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Rational and Generic Cohomology* **

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Let G be a semisimple algebraic group defined and split over $k_0 = GF(p)$. For $q = p^m$, let $G(q)$ be the subgroup of $GF(q)$-rational points. The main objective of this paper is to relate the cohomology of the finite groups $G(q)$ to the *rational* cohomology of the algebraic group G. Let V be a finite dimensional rational G-module, and, for a non-negative integer e, let $V(e)$ be the G-module obtained by "twisting" the original G-action on V by the Frobenius endomorphism $x \mapsto x^{[p^e]}$ of G. Theorem (6.6) states that, for sufficiently large q and e (depending on V and n), there are isomorphisms $H^n(G, V(e)) \cong H^n(G(q), V(e)) \cong H^n(G(q), V)$ where the first map is restriction. In particular, the cohomology groups $H^n(G(q), V)$ have a stable or "generic" value $H^n_{\text{gen}}(G, V)$. This phenomenon had been observed empirically many times (cf. [6, 20]). The computation of generic cohomology reduces essentially to the computation of rational cohomology. One (surprising) consequence is that $H^n_{\text{gen}}(G, V)$ does not depend on the exact weight lattice for a group G of a given type cf. (6.10), though this considerably affects the structure of $G(q)$. We also obtain that rational cohomology takes a stable value relative to twisting – i.e., for sufficiently large ε, we have semilinear isomorphisms $H^n(G, V(\varepsilon)) \cong H^n(G, V(e))$ for all $e \geqq \varepsilon$.

This paper contains many new results on rational cohomology beyond those required for the proof of the main theorem. We mention in particular the vanishing theorems (2.4) and (3.3), and especially the results (3.9) through (3.11) which relate $H^2(G, V)$ and $\mathrm{Ext}^1_G(V, W)$ to the structure of Weyl modules. These results explain for example the generic values of H^1 determined in [6], cf. (7.6). Also, it is shown in Theorem (3.12) that every finite dimensional rational G-module has a finite resolution by finite dimensional acyclic G-modules.

A key ingredient in the proofs is an important theorem of G. Kempf [19] on the vanishing of cohomology of certain homogeneous line bundles. This result is translated into the language of rational cohomology in (1.2), and is used in

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the proof of the transfer theorem (2.1) as well as the vanishing criterion (3.2). Another frequently used result is the calculation of the cohomology ring $H^*({}_k\mathrm{Add}, k)$ given in Section 4.

Finally we mention that this paper contains a short proof of the Mumford conjecture first proved by Haboush [11]; cf. (3.6).

§ 1. Some Preliminaries

Let k be an algebraically closed field of characteristic $p \geqq 0$, and let k_0 be the prime field of k. All algebraic groups in this paper are taken to be affine and defined over k. Our notation conforms closely to that of [8]; in particular, $R(G)$ denotes the affine coordinate ring of the algebraic group G. For $g \in G$, $f \in R(G)$, $g \cdot f$ and $f \cdot g$ are the elements of $R(G)$ defined by $(g \cdot f)(x) = f(xg)$ and $(f \cdot g)(x) = f(gx)$ for $x \in G$. Finally a G-module V is rational if it is a sum of finite dimensional submodules which are rational in the usual sense.

Given an algebraic group G and a rational G-module M, we can consider the rational (Hochschild) cohomology groups $H^n(G, M)$, $n \geqq 0$, defined in the usual manner from a rationally injective resolution of M (cf. [10] and [16]). Thus, if F_G is the fixed-point functor $M \mapsto M^G$, then $H^n(G, M) \cong R^n F_G(M)$, the n^{th} right derived functor. Similarly, in the category of rational G-modules we can talk about the derived functors Ext_G^n of Hom_G. When V is a finite dimensional rational G-module, $\mathrm{Ext}_G^n(V, M) \cong H^n(G, V^* \otimes M)$ for all rational G-modules M, where V^* is the rational G-module dual to V.

In what follows, use will be made of the Lyndon spectral sequence for rational cohomology. This is defined in the following manner: Let N be a closed normal subgroup of G. It is a formality that I^N is a rationally injective G/N-module when I is a rationally injective G-module. Hence, if M is a rational G-module, by [14; Thm. 2.4.1, p. 148], there exists a spectral sequence

$$E_2^{p,q} = H^p(G/N, R^q F_N(M)) \Rightarrow E_\infty^{p+q} = H^{p+q}(G, M).$$

Since G/N is an affine variety, [8; Thm. (4.3)] implies immediately that $R^q F_N(M) \cong H^q(N, M)$. This proves the following result of Haboush [12]:

(1.1) **Lemma.** *Let N be a closed normal subgroup of an affine algebraic group G, and let M be a rational G-module. Then there exists a Lyndon spectral sequence*

$$E_2^{p,q} = H^p(G/N, H^q(N, M)) \Rightarrow E_\infty^{p+q} = H^{p+q}(G, M).$$

As an immediate consequence, note that if N consists entirely of semisimple elements, then $H^*(G, M) \cong H^*(G/N, M^N)$. Dually, if G/N consists of semisimple elements, then $H^*(G, M) \cong H^*(N, M)^{G/N}$.

For purposes of making routine verifications, it is convenient to have this spectral sequence in a somewhat more explicit form. Let I^* (resp. J^*) be a resolution of the trivial module k by rationally injective G-modules (resp. G/N-modules). For example, we can take I^* to be the standard (homogeneous) resolution with $I^n = R(G^{n+1})$. Then $J^* \otimes (I^* \otimes M)^N$ is a G/N-injective resolution of the complex $(I^* \otimes M)^N$ in the sense required by Grothendieck [14], and so the spectral sequence above may be identified with the spectral sequence of the double complex $(J^* \otimes (I^* \otimes M)^N)^{G/N}$.

In particular, if G_0 is a closed subgroup of G and $N_0 \lhd G_0$ is a closed subgroup of N, then restriction induces a map of spectral sequences, which, most simply, is just that arising from the restriction maps

$$(R((G/N)^{p+1}) \otimes (R(G^{q+1}) \otimes M)^N)^{G/N} \to (R((G_0/N_0)^{p+1}) \otimes (R(G_0^{q+1}) \otimes M)^{N_0})^{G_0/N_0}.$$

The existence of the restriction map on the Lyndon spectral sequence will be heavily used in § 5.

Suppose k has positive characteristic, and let $\mathfrak{N}$ be a restricted ideal of the Lie algebra of G which is stable under the adjoint action of G. We denote by $G/\mathfrak{N}$ the quotient group of G by $\mathfrak{N}$ [2; § 5], and for a rational G-module M we denote by $M^{\mathfrak{N}}$ the subspace of M annihilated by $\mathfrak{N}$. Clearly, $M^{\mathfrak{N}}$ is a rational $G/\mathfrak{N}$-module (and as such is even rationally injective when M is a rationally injective G-module). Let $0 \to M \to I^*$ be a resolution of M by rationally injective G-modules I^n. If we assume $\mathfrak{N}$ is *toral* (i.e., consists of semisimple elements, and hence is abelian), we get that $0 \to M^{\mathfrak{N}} \to I^{*\mathfrak{N}}$ is a resolution of $M^{\mathfrak{N}}$ by rationally injective $G/\mathfrak{N}$-modules $I^{n\mathfrak{N}}$. Clearly, $(I^{n\mathfrak{N}})^{G/\mathfrak{N}} = I^{nG}$, and hence $H^*(G, M) \cong H^*(G/\mathfrak{N}, M)$[1]. We also obtain the following result which will be used in (2.7) below: Suppose $f: G \to G'$ is a surjective central isogeny of semisimple groups [3; § 2] and let M be a rational G'-module. We regard M as a G-module by means of f, and it follows, using [3; Cor. (2.19)] and the remark immediately following (1.1), that $H^*(G, M) \cong H^*(G', M)$.

When G is a semisimple algebraic group over k, the following notation will be used: B will be a Borel subgroup and T a fixed maximal torus of B. We denote by Σ the root system of T in G, viewed as a subset of the character group (or weight lattice) $\Lambda = X^*(B) = X^*(T)$. Let Δ be the set of fundamental roots defined by B, and let $\Lambda^+ \subseteq \Lambda$ be the corresponding set of dominant integral weights. Let $(\ ,\ )$ be a fixed positive definite symmetric bilinear form on $\mathbb{Q} \otimes X^*(T)$, invariant under the Weyl group, and for a root α, set $\langle \lambda, \alpha \rangle = 2(\lambda, \alpha)/(\alpha, \alpha)$. We will denote by Q the root lattice $\mathbb{Z}\Sigma$, and by Q^+ the set $\mathbb{Z}^+ \Sigma^+$ where Σ^+ is the set of positive roots defined by Δ, and $\mathbb{Z}^+$ is the set of *non-negative* integers. Finally, when G is defined and split over k_0 we will assume T (and hence B) is k_0-split. We write $B = T \cdot U$.

The theorem below is essentially due to G. Kempf [19]; our proof is merely a translation of his results into the language of rational cohomology. Let G be a semisimple group. For $\lambda \in \Lambda$, we also denote by λ the one-dimensional rational representation of B afforded by λ.

(1.2) **Theorem.** *For* $\lambda \in \Lambda^+$, $H^n(B, R(G) \otimes -\lambda) = 0$ *for all positive integers* n.

