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A MODULE STRUCTURE ON CERTAIN ORBIT SETS OF UNIMODULAR ROWS

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An algebraic version of cohomotopy groups is developed. Further the stabilization problem for the K_1 of Bass is studied for matrices that are much smaller than those treated classically.

1. Introduction

1.1. In this paper we consider two topics. The first concerns the kernel of the stabilization map $\mathrm{GL}_n(R)/E_n(R) \to \mathrm{GL}_{n+k}(R)/E_{n+k}(R)$ for n in a 'meta-stable range', viz. $\dim(R)+3 \le 2n \le 2\dim(R)+2$ ($k \ge 1$). Here our results generalize the prestabilization theorem of Vaserstein.

The second topic, somewhat related to the first, concerns the algebraic analogue of the following topological situation. Consider the fibration $\mathrm{SO}_{n-1} \to \mathrm{SO}_n \xrightarrow{p} S^{n-1}$ where $n \ge 3$ and p sends an orthogonal matrix to its first row. Let X be a finite CW complex of dimension at most $2n-4$. The set $[X, S^{n-1}]$ of homotopy classes of continuous maps from X to S^{n-1} is an abelian group, because the suspension theorem identifies it with a morphism set in the stable category. As we are interested in the 'stabilization' map $[X, \mathrm{SO}_{n-1}] \to [X, \mathrm{SO}_n]$, we look at the exact sequence of pointed sets $[X, \mathrm{SO}_{n-1}] \to [X, \mathrm{SO}_n] \to [X, S^{n-1}]$. It turns out that this sequence enjoys the following structure. One may view $[X, S^{n-1}]$ as a right module for the group $[X, \mathrm{SO}_n]$ in such a way that the usual action of $[X, \mathrm{SO}_n]$ on $[X, S^{n-1}]$ – induced by the action from the right of SO_n on S^{n-1} – takes the form $([v], g) \mapsto [v]g + p_*(g)$, where p_* is induced by p, $[v] \in [X, S^{n-1}]$, $g \in [X, \mathrm{SO}_n]$ (Theorem 7.8).

1.2. Classical algebraic K-theory has been designed to imitate certain parts of topology. The situation we have just described generalizes as follows: Let $n \ge 3$ and let R be a commutative ring with $\mathrm{sdim}(R)$ at most $2n-4$, where $\mathrm{sdim}(R)$ is the 'stable range dimension' of R, i.e. the 'dimension' which is detected by Bass' stable range conditions. (Thus $\mathrm{sdim}(R)$ is one less than the 'stable rank' $\mathrm{sr}(R)$ of [16].)

For example, if X is a finite CW complex of dimension d, then the stable range

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dimension of the ring $C(X)$ of continuous real valued functions on X is just d, by [16].

As in [4] we consider the orbit set $\mathrm{Um}_n(R)/E_n(R)$, which in the case of our example $R=C(X)$ is just $[X, S^{n-1}]$. We show that for $\mathrm{sdim}(R)\le 2n-4$ one may give $\mathrm{Um}_n(R)/E_n(R)$ the structure of a right $\mathrm{GL}_n(R)$ module in such a way that the ordinary action from the right of $\mathrm{GL}_n(R)$ on $\mathrm{Um}_n(R)/E_n(R)$ takes the form $([v], g)\mapsto [v]g+[g]$, where $[g]$ denotes the orbit of the first row of g, $g\in \mathrm{GL}_n(R)$, $[v]\in \mathrm{Um}_n(R)/E_n(R)$. (In the example $R=C(X)$ one may identify $\mathrm{SL}_n(R)/E_n(R)$ with $[X, \mathrm{SO}_n]$. See [8, §7].)

1.3. Before showing that $\mathrm{Um}_n(R)/E_n(R)$ is a right $\mathrm{GL}_n(R)$ module, we must first show that it has the structure of an abelian group. To this end we introduce weak (higher) Mennicke symbols. They are inspired both by the higher Mennicke symbols of Suslin [13] and by the 'symplectic' group structure which Vaserstein has put on $\mathrm{Um}_3(R)/E_3(R)$ when $\mathrm{sdim}(R)=2$ (see [15, Theorem 5.2]). We show, if $n\ge 3$ and R is a commutative ring with $\mathrm{sdim}(R)\le 2n-4$, that the universal weak Mennicke symbol $\mathrm{wms}: \mathrm{Um}_n(R)/E_n(R)\to \mathrm{WMS}_n(R)$ is a bijection with an abelian target, which provides $\mathrm{Um}_n(R)/E_n(R)$ with the desired structure of an abelian group. Here a weak Mennicke symbol (of order n) over R is a map wms from $\mathrm{Um}_n(R)/E_n(R)$ to a group such that – with some obvious abuse of notation – whenever $(q, v_2, \dots, v_n)$, $(1+q, v_2, \dots, v_n)$ are unimodular and $r(1+q)\equiv q \bmod (v_2R+\cdots+v_nR)$, one has

$$\mathrm{wms}(q, v_2, \dots, v_n) = \mathrm{wms}(r, v_2, \dots, v_n)\mathrm{wms}(1+q, v_2, \dots, v_n).$$

