Injective Stability for K_2

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§1. Introduction

1.1 We want to prove the following Theorem and some non-commutative variations on it.

Theorem 1 Let R be a commutative ring with noetherian maximal spectrum of dimension d, $d < \infty$. Let $n \geq d + 2$. Then the natural map $K_2(n,R) \to K_2(R)$ is surjective and the natural map $K_2(n + 1,R) \to K_2(R)$ is an isomorphism.

1.2 The proof of Theorem 1 is given in §§2,3,4,5. (In §5 we deal with a special case). In §6 we extend the Theorem to some non-commutative rings. In §7 we give some examples of non-stability for K_2, based on homotopy theory of real orthogonal groups. In §8 we recall the connection with second homology groups of $E(n,R)$ and with Quillen's non-stable K-groups.

1.3 The statement on surjectivity in Theorem 1 has been proved by Keith Dennis and also by L. N. Vaserstein. (See [5], [22], [28]). We refer to it as "surjective stability for K_2". In particular, this surjective stability implies that $K_2(n + 1,R) \to K_2(R)$ is surjective. (Use $n + 1 \geq d + 3 \geq d + 2$). So what we still have to prove is that it is an injective map. ("injective stability"). We will show that $St(n + 1,R) \to St(n + 2,R)$ is injective. This implies that $K_2(n + 1,R) \to K_2(n + 2,R)$ is injective, and, substituting n + k for n, one sees that $K_2(n + k + 1,R) \to K_2(n + k + 2,R)$ is injective for $k \geq 0$. Taking the limit gives the required result.

We will also need "injective stability for K_1", i.e. the fact that $K_1(n,R) \to K_1(n + 1,R)$ is injective for $n \geq d + 2$. This result is due to Bass and Vaserstein and also follows from the same arguments as surjective stability for K_2. Actually these earlier results are valid in the more general case of a ring satisfying Bass's stable range condition SR_n. (The ring R is the Theorem satisfies SR_n by a well known result of Bass). For our Theorem we will need more than SR_n however. In the case $n = 2$ we will use that we are dealing with semi-local rings ($d = 0$). When $n \geq 3$ we will use that the ring satisfies a very technical variation on the condition SR_n. We will prove our variation on the statement that R satisfies SR_n essentially by repeating the Eisenbud-Evans proof, following Swan. (see [9], [25]).

Once we have shown (in §2) that R satisfies the technical condition we start our construction of a map ρ from $St(n + 2,R)$ into a structure called <u>left</u>. The injective stability will follow from the fact that the composition of ρ with the homomorphism $St(n + 1,R) \to St(n + 2,R)$ is injective. The global features of the construction of ρ and <u>left</u> are based on Matsumoto's proof for his presentation of the K_2 of a field. (see [19], [20]). We modify Matsumoto's approach by the introduction of a chunk, in analogy with the construction of a group scheme from a group chunk, cf [2]. (I have used the same idea before, in presenting the K_2 of a "3-fold stable" ring, cf [16], [15]).

1.4 As I have mentioned before, I rely on earlier stability results. The proofs given by Keith Dennis for these results inspired some of the arguments in the present proof. What is more, I use the definition of the chunk which he suggested to me when we both attended Queen's Conference on Commutative Algebra in July 1975. I would like to thank him for the very instructive discussions we had there. I

am also indebted to E. Friedlander, D. Kahn and M. Barratt for telling me the basic facts of homotopy theory of orthogonal groups which I use in §7. And I thank Vicki Davis for typing the manuscript. I enjoyed the hospitality of Northwestern University during the time this research was done.

1.5 Let us now discuss in more detail the construction of ρ and left. First one defines a chunk C which is intended as a model for a piece of $St(n + 2,R)$. The building block for constructing C is $St(n + 1,R)$, which is considered to be "known". (In the case of 3-fold stable rings the building block was R^*, the group of units of R. That made it possible to find a presentation for $K_2(R)$. But in the present situation (i.e. for $d > 0$) the old chunk is too small and we don't get a presentation for $K_2(R)$.) The chunk allows a natural map $\pi: C \to St(n + 2,R)$ which is hoped to be injective. (If π is injective then C can be considered as a good model for $\pi(C)$. The problem of injective stability is actually equivalent to injectivity of π). The purpose of using C is to avoid the "unknown" set $\pi(C)$ which lies inside the "unknown" group $St(n + 2,R)$. Instead we now have the "known" set C constructed from the "known" group $St(n + 1,R)$. In $St(n + 2,R)$ one has for each element x a left multiplication $L_x: y \mapsto xy$. We can restrict its domain and codomain to $\pi(C)$ and obtain a partially defined map $\pi(C) \to \pi(C)$ which has domain $\pi(C) \cap (x^{-1}\pi(C))$. One now looks for its counterpart in the chunk, i.e. one looks for a partially defined map $\mathcal{L}(x)$ from C to C with $\pi \circ \mathcal{L}(x) = L_x \circ \pi$. For some x the choice of $\mathcal{L}(x)$ will be obvious, but not for all x. In any case, it is clear that one wants $\mathcal{L}(x)$ to be defined on the full set $\pi^{-1}(\pi(C) \cap x^{-1}\pi(C))$. Otherwise it gives incomplete information. Suppose one has a formula for $\mathcal{L}(x)$ which gives values on a domain that is too small. One way to enlarge the domain of $\mathcal{L}(x)$ is to use the counterparts $\mathcal{R}(y)$ of right multiplications R_y: $z \mapsto zy$. If the model is going to be correct then $\mathcal{L}(x)$ and $\mathcal{R}(y)$ will commute, because L_x and R_y do. That gives conditions for the values of $\mathcal{L}(x)$ at points where one doesn't yet have a formula. In order to define the extension of $\mathcal{L}(x)$ by means of these conditions, one has to find out whether the conditions are consistent with each other. That leads to the problem: Does $\mathcal{L}(x)$ commute with $\mathcal{R}(y)$ as far as the maps are defined? (That problem arises each time one introduces a new $\mathcal{L}(x)$ or $\mathcal{R}(y)$). We define <u>left</u> as the set of maps $\mathcal{L}$ which have domains of the proper size, satisfy $\pi \circ \mathcal{L} = L_x \circ \pi$ for some x, and commute with a selection from the maps $\mathcal{R}(y)$. Another way to enlarge the domain of a map $\mathcal{L}(x)$ is to use the fact that one wants $\mathcal{L}(p)\mathcal{L}(q)$ to coincide with $\mathcal{L}(pq)$ at points where the composite map $\mathcal{L}(p)\mathcal{L}(q)$ is defined. This leads to the problem: Do the $\mathcal{L}(x)$ combine in the expected way? Some of the answers will also be needed in the construction of ρ.

If $\mathcal{L}$, $\tilde{\mathcal{L}}$ are elements of <u>left</u> then $\mathcal{L} \circ \tilde{\mathcal{L}}$ denotes their composition as partially defined maps. We can show that there exists exactly one element $\mathcal{L} * \tilde{\mathcal{L}}$ of <u>left</u> which extends $\mathcal{L} \circ \tilde{\mathcal{L}}$. So <u>left</u> is now a set with composition $*$. This composition is associative. One expects <u>left</u> to be a group, isomorphic to $St(n + 2,R)$. Anyway, the units of <u>left</u> form a group <u>Uleft</u> with $*$ as composition. We look at those elements of <u>Uleft</u> which correspond to generators of $St(n + 2,R)$. They satisfy a set of defining relations for $St(n + 2,R)$. This yields a homomorphism ρ: $St(n + 2,R) \to$ <u>Uleft</u>. Because $St(n + 1,R)$ has been built into the chunk it is easy to check that the composition of ρ with $St(n + 1,R) \to St(n + 2,R)$ is injective. End of sketch.

1.6 Professor A. Suslin recently informed me that he obtained, in collaboration with M. Tulenbayev, a result similar to the main results of this paper. I quote from his letter:

"Let Λ be an associative ring. Then under $n \geq$ s.r. $\Lambda + 2$ the canonical map $St(n,\Lambda) \to St(n+1,\Lambda)$ is injective and consequently $K_{2,n}(\Lambda) \to K_2(\Lambda)$ is an isomorphism."

I presume that s.r. is the same as s.rk. in [27], but at this time no further information is available.

§2. Multiple Stable Range Conditions.

2.1 Rings are associative and have a unit. Let R be a ring. Recall that $(b_1,\dots,b_m)\in R^m$ is called <u>unimodular</u> if $\sum_{i=1}^{m} Rb_i = R$. If R is commutative then we may also say that $(b_1,\dots,b_m)$ is unimodular if $\sum_{i=1}^{m} b_iR = R$. We say that R satisfies SR_n if the following holds: Given a unimodular sequence (or column) $(b_1,\dots,b_n)$ there are $r_1,\dots,r_{n-1}\in R$ such that $(b_1 + r_1b_n,\dots,b_{n-1} + r_{n-1}b_n)$ is unimodular. One reason to recall this definition is that the literature is not unanimous: One also finds the notation SR_{n-1} for what we call SR_n.

2.2 <u>Definition</u> Let c,u,n,p be natural numbers with $c \geq u \geq n - 1$, $p \geq 1$. We say that R satisfies $SR_n^p(c,u)$ if the following holds: Let $A_1,\dots,A_p$ be matrices of size $(n-1) \times c$. For each i, let U_i be the submatrix of A_i consisting of the last u columns. Assume that for each i the matrix U_i can be completed, by adding rows, to a product of $u \times u$ elementary matrices. Then there is a column $\lambda\in R^{c-1}$ such that $A_i\binom{1}{\lambda}$ is a unimodular column for each i. So the property $SR_n^p(c,u)$ gives, for each set of matrices $A_1,\dots,A_p$, which satisfy the condition on the U_i, a column λ which behaves well with respect to $A_1,\dots,A_p$ simultaneously.

<u>Comment</u> If $c > u \geq n - 1$ then $SR_n^1(c,u)$ is automatic. One can show by an argument of Vaserstein that SR_n implies $SR_n^1(c,u)$ for any c,u with $c = u \geq n - 1$. This explains why we use the subscript n, given the convention in 2.1. (See also 3.37).

2.3 <u>Notation</u> We say that R satisfies $\widetilde{\widetilde{SR}}_n$ if it satisfies SR_n, $SR_n^3(n + 2, n + 1)$, $SR_{n+1}^4(n + 2, n + 1)$, $SR_{n+2}^3(n + 2, n + 2)$. So $\widetilde{\widetilde{SR}}_n$ is just shorthand for a list of conditions which we happen to need. It is not clear what the hierarchy is for the conditions in the list. It may be that $SR^3_n(n + 2, n + 1)$ actually implies $\tilde{\tilde{SR}}_n$.

2.4 THEOREM 2 Let R be a commutative ring with noetherian maximal spectrum of dimension d, $d < \infty$. Then R satisfies $\tilde{\tilde{SR}}_n$ for $n \geq \max(3, d + 2)$.

Comments Theorem 2 is certainly not the strongest result one can obtain along these lines. See for instance Theorem 3 below (in 2.11) and remark 2.12. One should also prove a non-commutative version of Theorem 2. This is done in Section 6.

2.5 The proof of Theorem 2 is given in the remainder of Section 2. (The idea is to copy §3 of [25], with minor adaptations). Instead of working with the maximal spectrum it is more convenient to work with the so-called j-spec. Its points are the prime ideals which are intersections of maximal ideals, and the topology on j-spec is (induced from) the Zariski topology. As the points of j-spec correspond to the irreducible closed subsets of the maximal spectrum, it is clear that j-spec has the same dimension as the maximal spectrum. Fix R as in Theorem 2. It is well known that R satisfies SR_n so we need not prove that. As an illustration we will prove $SR^3_n(n + 2, n + 1)$. Then we will indicate how to get $SR^3_{n+2}(n + 2, n + 2)$, $SR^4_{n+1}(n + 2, n + 1)$ and, more generally, how to prove Theorem 3 below.