Proof. For a rational B-module M, define a locally free $\mathcal{O}_{G/B}$-Module $\vartheta(M)$ as follows: for U open in G/B, set $\Gamma(U, \vartheta(M)) = (\Gamma(\pi^{-1}(U), \mathcal{O}_G) \otimes M)^B$, where

[1] We point out that this fact can also be obtained from a spectral sequence, similar to that of (1.1), which relates the cohomology of G to that of $G/\mathfrak{N}$ and $\mathfrak{N}$. The details are left to the interested reader. Still a third proof can be based on (2.1): $H^*(G, M) \cong H^*(B, M) \cong H^*(U, M)^T = H^*(U, M)^{T/\mathfrak{N}} \cong H^*(B/\mathfrak{N}, M) \cong H^*(G/\mathfrak{N}, M)$

$\pi: G \to G/B$ is the quotient morphism and B-fixed points are taken relative to the left diagonal action. If I is a rationally injective B-module, $\vartheta(I)$ is a Γ-acyclic $\mathcal{O}_{G/B}$-Module: it is enough to check this when $I = R(B)$, where it is easy. Thus, by (1.4) below, for any rational B-module M, $H^n(G/B, \vartheta(M)) \cong H^n(B, R(G) \otimes M)$ for all non-negative integers n. (This is a known result of Haboush [12].) Now by Kempf [19], we have $H^n(G/B, \vartheta(-\lambda)) = 0$ for all positive integers n if $\lambda \in \Lambda^+$. The theorem follows.

(1.3) *Remark.* Let H be a closed subgroup of an algebraic group G. If $0 \to M \to I_0 \to I_1 \to \cdots$ is a resolution of a rational H-module M by rationally injective H-modules, it is clear (almost by definition) that $H^n(H, R(G) \otimes M)$ is the n^{th} homology group of the complex $0 \to I_0|^G \to I_1|^G \to \cdots$ of induced modules. For the definition and properties of induced modules we refer the reader to [8]. In particular, if $H^n(H, R(G) \otimes M) = 0$ for all positive integers n, then $0 \to M|^G \to I_0|^G \to I_1|^G \to \cdots$ is a resolution of $M|^G$ by rationally injective G-modules.

For the rest of this paper, we assume the field k has *positive* characteristic p. This is mainly done for convenience, even though some of the results (e.g., Thm. (2.1)) clearly hold in characteristic zero. For an algebraic group G defined over the prime field k_0, if $q = p^n$, let $\sigma_q: G \to G$ be the Frobenius endomorphism induced by the field automorphism $x \mapsto x^q$. The fixed-points of σ_q will be denoted $G(q)$. Also, given a rational G-module V, and a non-negative integer e, we define $V(e)$ to be the rational G-module with underlying space V, but with a new G-action $g \cdot v = \sigma_{p^e}(g)\, v$. We remark that if G is semisimple and if V is irreducible of high weight λ then $V(e)$ is irreducible of high weight $p^e \lambda$.

We end this section by recalling a useful result of Grothendieck [14; Remark 3, p. 148]. Let $T: \mathcal{C} \to \mathcal{C}'$ be a left exact additive functor from an abelian category $\mathcal{C}$ with enough injectives to an abelian category $\mathcal{C}'$. Then we have

(1.4) **Lemma.** *Suppose $0 \to M \to A^*$ is a resolution of an object M of $\mathcal{C}$ by T-acyclics A^i (i.e., $R^p T(A^i) = 0$ if $p > 0$). Then $R^p T(M) \cong H^p(T(A^i))$ for all $p \geqq 0$.*

§ 2. A Transfer Theorem

Let G be semisimple over k, and let B be a Borel subgroup of G. The following was conjectured in [7; § 5, Remark]:

(2.1) **Theorem.** *If V is a rational G-module, then $H^n(G, V) \cong H^n(B, V)$ for all non-negative integers n.*

Proof. By (1.2) and (1.3), if $0 \to k \to I_0 \to I_1 \to \cdots$ is a resolution of k by rationally injective B-modules, then, inducing up to G, $0 \to k \to I_0|^G \to I_1|^G \to \cdots$ is a resolution of k by rationally injective G-modules. Thus, tensoring with V, we see $0 \to V \to V \otimes I_0 \to V \otimes I_1 \to \cdots$ and $0 \to V \to V \otimes (I_0|^G) \to V \otimes (I_1|^G) \to \cdots$ are rationally injective resolutions of V first as a B-module, then as a G-module. But $V \otimes (I_j|^G) \cong (V \otimes I_j)|^G$ by [8; Prop. (1.5)], and Frobenius reciprocity [8; Prop. (1.4)] yields an isomorphism $\mathrm{Hom}_G(k, V \otimes (I_j|^G)) \cong \mathrm{Hom}_B(k, V \otimes I_j)$. Q.E.D.

Our first application of this theorem occurs in the vanishing theorem proved below. We require two easy lemmas. Recall the root lattice is denoted by Q.

(2.2) **Lemma.** *Let η be a character of B. If $\eta \notin Q^+$, then $H^n(B,\eta)=0$ for all non-negative integers n.*

Proof. Let $0 \to \eta \to \eta \otimes R(U) \to \eta \otimes R(U) \otimes R(U) \to \cdots$ be the standard U-resolution, with T acting on $R(U)$ by conjugation. This is a rationally injective B-resolution by [8]. Clearly, $R(U)$ is a polynomial ring in generators y_α of T-weight $-\alpha$, α a positive root. Now our hypothesis implies that, on taking T-fixed points, one obtains the zero complex. Q.E.D.

Still another proof could be given using the Lyndon spectral sequence (1.1) and the precise computation of $H^*({}_k\mathrm{Add}, k)$ given in § 4.

Enumerate the fundamental system Δ as $\{\delta_1, \ldots, \delta_l\}$. Let $\eta \in \Lambda$, write $\eta = \Sigma r_i \delta_i$, and put $h(\eta) = \Sigma r_i$.

(2.3) **Lemma.** *For η as above, we have $H^n(B,\eta)=0$ for each $n>h(\eta)$.*

Proof. Let $\theta_1, \ldots, \theta_N$ be the elements of Σ^+, and write $\theta_j = \Sigma m_{jk}\delta_k$. We can compute $H^n(U,\eta)$ using the normalized n-cochains $C_0^n(U,\eta)$. These are rational maps $f: U \times \cdots \times U \to \eta$ (n copies) with $f(u_1, \ldots, u_n)=0$ whenever $u_i=1$ for some i. (See [13; p. 238] for a discussion – which carries over to rational cohomology – of normalized cochains for finite groups; cf. also [16].) Choose a basis of $C_0^n(U,\eta) \cong C_0^n(U,k) \otimes \eta$ consisting of monomials of degree $\geqq n$ in a set of T-weight vectors generating $R(U)$. The weight of such a monomial is $\mu = \eta - \Sigma a_j \theta_j$, where $a_j \in \mathbb{Z}^+$ and $\Sigma a_j \geqq n$. Since for each j there exists a k with $m_{jk}>0$, if we express η in terms of the δ's and sum over the coefficients we obtain

$$h(\eta) - \sum_{k,j} m_{jk} a_j \leqq h(\eta) - \sum_j a_j \leqq h(\eta) - n < 0.$$

Hence, $\mu \neq 0$. Q.E.D.

For a rational G-module V, let $h(V)$ denote the maximum of the $h(\eta)$'s where η runs over the weights of T in V.

(2.4) **Theorem.** a) *If no weight of T in V lies in Q^+, then $H^*(G,V)=0$. In particular, if V is irreducible of high weight $\lambda \notin Q$, then $H^*(G,V)=0$.*

b) *In any case, $H^n(G,V)=0$ for all $n>h(V)$. In particular, if V is irreducible of high weight λ, then $H^n(G,V)=0$ for all $n>h(\lambda)$.*

Proof. By (2.1), $H^*(G,V) \cong H^*(B,V)$. Clearly, we can assume V is finite dimensional. Using induction on the dimension of the B-module V and the long exact sequence of cohomology, we obtain part a) from (2.2), and part b) from (2.3). Q.E.D.

(2.5) *Remarks.* a) We record here the obvious fact that (2.4) holds for B as well as G.

b) It follows from (2.4b) and (1.4) that every finite dimensional rational G- (or B-)module has a finite resolution by acyclic modules (cf. § 3). In the next section we will show that the latter may be taken to be *finite dimensional.*

(2.6) *Example.* If $G=SL_2(k)$, $p=2$, and $\lambda = 2^n\lambda_1$, with λ_1 fundamental, then the bound in (2.4b) is sharp: If L is irreducible of high weight λ, then the weights of T in L are $\pm\lambda$. Since $H^*(B,-\lambda)=0$, by (2.2), the long exact sequence of cohomology and (2.1) yield

$$H^*(G, L) \cong H^*(B, \lambda) \cong H^*(U, \lambda)^T \cong H^*(U, k)_{-\lambda}.$$

If we apply (4.1), then we see $H^*(U, k) \cong S(V)$ where V is the vector space spanned by the T-weight vectors $a(-2^i\delta_1)$ of weight $-2^i\delta_1$, $i \geqq 0$. Now it is easily verified that the monomial $a(-\delta_1)^{2^n-1}$ spans $H^{2^n-1}(U, k)_{-\lambda}$.