Observe that $(q, v_2, \dots, v_n)$ and $(r(1+q), v_2, \dots, v_n)$ are in the same orbit, so that an ordinary higher Mennicke symbol defines a weak one. If n is even and $\mathrm{sdim}(R)\le 2n-5$, we further show that the universal (ordinary) higher Mennicke symbol induces a bijection $\mathrm{Um}_n(R)/(\mathrm{GL}_n(R)\cap E_{n+1}(R))\to \mathrm{MS}_n(R)$, just like for the classical case $n=2$, $\mathrm{sdim}(R)=1$, [1,20]. For further details we refer to the theorems in the paper. For more background see [4] and [20].

2. Pre-stabilization revisited

2.1. *Conventions.* A is an associative ring with unit, finitely generated as a module over a central subring R, and B is an ideal in A. We write I_m for the m by m identity matrix and drop the index m when the size is clear from the context. Recall that for $n\ge 3$ the relative elementary subgroup $E_n(A,B)$ of $\mathrm{GL}_n(A)$ is generated by elements $a^{ij}b^{ji}(-a)^{ij}$ with $b\in B$, $a\in A$, where y^{ij} (or $y^{i,j}$) denotes the elementary matrix with ones on the diagonal and y on the intersection of the ith row with the jth column $(i\ne j)$. The group $E_n(A,B)$ is normal in $\mathrm{GL}_n(A)$ [18]. When writing something like $\mathrm{GL}_n(A)\cap E_{n+i}(A,B)$ one means

$$\left\{x\in \mathrm{GL}_n(A):\ \begin{bmatrix} x & 0\\ 0 & I_i\end{bmatrix}\in E_{n+i}(A,B)\right\}.$$

As explained in 1.2 we write $\mathrm{sdim}(A)=\mathrm{sr}(A)-1$, where $\mathrm{sr}(A)$ is as in [16].

2.2. Theorem. *Let $n\geq 3$. Assume that A is commutative with $\mathrm{sdim}(A)\leq 2n-3$ or assume that the maximal spectrum of R is the union of finitely many noetherian subspaces of dimension at most $2n-3$. Let i,j be non-negative integers.*

For every $g\in \mathrm{GL}_{n+i}(A)\cap E_{n+i+j+1}(A,B)$ there are matrices u,v,w,M with entries in B and q with entries in A such that

$$\begin{pmatrix} I_{i+1}+uq & v \\ wq & I_{n-1}+M \end{pmatrix} \in gE_{n+i}(A,B),$$

$$\begin{pmatrix} I_{j+1}+qu & qv \\ w & I_{n-1}+M \end{pmatrix} \in E_{n+j}(A,B).$$

2.3. Comment. Conversely, if such u,v,w,M and q exist, then $g\in \mathrm{GL}_{n+i}(A)\cap E_{n+i+j+1}(A,B)$ by the appropriate version of the Whitehead lemma (see [4, 2.14]). Thus the theorem says that the Whitehead lemma describes all of $\mathrm{GL}_{n+i}(A)\cap E_{n+i+j+1}(A,B)$.

2.4. Corollary. (Pre-stabilization, cf. [17, 3.4; 6; 4, 2.8]). *Say A is commutative and $4m\geq 3\,\mathrm{sdim}(A)+5$. Then $\mathrm{GL}_m(A)\cap E(A,B)$ is generated by the fractions $(I_m+XY)(I_m+YX)^{-1}$ where X has entries in B, Y has entries in A and I_m+XY is invertible.*

Proof. Take n minimal so that $\mathrm{sdim}(A)\leq 2n-3$ and put $i=j=m-n$. Observe that $n+i+j+1\geq \mathrm{sdim}(A)+2$, so that $\mathrm{GL}_m(A)\cap E(A,B)=\mathrm{GL}_{n+i}(A)\cap E_{n+i+j+1}(A,B)$ by stability for K_1 ([17, Theorem 3.2] or [10, Theorem 2.2]). Now apply Theorem 2.2 and compare with [17].

2.5. Remark. To avoid further technicalities we do not consider the case $n=2$. See [20].

2.6. Before embarking on the proof of the theorem, let us briefly discuss the setting for the computations. We work with block matrices

$$\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix}$$

where p is $i+1$ by $i+1$, t is $n-1$ by $n-1$, x is $j+1$ by $j+1$.