2.6 Let $m \geq 1$, $s \geq 1$, $y \in$ j-spec. Let $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m)$, $(c_1, \ldots, c_m)$ be sequences of elements of R^s. (So $a_i \in R^s$ etc.) The letters a,b,c represent A_1, A_2, A_3 respectively, where A_i is as in 2.2.

Definitions Let V(y) be the irreducible subset of j-spec corresponding to y. So V(y) = closure of {y}, and y is the generic point of $V(y)$. We put $d(y)$, the "depth" of y, equal to the dimension of $V(y)$. Let $k(y)$ be the quotient field of R/y. There is a natural map $R^s \to k(y)^s$ which we denote by $f \mapsto \overline{f}$. We say that the system $(a_1,\ldots,a_m)$, $(b_1,\ldots,b_m)$, $(c_1,\ldots,c_m)$ is y-basic if (A) or (B) holds, where

(A): The field $k(y)$ has two or three elements and there are $\mu_i \in k(y)$ such that the three vectors $\overline{a}_1 + \mu_2\overline{a}_2 + \cdots + \mu_m\overline{a}_m$, $\overline{b}_1 + \mu_2\overline{b}_2 + \cdots + \mu_m\overline{b}_m$, $\overline{c}_1 + \mu_2\overline{c}_2 + \cdots + \mu_m\overline{c}_m$ are non-zero.

(B): The field $k(y)$ contains at least four elements; the vectors $\overline{a}_1,\ldots,\overline{a}_m \in k(y)^s$ form a system of rank $\geq \min(m, 1 + d(y))$ and the same holds for $\overline{b}_1,\ldots,\overline{b}_m$ and for $\overline{c}_1,\ldots,\overline{c}_m$.

We say that $(a_1,\ldots,a_m),(b_1,\ldots,b_m),(c_1,\ldots,c_m)$ is basic if it is y-basic for all $y \in$ j-spec. We use this definition for any pair of integers m,s with $m \geq 1$, $s \geq 1$. We call m the length.

2.7 Lemma Let $(a_1,\ldots,a_m)$, $(b_1,\ldots,b_m)$, $(c_1,\ldots,c_m)$ be basic, $m > 1$. Then there are $t_1,\ldots,t_{m-1} \in R$ such that $(a_1 + t_1a_m,\ldots,a_{m-1} + t_{m-1}a_m)$, $(b_1 + t_1b_m,\ldots,b_{m-1} + t_{m-1}b_m)$, $(c_1 + t_1c_m,\ldots,c_{m-1} + t_{m-1}c_m)$ is also basic, with length $m - 1$.

Proof We will first show that, at all but finitely many primes, the new system is automatically y-basic, regardless of the choice of the t_i. So suppose $(a_1 + t_1a_m,\ldots,a_{m-1} + t_{m-1}a_m)$, $(b_1 + t_1b_m,\ldots)$, $(c_1 + t_1c_m,\ldots)$ is not y-basic. If $k(y)$ has less than four elements, then $V(y)$ is an irreducible component of the closed set $\{z \in \text{j-spec}\ f^3 - f \in z \text{ for all } f \in R\}$, because this set only contains maximal ideals. So there are only finitely many y such that $k(y)$ has less than four elements. We may therefore assume that $k(y)$ has at least four elements. Without loss of generality (3 times finite is finite) we may assume that the rank of $\overline{a}_1 + \overline{t_1a_m},\ldots,\overline{a}_{m-1} + \overline{t_{m-1}a_m}$

is strictly smaller than $\min(m-1, 1+d(y))$. (Note that $m-1$ is the new length). As we also know that the rank of $\overline{a}_1,\ldots,\overline{a}_m$ is at least $\min(m, 1+d(y))$, the rank must have dropped when passing from $(a_1,\ldots,a_m)$ to $(a_1+t_1a_m,\ldots,a_{m-1}+t_{m-1}a_m)$. It cannot have dropped by more than one, so $m > 1+d(y)$ and the rank of $\overline{a}_1,\ldots,\overline{a}_m$ is $1+d(y)$. We therefore want to show that it occurs only for finitely many y's that at the same time $m > 1+d(y)$ and $1+d(y) = \operatorname{rank}(\overline{a}_1,\ldots,\overline{a}_m)$. As $0 \le d(y) \le d$ it is sufficient to show this for a fixed value of $d(y)$, say $d(y) = r-1$, $r\in\mathbb{N}$. We claim that y is a generic point of a component of the closed set
$X = \{ \mathbf{x}\in j\text{-spec} \mid$ the images of $a_1,\ldots,a_m$ in $k(\mathbf{x})^s$ form a system of rank $\le r\}$. (From this claim it follows that there are only finitely many possibilities for y). So suppose y is not such a generic point. Then there is $\mathbf{x} < y$ with $\mathbf{x}\in X$. One gets $d(\mathbf{x}) > d(y)$, so $\min(m, 1+d(\mathbf{x})) > r$. But $(a_1,\ldots,a_m)$, $(b_1,\ldots,b_m)$, $(c_1,\ldots,c_m)$ is $\mathbf{x}$-basic, so this is impossible. (Note that $k(\mathbf{x})$ is infinite).

We have proved now that it only can go wrong at finitely many primes, say $y_1,\ldots,y_g$ We may assume that $y_i < y_j$ implies $j < i$. (otherwise renumber). Then there exist $\tau_i\in R$ with $\tau_i\notin y_i$ but $\tau_i\in y_j$ for $j < i$. (Well known). Writing $t_i = \sum_j \rho_{ij}\tau_j$ we discuss the primes $y_1,\ldots,y_g$ one by one, starting with y_1, and choosing ρ_{ij} to fit the needs of y_j. In other words, we suppose ρ_{ij} to be given for $j < q$ and we look for ρ_{iq} such that the result will be y_q-basic. (this doesn't depend on the ρ_{ij} with $j > q$). So fix $y = y_q$. If $k(y_q)$ has less than four elements, we may as well assume that $\overline{\tau}_q = 1$, because $\overline{\tau}_q^2 = 1$. But then it is obvious from the definition of y-basic that one can choose the ρ_{ij} appropriately. If $k(y_q)$ has at least four elements then we have $m > 1+d(y_q)$ and we have $\operatorname{rank}(\overline{a}_1,\ldots,\overline{a}_m) = 1+d(y)$ or $\operatorname{rank}(\overline{b}_1,\ldots,\overline{b}_m) = 1+d(y)$ or $\operatorname{rank}(\overline{c}_1,\ldots,\overline{c}_m) = 1+d(y)$. The worst case is that all three of the ranks equal $1+d(y)$. (If rank $(\overline{b}_1,\ldots,\overline{b}_m) \ne 1+d(y)$ then

$\operatorname{rank}(\overline{b}_1,\ldots,\overline{b}_m) \geq 2 + d(y)$ and we don't have to look at the $\overline{b}_i$). We have to make sure that neither of the three ranks drops below $1 + d(y)$, when passing to the new system. This is achieved as follows. First one checks that we can choose the ρ_{iq}, with induction on i, such that, for any choice of those ρ's that are still to be considered, $\operatorname{rank}(\overline{a}_1 + \overline{t}_1\overline{a}_m,\ldots,\overline{a}_i + \overline{t}_i\overline{a}_m) \geq 2 + d(y) + i - m$. (This is an exercise in rank counting). One observes that at each step at most one value of $\overline{\rho}_{iq}$ fails to give the inequality. Now $k(y_q)$ has at least four elements, so R/y_q has at least four elements. Therefore we can avoid the failing values of $\overline{\rho}_{iq}$ for the a's, the b's, the c's simultaneously.

2.8 Corollary Let $(a_1,\ldots,a_m)$, $(b_1,\ldots,b_m)$, $(c_1,\ldots,c_m)$ be basic, $m > 1$. Then there are $t_2,\ldots,t_m$ such that $a_1 + t_2a_2 + \cdots + t_ma_m$, $b_1 + t_2b_2 + \cdots + t_mb_m$, $c_1 + t_2c_2 + \cdots + t_mb_m$ is basic with length 1.

Proof Apply Lemma 2.7 repeatedly.

2.9 We want to apply the Corollary to the columns of the matrices A_1,A_2,A_3 occurring in the conditions $SR^3_n(n + 2,n + 1)$ and $SR^3_{n+2}(n + 2,n + 2)$. Let us do $SR^3_n(n + 2,n + 1)$ first. So we have matrices A_1,A_2,A_3 of size $(n - 1) \times (n + 2)$ and the last $n + 1$ columns of A_i form a system of rank $n - 1$ for all $y \in j$-spec. Note that $n - 1 \geq 1 + d \geq 1 + d(y)$ for all $y \in j$-spec. Let $a_1,\ldots,a_{n+2}$ be the columns of A_1, let $b_1,\ldots,b_{n+2}$ be the columns of A_2 and let $c_1,\ldots,c_{n+2}$ be the columns of A_3. We want to show that this is a basic system. If $k(y)$ has at least four elements then the system is y-basic. So consider y which has a smaller $k(y)$. The question is whether there are $\mu_i \in k(y)$ such that the vectors

$\overline{a}_1 + \mu_2\overline{a}_2 + \cdots + \mu_{n+2}\overline{a}_{n+2}$, $\overline{b}_1 + \cdots + \mu_{n+2}\overline{b}_{n+2}$, $\overline{c}_1 + \cdots + \mu_{n+2}\overline{c}_{n+2}$

are non-zero. What choices of the μ_i are wrong for the first vector? They form a plane in $k(y)^{n+1}$, because $\mathrm{rank}(\bar{a}_2,\ldots,\bar{a}_{n+2}) = n - 1$. So to see whether we can get all three vectors non-zero, we just look whether $k(y)^{n+1}$ can be filled by three planes. It can't if $n \geq 3$, even if the field has only two elements. So that is the reason we have $n \geq 3$ in Theorem 2. We now apply the Corollary. It gives us $\lambda = (t_2,\ldots,t_m)$ such that $a_1 + t_2a_2 + \cdots + t_{n+2}a_{n+2}$, $b_1 + t_2b_2 + \cdots + t_{n+2}b_{n+2}$, $c_1 + \cdots + t_{n+2}c_{n+2}$ is a basic system with length 1. This means that we get three vectors which have non-zero images in $k(y)^{n-1}$ for all $y \in$ j-spec. In other words, we get three unimodular vectors. So that proves $SR^3_n(n + 2,n + 1)$. The proof of $SR^3_{n+2}(n + 2,n + 2)$ is similar: This time it boils down to checking the property $SR^3_{n+2}(n + 2,n + 2)$ for small fields $k(y)$. The wrong points in $k(y)^{n+1}$ fill at most three lines that don't pass through the origin or two lines and the origin. So not all points of $k(y)^{n+1}$ are wrong, even if $n = 2$. That proves $SR^3_{n+2}(n + 2,n + 2)$ for R. And, as we didn't need the restriction $n \geq 3$ here, we see that $SR^3_4(4,4)$ holds for a commutative semi-local ring. This is easy to prove anyway, but let us record it:

2.10 <u>Proposition</u> A commutative semi-local ring satisfies $SR^3_4(4,4)$.

2.11 What the method of proof actually shows is the following:

<u>THEOREM 3</u> Let R be a commutative ring with noetherian maximal spectrum of dimension $d < \infty$. Let $c \geq u \geq n - 1 \geq d + 1$ and $p \geq 1$. If $SR^p_n(c,u)$ holds for all residue fields of R then it holds for R.