(2.7) *Remark.* Given a rational G-module V, let V' be the submodule generated by the weight vectors having weights in the root lattice. Then by (2.4a) we obtain easily that $H^*(G, V) = H^*(G, V')$. Let $f\colon G \to G_1$ be a surjective central isogeny from G to the corresponding adjoint group G_1 (cf. [3; § 2]). The action of G on V' comes from one of G_1 on V', and it follows from the remarks of § 1 that $H^*(G, V') \cong H^*(G_1, V')$. Therefore, when treating only rational cohomology, we may always reduce to the case where G is of adjoint type. (A similar argument says we could also reduce to the simply connected case.)

§ 3. Acyclic Modules

A rational module M for an algebraic group G over k is called *acyclic* (or *G-acyclic*) if $H^n(G, M) = 0$ for all $n > 0$. If M is rationally injective, it is obviously acyclic, and the converse is true if G is unipotent (we leave the details to the reader).

(3.1) **Theorem.** (*Frobenius reciprocity for rational* Ext.) *Let H be a closed subgroup of an algebraic group G, and let M be a rational H-module. Assume that $R(G) \otimes M$ is H-acyclic. Then for any rational G-module V we have*

$$\operatorname{Ext}^n_G(V, M|^G) \cong \operatorname{Ext}^n_H(V|_H, M)$$

for all non-negative integers n.

Proof. Let $0 \to M \to I_0 \to I_1 \to \cdots$ be a resolution of M by rationally injective H-modules. Then, by (1.3), $0 \to M|^G \to I_0|^G \to I_1|^G \to \cdots$ is a resolution of the induced G-module $M|^G$ by rationally injective G-modules. Since, by Frobenius reciprocity [8; Prop. (1.4)], $\operatorname{Hom}_G(V, I_j|^G) \cong \operatorname{Hom}_H(V|_H, I_j)$, the result follows.

Let G be semisimple and let B be a Borel subgroup of G as before. Let $\psi \to \psi^*$ be the opposition involution on Λ. (That is, $\psi^* = -w_0\psi$ where w_0 is the element of the Weyl group with $w_0(\Sigma^+) = -\Sigma^+$.) Then $\psi^{**} = \psi$ and $(\Lambda^+)^* = \Lambda^+$. The character group Λ is partially ordered in the usual way. Clearly, $\psi \geqq \zeta$ iff $\psi^* \geqq \zeta^*$.

It is known [18] that $-\lambda|^G$ is the dual of the Weyl module $W(\lambda)$ obtained from reduction modulo p of the complex irreducible representation of high weight λ (of the corresponding complex semisimple Lie algebra) by a minimal $\mathcal{U}_{\mathbb{Z}}$-lattice (as explained in [23]). Essentially the argument is that [19] implies $\dim W(\lambda)^* = \dim -\lambda|^G$ (cf. for example the argument in [11; (1.1)]), while there is an obvious injection $W(\lambda)^* \to -\lambda|^G$ since $W(\lambda)^*$ has an irreducible socle.

(3.2) **Corollary.** *Let $\lambda \in \Lambda^+$, and let V be a finite dimensional rational G-module such that λ is not (strictly) less than $-\eta$ for any weight η of T in V. Then for all positive integers n, we have* $\operatorname{Ext}^n_G(V, -\lambda|^G) = 0$.

Proof. By (3.1) and (1.2), $\operatorname{Ext}^n_G(V, -\lambda|^G) \cong \operatorname{Ext}^n_B(V, -\lambda) \cong H^n(B, V^* \otimes -\lambda)$. Now the result follows from (2.4a) applied to B (cf. (2.5a)).

(3.3) **Corollary.** *For any pair* λ, $\mu \in \Lambda^+$, *the rational G-module* $-\lambda|^G \otimes -\mu|^G$ *is acyclic.*

Proof. Interchanging λ and μ if necessary we can assume that λ is not less than μ^*. The result now follows from (3.2) with $V = (-\mu|^G)^*$.

(3.3′) **Corollary.** *For any pair* λ, $\mu \in \Lambda^+$, *the rational B-module* $-\lambda|^G \otimes -\mu$ *is acyclic.*

(3.4) **Corollary.** *If* $\lambda \in \Lambda^+$, *then* $-\lambda|^G$ *is acyclic as a G- (or B-)module.*

Proof. Take $\mu = 0$ in (3.3) (resp. (3.3′)).

(3.5) **Corollary** *(Steinberg). We have* $H^2(G, k) = 0$.

Proof. Take $\lambda = 0$ in (3.4). (This also follows from (2.4b).)

(3.6) **Corollary.** *(Mumford conjecture* [11].*) Let V be a finite dimensional rational module for a simply connected semisimple group G having a non-zero fixed-point v. Then there exists a non-constant G-invariant homogeneous polynomial f on V with* $f(v) \neq 0$.

Proof. As in [11], it is enough to produce a G-map $h\colon V \to St(q) \otimes St(q)$ with $h(v) \neq 0$, where $St(q)$ is a Steinberg module for G (of high weight $(q-1)\rho$, where ρ is one-half the sum of the positive roots and q is a power of p). For this, note $St(q) \cong -(q-1)\rho|^G$, and so $\operatorname{Ext}^1_G(V/kv, St(q) \otimes St(q)) \cong \operatorname{Ext}^1_B((V/kv) \otimes St(q), -(q-1)\rho)$ (by (3.1) and (1.2)) $\cong H^1(B, (V/kv)^* \otimes -(q-1)\rho \otimes St(q)) = 0$ for sufficiently large q by (3.3′), applied to the composition factors of $(V/kv)^* \otimes -(q-1)\rho$. The result follows easily.[2]

As a corollary of the proof of (3.6), and using (2.7), we have

(3.7) **Corollary.** *Let V be a finite dimensional rational G-module. Then* $V \otimes St(q) \otimes St(q)$ *is G-acyclic for sufficiently large* $q = p^n$. *Also, if W is a finite dimensional rational B-module,* $W \otimes St(q) \otimes -(q-1)\rho$ *is B-acyclic for sufficiently large q.*

(3.8) **Corollary.** *Let V be a finite dimensional rational G-module (resp. B-module). For any non-negative integer n, we have that* $H^n(G, V)$ *(resp.* $H^n(B, V)$*) is a finite dimensional k-space.*

Proof. This clearly holds for $n = 0$. It follows for general n by using dimension shifting and (3.7), which implies finite dimensional modules can be embedded in finite dimensional acyclic modules.

The following results give effective methods of computing $H^2(G, V)$ and $\operatorname{Ext}^1_G(V, W)$ for many common irreducible modules V, W.

(3.9) **Corollary.** *Let* $V(\lambda)$ *be the irreducible G-module of high weight* λ. *Then*

$$H^2(G, V(\lambda)) \cong H^1(G, M(\lambda^*)^*)$$

where $M(\lambda^*)$ *is the unique maximal submodule of the Weyl module* $W(\lambda^*)$.

[2] The use of (1.2) here is not actually necessary: The discussion preceeding [11; (1.1)] and the proof of (1.2) show that $R(G) \otimes -(q-1)\rho$ is B-acyclic for large q. Also, [11; (1.1)] implies $-(q-1)\rho|^G \cong St(q)$ for large q

Proof. By (3.4), the long exact sequence of cohomology, and the fact that $-\lambda^*|^G \cong W(\lambda^*)^*$.

(3.10) **Corollary.** *Suppose λ, $\eta \in \Lambda^+$ with λ not less than η. If $V(\lambda)$, $V(\eta)$ are the irreducible modules of high weights λ, η respectively, then* $\mathrm{Ext}^1_G(V(\eta, V(\lambda)) \cong \mathrm{Hom}_G(V(\eta), M(\lambda^*)^*)$, *where $M(\lambda^*)$ is as in* (3.9).

Proof. Apply the covariant long exact sequence for Ext_G to the sequence $0 \to V(\lambda) \to -\lambda^*|^G \to M(\lambda^*)^* \to 0$ and use (3.2).

(3.11) *Remarks.* a) In view of the isomorphisms

$$\mathrm{Ext}^1_G(V(\eta), V(\lambda)) \cong \mathrm{Ext}^1(V(\lambda)^*, V(\eta)^*) \cong \mathrm{Ext}^1_G(V(\lambda^*), V(\eta^*))$$

and the fact that the opposition involution preserves the natural ordering on weights, (3.10) can be applied when $\lambda < \eta$.

b) In view of the isomorphism $\mathrm{Hom}_G(V(\eta), M(\lambda^*)^*) \cong \mathrm{Hom}_G(M(\lambda^*), V(\eta)^*)$, we may express (3.10) in terms of the structure of the Weyl modules:

$$\mathrm{Ext}^1_G(V(\lambda), V(\eta)) \cong \mathrm{Hom}_G(M(\lambda), V(\eta)).$$

In other words, if λ is not less than η, then the dimension of $\mathrm{Ext}^1_G(V(\lambda), V(\eta))$ is the multiplicity of $V(\eta)$ in the Frattini quotient of the unique maximal submodule $M(\lambda)$ of the Weyl module $W(\lambda)$.

c) The lemma (3.10) together with the structure theorem [4] allows an easy explicit determination of the extensions of irreducibles by irreducibles when $G = SL_2(k)$. These results will appear in [5].