More specifically, we work with expressions built from factors

$$\begin{pmatrix} p & * \\ 0 & * \end{pmatrix}=\begin{pmatrix} p & q & r \\ 0 & t & u \\ 0 & w & x \end{pmatrix}, \qquad \begin{pmatrix} * & * \\ 0 & x \end{pmatrix}=\begin{pmatrix} p & q & r \\ s & t & u \\ 0 & 0 & x \end{pmatrix}$$

(the upper block triangular types) and

$$\begin{pmatrix} p & 0 \\ * & * \end{pmatrix} = \begin{pmatrix} p & 0 & 0 \\ s & t & u \\ v & w & x \end{pmatrix}, \qquad \begin{pmatrix} * & 0 \\ * & x \end{pmatrix} = \begin{pmatrix} p & q & 0 \\ s & t & 0 \\ v & w & x \end{pmatrix}$$

(the lower block triangular types). In products of such matrices we freely move factors back and forth, keeping in mind that matrices in $\mathrm{GL}_{n+i+j+1}(A)$ of the shape $\begin{pmatrix} I_{i+1} & * \\ 0 & * \end{pmatrix}$ form a subgroup, just like matrices of the shape $\begin{pmatrix} I_{i+1} & 0 \\ * & * \end{pmatrix}$, or those of the shape $\begin{pmatrix} * & * \\ 0 & I_{j+1} \end{pmatrix}$, or those of the shape $\begin{pmatrix} * & 0 \\ * & I_{j+1} \end{pmatrix}$.

Example 1. Consider the product

$$\begin{pmatrix} c & 0 \\ * & I_{j+1} \end{pmatrix} \begin{pmatrix} I_{i+1} & 0 \\ * & I_{n+j} \end{pmatrix} \begin{pmatrix} d & 0 \\ * & I_{j+1} \end{pmatrix}$$

where $c, d \in E_{n+i}(A, B)$. Rewrite it as

$$\begin{pmatrix} c & 0 \\ * & I_{j+1} \end{pmatrix} \begin{pmatrix} I_{i+1} & 0 & 0 \\ * & I_{n-1} & 0 \\ 0 & 0 & I_{j+1} \end{pmatrix} \begin{pmatrix} I_{n+i} & 0 \\ * & I_{j+1} \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & I_{j+1} \end{pmatrix}$$

and then as

$$\begin{pmatrix} c' & 0 \\ * & I_{j+1} \end{pmatrix} \begin{pmatrix} I_{i+1} & 0 & 0 \\ * & I_{n-1} & 0 \\ * & * & I_{j+1} \end{pmatrix}$$

where c' is the product of c with a conjugate of d (and thus $c' \in E_{n+i}(A, B)$). Thus d has been 'moved over to c'.

Example 2. Let

$$g = \begin{pmatrix} I_{i+1} & t & u \\ 0 & I_{n-1} & 0 \\ 0 & 0 & I_{j+1} \end{pmatrix} \begin{pmatrix} I_{i+1} & 0 & 0 \\ v & I_{n-1} & 0 \\ 0 & 0 & I_{j+1} \end{pmatrix}$$

where t, u have entries in B and v has entries in A. Both factors have shape $\begin{pmatrix} * & * \\ 0 & I_{j+1} \end{pmatrix}$. We may 'interchange' the two factors and rewrite g as

$$\begin{pmatrix} I_{i+1} & 0 & 0 \\ v & I_{n-1} & 0 \\ 0 & 0 & I_{j+1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & I_{j+1} \end{pmatrix} \begin{pmatrix} I_{i+1} & 0 & u \\ 0 & I_{n-1} & w \\ 0 & 0 & I_{j+1} \end{pmatrix}$$

where w has entries in B and $a \in E_{n+i}(A, B)$ because

$$\begin{pmatrix} I & t \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ v & I \end{pmatrix} \equiv \begin{pmatrix} I & 0 \\ v & I \end{pmatrix} \quad \mathrm{mod}\, E_{n+i}(A, B).$$

Thus v has been 'moved over to the left'. Observe that one could just as well write g as

$$\begin{bmatrix} c & 0 \\ 0 & I_{j+1} \end{bmatrix}\begin{bmatrix} I_{i+1} & 0 & 0 \\ v & I_{n-1} & 0 \\ 0 & 0 & I_{j+1} \end{bmatrix}\begin{bmatrix} I_{i+n} & * \\ 0 & I_{j+1} \end{bmatrix}$$

with $c\in E_{n+i}(A,B)$.

In this section we will often leave long sequences of such manipulations to the reader, without much further comment.