<u>Proof</u> If $k(y)$ has at most p elements then $\bar{f}^{(p-1)!}$ is zero or one for $\bar{f} \in k(y)$. So y will be a generic point of a component of $\{ z \in \text{j-spec} \mid f(f^{(p-1)!} - 1) \in z$ for all $f \in R\}$. There are only finitely

many such primes y. One treats them as the y's whose residue fields have less than four elements in 2.6 through 2.9. The remaining y's are treated as the y's whose residue fields have at least four elements in 2.6 through 2.9. And instead of using three sequences $(a_1,\ldots,a_m)$, $(b_1,\ldots,b_m)$, $(c_1,\ldots,c_m)$ one now uses p sequences of length m.

2.12 Remark One can refine the result, cf. Bass, as follows: Say one has finitely many subspaces of the maximal spectrum (not j-spec) with the full maximal spectrum as the union. Then d can be replaced by the maximum dimension of these subspaces. This is not always the same as the original d. (See page 173, §2, Ch. IV in [4]). One adapts the proof by defining for each of the subspaces the analogues of j-spec and the depth function $d(y)$.

2.13 For proving Theorem 2 we still have to show that $SR^4_{n+1}(n + 2, n + 1)$ holds for fields, when $n \geq 3$. This follows from the fact that one cannot fill $k(y)^{n+1}$ with four lines.

2.14 Remark. Note that $\tilde{\tilde{SR}}_2$ holds for a semi-local commutative ring which doesn't have any residue field with 2 or 3 elements.

§3. The Chunk

3.1 In Sections 3 and 4 we will prove

THEOREM 4 Let R satisfy $\tilde{\tilde{SR}}_n$, $n \geq 2$. Then the natural map $K_2(n + 1,R) \to K_2(n + 2,R)$ is an isomorphism.

Comments We don't require R to be commutative. As surjective stability is known even under SR_n (or SR_{n+1}) we only have to prove that the map is injective. In most of the proof we only use SR_n, $SR^3_{n+2}(n + 2, n + 2)$. So most of the proof also works for commutative semi-local rings. In Section 5 we take a closer look at the case of commutative semi-local rings. There we will repair the proofs which involve $SR^4_{n+1}(n + 2, n + 1)$ or $SR^3_n(n + 2, n + 1)$, using properties of commutative semi-local rings instead. We only need to repair proofs for n = 2 because this is the case of Theorem 1 which is not covered by Theorems 2 and 4. It turns out that our proofs in Section 5 are at least as complicated as the proofs they are replacing. So in that sense the higher dimensional case is easier! (Of course multiple stable range conditions, if true, are much easier to prove in the semi-local case).

3.2 In the proof of Theorem 4 we never use SR_n directly, but only some of its known consequences. If we take that into account we get the following version of Theorem 4:

THEOREM 4' Let R be a ring, n an integer, $n \geq 2$. Assume that (i),(ii),(iii) are satisfied, where

(i) $E(n,R)$ acts transitively on the set of unimodular columns of length n and $E(n+1,R)$ acts transitively on the set of unimodular columns of length n + 1.

(ii) The natural map $K_1(n,R) \to K_1(n+2,R)$ is injective.

(iii) $SR^3_n(n+2,n+1)$, $SR^4_{n+1}(n+2,n+1)$, $SR^3_{n+2}(n+2,n+2)$ hold for R.

Then the natural map $K_2(n+1,R) \to K_2(n+2,R)$ is injective.

Comment It is not clear whether Theorem 4' is actually sharper than Theorem 4. We will not mention Theorem 4' after this, but just prove Theorem 4. Note that (i),(ii) imply that the natural map $K_1(n,R) \to K_1(n+1,R)$ is an isomorphism, so that the map $K_1(n+1,R) \to K_1(n+2,R)$ is also injective. (See [4],Ch.V,(3.3) (iii)).

3.3 So let us assume that R satisfies $\tilde{\tilde{SR}}_n$, $n \geq 2$. (We will indicate which arguments use more than SR_n, $SR^3_{n+2}(n+2, n+2)$).

3.4 Notations Let I and J be sets. Then $St(I \times J,R)$, or just $St(I \times J)$, is the group with generators $x_{ij}(r)$, where $i \in I$, $j \in J$, $i \neq j$, $r \in R$, and defining relations

(1) $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$. (Here one assumes, of course, that $i \in I$, $j \in J$, $i \neq j$, $r \in R$, $s \in R$).

(2) $[x_{ij}(r),x_{jk}(s)] = x_{ik}(rs)$, if this makes sense, where $[p,q]$ stands for $pq\,p^{-1}q^{-1}$. (We need $j \in I \cap J$ and $i \neq k$ among other things).

(3) $[x_{ij}(r),x_{k\ell}(s)] = 1$ if i,j,k,ℓ are distinct and the expression makes sense.

(4) $[x_{ij}(r),x_{ik}(s)] = 1$ if this makes sense.

(5) $[x_{ij}(r),x_{kj}(s)] = 1$ if this makes sense.

In the case that $I = J = \{1,\ldots,m\}$ we just write $St(m,R)$ for

$St(I \times J,R)$, as usual. We also write $St(m)$ for it. If $I \subset I'$, $J \subset J'$, there is an obvious map from $St(I \times J)$ into $St(I' \times J')$. We will abuse notations and denote both the generators of $St(I \times J)$ and the generators of $St(I' \times J')$ by $x_{ij}(r)$. This is a major abuse because the natural map need not be injective. In fact, that is what this paper is about. Instead of using different notations for an element x of $St(I \times J)$ and its image in $St(I' \times J')$, we will indicate in what group the notation is to be interpreted. So if $x,y \in St(I \times J)$, the statement "$x = y$ in $St(I' \times J')$" will mean that the images of x and y in $St(I' \times J')$ are equal. We use this convention in order to avoid complicated notations. Let us give one more example to show how the convention works: Consider $x = x_{12}(r)$ in $St(2)$, $y = x_{23}(s)$ in $St(\{2,3\} \times \{2,3\})$. Then $[x,y] = x_{13}(rs)$ in $St(3)$. Here x stands for the image of $x_{12}(r)$ in $St(3)$, y stands for the image of $x_{23}(s)$, under a different map!, and $x_{13}(rs)$ is just a generator of $St(3)$.

It will be convenient to have notations for certain subsets of $\{1,2,\ldots,n + 2\}$. (The convention which we just introduced forces us to mention groups of type $St(I \times J)$ all the time). We use $[k]$ for the set $\{1,\ldots,k\}$ and stars for complements: $\{1\}^* = \{2,3,\ldots,n + 2\}$, $\{n + 2\}^* = [n + 1]$ etcetera. Notice that the groups $St(\{1\} \times [n + 2])$ and $St(\{1\} \times \{1\}^*)$ are identical. We will use both notations.

If $I,J \subseteq [n + 2]$ then there is a natural map <u>mat</u> from $St(I \times J)$ into the elementary group $E(n + 2,R) = E_{n+2}(R)$. (cf.[20]). We call its image $E(I \times J)$. The image of $St(m)$ is called $E(m)$, for $m \leq n + 2$. (We never go beyond $n + 2$). We will say that $[x,y] = x_{13}(rs)$ in $E(3)$, where $x,y,x_{13}(rs)$ are as in the example above. So we could as well write $\underline{mat}[x,y] = \underline{mat}(x_{13}(rs))$ or $[\underline{mat}(x),\underline{mat}(y)] = \underline{mat}(x_{13}(rs))$. (The map <u>mat</u> is a homomorphism and $E(3)$ is an honest subset of $E(n + 2)$).

3.5 Consider $St(I \times J)$ when $I \cap J = \emptyset$. One easily sees that $(a_{ij})_{i\in I, j\in J} \mapsto \prod_{i\in I, j\in J} x_{ij}(a_{ij})$ provides an isomorphism from $R^{I\times J}$ onto $St(I \times J)$. The homomorphism mat: $St(I \times J) \to E(I \times J)$ is an isomorphism in this case, because one can still read the a_{ij} off from the image in $E(I \times J)$. More generally, say K,L,M are disjoint subsets of $[n + 2]$ and $I = K \cup L$, $J = L \cup M$. Then the map $St(K \times J) \to St(I \times J)$ is injective, because mat: $St(K \times J) \to E(n+2)$ is injective. So we may denote the image of $St(K \times J)$ in $St(I \times J)$ by $St(K \times J)$ again. It is a normal subgroup. Similarly $St(I \times M)$ can be identified with a normal subgroup of $St(I \times J)$. The action by conjugation of $St(I \times J)$ on $St(K \times J)$ can be studied inside $E(n+2)$, using the isomorphism mat: $St(K \times J) \to E(K \times J)$. (Same for action on $St(I \times M)$). Sending $x_{ij}(r)$ to $x_{ij}(r)$ for $r\in R$, $i\in L$, $j\in J$ and $St(K \times J)$ to 1 gives a homomorphism $\pi_{L\times J}$: $St(I \times J) \to St(L \times J)$ with the natural map $St(L \times J) \to St(I \times J)$ as a cross section. One sees that $St(I \times J)$ is the semi-direct product of $St(L \times J)$ and $E(K \times J)$, with the action coming from conjugation in $E(n+2)$. Recall that a semi-direct product $H \rtimes G$ is given by three data: A group G, a group H and an action of G on H. Say ${}^g h$ denotes the value resulting from the action of $g\in G$ on $h\in H$. Then $H \rtimes G$ consists of pairs $(h,g), h\in H, g\in G$, with multiplication $(h,g)(h_1,g_1) = (h\,{}^g h_1, g\,g_1)$. We can summarize the discussion as follows:

$$St(I \times J) = St(K \times J) \rtimes St(L \times J) \cong E(K \times J) \rtimes St(L \times J)$$

$$St(I \times J) = St(I \times M) \rtimes St(I \times L) \cong E(I \times M) \rtimes St(I \times L).$$

3.6 Definitions Low $= St([n+2] \times \{n+2\}^*)$, Up $= St([n+2] \times \{1\}^*)$ and the mediator is Med $= St([n+2] \times \{1,n+2\}^*)$. The chunk C consists of the orbits of Med in the set Low $\times$ Up under the action shift which is defined as follows: shift $(g)(X,Y) = (Xg^{-1}, gY)$ for $g\in$Med, $X\in$Low, $Y\in$Up, where we abuse notation, as promised. (From the context it follows that (Xg^{-1}, gY) must be an element of Low × Up, so g^{-1} must stand for the inverse of the image of g in Low and the other g must stand for the image in Up). We denote the orbit of (X,Y) by $\langle X,Y\rangle$. One can also say that $\langle X,Y\rangle$ is the equivalence class for the relation: $(X,Y) \sim (X',Y')$ if there is $g \in$ Med such that $X' = Xg^{-1}$ in Low and $Y' = gY$ in Up.

Digression (This piece will not be used).

The proofs have been written without pictures, but of course they were not found that way. In order to understand what is going on, one may want to picture the elements of the $St(I \times J)$ like matrices: Say $n = 3$, so $n + 2 = 5$. Then one would picture an arbitrary element x of $St([5] \times \{5\}^*)$ as

$$\left(\begin{array}{cccc|c} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ \hline * & * & * & * & 1 \end{array}\right) \text{ or } \left(\begin{array}{c|c} A & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline b & 1 \end{array}\right)$$

with $A \in St(4)$, $b \in R^4 \simeq St(\{5\} \times [5])$.