Finally, we mention the following result.

(3.12) **Theorem.** *Any finite dimensional rational G-module (resp. B-module) V has a finite resolution by finite dimensional G-acyclic (resp. B-acyclic) modules.*

Proof. This is clear from (3.7) (cf. proof of (3.8)), and (2.4b).

§ 4. The ring $H^*({}_{k_0}\mathbf{Add}, k)$

Let T be a k_0-split torus acting on $U = {}_{k_0}\mathbf{Add}$ with weight α. We shall describe $H^*(U, k)$ as a T-algebra (under cup product, as defined in the usual way). Let V be a vector space with basis $a(-p^i\alpha)$, $i = 0, 1, \ldots$, and let T act on V by $t\,a(\mu) = \mu(t)\,a(\mu)$, $t \in T$. If p is odd, we shall require also a vector space W with basis $b(-p^i\alpha)$, $i = 1, 2, \ldots$, again with $t\,b(\mu) = \mu(t)\,b(\mu)$. Let $\beta\colon V \to W$ be the isomorphism defined by $\beta\,a(\mu) = b(p\,\mu)$, and note that $\beta(t^p v) = t\,\beta(v)$ for $v \in V$, $t \in T$. For any power q of p, let $V(q)$ be the subspace spanned by the vectors $a(-p^i\alpha)$, $p^i < q$. Set $W(q) = \beta V(q)$.

Let $S(V)$ (resp. $\Lambda(V)$) denote the symmetric algebra (resp. the exterior algebra) of V, viewed as graded T-algebras in the obvious sense.

(4.1) **Theorem.** *We have isomorphisms of graded T-algebras*

$H^*(U, k) \cong S(V)$ *if* $p = 2$;

$H^*(U, k) \cong \Lambda(V) \otimes S(W)$ *if* $p \neq 2$, *where V, W have degrees* 1, 2 *respectively. Moreover, the restriction map* $H^*(U,k) \to H^*(U(q),k)$ *induces* $T(q)$*-isomorphisms*

$$S(V(q)) \cong H^*(U(q),k) \quad \text{if } p=2;$$

$$\Lambda(V(q)) \otimes S(W(q)) \cong H^*(U(q),k) \quad \text{if } p \neq 2.$$

Finally, when $p \neq 2$, *the map* $\beta|_{V(q)}$ *defines the Bockstein homomorphism from* $H^1(U(q),k)$ *to* $H^2(U(q),k)$.

Proof. We can obtain $H^*(U,k)$ as the cohomology of the complex $C^*(U,k)$ of nonhomogeneous rational cochains (cf. [10; p. 185], [16; p. 497], and [13; p. 238]). The n^{th} term $C^n(U,k)$ consists of all rational maps from $U \times \cdots \times U$ (n copies) into k, and the differential on $C^n(U,k)$ is given by the formula

$$(\partial f)(u_1, \ldots, u_{n+1}) = f(u_2, \ldots, u_{n+1}) + \sum_{i=1}^{n} (-1)^i f(u_1, \ldots, u_i + u_{i+1}, \ldots, u_{n+1}) + (-1)^{n+1} f(u_1, \ldots, u_n).$$

We regard $C^n(U,k)$ as $R(U) \otimes \cdots \otimes R(U) = k[w_1, \ldots, w_n]$ in an obvious notation. Observe that the polynomials of degree $<q$ in each indeterminate w_i form a subcomplex $C^n(U,k;q)$. Moreover, this subcomplex is even closed under cup multiplication

$$(f \cup g)(w_1, \ldots, w_n) = f(w_1, \ldots, w_k)\, g(w_{k+1}, \ldots, w_n)$$

for f of degree k, g of degree $n-k$ (see [22; Ex. 1, p. 248] and [15; Ex. 13.6, p. 219]).

Next, consider the restriction map $C^*(U,k) \to C^*(U(q),k)$ from the nonhomogeneous complex for U to that for $U(q)$. This clearly induces an isomorphism $C^*(U,k;q) \cong C^*(U(q),k)$. In particular, the inclusion $C^*(U,k;q) \subseteq C^*(U,k)$ is split, and so remains an inclusion $H^*(U,k;q) \subseteq H^*(U,k)$ upon taking cohomology. Also, $H^*(U,k;q) \cong H^*(U(q),k)$.

We calculate $H^1(U,k) = Z^1(U,k)$ to be the span of the elements $w_1{}^r$ for r a power of p. If $p=2$, $H^*(U(q),k)$ is a symmetric algebra $S(H^1(U(q),k))$. Thus, the same is true for the subalgebra $H^*(U,k;q)$ of $H^*(U,k)$ – i.e., it is a polynomial ring generated by a basis for its degree 1 terms, namely,

$$H^*(U,k;q) \cong k[w_1{}^r; r|q, r<q].$$

The theorem follows in this case by setting $a(-r\alpha) = w_1{}^r$.

If p is odd, then it follows as above that $H^*(U,k) \cong \Lambda(V) \otimes S(W)$, where W consists of those terms corresponding to the degree 2 generators in the finite cases. These can be described in terms of the degree 1 generators by means of the Bockstein operator:

$$f(u) \mapsto \frac{(f(u)+f(v))^p - f(u)^p - f(v)^p}{p}.$$

This completes the proof of the theorem.

(4.2) *Remarks.* a) We record here the following fact, obtained from the proof of the theorem. Suppose $i, j \geq 0$ and $p^i \equiv p^j \pmod{(q-1)}$. Then the elements

$a(-p^i\alpha)$, $a(-p^j\alpha)$ have the same restriction to $H^*(U(q),k)$ as do $b(-p^i\alpha)$, $b(-p^j\alpha)$ when $i, j \geqq 1$.

b) It follows from the theorem that the restriction map $H^*(U,k) \to H^*(U(q),k)$ is surjective. This is a known result of [10; II, § 3, no. 4], where a description of $H^1(U,k)$ and $H^2(U,k)$ also appears.

§ 5. Some Reductions

We begin with some remarks on spectral sequences.

Suppose $f=\{f_r^{s,t}\}$ is a map $E \to \bar{E}$ of first quadrant cohomology spectral sequences, and assume $f_2^{s,t}$ is an isomorphism for $s+t \leqq m$ and is an injection for $s+t=m+1$. Then the same is true for all $f_r^{s,t}$ with $r \geqq 2$, and hence for $r=\infty$. Moreover, if $E_\infty^k = \bigoplus_{r+s=k} E_\infty^{r,s}$ and $\bar{E}_\infty^k = \bigoplus_{r+s=k} \bar{E}_\infty^{r,s}$ can be regarded as the graded objects associated with certain filtrations of abelian groups E_{tot}^k and $\bar{E}_{\text{tot}}^k$ in such a way that the maps $f_2^{r,s}$ with $r+s=k$ arise from filtration preserving maps $f_{\text{tot}}^k : E_{\text{tot}}^k \to \bar{E}_{\text{tot}}^k$, then f_{tot}^k is an isomorphism for $k \leqq m$ and an injection for $k=m+1$. We leave the details to the reader (cf. also [24]).

The long exact sequence of cohomology has a similar property with respect to isomorphism and injection: Suppose we have a map of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & A^k & \longrightarrow & B^k & \longrightarrow & C^k & \longrightarrow & A^{k+1} & \longrightarrow & B^{k+1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f^k} & & \downarrow{\scriptstyle g^k} & & \downarrow{\scriptstyle h^k} & & \downarrow{\scriptstyle f^{k+1}} & & \downarrow{\scriptstyle g^{k+1}} & \\
\cdots \longrightarrow & \bar{A}^k & \longrightarrow & \bar{B}^k & \longrightarrow & \bar{C}^k & \longrightarrow & \bar{A}^{k+1} & \longrightarrow & \bar{B}^{k+1} & \longrightarrow \cdots
\end{array}$$

such that f^k, h^k are isomorphisms for $k \leqq m$ and are injections for $k=m+1$. Then g^k is an isomorphism for $k \leqq m$ and is an injection for $k=m+1$. This may be checked by diagram-chasing, or application of the 5-lemma as stated in [21; p. 14].

Finally, we mention the following: Suppose $E=(E_r^{s,t}, d_r)$ is a spectral sequence whose terms are k-vector spaces. Suppose T is either a torus or a finite abelian p'-group which acts (rationally) as a group of automorphisms of E. If $\lambda \in X^*(T)$, then the weight spaces $(E_r^{s,t})_\lambda$, together with the restrictions of the differentials d_r, form a spectral sequence E_λ and $E=\bigoplus_\lambda E_\lambda$. We refer to [22] for details. We call E_λ the λ-component of E.