It is usually not hard to guess which factors need to be broken up, which factors are better taken together, and in which direction something must be moved to improve the situation. The formulas underlying the computations are those of block multiplication, such as

$$\begin{bmatrix} a & u \\ 0 & I_{j+1} \end{bmatrix}\begin{bmatrix} c & v \\ 0 & I_{j+1} \end{bmatrix}=\begin{bmatrix} ac & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I_{n+i} & c^{-1}a^{-1}u+c^{-1}v \\ 0 & I \end{bmatrix}.$$

2.7. Let $M_{m,n}(B)$ denote the set of m by n matrices over B. Theorem 2.2 will be derived from

Proposition. *Under the conditions of* 2.2 *every element of* $E_{n+i+j+1}(A,B)$ *may be written in the form*

$$\begin{bmatrix} a & p \\ 0 & I_{j+1} \end{bmatrix}\begin{bmatrix} I_{i+1} & 0 \\ q & I_{n+j} \end{bmatrix}\begin{bmatrix} I_{i+1} & r \\ 0 & b \end{bmatrix}\begin{bmatrix} I_{n+i} & 0 \\ s & I_{j+1} \end{bmatrix}\begin{bmatrix} c & t \\ 0 & I_{j+1} \end{bmatrix}$$

where $a,c\in E_{n+i}(A,B)$, $b\in E_{n+j}(A,B)$, $p,t\in M_{n+i,j+1}(B)$, $r\in M_{i+1,n+j}(B)$, $q\in M_{n+j,i+1}(A)$, $s\in M_{j+1,n+i}(A)$.

Remark. Computing modulo B one finds that in fact the top $n-1$ rows of q have entries in B. That part of q may thus be absorbed in a. Similarly the last $n-1$ columns of s have entries in B so that one may assume them to be zero (absorb them in b).

2.8. First assume that the proposition has been proved for $i=0$. Under that assumption we now discuss how to derive the proposition by induction on i. Let $j\geq 1$ and let an element of $E_{n+i+j+1}(A,B)$ be written as in the proposition. We want to increase i by 1 and thus decrease j by 1. We assume that the last $n-1$ columns of s vanish, cf. Remark 2.7. Use the case $i=0$ to decompose b as

$$\begin{bmatrix} a' & p' \\ 0 & I_j \end{bmatrix}\begin{bmatrix} 1 & 0 \\ q' & I_{n+j-1} \end{bmatrix}\begin{bmatrix} 1 & r' \\ 0 & b' \end{bmatrix}\begin{bmatrix} I_n & 0 \\ s' & I_j \end{bmatrix}\begin{bmatrix} c' & t' \\ 0 & I_j \end{bmatrix}$$

and plug this into the expression for g where g is our element.

The resulting expression may be reorganized to obtain the desired form

$$\begin{pmatrix} a'' & p'' \\ 0 & I_j \end{pmatrix}\begin{pmatrix} I_{i+2} & 0 \\ q'' & I_{n+j-1} \end{pmatrix}\begin{pmatrix} I_{i+2} & r'' \\ 0 & b'' \end{pmatrix}\begin{pmatrix} I_{n+i+1} & 0 \\ s'' & I_j \end{pmatrix}\begin{pmatrix} c'' & t'' \\ 0 & I_j \end{pmatrix}$$

by persistent application of small modifications as discussed in 2.6 (with i replaced by $i+1$, j by $j-1$). But the reader may need the following hint. Split

$$\begin{pmatrix} I_{n+1} & 0 \\ s & I_{j+1} \end{pmatrix} = \begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & I_{n-1} & 0 \\ * & 0 & I_{j+1} \end{pmatrix}$$

into factors

$$\begin{pmatrix} I_{i+1} & 0 \\ u & I_{n+j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_{i+1} & 0 & 0 \\ v & I_n & 0 \\ * & 0 & I_j \end{pmatrix}$$

where v has entries in B and u is such that

$$\begin{pmatrix} I_{i+1} & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 \\ u & I_{n+j} \end{pmatrix}$$

$$= \begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & a' & p' \\ 0 & 0 & I_j \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & q' & I_{n+j-1} \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ * & I_n & 0 \\ 0 & 0 & I_j \end{pmatrix}$$

$$\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & 1 & r' \\ 0 & 0 & b' \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & s' & I_j \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & c' & t' \\ 0 & 0 & I_j \end{pmatrix}.$$

Observe that this hint is superfluous if $A=B$, as in that case $u=0$ will do. Now first show that g may be written as

$$\begin{pmatrix} * & * \\ 0 & I_j \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ * & 1 & 0 \\ * & * & I_{n+j-1} \end{pmatrix}\begin{pmatrix} I_{i+1} & * & * \\ 0 & I_n & 0 \\ 0 & 0 & I_j \end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ * & I_n & 0 \\ 0 & 0 & I_j \end{pmatrix}$$

$$\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & 1 & r' \\ 0 & 0 & b' \end{pmatrix}\begin{pmatrix} I_{n+i+1} & 0 \\ * & I_j \end{pmatrix}\begin{pmatrix} * & * \\ 0 & I_j \end{pmatrix},$$

trace what happened modulo B and give a qualitative description of the stars.