Here we use the semi-direct product: $St([5] \times \{5\}^*) \simeq St(\{5\} \times [5]) \rtimes St(4)$. We can also write

$$x = \left(\begin{array}{c|c} A & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline 0\;0\;0\;0 & 1 \end{array}\right)\left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ \hline & & b & & 1 \end{array}\right).$$

In

$$\left(\begin{array}{c|c} A & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline 0\;0\;0\;0 & 1 \end{array}\right)$$

we do not mean to say that A is a matrix. We mean to say that A lives on the indices on which it is pictured. (So this is more than saying that the entries of <u>mat</u>(A) fit the picture). Some of the rules for matrix multiplication are still valid. For instance,

$$\left(\begin{array}{c|c} A & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline a & 1 \end{array}\right)\left(\begin{array}{c|c} B & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline b & 1 \end{array}\right) = \left(\begin{array}{c|c} AB & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline c & 1 \end{array}\right) \text{ with } c = b + a\,\underline{\text{mat}}(B).$$

(Notice that one multiplies block-wise).

The following division into blocks will play an important role

$$\left(\begin{array}{c|ccc|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right).$$

The big block in the middle corresponds to St({1,5}* x {1,5}*) which is isomorphic to St(3).

We have
$$\left(\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$
corresponding to St(4), and

$$\left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right)$$
corresponding to St({1}* x {1}*), which is isomorphic to St(4).

We will use elements of type
$$\left(\begin{array}{c|c} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ \hline & 1 \end{array}\right).$$
They form <u>Low</u>.

The elements $\begin{pmatrix} 1 & * \\ 0 & \\ 0 & * \\ 0 & \\ 0 & \end{pmatrix}$ form <u>Up</u>, and the

$$\begin{pmatrix} 1 & * & 0 \\ 0 & & 0 \\ 0 & * & 0 \\ 0 & & 0 \\ 0 & * & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & 0 \\ 0 & & * & & 0 \\ 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & * & & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & & * & & 1 \end{pmatrix}$$ form <u>Med</u>.

"Complementary" to <u>Low</u> one has the

$$x_5(v) = \begin{pmatrix} 1 & & & 0 & v_1 \\ & 1 & & & v_2 \\ & & 1 & & v_3 \\ & & & 1 & v_4 \\ & 0 & & & 1 \end{pmatrix}$$

and complementary to <u>Up</u> one has the $x_1(w) = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ w & & 1 & & \\ & & & 1 & \\ & 0 & & & 1 \end{pmatrix}$,

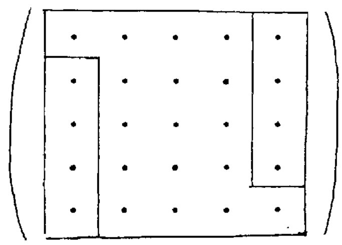
$w = (w_2, w_3, w_4, w_5)$.

(We also write columns in the form $(*,\ldots,*)$, which is of course more suitable for rows).

The $x_5(v)$, $x_1(w)$ will be very important. We will apply the multiple stable range conditions to v, w. For instance, say one has some freedom of choice for v in

$$\begin{pmatrix} 1 & & & & \\ & 1 & 0 & & \\ & & 1 & & v \\ & & & 1 & \\ & 0 & & & 1 \end{pmatrix} \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & T & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & T & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & 1 & 0 & & \\ & & 1 & & z \\ & & & 1 & \\ & 0 & & & 1 \end{pmatrix}$$

Then one may arrange that (z_2,z_3,z_4) is unimodular. (If one has full freedom of choice then this is obvious. The conditions come in if one wants to achieve more with v at the same time).

The basic pattern thus is  . One sees in it

the outlines of Up, Low, Med, and therefore also the places where the co-ordinates of the "complementary" groups are situated, where $\{x_5(v) \mid v \in R^4\}$ is the group complementary to Low, for instance. End of digression.

3.7 Lemma Med → Low and Med → Up have the same kernel N.

Comment So we may identify Med/N with a subgroup of Low and also with a subgroup of Up. Then we can say that $(X,Y) \sim (X',Y')$ is equivalent to: There is $g \in$ Med/N with $X' = Xg^{-1}$, $Y' = gY$. Now there is less abuse of notation. In particular, $(X,Y) \sim (X',Y)$ implies $X = X'$.

Proof of Lemma Let g be in one of the two kernels. Then $\underline{\text{mat}}(g) = 1$ in $E([n+2] \times \{1,n+2\}^*)$. The group Med is a semi-direct product of $St(\{1,n+2\} \times \{1,n+2\}^*)$ and $St(\{1,n+2\}^* \times \{1,n+2\}^*)$, and a similar decomposition holds for $E([n+2] \times \{1,n+2\}^*)$. The restriction of mat to the normal subgroup is an isomorphism, so g must be in the subgroup $St(\{1,n+2\}^* \times \{1,n+2\}^*)$. (We will use this argument often). Say the image of g in Low is trivial. Then its image in $St(n+1)$ must be trivial, again because of the semi-direct product structure, this time of Low. We claim that its image in $St(\{1\}^* \times \{1\}^*)$ must also be trivial. One passes from $St(\{1,n+2\}^* \times \{1,n+2\}^*)$ to $St(n+1)$ by adding an index 1, and one passes from $St(\{1,n+2\}^* \times \{1,n+2\}^*)$ to $St(\{1\}^* \times \{1\}^*)$ by adding an index $n+2$. But it can make no difference for g whether the new index is called 1 or $n+2$, whence the claim. As Up contains $St(\{1\}^* \times \{1\}^*)$, the image of g in Up is trivial. The other part of the proof is similar.

3.8 Notations We have a natural map $\pi: C \to St(n+2)$ given by $\pi\langle X,Y\rangle = XY$. It is clear that π is well-defined. We denote the composition of π and mat simply by mat again. So now we also have mat: $C \to E(n+2)$ with mat $\langle X,Y\rangle$ = mat(X)mat(Y).

3.9 Definitions For $g \in$ Low we define $L(g)$: Low × Up → Low × Up by $L(g)(X,Y) = (gX,Y)$. And we define $\mathcal{L}(g): C \to C$ by $\mathcal{L}(g)\langle X,Y\rangle = \langle gX,Y\rangle$. So $\mathcal{L}(g)\langle X,Y\rangle$ is the class of $L(g)(X,Y)$. It is easy to see that $\mathcal{L}(g)$ is well-defined, that it is a permutation of C, that $\mathcal{L}(gh) = \mathcal{L}(g)\mathcal{L}(h)$ for $g,h \in$ Low. Similarly, for $f \in$ Up, we put $R(f)(X,Y) = (X,Yf)$ and $\mathcal{R}(f)\langle X,Y\rangle = \langle X,Yf\rangle$. Taking $\mathcal{L}$ and $\mathcal{R}$ together one gets, for $g \in$ Med the permutation $\mathfrak{Ins}(g)$ of C given by $\mathfrak{Ins}(g)\langle X,Y\rangle = \langle gX,Yg^{-1}\rangle$.

3.10 Proposition (The squeezing principle).
Let $1 < i < n+2$, $X \in St(\{i\}^* \times \{n+2\}^*)$, $Y \in St(\{i\}^* \times \{1\}^*)$. Suppose that $XY = 1$ in $St(\{i\}^* \times [n+2])$. Then $\langle X,Y\rangle = \langle 1,1\rangle$ in C.

Comment We call it the squeezing principle because it shows how one can prove an equality in the chunk by squeezing the problem into some $St(\{i\}^* \times [n+2])$.

<u>Proof of Proposition</u> Write X as X_1X_2 with $X_1 \in St(\{i\}^* \times \{i,n+2\}^*)$, $X_2 \in St(\{i\}^* \times \{i\})$. (This is a new form of the abuse introduced in 3.4. We really mean $X_1 \in St(\{i\}^* \times \{i,n+2\}^*)$, and we refer to an element of $St(\{i\}^* \times \{n+2\}^*)$ when writing X_1X_2). Write Y as Y_1Y_2 with $Y_1 \in St(\{i\}^* \times \{i\})$, $Y_2 \in St(\{i\}^* \times \{1,i\}^*)$. It follows from the decomposition of $E(\{i\}^* \times [n+2])$ as a semi-direct product that $X_2Y_1 = 1$ in $E(\{i\}^* \times \{i\})$, hence in $St(\{i\}^* \times \{i\})$, which is isomorphic to it. So $\langle X,Y\rangle = \langle X_1,Y_2\rangle$ and we may as well assume $X = X_1$, $Y = Y_2$. As $\underline{mat}(X) = \underline{mat}(Y^{-1})$, the matrix $\underline{mat}(X)$ has trivial columns at positions $1,i,n+2$. (A column or row is called trivial if it is the same as in the identity matrix). It easily follows that there is $m \in St(\{1,n+2\} \times \{1,i,n+2\}^*)$ such that $\underline{mat}(X)\underline{mat}(m)$ also has trivial <u>rows</u> at positions 1 and $n+2$. (and at position i of course). Replacing (X,Y) by $(Xm,m^{-1}Y)$ we may now assume that $\underline{mat}(X)$ has trivial rows and columns at positions $1,i,n+2$. The same will hold for its inverse $\underline{mat}(Y)$. So now we have $X \in St(\{i\}^* \times \{i,n+2\}^*)$, $Y \in St(\{i\}^* \times \{1,i\}^*)$ with $XY = 1$ in $St(\{i\}^* \times [n+2])$ and the matrices have these trivial rows and columns. Because $St(\{i\}^* \times [n+2])$ is a semi-direct product it is easy to see that actually $XY = 1$ in $St(\{i\}^* \times \{i\}^*)$. Write X as X_3X_4 with $X_4 \in St(\{n+2\} \times \{i,n+2\}^*)$, $X_3 \in St(\{i,n+2\}^* \times \{i,n+2\}^*)$. As $\underline{mat}(X)$ has a trivial row at position $n+2$ the factor X_4 has to be 1. So now we have $X = X_3$ and we may say $X \in St(\{i,n+2\}^* \times \{i,n+2\}^*)$. Similarly we get $Y \in St(\{1,i\}^* \times \{1,i\}^*)$.