These remarks are applied as follows: Let $B=T \cdot U$ be the Borel subgroup of the semisimple group G, defined and split over $k_0=GF(p)$ as in § 1, and let V be a finite dimensional rational G-module. Let q be a power of p. We are interested in conditions which guarantee that a restriction map f^m: $H^m(B,V) \to H^m(B(q),V)$ is an isomorphism or is an injection. A first reduction is to the case where $V=\lambda$ is one dimensional: If, on each composition factor of V, f^n is an isomorphism for $n \leqq m$ and an injection for $n=m+1$, then f^n has these properties for V itself. Next, regarding $H^n(B,\lambda) \cong H^n(U,\lambda)^T \cong H^n(U,k)_{-\lambda}$, we take a T-stable central subgroup Z_1 of U and consider the $-\lambda$-component of the Lyndon spectral sequence (cf. § 1) for Z_1, U. The E_2 term is $(H^s(U/Z_1,k) \otimes H^t(Z_1,k))_{-\lambda}$. For fixed $t=t_1$, the abelian groups involved are in turn obtainable (up to a filtration) by a spectral sequence with E_2^{s,t_2} term given by $(H^s(U/Z_2,k)\otimes H^{t_2}(Z_2/Z_1,k)\otimes H^{t_1}(Z_1,k))_{-\lambda}$, where Z_2/Z_1 is a T-stable central subgroup of U/Z_1. This is just the $-\lambda$-component of the tensor product with $H^{t_1}(Z_1,k)$ of the E_2 term of a Lyndon spectral sequence. At this point we pause to note that similar constructions apply for $U(q)$ and $-\lambda|_{T(q)}$, and that restriction induces a map (cf. § 1) from the two spectral sequences considered so far to their counterparts for $U(q)$. (The groups Z_1, Z_2 are automatically σ_q-stable, being products of root groups.) If restriction is an isomorphism on the E_2^{s,t_2} terms above when $s+t_2+t_1\leqq m$ and an injection for $s+t_2+t_1=m+1$, then restriction is an isomorphism on the $E_2^{s,t}$ terms of the first spectral sequence when $s+t\leqq m$ and an injection for $s+t=m+1$. This in turn gives an isomorphism $H^n(B,\lambda)\to H^n(B(q),\lambda)$ for $n\leqq m$ and an injection for $n=m+1$.

Obviously we can continue this procedure, further refining the T-stable central series until the successive quotients are each covered by a single root group. Let $\theta_1,\dots,\theta_N$ be the positive roots, and denote by U_i the root group corresponding to a root θ_i. We have proved the following:

(5.1) **Lemma.** *Let V be a finite dimensional rational B-module. Suppose, for each weight λ of T in V, that restriction induces an isomorphism*

$$(\$)\quad (H^{s_1}(U_1,k)\otimes\cdots\otimes H^{s_N}(U_N,k))_{\lambda}\to(H^{s_1}(U_1(q),k)\otimes\cdots\otimes H^{s_N}(U_N(q),k))_{(\lambda|_{T(q)})}$$

for $s_1+\cdots+s_N\leqq m$ and an injection for $s_1+\cdots+s_N=m+1$. Then restriction induces an isomorphism $H^n(B,V)\to H^n(B(q),V)$ for $n\leqq m$ and an injection for $n=m+1$.

In view of (4.1), the question as to whether ($) is an isomorphism or an injection is purely arithmetic.

For example, let $p=2$. Let $L(s_1,\dots,s_N)$, $R(s_1,\dots,s_N)$ denote the left and right hand sides respectively of ($). Put $L(n)=\bigoplus_{s_1+\cdots+s_N=n} L(s_1,\dots,s_N)$ and define $R(n)$ similarly. By (4.1), $L(n)$ has a basis consisting of monomials $a(\mu_1)\dots a(\mu_n)$ with $\mu_k=-p^{i_k}\alpha_k$, $i_k\geqq 0$, $\alpha_k\in\Sigma^+$, and $\sum p^{i_k}\alpha_k=\lambda$. If $\mu=-p^i\alpha$ with $i\geqq 0$ and $\alpha\in\Sigma^+$, let $\bar{a}(\mu)$ denote the image of $a(\mu)$ under restriction. By (4.2a) we have that $\bar{a}(-p^i\alpha)=\bar{a}(-p^j\alpha)$ if $p^i\equiv p^j\ (\mathrm{mod}\ (q-1))$. Also, it follows from (4.1) that a basis for $R(n)$ consists of monomials $\bar{a}(\mu_1)\dots\bar{a}(\mu_n)$ with $\mu_k=-p^{i_k}\alpha_k$, $1\leqq p^{i_k}<q$, $\alpha_k\in\Sigma^+$, and $\sum_1^n p^{i_k}\alpha_k\equiv\lambda\ (\mathrm{mod}\ (q-1)\Lambda)$. Thus, the image of any basis monomial for $L(n)$ is one of the monomials of $R(n)$. In particular, ($) is injective on $L(n)$ if and only if no two basis monomials map to the same basis monomial in $R(n)$. A somewhat simpler sufficient condition is the following:

(5.2) **Injectivity Condition ($p=2$).** *In every equation*

$$\sum_{k=1}^n p^{i_k}\alpha_k=\lambda$$

with $i_k\geqq 0$ and $\alpha_k\in\Sigma^+$ for all k, we have $p^{i_k}<q$ for all k.

Similarly, ($) is an isomorphism if and only if it induces a bijection on basis monomials. A sufficient condition for this is the following:

(5.3) **Isomorphism Condition ($p=2$).** *Every congruence*

$$\sum_{k=1}^{n} p^{i_k} \alpha_k \equiv \lambda \qquad (\mathrm{mod}\,(q-1)\,\Lambda)$$

with $1 \leqq p^{i_k} < q$ *and* $\alpha_k \in \Sigma^+$ *for all* k, *is an equality*

$$\sum_{k=1}^{n} p^{i_k} \alpha_k = \lambda.$$

We give analogous conditions for $p \neq 2$ as follows. The details are similar to the above, and are left to the reader.

(5.4) **Injectivity Condition ($p \neq 2$).** *In every equation*

$$\sum_{k=1}^{n_1} p^{i_k} \alpha_k + \sum_{l=1}^{n_2} p^{j_l} \alpha_l = \lambda$$

with $n_1 + 2n_2 = n$, $i_k \geqq 0$, $j_l \geqq 1$, *and* α_k, $\alpha_l \in \Sigma^+$ *for all* k, l, *we have* $p^{i_k} < q$ *and* $p^{j_l} \leqq q$ *for all* k, l.

(5.5) **Isomorphism Condition ($p \neq 2$).** *Every congruence*

$$\sum_{k=1}^{n_1} p^{i_k} \alpha_k + \sum_{l=1}^{n_2} p^{j_l} \alpha_l \equiv \lambda \qquad (\mathrm{mod}\,(q-1)\,\Lambda)$$

with $n_1 + 2n_2 = n$, $1 \leqq p^{i_k} < q$, $1 < p^{j_l} \leqq q$, *and* α_k, $\alpha_l \in \Sigma^+$ *for all* k, l, *is an equality*

$$\sum_{k=1}^{n_1} p^{i_k} \alpha_k + \sum_{l=1}^{n_2} p^{j_l} = \lambda.$$

We will analyze these conditions in the next section. It turns out that when $\lambda = p^e \mu$, $\mu \in \Lambda$, and e is sufficiently large, that ($) is an isomorphism for all sufficiently large q.

We conclude this section with the observation that the corresponding results for G come immediately from those for B.

(5.6) **Lemma.** *Let* V *be a finite dimensional rational* G*-module and let* n *be a non-negative integer. Suppose the* p*-power* q *is such that restriction* $H^n(B, V) \to H^n(B(q), V)$ *is an isomorphism (resp. an injection). Then restriction* $H^n(G, V) \to H^n(G(q), V)$ *is an isomorphism (resp. an injection).*

Proof. Consider the commutative diagram of restriction maps:

$$\begin{array}{ccc} H^n(G, V) & \longrightarrow & H^n(G(q), V) \\ \downarrow & & \downarrow \\ H^n(B, V) & \longrightarrow & H^n(B(q), V). \end{array}$$

By (2.1) the left side is an isomorphism, and the right side is an injection since the index of $B(q)$ in $G(q)$ is prime to p. The lemma is now immediate.

(5.7) *Remark.* We note that in the isomorphism case of the above lemma we also get that $H^n(G(q), V) \cong H^n(B(q), V)$. This holds for all finite dimensional rational V and sufficiently large q, since we can "twist" V so that the arithmetic results of the next section apply (cf. (6.6)). This result was previously known [7], and to a large extent suggested the present investigation.

§ 6. Arithmetic and the Main Theorem

In this section we first give arithmetic conditions which guarantee that the conditions (5.2) through (5.5) are satisfied. We then prove the main theorem and some related results.