Then rewrite g as

$$\begin{pmatrix} a'' & p'' \\ 0 & I_j \end{pmatrix}\begin{pmatrix} I_{i+2} & 0 & 0 \\ 0 & I_{n-1} & 0 \\ * & * & I_j \end{pmatrix}\begin{pmatrix} I_{i+2} & * \\ 0 & * \end{pmatrix}\begin{pmatrix} I_{n+i+1} & 0 \\ * & I_j \end{pmatrix}\begin{pmatrix} * & * \\ 0 & I_j \end{pmatrix}$$

and finally finish the induction step.

2.9. It remains to prove the proposition for $i=0$. We start with some lemmas. Recall that R lies in the center of A, with A finite over R.

Lemma. *Let $x={}^{\mathrm{T}}(x_1,\dots,x_{n+j})$ be a column over A* (T *denotes 'transpose'), let $q\in M_{1,j+1}(B)$, $z\in A$, $y\in R\cap(Ax_1)+R\cap(Ax_2+\cdots+Ax_{n-1})$. Then there are $a\in E_n(A,B)$, $b\in E_{n+j}(A,B)$ with*

$$\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ xyz & I \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ xyz & I \end{pmatrix}\begin{pmatrix} a & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & b \end{pmatrix}.$$

First proof. This proof only works if A is commutative. Say $A=R$. For simplicity also take $B=A$. Write $y=t_1x_1+\cdots+t_{n-1}x_{n-1}$ and write $q=(q_n,\dots,q_{n+j})$. It suffices to treat the case where A is a polynomial ring over $\mathbb{Z}$ in variables t_m, x_m, q_m, z, as the general case then follows by substitution. We write qx for $q_nx_n+\cdots+q_{n+j}x_{n+j}$. Now ${}^{\mathrm{T}}(1+qxyz,x_1yz,\dots,x_{n-1}yz)$ is unimodular (go local or look at $(1-qxyz)\times(1+qxyz)-1$) and therefore it equals the first column of some $g\in E_n(A)$, by [11, Theorem 2.6]. Choose $\left(\begin{smallmatrix} g & 0 \\ u & I \end{smallmatrix}\right)$ such that its first column is the first column of

$$h=\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ xyz & I \end{pmatrix}$$

and write h as $\left(\begin{smallmatrix} g & 0 \\ u & I \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & * \\ 0 & b \end{smallmatrix}\right)$. Then $b\in E_{n+j}(A)$ by [11, Corollary 6.5] and the lemma follows.

Second proof. This proof is inspired by earlier work with the Steinberg group, cf. [5, Lemma 4.7]. Again write q as $(q_n,\dots,q_{n+j})$. Let us define $U_n^j(q)$ to consist of the columns $v={}^{\mathrm{T}}(v_1,\dots,v_{n+j})$ for which ${}^{\mathrm{T}}(1+qv,v_1,\dots,v_{n-1})$ is unimodular, where qv denotes $q_nv_n+\cdots+q_{n+j}v_{n+j}$. Let V be the subset of $U_n^j(q)$ consisting of the v for which

$$\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ v & I \end{pmatrix}$$

may be written as

$$\begin{pmatrix} 1 & 0 \\ v & I \end{pmatrix}\begin{pmatrix} a & 0 \\ * & I \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & b \end{pmatrix}$$

with $a \in E_n(A,B)$, $b \in E_{n+j}(A,B)$.

We now define some operations that leave V invariant (and whose inverses also leave V invariant). The first operation adds pv_r to v_s, with $r \neq s$, $r<n$, $s<n$, $p \in A$. To see that it leaves V invariant, conjugate

$$\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ v & I \end{pmatrix}$$

by $p^{s+1,r+1}$ (and then reorganize, as always).

The next operation adds pv_r to v_s with $p \in A$, $r<n \leq s$. To see that it leaves V invariant, multiply

$$\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 1 & 0 \\ v & I \end{pmatrix}$$

from the left by $(q_s\, p)^{1,r+1}\, p^{s+1,r+1}$, from the right by $(-p)^{s+1,r+1}$. The last operation adds $p(1+qv)$ to v_r where $p \in A$, $r<n$. To see that it leaves V invariant, multiply from the left by $p^{r+1,1}$ and from the right by the commutator of

$$\begin{pmatrix} 1 & 0 & q \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $p^{r+1,1}$.

