

[5] Shabat, G., V. Voevodsky, *Drawing curves over number fields* in *The Grothendieck Festschrift Volume III*, P. Cartier et al. (eds.) Progress in Math. 88, Birkhäuser, Boston, 1990.