For the convenience of the reader, we begin with some preliminary remarks concerning the arithmetic results. Assume $p=2$ for simplicity. Suppose we have a congruence

$$\sum_{k=1}^{n} p^{i_k} \alpha_k \equiv p^e \lambda \quad (\operatorname{mod} (p^{e+f}-1) \Lambda) \tag{A}$$

with $1 \leqq p^{i_k} < p^{e+f}$ and $\alpha_k \in \Sigma^+$ for all k. We would like to conclude (cf. (5.3)) that (A) must be an equality when e, f are sufficiently large. Rewrite (A) as

$$\sum_{k=1}^{n} p^{i_k} \alpha_k = p^e \lambda + (p^{e+f}-1) \mu \tag{B}$$

where $\mu \in \Lambda$. We can assume λ, μ belong to the root lattice Q by multiplying this expression by a suitable positive integer (and increasing n). Decompose the sum in (B) into two parts $\Sigma_1 + p^e \Sigma_2$, where the terms $p^{i_k} \alpha_k$ in Σ_1 are those with $i_k < e$. Now multiply (B) by p^f and regroup to obtain

$$p^f \Sigma_1 + \Sigma_2 = \lambda + (p^{e+f}-1)(\lambda + p^f \mu - \Sigma_2). \tag{C}$$

The "small" terms on the left hand side of (C) all lie in Σ_2, whereas the "small" part of the right hand side is $\lambda - (\lambda + p^f \mu - \Sigma_2) = -p^f \mu + \Sigma_2$. Thus, if we can show the two "small" parts are the same, we have $\mu = 0$, and equality holds in (A). So reformulate the problem:

$$\sum_{k=1}^{n} p^{i_k} \alpha_k = \lambda + (p^{e+f}-1) \mu, \tag{D}$$

with $\lambda, \mu \in Q$ and $1 \leqq p^{i_k} < p^{e+f}$. We want to show that the terms $p^{i_k} \alpha_k$ with $i_k < f$ sum to $\lambda - \mu$ when e, f are sufficiently large. Clearly, we will be able to force $\mu \geqq 0$; also, we can get $\lambda - \mu \geqq 0$: consider an expression

$$\sum_{k=1}^{r} p^{i_k} + s = p^t u$$

with s, t, u positive integers, $0 \leqq i_k < t$, and s small. If we try to put the left hand side into a p-adic expansion, starting with s, then we discover that we need a lot of terms in the sum to prevent non-zero coefficients of powers of p smaller than

p^t. (This type of argument is applied many times below via (6.1), which would be applied here by noting that $p^t u - 1 = p^t(u-1) + p^t - 1$ has a large number of "digits".) This remark is applied by taking coefficients at a fundamental root of both sides of (D), the coefficient of $\lambda - \mu$ playing the role of $-s$. Since n is fixed, r is bounded, and we force a contradiction if s is positive. Once we know $\lambda - \mu \geqq 0$, it is an easy matter to show that a subsum in (D) is actually equal to $\lambda - \mu$ (cf. the first paragraph of the proof of (6.4)). Then another "digit counting" argument shows this subsum consists of the terms with $i_k < f$.

(6.1) **Lemma.** *(Digit counting.) Let r be a non-negative integer, and consider a sum*

$$\sum_{k=1}^{r} p^{j_k} = A p^a + B$$

where $0 \leqq j_k \leqq a$, $A \geqq 0$, $0 \leqq B < p^a$ are all integers. Let $\operatorname{dig}(B)$ be the sum of the coefficients in the p-adic expansion of B. Then $r \geqq A + \operatorname{dig}(B)$.

Proof. If we collect terms, starting with the smallest values of p^{j_k}, and write the sum in the form $A p^a + B$ as above, we can only decrease coefficients. Q.E.D.

(6.2) *Remark.* We alert the reader that this lemma will most often be applied with the sum replaced by $\sum_{k=1}^{r} p^{j_k} + s$, and r replaced in the conclusion by $r + s$.

Now define integer functions e and f by

$$e(r) = \left[\frac{r-1}{p-1}\right]$$

and

$$f(r) = [\log_p(|r|+1)] + 2$$

where [ ] denotes the greatest integer function.

(6.3) **Lemma.** *Let r be a non-negative integer. Consider an expression*

$$\sum_{k=1}^{r} p^{j_k} = L + (p^{e+f} - 1) M$$

for integers L, M, $e \geqq e(r)$, $f \geqq f(L)$, and $0 \leqq j_k < e + f$. Then $L \geqq M \geqq 0$.

Proof. Increasing f if necessary, we can assume $e = e(r)$. Clearly, by hypothesis, $|L| < p^{f-1} - 1 \leqq p^{e+f} - 1$, so M must be non-negative. This inequality also implies $L = M = 0$ when $e = -1$ (and so $r = 0$). Thus, we can assume $r > 0$ and $e \geqq 0$.

Now suppose $e = 0$. Then the sum in our expression is at most $(p-1) p^{f-1}$. Thus,

$$\begin{aligned}(p-1) p^{f-1} \geqq L + (p^f - 1) M &> -(p^{f-1} - 1) + (p^f - 1) M \\ &= (p^f - p^{f-1}) + (p^f - 1)(M - 1).\end{aligned}$$

This implies $L > M = 0$, since $M > 0$ gives a contradiction. Thus, we can assume that e is positive.

Suppose $M > L \geqq 0$. If we digit count in the expression

$$\sum_{k=1}^{r} p^{j_k} + (M - L - 1) = (pM - 1)\, p^{e+f-1} + (p^{e+f-1} - 1)$$

(see (6.2)), we obtain that $r + M - L - 1 \geqq pM - 1 + (p-1)(e+f-1)$. Since $(p-1)\,e \geqq r - p + 1$, $L \geqq 0$, and $f \geqq 2$, this implies that $r + M - 1 \geqq pM + r - 1$. Thus, $M(p-1) \leqq 0$, a contradiction.

Therefore, if the lemma fails, we have $M > 0 > L$. Now apply digit counting to the expression

$$\sum_{k=1}^{r} p^{j_k} + M - 1 = (pM - 1)\, p^{e+f-1} + (p^{e+f-1} + L - 1).$$

Since $p^{e+f-1} + L - 1 = p^f(p^{e-1} - 1) + (p^f + L - 1)$ and $p^f > p^f + L - 1 > (p-1)\, p^{f-1}$, we see that $\operatorname{dig}(p^{e+f-1} + L - 1) \geqq (p-1)(e-1) + p$. Thus, $r + M - 1 \geqq (pM - 1) + (p-1)(e-1) + p$. Again since $e(p-1) \geqq r - p + 1$, this leads to $p - 2 \geqq (p-1)\, M$, a contradiction.

(6.4) **Lemma.** *Consider an expression*

$$\sum_{k=1}^{r} p^{j_k} = L + (p^{e+f} - 1)\, M$$

satisfying the hypotheses of (6.3). *Then there are no terms with* $L < p^{j_k} < p^f$, *and the terms with* $p^{j_k} \leqq L$ *sum to* $L - M$.

Proof. We can assume the p^{j_k} are arranged in increasing order, and we let

$$s_n = \sum_{k=1}^{n} p^{j_k}.$$

The result is clear when $M = 0$, so we can assume $M \geqq 1$. Choose $b \geqq 0$ maximal with $s_b \leqq L - M$. If we write $s_{b+1} - s_b = p^j$, then $0 \leqq L - M - s_b < p^j$ and also $L - M \equiv s_b \pmod{p^j}$, so it follows that $L - M = s_b$. It remains only to show that $p^j \geqq p^f$. Applying digit counting to the expression

$$\sum_{k=b+2}^{r} p^{j_k} = (pM - 1)\, p^{e+f-1} + (p^{e+f-1} - p^j),$$

we obtain $r - b - 1 \geqq (pM - 1) + (p-1)(e + f - 1 - j) \geqq (p-1)(e + f - j)$. Thus, $r - 1 \geqq (p-1)(e + f - j)$, so $e \geqq e + f - j$ and $j \geqq f$. Q.E.D.

Let $\omega = \Sigma\, n_\delta \delta$ be the maximal root in Σ^+, and put $c = \max n_\delta$. For ζ in the root lattice Q, write $\zeta = \Sigma\, m_\delta \delta$ and put $c(\zeta) = \max |m_\delta|$.

For $\lambda \in \Lambda$, let $t(\lambda)$ be the order of the image of λ in the abelian group Λ/Q, and let $t = t(G)$ be the torsion exponent of Λ/Q. Also, we write $\bar{\lambda} = t\lambda$, and we define $t_p(\lambda)$ to be the p-part of $t(\lambda)$.

(6.5) **Proposition.** *Let n be a non-negative integer,* $\lambda \in \Lambda$, $e \geqq e(ctn)$. *Let* $f \geqq f(c(\bar{\lambda}))$. *Suppose we have an expression*

$$\sum_{k=1}^{n} p^{i_k} \alpha_k = \lambda + (p^{e+f} - 1)\, \mu, \tag{$*$}$$

where α_k are positive roots, $0 \leqq i_k < e+f$, and $\mu \in \Lambda$. Then $\bar{\lambda} \geqq \bar{\mu} \geqq 0$, and the terms in $(*)$ with $p^{i_k} \leqq c(\bar{\lambda})$ sum to $\lambda - \mu$.

Proof. Multiply the expression $(*)$ through by t to obtain

$$\sum_{k=1}^{n} t\, p^{i_k} \alpha_k = \bar{\lambda} + (p^{e+f} - 1)\, \bar{\mu}. \qquad (**)$$

Expressing $\bar{\lambda}$, $\bar{\mu}$, and the α_k in terms of the fundamental roots $(**)$ leads to expressions of the form

$$\sum_{k=1}^{r} p^{j_k} = L + (p^{e+f} - 1)\, M$$

where $r \leqq c\,t\,n$, $f \geqq f(c(\bar{\lambda}))$, and $e \geqq e(r)$. The proposition now is an immediate consequence of (6.4).

Let G be a semisimple group defined and split over $k_0 = GF(p)$ as before.