Consider $[X,x_{i,1}(t)]$ in $St(\{n+2\}^* \times \{i,n+2\}^*)$. It lies in the normal subgroup $St(\{i\} \times \{i,n+2\}^*)$, which is mapped isomorphically into $E(n+2)$. But in $E(n+2)$ we know that $\underline{mat}(X)$ has trivial rows and columns at positions 1 and i, so the commutator is trivial. Similarly $[X,x_{1,i}(t)] = 1$ in $St(\{i,n+2\}^* \times \{n+2\}^*)$. So X commutes with $w_{i,1}(1) = x_{i,1}(1)x_{1,i}(-1)x_{i,1}(1)$ in

$St(\{n + 2\}^* \times \{n + 2\}^*)$. On the other hand
$w_{i,1}(1)x_{1,k}(a)w_{i,1}(1)^{-1} = x_{i,k}(a)$,
$w_{i,1}(1)x_{k,1}(a)w_{i,1}(1)^{-1} = x_{k,i}(a)$, $w_{i,1}(1)x_{k,\ell}(a)w_{i,1}(1)^{-1} = x_{k,\ell}(a)$
if $a \in R$, $k \neq 1$, $k \neq \ell$, $\ell \neq 1$, $k \neq i$, $\ell \neq i$, $k,\ell \in [n + 1]$. It follows that conjugation by $w_{i,1}(1)$ corresponds to the automorphism <u>switch</u>$(1,i)$ of $St(n + 1)$ which is induced by the permutation of $[n + 1]$ which switches 1 and i, and leaves the other indices fixed. We see that $X = w_{i,1}(1)Xw_{i,1}(1)^{-1} =$ <u>switch</u>$(1,i)X$ in $St(n + 1)$. Let X' be the counterpart of X in $St(\{1,n + 2\}^* \times \{1,n + 2\}^*)$ which one gets from $X \in St(\{i,n+2\}^* \times \{i,n+2\}^*)$ by replacing the indices 1 by indices i. We have $X = X'$ in <u>Low</u>, so $\langle X,Y\rangle = \langle X',Y\rangle = \langle 1,X'Y\rangle$. To prove the Proposition it suffices to prove that $X'Y = 1$ in <u>Up</u>. (This is also necessary, by 3.7). The relation $X' =$ <u>switch</u>$(1,i)X'$ in $St(n + 1)$ has a counterpart $X' =$ <u>switch</u>$(n + 2,i)X'$ in $St(\{1\}^* \times \{1\}^*)$, because for X' there is no difference between 1 and n + 2. (Here <u>switch</u>$(n + 2,i)$ is an automorphism of $St(\{1\}^* \times \{1\}^*)$). Replacing the indices i by indices n + 2 one gets from X' to an element X'' in $St(\{1,i\}^* \times \{1,i\}^*)$ with $X' = X''$ in $St(\{1\}^* \times \{1\}^*)$. So we have to show that $X''Y = 1$ in $St(\{1\}^* \times \{1\}^*)$. This amounts to the same as proving $X''Y = 1$ in $St(\{i\}^* \times \{i\}^*)$, because for X'' and Y there is no difference between 1 and i. But in $St(\{i\}^* \times \{i\}^*)$ we know that $XY = 1$. And we also know that $X =$ <u>switch</u>$(1,i)X$ in $St(n + 1)$. Again, as for X there is no difference between i and n + 2 we also have $X =$ <u>switch</u>$(1,n + 2)X$ in $St(\{i\}^* \times \{i\}^*)$, i.e. $X = X''$ in $St(\{i\}^* \times \{i\}^*)$. So $X''Y = 1$ in $St(\{i\}^* \times \{i\}^*)$ indeed.

3.11 <u>Corollary</u> (Squeezing principle reformulated).
Let $X,X' \in St(\{i\}^* \times [n + 1])$, $Y,Y' \in St(\{i\}^* \times \{1\}^*)$ for some $1 < i < n + 2$. Suppose that $XY = X'Y'$ in $St(\{i\}^* \times [n + 2])$. Then $\langle X,Y\rangle = \langle X',Y'\rangle$ in C.

Proof One has $\langle X^{-1}X', Y'Y^{-1}\rangle = \langle 1,1\rangle$ by the Proposition. Now apply $\mathcal{L}(X)$ and $\mathcal{R}(Y)$.

3.12 Notation Let $v \in R^{n+1}$, say $v = (v_1, \ldots, v_{n+1})$. (One should really write v as a column). Then we write $x_{n+2}(v)$ for the product of the $x_{i,n+2}(v_i)$. Similarly, if $w = (w_2, \ldots, w_{n+2})$ then $x_1(w)$ stands for the product $x_{21}(w_2) \cdots x_{n+2,1}(w_{n+2})$. We also write $x_{n+2}(v_1, \ldots, v_{n+1})$ for $x_{n+2}(v)$ and we write $x_1(w_2, \ldots, w_{n+2})$ for $x_1(w)$.

3.13 We want to define maps $\mathcal{L}(x_{n+2}(v))$ for $v \in R^{n+1}$. The general case is too difficult to do right now. But let us look at the case $v_1 = 0$, $X = x_{n+2,1}(q)$, $Y \in \underline{Up}$. In $St(n + 2)$ one has $[x_{n+2}(v), x_{n+2,1}(q)] = x_1(vq)$. So it is reasonable to put $L(x_{n+2}(v))(x_{n+2,1}(q), Y) = (x_1(v_2q, \ldots, v_{n+1}q, q), x_{n+2}(v)Y)$, and $\mathcal{L}(x_{n+2}(v))\langle x_{n+2,1}(q), Y\rangle$ = class of this element $L(x_{n+2}(v))(x_{n+2,1}(q), Y)$.

We have to show that the resulting class only depends on the class of $(x_{n+2,1}(q), Y)$. So suppose $\langle x_{n+2,1}(q), Y\rangle = \langle x_{n+2,1}(r), Y'\rangle$. Then $x_{n+2,1}(q-r) \in \underline{mat}(\underline{Med})$, so $q = r$. But then also $Y = Y'$ by 3.7. So there is no other element of the same class which assumes this simple form.

We have now defined $\mathcal{L}(x_{n+2}(v))\langle X, Y\rangle$ in the case that $v_1 = 0$ and that $\langle X, Y\rangle$ contains a representative of a specific form.

3.14 Now let $T \in St(n + 1)$, $Y \in \underline{Up}$, $q \in R$, $v \in R^{n+1}$. Suppose that $x_{n+2}(v)T = Tx_{n+2}(0, w_2, \ldots, w_{n+1})$ in $St(n + 2)$. We put $w = (0, w_2, \ldots, w_{n+1})$. So $w_1 = 0$ and $w = \underline{mat}(T^{-1})v$.
One is tempted to define $\mathcal{L}(x_{n+2}(v))\langle Tx_{n+2,1}(q), Y\rangle$ as being $\mathcal{L}(T)\mathcal{L}(x_{n+2}(w))\langle x_{n+2,1}(q), Y\rangle$, where the latter is defined by 3.9 and 3.13. (Its image in $St(n + 2)$ is like we want it). However, it is not easy to check that this is a consistent definition: What

happens if $\langle Tx_{n+2,1}(q),Y\rangle = \langle T'x_{n+2,1}(q'),Y'\rangle$ with $x_{n+2}(v)T' = T'x_{n+2}(0,*,\ldots,*)$? (Stars stand for things which don't need names. Two stars need not stand for the same thing).

3.15 Notation In the situation of 3.14 we put $L(x_{n+2}(v))(Tx_{n+2,1}(q),Y) = L(T)(x_1(w_2q,\ldots,w_{n+1}q,q),x_{n+2}(w)Y)$. So we do with the representatives what we wanted to do with the classes. And we extended the definition in 3.13.

3.16 Lemma Let $A\in St(\{1,n+2\}^* \times \{1,n+2\}^*)$, $T\in St(n+1)$, $Y\in \underline{Up}$, $q\in R$, $v\in R^{n+1}$, such that both $L(x_{n+2}(v))(Tx_{n+2,1}(q),Y)$ and $L(x_{n+2}(v))(Tx_{n+2,1}(q)A,A^{-1}Y)$ are defined (as in 3.15). Then they are in the same class.

Proof Using $L(T)$ and $R(Y)$ one reduces to $L(x_{n+2}(w))(x_{n+2,1}(q),1)$ versus $L(x_{n+2}(w))(x_{n+2,1}(q)A,A^{-1}) = L(x_{n+2}(w))(Ax_{n+2,1}(q),A^{-1})$. One shows that they determine the same element of the chunk by executing $L(x_{n+2}(w))$ in both cases and then using A to transform one representative into the other. The semi-direct product structures of $St(\{1\}^* \times \{n+2\}^*)$ and $St(\{n+2\}^* \times \{1\}^*)$ make this easy.

Remark We only needed to require that one of the two is defined as in 3.15. Then the other one is also defined.

3.17 Lemma Suppose in 3.14 that $w_1 = w_2 = 0$. Let $A\in St([n+1] \times \{1,n+2\}^*)$ such that $x_{n+2}(w)A = Ax_{n+2}(0,*,\ldots,*)$. Let $B\in St(\{n+2\} \times \{1\}^*)$ such that $Tx_{n+2,1}(q)AB = T'x_{n+2,1}(*)$ in $\underline{Low}$ for some $T'\in St(n+1)$. (See 3.14 for notations). Then $L(x_{n+2}(v))(Tx_{n+2,1}(q),Y) \sim L(x_{n+2}(v))(Tx_{n+2,1}(q)AB,B^{-1}A^{-1}Y)$.

Proof First note that, given A, the element B is unique, because

B can be computed in $E([n+2] \times \{n+2\}^*)$. Write $A = A_1A_2$ with $A_1 \in St(\{1\} \times \{n+2\}^*)$, $A_2 \in St(\{1,n+2\}^* \times \{1,n+2\}^*)$. In Med we can write AB as $A_1B_1A_2$ with $B_1 \in St(\{n+2\} \times \{1\}^*)$. By the previous Lemma we can assume that $A_2 = 1$. And using $R(Y)$ we can get rid of Y. Now "execute" $L(x_{n+2}(v))$ in both cases and use $L(T)$ to get rid of T. Then we have to deal with the case $T = 1$, $Y = 1$, $A_2 = 1$, $v = w$. But there we can apply the squeezing principle, with $i = 2$. (see 3.11).

3.18 Definition Let $T \in St(n+1)$, $v \in R^{n+1}$, $q \in R$, $Y \in \underline{Up}$ be such that $x_{n+2}(v)T = Tx_{n+2}(0,0,*\ldots,*)$. Then we put $\mathcal{L}(x_{n+2}(v))\langle Tx_{n+2,1}(q),Y\rangle$ = class of $L(x_{n+2}(v))(Tx_{n+2,1}(q),Y)$. It is easy to see from Lemma 3.17 that this is a consistent definition. So now we have defined $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ for some more v and X. One checks that our new definition is compatible with the one in 3.13. We say that $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ is defined at the bottom if there is $T \in St(n+1)$ with $\langle X,Y\rangle = \langle Tx_{n+2,1}(*),*\rangle$, $x_{n+2}(v)T = Tx_{n+1,n+2}(*)$. In particular $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ is then defined by the definition above. (We say that it is defined at the bottom because the relevant entries of $\underline{mat}(x_{n+1,n+2}(*))$ and $\underline{mat}(x_{n+2,1}(*))$ are in the bottom two rows). We will prefer to talk about the case that $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ is defined at the bottom rather than the more general case covered by the definition. The reason is that the notion "defined at the bottom" has a constant meaning, while "defined" will have a different meaning when we will introduce $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ in cases not covered by the present definition.

3.19 Given $\langle X,Y\rangle \in C$, what are the $T \in St(n+1)$ such that we can write $\langle X,Y\rangle = \langle Tx_{n+2,1}(*),*\rangle$? Write X as X_1X_2 with $X_1 \in St(n+1)$, $X_2 \in St(\{n+2\} \times [n+1])$. Using the semi-direct product structure of Low one sees that a necessary and sufficient condition is that there is $g \in \underline{Med}$ with $X_1 g = T$. So it only depends on X_1. In particular, if $v \in R^{n+1}$, $T \in St(n + 1)$, $B \in St(\{n + 2\} \times [n + 1])$ then $\mathcal{L}(x_{n+2}(v))\langle TB,*\rangle$ is defined at the bottom if and only if $\mathcal{L}(x_{n+2}(v))\langle Tx_{n+2,1}(*),*\rangle$ is.

3.20 Lemma Let $v \in R^{n+1}$, $T \in St(n + 1)$, $r \in R$, such that $x_{n+2}(v)T = Tx_{n+2}(w_1,w_2,\ldots,w_{n+1})$ with $(w_2,\ldots,w_{n+1})$ unimodular. Then $\mathcal{L}(x_{n+2}(vr))\langle Tx_{n+2,1}(*),*\rangle$ is defined at the bottom.