We claim there is a sequence of these operations that sends ${}^{\mathrm{T}}(x_1yz, \ldots, x_{n+j}yz)$ to zero. This will establish the lemma, as zero is clearly in V. Use the Nakayama Lemma and localization to find $g \in R$ with $1+yg \in A(1+qxyz)$. Put $f=1+yg$. The operations enable us to replace $\binom{x_1yz}{x_2yz}$ by

$$\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}\begin{pmatrix} x_1yz \\ x_2yz \end{pmatrix} - \begin{pmatrix} yf & 0 \\ f^2 & yf \end{pmatrix}\begin{pmatrix} x_1z \\ x_2z \end{pmatrix} = \begin{pmatrix} x_2y^2z \\ -x_1z \end{pmatrix}.$$

In ${}^{\mathrm{T}}(x_2y^2z, -x_1z, x_3yz, \ldots, x_{n+j}yz)$ we may further replace y by an element of $R \cap (Ax_2 + \cdots + Ax_{n-1})$, as we may add multiples of the entry $-x_1z$ to the other entries.

With such a new y the column ${}^{\mathrm{T}}(1+qxyz, x_2y^2z, x_3yz, \ldots, x_{n-1}yz)$ is unimodular (go local or look at $1-(qxyz)^4$), so that we may moreover replace x_1 by the new y. Finish the proof.

2.10. Remark. So far we did not use sdim. We used only that A is module finite over R, when applying the Nakayama Lemma.

2.11. Recall that a triangular matrix is called unipotent if it has ones on the diagonal.

Lemma (cf. [6, 3.4]). *Let $U \in E_{n+j}(B)$ be unipotent upper triangular, let $N \in E_{n-1}(A)$ and $W \in \mathrm{GL}_{n+j}(A)$. Let $x = {}^{\mathrm{T}}(x_1, \ldots, x_{n+j})$ be the first column of $\left(\begin{smallmatrix} N & 0 \\ 0 & I \end{smallmatrix}\right) UW$ and let $y \in R \cap (Ax_1) + R \cap (Ax_2 + \cdots + Ax_{n-1})$, $z \in A$. Finally let $a \in E_n(A, B)$, $p \in M_{n,j+1}(B)$, $q \in M_{n+j,1}(A)$, $r \in M_{1,n+j}(B)$. Then*

$$\begin{bmatrix} a & p \\ 0 & I_{j+1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q & I_{n+j} \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & W \end{bmatrix} (yz)^{2,1} = \begin{bmatrix} a' & p' \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ * & I \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & bW \end{bmatrix}$$

with $a' \in E_n(A, B)$, $b \in E_{n+j}(A, B)$, $p' \in M_{n,j+1}(B)$.

Proof. We may assume $a = I$, $p = 0$. Then

$$\begin{bmatrix} 1 & 0 \\ q & I_{n+j} \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & W \end{bmatrix} (yz)^{2,1}$$

$$= \begin{bmatrix} 1 & 0 \\ * & U^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & N^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 1 & * & 0 \\ 0 & I & 0 \\ 0 & 0 & I_{j+1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & k \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ xyz & I \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & UW \end{bmatrix}$$

with $k \in M_{1,j+1}(B)$. By 2.9 we may write

$$\begin{bmatrix} 1 & 0 & k \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ xyz & I \end{bmatrix} \quad \text{as} \quad \begin{bmatrix} 1 & 0 \\ xyz & I \end{bmatrix} \begin{bmatrix} a'' & 0 \\ * & I \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & b' \end{bmatrix}$$

with $a'' \in E_n(A, B)$, $b'' \in E_{n+j}(A, B)$. Plug this in and rearrange.

2.12. Now we are going to use the dimension hypothesis of the theorem.

Lemma. *Under the conditions of 2.2 let ${}^{\mathrm{T}}(w_1, \ldots, x_{n+j})$, ${}^{\mathrm{T}}(w'_1, \ldots, w'_{n+j})$ be unimodular columns with $w_1 - 1 \in B$, $w'_1 - 1 \in B$. There are unipotent upper triangular matrices U, U' in $E_{n+j}(B)$ and elements N, N' of $E_{n-1}(A)$ such that if one puts*

$$x = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} Uw, \qquad x' = \begin{bmatrix} N' & 0 \\ 0 & I \end{bmatrix} U'w',$$

then ${}^{\mathrm{T}}(x_1,\dots,x_{n-1},x_1',\dots,x_{n-1}')$ *is unimodular. Moreover, if* A *is commutative one may take* $N=N'=I$ *and if* A *is not commutative one may arrange that*