(6.6) **Main Theorem.** *Let V be a finite dimensional rational G-module and let m be a non-negative integer. Let e, f be non-negative integers with $e \geqq e(c\,t\,m)$, $f \geqq f(c(\bar{\lambda}))$ for every weight λ of T in V. If $p \neq 2$ assume also $e \geqq e(c\,t_p(\lambda)(m-1)) + 1$.*

Then the restriction map $H^n(G, V(e)) \to H^n(G(p^{e+f}), V(e))$ *is an isomorphism for $n \leqq m$ and is an injection for $n = m+1$.*

Proof. First suppose $p = 2$. By § 5, we have just to check that the hypotheses of (5.2) and (5.3) are satisfied for the weight $p^e \lambda$:

If $\sum_{k=1}^{n} p^{i_k} \alpha_k = p^e \lambda$, then for (5.2) we have to check that $i_k < e+f$ for each k. If not, then certainly $t\, p^{e+f} \leqq p^e c(\bar{\lambda})$ and so $p^f \leqq t\, p^f \leqq c(\bar{\lambda}) < p^f$, a contradiction.

For (5.3), suppose we have a congruence

$$\sum_{k=1}^{n} p^{i_k} \alpha_k \equiv p^e \lambda \qquad (\mathrm{mod}\ (p^{e+f} - 1)\, \Lambda)$$

with $n \leqq m$, $1 \leqq p^{i_k} < p^{e+f}$, and $\alpha_k \in \Sigma$ for all k. Write this in the form

$$\sum_{k=1}^{n} p^{i_k} \alpha_k = p^e \lambda + (p^{e+f} - 1)\, \mu$$

with $\mu \in \Lambda$ as in the discussion at the beginning of this section. Now $\mu = 0$ follows as indicated there from (6.5). This completes the proof when $p = 2$.

Now suppose $p \neq 2$. The verification of (5.4) is exactly as in the $p = 2$ case for (5.2). To check (5.5) suppose we have a congruence

$$\sum_{k=1}^{n_1} p^{i_k} \alpha_k + \sum_{l=1}^{n_2} p^{j_l} \alpha_l \equiv p^e \lambda \qquad (\mathrm{mod}\ (p^{e+f} - 1)\, \Lambda)$$

with $n_1 + 2 n_2 \leqq m$, $1 \leqq p^{i_k} < p^{e+f}$, $1 < p^{j_l} \leqq p^{e+f}$, and α_k, $\alpha_l \in \Sigma^+$ for all k, l. The argument in the $p = 2$ case gives here an equality

$$\sum_{k=1}^{n_1} p^{i_k} \alpha_k + \sum_{l=1}^{n_2} p^{j_l} \alpha_l = p^e \lambda$$

where

$$\bar{j}_l=\begin{cases} j_l & \text{if } j_l<e+f \\ 0 & \text{if } j_l=e+f. \end{cases}$$

If $\bar{j}_l=0$ for some $l=l_0$, then multiply the above expression by $t_p(\lambda)$, subtract a fundamental root δ_0 appearing in α_{l_0} from both sides, and digit count at the coefficient of δ_0. This leads to $(m-1)\,t_p(\lambda)\,c-1\geqq(p-1)\,e\geqq(p-1)([(m-1)\,t_p(\lambda)\,c-1)/(p-1)]+1)$, a contradiction. Hence, $\bar{j}_l=j_l$ for all l, and the proof is complete.

(6.7) *Remarks.* a) The above theorem of course holds if G is replaced by B, and V by any rational B-module. We can even replace G by any connected closed subgroup (or section) H of G containing T, and V by a rational H-module. The proof is essentially the same.

b) In the statement of the theorem, $V(e)$ can be replaced by V, provided the weights of T in V are all divisible by a sufficiently high power of p. We leave the reformulation to the interested reader. Also, it should be noted that in general $H^m(H(q), V)\approx H^m(H(q), V(e))$ for any algebraic group H defined over k_0 and V a finite dimensional rational H-module. Thus, the theorem implies that $H^m(H(q), V)$ is stable for sufficiently large q when $H=G$ (or any H as in a) above). We denote this stable value of $H^m(H(q), V)$ by $H^m_{\text{gen}}(H, V)$ and we call this the m^{th} *generic* cohomology group of H in V. Note that for any $e\geqq 0$ we have a semilinear isomorphism $H^m_{\text{gen}}(G, V)\cong H^m_{\text{gen}}(G, V(e))$ obtained from twisting, as well as a map $H^m(G, V)\to H^m_{\text{gen}}(G, V)$ obtained from restriction. It would be interesting to have examples where this map is not injective[3]. It is easy to give examples where it is not surjective (cf. (6.15a)). The above theorem says of course that it becomes an isomorphism when V is sufficiently twisted.

c) In verifying the hypotheses of the theorem, it is not necessary to check each weight λ individually. For example, suppose V is an irreducible G-module with high weight μ. Then $c(\bar{\lambda})\leqq c(\bar{\mu})$ for every weight λ of T in V, and $t_p(\lambda)\leqq t_p(\mu)$; in particular, if the hypotheses of the theorem are satisfied for μ, they are automatically satisfied for λ.

d) When G is adjoint, we have $t=t_p(\lambda)=1$ in the hypotheses of the theorem. Of course $G(q)$ need not be a Chevalley group in the sense of [23], and its cohomology groups may be smaller than those of the corresponding Chevalley group. Nevertheless, if one is interested solely in rational cohomology (as in (6.8)) it is permissible (and always possible, cf. (2.7)) to use the adjoint group. (Also, Cor. (6.9) shows that the cohomology groups of $G(q)$ are *not* smaller for q large.)

e) It would be interesting to have a version of the theorem with $G(q)$ replaced by G_σ for σ a rational endomorphism with finite fixed-point set. It appears likely from [7] and [1] that a similar result holds in essentially the same form. The reductions in §5 generalize easily to this case, though the arithmetic appears more formidable. It would also be interesting to have versions of the theorem for more general groups H than those described in a) above.

(6.8) **Corollary** *(Rational stability). Let G be a semisimple group and V a finite dimensional rational G-module. Let m be a non-negative integer. Then there is a non-negative integer ε such that, for each integer $e\geqq\varepsilon$, twisting induces a semilinear isomorphism $H^m(G, V(\varepsilon))\cong H^m(G, V(e))$.*

[3] See Section 7

(6.9) **Corollary.** *Let G, V, m, ε be as in* (6.8). *Then there is a semilinear isomorphism* $H^m_{\text{gen}}(G, V) \cong H^m(G, V(\varepsilon))$.

(6.10) **Corollary.** *Let $f: G \to G'$ be a surjective central k_0-isogeny of semisimple groups (defined and split over k_0 as above). Let V be a finite dimensional rational G'-module. Then there is a (linear) isomorphism $H^m_{\text{gen}}(G, V) \cong H^m_{\text{gen}}(G', V)$ for all non-negative integers m.*

We also have the following generic version of (2.4a); it may also be viewed as a generic version of the fact that "the center kills cohomology":

(6.11) **Corollary.** *Let G, V, m be as in* (6.8). *Assume for each weight λ of T in V and non-negative integer e that $p^e \lambda$ is not in the root lattice. Then $H^m_{\text{gen}}(G, V) = 0$. (This happens in particular when V is irreducible and its high weight λ satisfies the condition.)*

All the corollaries above hold for B, etc. as in (6.7a). The following corollary, which is a generic version of (2.2), follows from the proof of (6.6) and the non-negativity condition in (6.5).

(6.12) **Corollary.** *Suppose $\lambda \in \Lambda$ and $H^m_{\text{gen}}(B, \lambda) \neq 0$ for some non-negative integer m. Then $\bar{\lambda} \geqq 0$.*

§ 7. 1- and 2-Cohomology

Suppose we wish to apply (6.9) and (3.6) through (3.10) to compute $H^m_{\text{gen}}(G, V)$ for $m = 1, 2$. Since the Weyl modules associated with $V(\varepsilon)$ generally get more complicated as ε becomes large, it is desirable to be able to take ε as small as possible. Say $m = 1$. For large p, the estimate in (6.6) allows us to take $\varepsilon = 0$ (note $e(0) = -1$). For small p however the situation is less satisfactory. The theorems below give sharper results for $m = 1, 2$.

First, though, a general remark on rational cohomology is in order. The approach of Section 5 gives a way of directly determining the value of ε which works in (6.8) and (6.9): There is an obvious analogue of (5.1) for guaranteeing that $H^n(B, \lambda) \to H^n(B, p\lambda)$ is an isomorphism for $n \leqq m$ and an injection for $n = m + 1$. The analogues of (5.2), (5.4) result in no conditions at all, while (5.3), (5.5) become:

($p = 2$) In every equation $\sum_{k=1}^{n} p^{i_k} \alpha_k = p\lambda$ with $i_k \geqq 0$, $\alpha_k \in \Sigma^+$, we have $i_k \geqq 1$ for each k, ($n \leqq m$);

($p \neq 2$) In every equation $\sum_{k=1}^{n_1} p^{i_k} \alpha_k + \sum_{k=1}^{n_2} p^{j_l} \alpha_l = p\lambda$ with $n_1 + 2n_2 = n$, $i_k \geqq 0$, $j_l \geqq 1$, $\alpha_k, \alpha_l \in \Sigma^+$, we have $i_k \geqq 1$ and $j_l \geqq 2$ for each k, l, ($n \leqq m$).