Proof By "linearity" it is sufficient to do the case $r = 1$. As $St(n)$ acts transitively on unimodular columns of length n (see [4], Ch. V, Thm. (3.3)), there is $T' \in St(\{1,n + 2\}^* \times \{1,n + 2\}^*)$ with $x_{n+2}(v)TT' = TT'x_{1,n+2}(w_1)x_{n+1,n+2}(1)$. Choose $T'' = x_{1,n+1}(+w_1)$. Then $x_{n+2}(v)TT'T'' = TT'T''x_{n+1,n+2}(1)$, as required.

3.21 Remark Note that this is the first time that we use a stable range condition.

3.22 Lemma Let $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ be defined at the bottom and let $\langle P,Q\rangle \in C$ with $\underline{mat}\langle X,Y\rangle = \underline{mat}\langle P,Q\rangle$. Then $\mathcal{L}(x_{n+2}(v))\langle P,Q\rangle$ is defined at the bottom.

Proof Put $M = \underline{mat}(X^{-1}P) = \underline{mat}(YQ^{-1})$. Then $M \in E(n + 2)$ and M has trivial columns at positions 1 and $n + 2$. Choose $B \in \underline{Med}$ such that $M\,\underline{mat}(B)$ has trivial rows at those positions too. Using injective stability for K_1 we see that
$M\,\underline{mat}(B) \in \underline{mat}(St(\{1,n + 2\}^* \times \{1,n + 2\}^*))$, so in particular $M \in \underline{mat}(\underline{Med})$. Therefore $\langle X,Y\rangle = \langle X',Y'\rangle$ with $\underline{mat}(X') = \underline{mat}(P)$, $\underline{mat}(Y') = \underline{mat}(Q)$. But it is not hard to see (and may have been

noted by the reader) that being defined at the bottom only depends on the matrices, not on the finer structure of Low.

3.23 Lemma Let $v,w \in R^{n+1}$, $T \in St(n+1)$, $\langle X,Y\rangle \in C$ such that $x_{n+2}(v)T = Tx_{n+2}(w)$ and such that $\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$ is defined at the bottom. Then $\mathcal{L}(T)\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(v))\langle TX,Y\rangle$ and the latter is also defined at the bottom.

Proof Actually it is easy to see that they are equal if $\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$ is defined (as in 3.18): Write $\langle X,Y\rangle = \langle T'x_{n+2,1}(*),*\rangle$ with $T' \in St(n+1)$ such that one can execute $L(x_{n+2}(w))(T'x_{n+2,1}(*),*)$. Then compare.

3.24 Lemma If $L(x_{n+1}(v))(X,Y)$ is defined (see 3.15) and if $\mathcal{L}(x_{n+1}(v))\langle X,Y\rangle$ is defined at the bottom, then the former is a representative of the latter.

Proof By the previous Lemma we may assume $X = x_{n+2,1}(q)$, $q \in R$. As $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ is defined at the bottom, there must be $A \in St([1,n+2]^* \times [1,n+2]^*)$ with $x_{n+2}(v)A = Ax_{n+1,n+2}(*)$. Then we may replace (X,Y) by $(XA,A^{-1}Y)$, because of Lemma 3.16. But then it is obvious.

3.25 Lemma (Additivity, first case).
Let $v = (0,v_2,\ldots,v_{n+1})$, $w = (0,w_2,\ldots,w_{n+1})$. Let $\langle X,Y\rangle = \mathcal{L}(x_{n+2}(w))\langle x_{n+2,1}(q),Z\rangle$ and let $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ be defined at the bottom. Then $\mathcal{L}(x_{n+2}(v))\mathcal{L}(x_{n+2}(w))\langle x_{n+2,1}(q),Z\rangle = \mathcal{L}(x_{n+2}(v+w))\langle x_{n+2,1}(q),Z\rangle$.

Proof $L(x_{n+2}(v))L(x_{n+2}(w))(x_{n+2,1}(q),Z) = L(x_{n+2}(v+w))(x_{n+2,1}(q),Z)$ and the left hand side is relevant by the previous Lemma.

3.26 <u>Definition</u> We say that $\mathcal{L}(x_{n+2}(v))$, $\mathcal{L}(x_{n+2}(w))$ <u>slide past each other at</u> $\langle \underline{X},\underline{Y}\rangle$ if there is $T\in St(n+1)$ such that
$\langle X,Y\rangle = \langle Tx_{n+2,1}(*),*\rangle$ and such that
$x_{n+2}(v)\ T = Tx_{n+2}(0,z_2,\ldots,z_k,0,\ldots,0)$,
$x_{n+2}(w)T = Tx_{n+2}(0,\ldots\ldots,0,z_{k+1},\ldots,z_{n+1})$ for some $2 \leq k \leq n+1$ and some $z_2,\ldots,z_{n+1}$ in R.

The relation is symmetric because there is an element T" of $St(\{1,n+2\}^* \times \{1,n+2\}^*)$ with $x_{i,n+2}(*)T'' = T''x_{n+3-i,n+2}(*)$ for $2 \leq i \leq n+1$. (Use a product of the elements $w_{p,q}(1)$). From the same observation it follows that both $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$, $\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$ are defined in the fashion described in 3.18. (For the second one this is obvious). Executing the maps one actually sees that both steps in $\mathcal{L}(x_{n+2}(w))\,\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ are defined in the way described by 3.18. And again we can use T" to show from this that both steps in $\mathcal{L}(x_{n+2}(v))\,\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$ are defined that way.

3.27 <u>Lemma</u> (Additivity, second case).
Let $\mathcal{L}(x_{n+2}(v))$, $\mathcal{L}(x_{n+2}(w))$ slide past each other at $\langle X,Y\rangle$, and let both $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ and $\mathcal{L}(x_{n+2}(v+w))$ be defined at the bottom. Then $\mathcal{L}(x_{n+2}(v))\,\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$.

<u>Proof</u> We choose a representative $(Tx_{n+2,1}(*),*)$ of $\langle X,Y\rangle$, as in 3.26. One checks that the first execution of the expression $L(x_{n+2}(v))L(x_{n+2}(w))(Tx_{n+2,1}(*),*)$ leaves a result of the form $L(x_{n+2}(v))(P,Q)$ with $P = TAx_{n+2,1}(*)$, $A\in St([n+1] \times \{1\})$. It easily follows that $\mathcal{L}(x_{n+2}(v))\langle P,Q\rangle$ is defined at the bottom and therefore, by Lemma 3.24, we can use
$L(x_{n+2}(v))L(x_{n+2}(w))(Tx_{n+2,1}(*),*)$ for representing

$\mathcal{L}(x_{n+2}(v))\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$. Similarly one can use $L(x_{n+2}(v+w))(Tx_{n+2,1}(*),*)$ for $\mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$. But then it is easy. (Compare Lemma 3.25).

3.28 <u>Lemma</u> Let $v,w,z\in R^{n+1}$, $T\in St(n+1)$ such that $x_{n+2}(w)T = Tx_{n+1,n+2}(*)$ and $x_{n+2}(v)T = Tx_{n+2}(z)$ with $(z_2,\ldots,z_n)$ unimodular. Then Lemma 3.27 applies for $\langle X,Y\rangle = \langle Tx_{n+2,1}(*),*\rangle$.

<u>Proof</u> By Lemma 3.20 we only have to show that $\mathcal{L}(x_{n+2}(v))$ and $\mathcal{L}(x_{n+2}(w))$ slide past each other at $\langle X,Y\rangle$. There are $a_2,\ldots,a_n$ such that $a_2z_2+\cdots+a_nz_n = z_1$. We have a representative of $\langle X,Y\rangle$ which takes the form $(Tx_{12}(a_2)\cdots x_{1n}(a_n)x_{n+2,1}(*),*)$. Replacing $(Tx_{n+2,1}(*),*)$ by this representative we reduce to the case $z_1 = 0$. Similarly we can reduce to the case $z_{n+1} = 0$ by using a representative of the form
$(Tx_{n+1,2}(b_2)\cdots x_{n+1,n}(b_n)x_{n+2,1}(*),*)$. But if $z_1 = z_{n+1} = 0$ then the situation is the one described in definition 3.26.

3.29 What Lemma 3.28 tells us is that, if $\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$ is defined at the bottom, one has additivity for those v which make a certain piece of a column unimodular. This is the sort of situation the conditions $SR_n^p(c,u)$ refer to. So let us apply them. (Until now we only used SR_n). The case of commutative semi-local rings diverges from the "general" case at this point (for the first time).

3.30 <u>Proposition</u> (Additivity for maps defined at the bottom). Let $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ be defined at the bottom, with value $\langle P,Q\rangle$, let $\mathcal{L}(x_{n+2}(w))\langle P,Q\rangle$ be defined at the bottom and let $\mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$ be defined at the bottom. Then
$\mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(w))\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$.

<u>Proof</u> We assume $\tilde{\tilde{SR}}_n$ and therefore $SR_n^3(n+2,n+1)$. If $v+w=0$ then it is easy: apply Lemma 3.23 and Lemma 3.25 for instance. (Or just apply the definitions.) In the general case the idea is to chose z such that repeated application of Lemma 3.28 yields:
$\mathcal{L}(x_{n+2}(z))\mathcal{L}(x_{n+2}(w))\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle =$
$\mathcal{L}(x_{n+2}(z+w))\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(z+v+w))\langle X,Y\rangle =$
$\mathcal{L}(x_{n+2}(z))\mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$. Then one applies $\mathcal{L}(x_{n+2}(-z))$ to both ends of the string. So say $\langle P,Q\rangle = \langle Tx_{n+2,1}(*),*\rangle$ with $x_{n+2}(w)T = Tx_{n+1,n+2}(*)$. Then we want that $x_{n+2}(z)T = Tx_{n+2}(*,a_2,\ldots,a_n,*)$ with $(a_2,\ldots,a_n)$ unimodular. (That will guarantee that the first two members of the string exist and that they are equal). There is an elementary matrix $M_1 = \underline{mat}(T^{-1})$ in $E(n+1)$ such that $(*,a_2,\ldots,a_n,*)$ is just M_1z. Choose A_1 to be the matrix ($n-1$ by $n+2$) with first column zero and the remaining part taken from rows 2 through n of M_1. Then A_1 is a matrix of the type which one considers in $SR_n^3(n+2,n+1)$. (It doesn't matter that the obvious way to add rows to the "U_1-part" of A_1 is not the way the reader expected when reading 2.2: If a matrix is in $E(n+1)$ up to a permutation of rows, then the matrix is in $E(n+1)$ up to multiplication of one row by a sign. (Use the $w_{ij}(1)$)). We will have unimodular $(a_2,\ldots,a_n)$ if and only if $A_1\binom{1}{z}$ is unimodular. Next we want the third member of the string to be defined and we want it to be equal to the second member. Now there is a matrix $M_2\epsilon E(n+1)$ such that it works if $M_2(z+w) = (*,b_2,\ldots,b_n,*)$ with $(b_2,\ldots,b_n)$ unimodular. Choose A_2 to be the matrix ($n-1$ by $n+2$) with the first column equal to the middle part of M_2w and the remainder taken from row 2 through n of M_2. Again we have translated what we want into the form "$A_i\binom{1}{z}$ is unimodular", with A_i of the proper type. Finally there is A_3 which stands for the wish to have the fourth member exist and to have it equal to the third. By $SR_n^3(n+2,n+1)$

we can choose z such that it works. It is easy to see (from the fact that the executions of $\mathcal{L}(x_{n+2}(z))$ are defined at the bottom) that one can apply $\mathcal{L}(x_{n+2}(-z))$ to both ends. This is an instance of the case $v + w = 0$ and we see that the $\mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(z))$ cancel out.