$$R\cap(Ax_1)+\cdots+R\cap(Ax_{n-1})+R\cap(Ax_1')+\cdots+R\cap(Ax_{n-1}')=R.$$

Proof. First consider the case that A is commutative, $\mathrm{sdim}(A)\le 2n-3$. Choose z in $Aw_n+\cdots+Aw_{n+j}$ and z' in $Aw_n'+\cdots+Aw_{n+j}'$ such that ${}^{\mathrm{T}}(w_1,\dots,w_{n-1},z)$ and ${}^{\mathrm{T}}(w_1',\dots,w_{n-1}',z')$ are unimodular. The column ${}^{\mathrm{T}}(w_1,\dots,w_{n-1},w_1',\dots,w_{n-1}',z(1-w_1)z'(1-w_1'))$ is unimodular of length $2n-1$ (inspect it modulo an arbitrary maximal ideal), so that we may apply the stable range condition to it. We may add multiples of the last entry to the other entries and get a shorter unimodular column. Translate this into matrix language to get the lemma.

In the non-commutative case the lemma just encodes one step in one of the easier proofs of the relevant stable range condition relative to the ideal B. Compare [4, 2.4] and [17, 2.5]. As the products of type $\left(\begin{smallmatrix} N & 0\\ 0 & I\end{smallmatrix}\right)U$ form a group (N, U as in the lemma), we may first reduce to the case where w_r-1, $w_r'-1$ are in B for all $r<n$. In that case we will only need to add elements of Bw_r to w_s, of Bw_r' to w_s' with $s<n$, $r\ne s$. (Such additions translate into multiplications by elementary matrices, of course.) With such additions modify $w_1,w_2,\dots,w_{n-1},w_1',\dots,w_{n-1}'$ consecutively (thus creating $x_1,\dots,x_{n-1}'$ consecutively) such that for each k the first k of the ideals $R\cap(Ax_1),\dots,R\cap(Ax_{n-1}')$ add up to an ideal of 'codimension' at least k, as in the proof of [4, 2.4].

2.13. Lemma. *Under the assumptions of* 2.2 *the matrices in* $\mathrm{GL}_{n+j+1}(A)$ *that can be written as*

$$\begin{pmatrix} a & p\\ 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ q & I_{n+j}\end{pmatrix}\begin{pmatrix} 1 & r\\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & 0\\ s & I_{n+j}\end{pmatrix}\begin{pmatrix} c & t\\ 0 & I_{j+1}\end{pmatrix}$$

with $a,c\in E_n(A,B)$, $b\in E_{n+j}(A,B)$, $p,t\in M_{n,j+1}(B)$, $q,s\in M_{n+j,1}(A)$, $r\in M_{1,n+j}(B)$, *form a subgroup and this subgroup contains* $E_{n+j+1}(A,B)$.

Proof. The last fact follows from the 'transpose' of [4, 2.2]. For the same reason it suffices to show that the set V of matrices that can be written as in the lemma is invariant under left multiplication by elements $\left(\begin{smallmatrix} 1 & u\\ 0 & I\end{smallmatrix}\right)$ with $u\in M_{1,n+j}(B)$ (easy) and by elements $\left(\begin{smallmatrix} 1 & 0\\ * & I\end{smallmatrix}\right)$. If $g\in V$ we therefore consider $h=\left(\begin{smallmatrix} 1 & 0\\ * & I\end{smallmatrix}\right)g$ and rewrite it as

$$\begin{pmatrix} a_1 & p_1\\ 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0\\ q_1 & I_{n+j}\end{pmatrix}\begin{pmatrix} 1 & r_1\\ 0 & b_1\end{pmatrix}\begin{pmatrix} 1 & 0\\ s & I\end{pmatrix}$$
$$\begin{pmatrix} 1 & r_2\\ 0 & b_2\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0\\ q_2 & I_{n+j}\end{pmatrix}^{-1}\begin{pmatrix} a_2 & p_2\\ 0 & I_{j+1}\end{pmatrix}^{-1}$$

with $a_1,a_2\in E_n(A,B)$, $b_1,b_2\in E_{n+j}(A,B)$, $p_1,p_2\in M_{n,j+1}(B)$ (in fact it is easiest to take p_1 zero), $q_1, q_2, s \in M_{n+j,1}(A)$, $r_1, r_2 \in M_{1,n+j}(B)$. The idea is now to follow Vaserstein [20, §7] and to get rid of the factor $\left(\begin{smallmatrix}1 & 0\\ s & I\end{smallmatrix}\right)$ by repeated application of 2.11 with the auxiliary matrices coming from 2.12. First we show how $\left(\begin{smallmatrix}1 & 0\\ s & I\end{smallmatrix}\right)$ may be replaced by $\left(\begin{smallmatrix}1 & 0\\ s & I\end{smallmatrix}\right)z^{2,1}$ for any $z \in A$, where the values of the other factors change – of course – but not their general shape and not the total product h. Apply 2.12 with ${}^{\mathrm{T}}(w_1, \dots, w_{n+j})$ = first column of b_1, ${}^{\mathrm{T}}(w_1', \dots, w_{n+j}')$ = first column of b_2. Say A is not commutative (otherwise read $A = R$, etc.). Choose y_1 in $R \cap (Ax_1) + \cdots + R \cap (Ax_{n-1})$ and y_2 in $R \cap (Ax_1') + \cdots + R \cap (Ax_{n-1}')$ with $y_1 + y_2 = 1$, where x_i, x_i' come from 2.12.