If these equations are satisfied for every weight λ in a finite dimensional rational G-module V, we have $H^n(G, V) \cong H^n(G, V(1))$ for $n \leqq m$, and an injection $H^{m+1}(G, V) \to H^{m+1}(G, V(1))$. Now we are in a position to determine some good values for ε for (6.8) when $m = 1, 2$.

(7.1) **Theorem.** *Let G, V be as in (6.8). Then $H^1(G, V(1)) \cong H^1_{\text{gen}}(G, V(1))$. Moreover, this holds with $V(1)$ replaced by V, except possibly when $p=2$, Σ has an indecomposable component of type C_l, $l \geqq 1$, and $1/2\alpha$ is a weight of V for some root α. Finally, the map $H^2(G, V) \to H^2_{\text{gen}}(G, V)$ is an injection.*

Proof. For the first assertion it is enough to show $H^1(G, V(1)) \cong H^1(G, V(2))$. This follows from the fact that the equation $\alpha = p^2\lambda$ for $\alpha \in \Sigma$, $\lambda \in \Lambda$ is impossible. For the second assertion, it is enough to observe that an equation $\alpha = p\lambda$ is possible if and only if $p=2$, and α is a long root in a component of type C_l, $l \geqq 1$. The last assertion follows easily from the second assertion, the remarks preceeding (7.1), and (2.4a). Q.E.D.

(7.2) **Theorem.** *Let G, V be as in (6.8). Then $H^2(G, V(2)) \cong H^2_{\text{gen}}(G, V(2))$ and the map $H^3(G, V(2)) \to H^3_{\text{gen}}(G, V(2))$ is injective. Moreover, we can replace $V(2)$ by $V(1)$ in this statement except possibly when $p=2$, Σ contains an indecomposable component of type C_l $(l \geqq 1)$, and $1/2\alpha$ is a weight of V for some root α. Finally, if $p \neq 2, 3$, and no root is a weight of V, then $V(2)$ may even be replaced by V.*

Proof. The first assertion follows from the impossibility of equations $p^a\alpha + \beta = p^3\lambda$, $p\alpha = p^3\lambda$ for $\alpha, \beta \in \Sigma$, λ a weight in V: The second equation is clearly not possible, and in the first, $a=0$ (otherwise $p=2$, β belongs to a component of type C_l, and we get in fact $\beta \in pQ$, an absurdity). Thus, $\alpha + \beta = p^3\lambda$, and $\langle \alpha, \beta \rangle + \langle \beta, \beta \rangle \equiv 0 \pmod{p^3}$. This is impossible since we may arrange $|\langle \alpha, \beta \rangle| \leqq 1$ if $\alpha \neq \beta$.

The remaining assertions follow similarly. Q.E.D.

We also have the following two sharp criteria for injectivity.

(7.3) **Proposition.** *Let G, V be as in (6.8). If $V^T = V^{T(q)}$ for a given p-power q, then restriction $H^1(G, V) \to H^1(G(q), V)$ is injective.*

Proof. Let γ be a rational 1-cocycle such that $\gamma|_{G(q)}$ is cohomologous to 0. Adjusting γ we can assume $\gamma|_T = 0$. There exists a vector v such that $\gamma(g) = gv - v$, $g \in G(q)$. It follows $v \in V^{T(q)} = V^T$. Now adjusting γ by the coboundary defined by v on G, we have $\gamma|_T = 0$ and $\gamma|_{G(q)} = 0$. Since $G = \langle T, G(q) \rangle$, it follows from the cocycle condition that $\gamma = 0$. Q.E.D.

(7.4) **Theorem.** *Let G, V be as in (6.8), and assume G is simply connected. Suppose $V = \operatorname{Hom}(U, W)$ where U, W are finite dimensional rational G-modules with $\langle \lambda, \delta \rangle \leqq q-1$ for every high weight λ of every irreducible composition factor of U or W and every fundamental root δ. Then $H^0(G, V) \cong H^0(G(q), V)$ and the map $H^1(G, V) \to H^1(G(q), V)$ is injective.*

Proof. First suppose U, W are irreducible. In this case, the statement regarding H^0 is a well-known result of Steinberg. By (3.10) the group $H^1(G, \operatorname{Hom}(U, W)) \cong \operatorname{Ext}^1_G(U, W) \cong \operatorname{Ext}^1_G(W^*, U^*)$ is isomorphic to $\operatorname{Hom}_G(M, W)$ (or else $\operatorname{Hom}_G(M, U^*)$) where M is the maximal submodule of the Weyl module associated with U (or W^*). By [25; Theorem 2D] this M remains the unique maximal submodule upon restriction to $G(q)$, which implies that the map $\operatorname{Hom}_{G(q)}(M, W) \to \operatorname{Ext}^1_{G(q)}(U, W)$ is injective. The reader may check the proof of (3.10) to see that the injective composite $\operatorname{Ext}^1_G(U, W) \cong \operatorname{Hom}_G(M, W) \subseteq \operatorname{Hom}_{G(q)}(M, W) \to \operatorname{Ext}^1_{G(q)}(U, W)$ is indeed restriction. (There is of course the other case with W replaced by U, which is handled similarly.)

Now just assuming U is irreducible, the validity of the theorem for arbitrary W is established using induction, and the observation on the long exact sequence in Section 5. A similar argument now establishes the theorem for arbitrary U, W.

Notice the conclusion of (7.4) is interesting even for $n=0$. We have

(7.5) **Corollary.** *Under the hypotheses of* (7.4), *every* $G(q)$*-submodule of* W *is a* G*-submodule.*

Proof. Let L be a $G(q)$-submodule of W. By induction we may assume L is irreducible. Let U be an irreducible G-module isomorphic to L as a $G(q)$-module and having high weight λ satisfying $\langle\lambda, \delta\rangle \leqq q-1$ for each fundamental root δ. By (7.4) we have $\mathrm{Hom}_G(U, W) = \mathrm{Hom}_{G(q)}(U, W)$, and the desired result follows.

(7.6) *Examples.* a) First we consider a very simple case, to illustrate some of the ideas. Let $G=SL_2(k)$, and let V be the standard 2-dimensional module with high weight λ_1, the fundamental weight. Assume $p=2$. Then λ_1 is not in the root lattice, so $H^m(G, V)=0$ for all $m \geqq 0$. Moreover, the Weyl module associated with the twisted irreducible module $V(1)$ with high weight $2\lambda_1$ is 3-dimensional ($=2\times 2$ matrices of trace 0). Thus, $H^1(G, V(1)) \cong k$ by (3.11b). By (6.7b) and (7.1), it follows that $H^1_{\mathrm{gen}}(G, V) \cong H^1_{\mathrm{gen}}(G, V(1)) \cong k$. In particular, $H^1_{\mathrm{gen}}(G, V)$ is not isomorphic to $H^1(G, V)$ without twisting. Also, by (3.9), we obtain that $H^2(G, V(1)) = H^1(G, k) = 0$. On the other hand, Example (2.6) shows that $H^2(G, V(2)) \neq 0$. In particular, this means it is *not* true that we can take $\varepsilon=0$ in (6.8) when all the weights lie in the root lattice. Even larger powers of 2 are required for higher dimensional cohomology groups in (6.8) as Example (2.6) shows: indeed, Theorem (6.6) gives the best possible estimate for ε (taking $t=1$ as per (6.7d) after replacing V by $V(1)$) in this case.

b) Assume G is simply connected and that V is a "minimal" module in the (somewhat unfortunate) sense of [6]. Ignoring type C_l in characteristic 2, we can compute $H^1_{\mathrm{gen}}(G, V)$ from (7.1) and (3.11b). These results agree with those of [6]; moreover, the argument in [6] used to obtain lower bounds exactly parallels (3.11b) and originally suggested this result.

c) Let G, V be as in b), but assume the high weight λ does not lie in the root lattice. If $p \neq 2, 3$ then $H^2_{\mathrm{gen}}(G, V)=0$ by (7.2) and (2.4) in agreement with Landazuri [20]. Moreover, Landazuri refers to work of McLaughlin and others which shows that $H^2_{\mathrm{gen}}(G, V) \neq 0$ in many cases where λ can be twisted into the root lattice and $p=2$ or 3. We illustrate (7.2) by computing $H^2_{\mathrm{gen}}(G, V)$ in one of these cases:

Suppose $G=SL_4(k)$, $p=2$, and $V=S\wedge S$, where S is the 4-dimensional standard module. Then V is irreducible of dimension 6, and $H^2_{\mathrm{gen}}(G, V) \cong H^2_{\mathrm{gen}}(G, V(1)) \cong H^2(G, V(1))$ by (7.2). One easily determines from the characteristic zero theory that the Weyl module W associated to $V(1)$ has dimension 20. The dominant weights in W other than the high weight are 0 and the maximal root ω. Also, if $p=2$, the irreducible module M of high weight ω has dimension 14. Since $\dim V(1)=6$, it follows that the maximal submodule of W is isomorphic to M. Since M is self-dual, (3.10) yields $H^2(G, V(1)) \cong H^1(G, M)$ which has dimension 1 (say by [6] and (7.1), or by (3.10) again).

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