3.31 Definition Let $v \epsilon R^{n+1}$, $\langle X,Y \rangle \epsilon C$ such that $\underline{mat}(x_{n+2}(v))XY) \epsilon \underline{mat}(C)$. We claim there is $z \epsilon R^{n+1}$ such that both steps in $\mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(v + z))\langle X,Y\rangle$ are defined at the bottom. We define $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ to be equal to the result. We need to do some checks to see that this definition is consistent. First of all it is consistent with the earlier definitions in case $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ is defined at the bottom, by Proposition 3.30. One can also check that it is consistent with the other earlier definitions, using Lemma 3.24, but those definitions have served their purpose anyway. (So let's overrule them).

3.32 So let us prove that the definition makes sense. First let us show that z exists. As in 3.30 we see that there is an n by $n + 2$ matrix A_1 with the property that $\mathcal{L}(x_{n+2}(v + z))\langle X,Y\rangle$ is defined at the bottom (in the way described by Lemma 3.20) if $A_1\binom{1}{z}$ is unimodular. Moreover, this matrix is of the type one considers in $SR^4_{n+1}(n + 2, n + 1)$. (This proof doesn't work for arbitrary commutative semi-local rings). Choose $\langle P',Q'\rangle \epsilon C$ such that $\underline{mat}(P'Q') = \underline{mat}(x_{n+2}(v)XY)$. There is an n by $n + 2$ matrix A_2, of the proper type, such that $\mathcal{L}(x_{n+2}(z))\langle P',Q'\rangle$ is defined at the bottom if $A_2\binom{1}{z}$ is unimodular. But if $\mathcal{L}(x_{n+2}(z))\langle P',Q'\rangle$ is defined at the bottom then the second step in $\mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(z))\langle P',Q'\rangle$ is also defined at the bottom. Therefore, if both $A_1\binom{1}{z}$ and $A_2\binom{1}{z}$ are unimodular, both steps in $\mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(v + z))\langle X,Y\rangle$ are defined at the bottom. (Use

Lemma 3.22). So z exists. We have only used A_1 and A_2, so we might still choose A_3, A_4 because we have $SR^4_{n+1}(n+2, n+1)$ available. (So far it was just $SR^2_{n+1}(n+2, n+1)$ which actually holds for commutative semi-local rings too). We need to show that another z gives the same answer. So suppose that z' also fits, i.e. that both steps in $\mathcal{L}(x_{n+2}(-z'))\mathcal{L}(x_{n+2}(v+z'))\langle X,Y\rangle$ are defined at the bottom. The idea is to choose $t \in R^{n+1}$ such that all steps in $\mathcal{L}(x_{n+2}(t-z))\mathcal{L}(x_{n+2}(v+z))\langle X,Y\rangle$, $\mathcal{L}(x_{n+2}(t+v))\langle X,Y\rangle$, $\mathcal{L}(x_{n+2}(t-z'))\mathcal{L}(x_{n+2}(v+z'))\langle X,Y\rangle$, $\mathcal{L}(x_{n+2}(t))\mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(v+z))\langle X,Y\rangle$ are defined at the bottom. That amounts to four conditions on t and they can simultaneously be satisfied because of $SR^4_{n+1}(n+2, n+1)$. (so we don't use the old A_1, A_2 but a new set of four matrices, chosen after z, z'). Using Lemma 3.22 we see that all steps in the following computation are defined at the bottom, which makes that Proposition 3.30 applies: $\mathcal{L}(x_{n+2}(t))\mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(v+z))\langle X,Y\rangle =$
$\mathcal{L}(x_{n+2}(t-z))\mathcal{L}(x_{n+2}(v+z))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(t+v))\langle X,Y\rangle =$
$\mathcal{L}(x_{n+2}(t-z'))\mathcal{L}(x_{n+2}(v+z'))\langle X,Y\rangle =$
$\mathcal{L}(x_{n+2}(t))\mathcal{L}(x_{n+2}(-z'))\mathcal{L}(x_{n+2}(v+z'))\langle X,Y\rangle$. Now apply $\mathcal{L}(x_{n+2}(-t))$ to both ends.

3.33 <u>Proposition</u> (Additivity).
$\mathcal{L}(x_{n+2}(v))\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$ whenever the left hand side is defined.

<u>Proof</u> Obviously the right hand side is defined if the left hand side is. (Read 3.31). So assume this is the case. Say $\langle P,Q\rangle = \mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$. If both $\mathcal{L}(x_{n+2}(v))\langle P,Q\rangle$ and $\mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$ are defined at the bottom then it is an easy consequence of the definition of $\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$. Choose z such that $\mathcal{L}(x_{n+2}(v))\langle P,Q\rangle = \mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(v+z))\langle P,Q\rangle$, with both

steps at the right hand side defined at the bottom, and such that $\mathcal{L}(x_{n+2}(z+v+w))\langle X,Y\rangle$ is defined at the bottom. (This is an $SR^3_{n+1}(n+2,n+1)$ type problem so it can be solved by virtue of $SR^4_{n+1}(n+2,n+1)$. We still are doing things that don't work for some commutative semi-local rings). We have to show that $\mathcal{L}(x_{n+2}(v+z))\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(z))\mathcal{L}(x_{n+2}(v+w))\langle X,Y\rangle$. But the right hand side is $\mathcal{L}(x_{n+2}(v+w+z))\langle X,Y\rangle$ by the case discussed above. But then we are back at just this same case.

3.34 <u>Lemma</u> Let $v,w \in R^{n+1}, T \in St(n+1)$, $X \in$ <u>Low</u>, $Y \in$ <u>Up</u> such that $x_{n+2}(v)T = Tx_{n+2}(w)$ and such that $\mathcal{L}(x_{n+2}(v))\langle TX,Y\rangle$ is defined. Then $\mathcal{L}(x_{n+2}(v))\langle TX,Y\rangle = \mathcal{L}(T)\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle$.

<u>Proof</u> By definition $\mathcal{L}(x_{n+2}(w))\langle X,Y\rangle = \mathcal{L}(x_{n+2}(-z))\mathcal{L}(x_{n+2}(w+z))\langle X,Y\rangle$ with both steps at the right hand side defined at the bottom. Now apply Lemma 3.23.

3.35 In Section 5 we will have to find an alternative for 3.30, 3.31, 3.32 such that 3.33, 3.34 still hold and such that $\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ is defined if <u>mat</u>$(x_{n+2}(v)XY) \in$ <u>mat</u>(C). The next proofs and definitions will then go through for commutative semi-local rings too.

3.36 <u>Lemma</u> The set <u>mat</u>(C) (see 3.8) consists of all matrices in $E(n+2)$ whose first column is of the form $(a_1,\ldots,a_{n+2})$ with $(a_1,\ldots,a_{n+1})$ unimodular.

<u>Proof</u> Obviously every element of <u>mat</u>(C) looks like that. Conversely, let M be such a matrix. Multiplying M from the left by a matrix in $E(n+1)$ one reduces to the case $(a_1,\ldots,a_{n+2}) = (1,0,\ldots,0,a_{n+2})$, because $E(n+1)$ acts transitively

on unimodular columns of length n + 1 (see [4], Ch. V, Thm. (3.3)). So multiplying M from the left by an element of <u>mat</u>(<u>Low</u>) we reduce to the case that the first column of M is trivial. Multiplying from the right by an element of <u>mat</u>(<u>Up</u>) we can get the first row trivial too. By injective stability for K_1 the matrix M then is in <u>mat</u> St({1}* x {1}*), so certainly in <u>mat</u>(<u>Up</u>).

3.37 For the sake of completeness we include the following Lemma. (Compare with 2.2).

<u>Lemma</u> Let $n \geq m \geq 2$. Let A be an $(m-1) \times n$ matrix over a ring S, where S satisfies SR_m. Suppose that A can be completed to an invertible $n \times n$ matrix. Then there is $\lambda \in S^{n-1}$ such that $A\binom{1}{\lambda}$ is unimodular.

<u>Proof</u> We use the notations of 3.4, except that the base ring is now S, not R. Let $M \in GL(n,S)$ be such that A consists of the top $m-1$ rows of M. It is enough to show the following. There is $L \in GL(n,S)$, with trivial first row, such that in the first column $(a_1,\ldots,a_n)$ of ML the piece $(a_1,\ldots,a_{m-1})$ is unimodular. Using SR_n, which follows from SR_m as $n \geq m$, we choose $L_1 \in E([n] \times \{1\})$, $U \in E(\{1\} \times [n])$ so that the $(1,n)$-entry of UL_1M^{-1} is zero. As $(UL_1M^{-1})(ML_1^{-1}) = U$, one sees that in the first column $(b_1,\ldots,b_n)$ of ML_1^{-1} the piece $(b_1,\ldots,b_{n-1})$ is unimodular. So if $m = n$ we are done. Otherwise $n - 1 \geq m$ and $E(n - 1)$ acts transitively on unimodular columns of length $n - 1$. So choose $T \in E([n] \times [n-1])$ so that the first column of TML_1^{-1} is trivial. Then choose $B \in E(\{1\} \times [n])$ so that the first row of $L_2 = BTML_1^{-1}$ is also trivial. We may replace M by $ML_1^{-1}L_2^{-1}$. But $ML_1^{-1}L_2^{-1} = T^{-1}B^{-1}$ is an $n \times n$ matrix with an invertible $(n-1) \times (n-1)$ submatrix in the upper left hand corner. Thus the problem has been reduced in size and we can apply induction.

§4. The Group <u>Uleft</u>.

4.1 There is a left-right symmetry in the chunk: Consider the anti-homomorphism <u>inv</u> which sends $x_{ij}(r)$ to $x_{n+3-i,n+3-j}(-r)$. (It is the composite of the map $z \mapsto z^{-1}$ with a homomorphism). If $I,J \subseteq [n+2]$ then we get <u>inv</u>: $St(I \times J) \to St(I' \times J')$ where I' is the set $\{a \in [n+2] | n+3-a \in I\}$ or $\{n+3-a | a \in I\}$ and J' is the set $\{n+3-b | b \in J\}$. One has <u>inv</u>•<u>inv</u> = id, <u>inv</u>(<u>Low</u>) = <u>Up</u>, <u>inv</u>(<u>Med</u>) = <u>Med</u>, <u>inv</u>(<u>Up</u>) = <u>Low</u>. (We should write $\underline{inv}_{I\times J}$: $St(I \times J) \to St(I' \times J')$). It is easy to see that $\underline{inv}\langle X,Y\rangle = \langle \underline{inv}(Y), \underline{inv}(X)\rangle$ defines an involution of C, i.e. a map $C \to C$ which is its own inverse. We have defined in Section 3 what $\mathcal{L}(X)\langle P,Q\rangle$ means if $X,P \in$ <u>Low</u>, $Q \in$ <u>Up</u>. We also defined $\mathcal{R}(Y)\langle P,Q\rangle$ if $P \in$ <u>Low</u>, $Y,Q \in$ <u>Up</u>. The connection between the two notions is as follows: $\mathcal{R}(Y) = \underline{inv} \bullet \mathcal{L}(\underline{inv}(Y)) \bullet \underline{inv}$. This suggests to define $\mathcal{R}(x_1(v)) = \underline{inv} \bullet \mathcal{L}(\underline{inv}\ x_1(v)) \bullet \underline{inv}$, i.e. $\mathcal{R}(x_1(v_2,\dots,v_{n+2}))\langle X,Y\rangle = \underline{inv}(\mathcal{L}(x_{n+2}(v_{n+2},\dots,v_2))\langle \underline{inv}(Y), \underline{inv}(X)\rangle)$ whenever the right hand side is defined. So let us do that. Then $\mathcal{R}(x_1(v))\langle X,Y\rangle$ is defined if and only if $\underline{mat}(XYx_1(v)) \in \underline{mat}(C)$. This is just one of the properties we get by translating earlier results by means of <u>inv</u>. Other ones are: $\mathcal{R}(T)\mathcal{R}(x_1(v))\langle X,Y\rangle = \mathcal{R}(x_1(w))\mathcal{R}(T)\langle X,Y\rangle$ if both sides are defined and $T \in St(\{1\}^* \times \{1\}^*)$, $v = (v_2,\dots,v_{n+2})$, $w = (w_2,\dots,w_{n+2})$ with $Tx_1(w) = x_1(v)T$. And additivity: $\mathcal{R}(x_1(v))\mathcal{R}(x_1(w))\langle X,Y\rangle = \mathcal{R}(x_1(v+w))\langle X,Y\rangle$ if both sides are defined. It may seem more convenient to write $\langle X,Y\rangle\mathcal{R}(x_1(v))$ instead of $\mathcal{R}(x_1(v))\langle X,Y\rangle$. We don't do that because we want to emphasize the order of execution in expressions like $\mathcal{R}(x_1(w))\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$. In the alternative notation it would read $(\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle)\mathcal{R}(x_1(w))$. The reader may find however that certain arguments are better understood when one writes $\mathcal{R}$'s at the right.

4.2 We want to show that $\mathcal{L}(X)$, $\mathcal{L}(x_{n+2}(v))$ commute with $\mathcal{R}(T)$, $\mathcal{R}(x_1(w))$ if $X \in$ <u>Low</u>, $v \in R^{n+1}$, $w = (w_2,\ldots,w_{n+2})$, $T \in St(\{1\}^* \times \{1\}^*)$. (So T is less arbitrary than X). As some of these maps are only defined on part of C it only makes sense to prove, for instance, that $\mathcal{L}(x_{n+2}(v))\mathcal{R}(x_1(w))$ equals $\mathcal{R}(x_1(w))\mathcal{L}(x_{n+2}(v))$ where both compositions are defined.

4.3 <u>Notation</u> If f is a map defined on part of C and with values in C and if g is also such a map then $f \bullet g$ is the map which sends $\langle X,Y\rangle \in C$ to $f(g\langle X,Y\rangle)$ whenever the latter is defined. We say that $f \approx g$ if $f\langle X,Y\rangle = g\langle X,Y\rangle$ whenever both sides are defined. This is <u>not</u> an equivalence relation.

4.4 So we want to prove that $\mathcal{L}(X)\bullet\mathcal{R}(T) \approx \mathcal{R}(T)\bullet\mathcal{L}(X)$ etc. (See 4.2). In fact $\mathcal{L}(X)\bullet\mathcal{R}(T) \approx \mathcal{R}(T)\bullet\mathcal{L}(X)$ is a triviality (T and X as in 4.2). The non-trivial case to consider is the one of $\mathcal{L}(x_{n+2}(v))\bullet\mathcal{R}(x_1(w))$ versus $\mathcal{R}(x_1(w))\bullet\mathcal{L}(x_{n+2}(v))$.

4.5 <u>Definition</u> We say that $\mathcal{L}(x_{n+2}(v))$, $\mathcal{R}(x_1(w))$ <u>slide past each other at</u> $\langle \underline{X},\underline{Y}\rangle$ if there are $T \in St(n+1)$, $U \in St(\{1\}^* \times \{1\}^*)$, $A \in St(\{n+2\} \times [n+2])$, $B \in St(\{1\} \times [n+2])$, $2 \leq k \leq n$, such that $\langle X,Y\rangle = \langle TA,BU\rangle$ and such that
$x_{n+2}(v)T = Tx_{n+2}(0,0,\ldots,0,f_{k+1},\ldots,f_{n+1})$,
$Ux_1(w) = x_1(g_2,\ldots,g_k,0,\ldots,0)U$. We could also require that actually $A \in St(\{n+2\} \times [k])$ because one can take the part of A which comes from $St(\{n+2\} \times [k]^*)$ to the right without spoiling anything. Having done that one can do the same sort of thing to B and reduce to the case that $B \in St(\{1\} \times [k]^*)$. So if we want to prove that $\mathcal{L}(x_{n+2}(v))$, $\mathcal{R}(x_1(w))$ slide past each other at $\langle X,Y\rangle$

we only need to have $A \in St(\{n + 2\} \times [n + 2])$, $B \in St(\{1\} \times [n + 2])$. But if we apply that they slide past each other we usually take $A \in St(\{n + 2\} \times [k])$, $B \in St(\{1\} \times [k]^*)$. We refer to this particular choice by saying that (TA,BU) is separated with respect to $x_{n+2}(v)$, $x_1(w)$.

4.6 Proposition Let $\mathcal{L}(x_{n+2}(v))$, $\mathcal{R}(x_1(w))$ slide past each other at $\langle X,Y\rangle$. Then $\mathcal{L}(x_{n+2}(v))\mathcal{R}(x_1(w))\langle X,Y\rangle = \mathcal{R}(x_1(w))\mathcal{L}(x_{n+2}(v))\langle X,Y\rangle$ (Both sides are defined).

Proof So take $(TA,BU) \in \langle X,Y\rangle$ separated with respect to $x_{n+2}(v)$, $x_1(w)$. One easily sees that $\mathcal{R}(U) \bullet \mathcal{L}(x_{n+2}(v)) \approx \mathcal{L}(x_{n+2}(v)) \bullet \mathcal{R}(U)$. (Compare 4.4). Its counterpart at the other side states $\mathcal{L}(T) \bullet \mathcal{R}(x_1(w)) \approx \mathcal{R}(x_1(w)) \bullet \mathcal{L}(T)$. From this and Lemma 3.34 one sees that we may assume $T = 1$. Similarly we may assume $U = 1$. Say $A = x_{n+2,1}(a_1) \cdots x_{n+2,k}(a_k)$ and $B = x_{1,k+1}(b_{k+1}) \cdots x_{1,n+2}(b_{n+2})$. We just compute both $\mathcal{L}(x_{n+2}(v))\mathcal{R}(x_1(w))\langle A,B\rangle$ and $\mathcal{R}(x_1(w))\mathcal{L}(x_{n+2}(v))\langle A,B\rangle$ and compare: $\mathcal{L}(x_{n+2}(v))\mathcal{R}(x_1(w))\langle A,B\rangle = \mathcal{L}(x_{n+2}(v))\mathcal{R}(x_1(w))\langle Ax_{1,k+1}(b_{k+1}) \cdots x_{1,n+1}(b_{n+1}), x_{1,n+2}(b_{n+2})\rangle = \mathcal{L}(x_{n+2}(v))\langle Ax_{1,k+1}(b_{k+1}) \cdots x_{1,n+1}(b_{n+1})x_1(w), x_{n+2}(b_{n+2}, -w_2b_{n+2}, \ldots, -w_kb_{n+2}, 0, \ldots, 0)\rangle =$ $\mathcal{L}(x_{n+2}(v))\langle Ax_1(w), PBx_{n+2}(0, -w_2b_{n+2}, \ldots, w_kb_{n+2}, 0, \ldots, 0)\rangle$ with P = product of the $x_{ij}(-w_ib_j)$ with $2 \leq i \leq k$, $k + 1 \leq j \leq n + 1$. So we get $\mathcal{L}(x_{n+2}(v))\langle x_1(w)x_{n+2,1}(a_1 + a_2w_2 + \cdots + a_kw_k)$, $x_{n+2,2}(a_2) \cdots x_{n+2,k}(a_k)PBx_{n+2}(0, -w_2b_{n+2}, \ldots, -w_kb_{n+2}, 0, \ldots, 0)\rangle$. Say $q = a_1 + a_2w_2 + \cdots + a_kw_k$. Then it equals $\langle x_1(w_2, \ldots, w_k, v_{k+1}q, \ldots, v_{n+1}q, q), x_{n+2}(v)x_{n+2,2}(a_2) \cdots x_{n+2,k}(a_k)PBx_{n+2}(0, -w_2b_{n+2}, \ldots, -w_kb_{n+2}, 0, \ldots, 0)\rangle$, or $\langle x_1(w_2, \ldots, w_k, v_{k+1}q, \ldots, v_{n+1}q, q)x_{n+2,2}(a_2) \cdots x_{n+2,k}(a_k)Q$, $x_{n+2}(v)PBx_{n+2}(0, -w_2b_{n+2}, \ldots, -w_kb_{n+2}, 0, \ldots, 0)\rangle$ where Q = product of the $x_{ij}(v_ia_j)$ with $k + 1 \leq i \leq n + 1$, $2 \leq j \leq k$. That again is the same as $\langle x_1(w_2,\ldots,w_k,v_{k+1}q,\ldots,v_{n+1}q,q)x_{n+2,2}(a_2)\cdots x_{n+2,k}(a_k)Q,$ $Px_{1,k+1}(b_{k+1})\cdots x_{1,n+1}(b_{n+1})x_{n+2}(p,-w_2p,\ldots,-w_kp,v_{k+1},\ldots,v_{n+1})\rangle$ where $p = b_{n+2} - b_{k+1}v_{k+1} - \cdots - b_{n+1}v_{n+1}$. So $\mathcal{L}(x_{n+2}(v))\,\mathcal{R}(x_1(w))\langle A,B\rangle$ is equal to this very symmetric expression. Using inv or doing the same sort of computation for $\mathcal{R}(x_1(w))\,\mathcal{L}(x_{n+2}(v))\langle A,B\rangle$ one sees that the results are the same. Other proof: Use that $\mathcal{L}(x_{n+2}(v))\,\mathcal{R}(x_1(w))\langle A,B\rangle$ apparently can be written in the form $\langle$(product of $x_{ij}(r)$'s with $i > j$), (product of $x_{ij}(r)$'s with $i < j$)$\rangle$. Then, applying inv, derive the same result for $\mathcal{R}(x_1(w))\,\mathcal{L}(x_{n+2}(v))\langle A,B\rangle$. And show that two elements of this particular form are equal as soon as their images under mat are equal. (Reduce for instance to the case that one of the two elements is trivial). For $n > 2$ there still is another proof, based on writing $v = z + (v-z)$ where z and $v - z$ have more zeroes than we assumed for v. The squeezing principle will then do the job. At any rate, the computation may look horrendous but there really is no problem.

4.7 Lemma Let $A\in St(\{n+2\} \times [n+2])$, $B\in (St\{1\} \times [n+2])$, $T\in St(n+1)$, $U\in St(\{1\}^* \times \{1\}^*)$, $v = (v_1,\ldots,v_{n+1})$, $w = (w_2,\ldots,w_{n+2})$, $z = (z_2,\ldots,z_{n+2})$ such that $x_{n+2}(v)T = Tx_{n+1,n+2}(*)$, $Ux_1(w) = x_1(z)U$, $(z_2,\ldots,z_n)$ is unimodular. Then $\mathcal{L}(x_{n+2}(v))$, $\mathcal{R}(x_1(w))$ slide past each other at $\langle TA,BU\rangle$.

Proof We may assume $T = 1$ and $U = 1$. We want to get rid of w_{n+1} and w_{n+2}. As $(w_2,\ldots,w_n)$ is unimodular, there is $P\in St(\{n+1,n+2\} \times \{1,n+2\}^*)$ with $Px_1(w) = x_1(w_2,\ldots,w_n,0,0)P$. We have $\langle A,B\rangle = \langle AP^{-1},PB\rangle$ and inspection shows that $\mathcal{L}(x_{n+2}(v))$, $\mathcal{R}(x_1(w))$ slide past each other at $\langle AP^{-1},PB\rangle$.

4.8 So now we are in a situation comparable with 3.29. We can