By 2.11 one may absorb a factor $(-y_1 z)^{2,1}$ in

$$\begin{bmatrix} a_1 & p_1 \\ 0 & I_{j+1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_1 & I_{n+j} \end{bmatrix} \begin{bmatrix} 1 & r_1 \\ 0 & b_1 \end{bmatrix}$$

and similarly one may absorb a factor $(-y_2 z)^{2,1}$ in

$$\begin{bmatrix} 1 & r_2 \\ 0 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ q_2 & I_{n+j} \end{bmatrix}^{-1} \begin{bmatrix} a_2 & p_2 \\ 0 & I_{j+1} \end{bmatrix}^{-1}.$$

(Take inverses and apply 2.11 again.) Write $\left(\begin{smallmatrix}1 & 0\\ s & I\end{smallmatrix}\right)$ as

$$(-y_1 z)^{2,1} \begin{bmatrix} 1 & 0 \\ s & I \end{bmatrix} z^{2,1} (-y_2 z)^{2,1}$$

and do indeed absorb the factors $(-y_1 z)^{2,1}$, $(-y_2, z)^{2,1}$ in the rest of our expression for h. The effect is that $\left(\begin{smallmatrix}1 & 0\\ s & I\end{smallmatrix}\right)$ is replaced by $\left(\begin{smallmatrix}1 & 0\\ s & I\end{smallmatrix}\right)z^{2,1}$, as desired. It follows that we may further assume that the top coordinate of s equals 1. Under that assumption consider $L = \left(\begin{smallmatrix}1 & 0\\ * & I\end{smallmatrix}\right)$ with first columns s. We have

$$\begin{bmatrix} 1 & 0 \\ s & I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} 1^{2,1} \begin{bmatrix} 1 & 0 \\ 0 & L^{-1} \end{bmatrix}.$$

Plug this into the expression for h.

Now apply 2.12 with ${}^{\mathrm{T}}(w_1, \dots, w_{n+j})$ = first columns of $b_1 L$, ${}^{\mathrm{T}}(w_1', \dots, w_{n+j}')$ = first column of $b_2 L$, and proceed as before to find y_1 and y_2 so that $(y_1^{2,1})$ may be absorbed in

$$\begin{bmatrix} a_1 & p_1 \\ 0 & I_{j+1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_1 & I_{n+j} \end{bmatrix} \begin{bmatrix} 1 & r_1 L \\ 0 & b_1 L \end{bmatrix},$$

$(y_2)^{2,1}$ may be absorbed in

$$\begin{bmatrix} 1 & r_2 L \\ 0 & b_2 L \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ q_2 & I_{n+j} \end{bmatrix}^{-1} \begin{bmatrix} a_2 & p_2 \\ 0 & I_{j+1} \end{bmatrix}^{-1},$$

$y_1 + y_2 = 1$. One thus gets rid of $1^{2,1}$. After that, cancel L and finish the proof.

2.14. Proof of Proposition 2.7. Recall we still have to prove the case $i=0$. Let $g\in E_{n+j+1}(A,B)$. Write it as in Lemma 2.13, next as

$$\begin{pmatrix} a & p \\ 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & I_{n+j}\end{pmatrix}\begin{pmatrix} 1 & r_1 & r_2 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ s_1 & I_{n-1} & 0 \\ s_2 & 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} c & t \\ 0 & I_{j+1}\end{pmatrix}$$

and observe that q is congruent to $\binom{-s_1}{-s_2}$ modulo B. Rewrite g as

$$\begin{pmatrix} a' & p' \\ 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} I_n & 0 \\ * & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0 & r_2 \\ 0 & I & r_3 \\ 0 & 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-1} & 0 \\ s_2 & 0 & I\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} c & t \\ 0 & I_{j+1}\end{pmatrix}$$

where r_2, r_3, p' have entries in B, $a'\in E_n(A,B)$. Move b back to the middle and finish the proof.

2.15. Proof of Theorem 2.2. Let $g\in \mathrm{GL}_{n+i}(A)$ with $\left(\begin{smallmatrix} g & 0 \\ 0 & I_{j+1}\end{smallmatrix}\right)\in E_{n+i+j+1}(A,B)$. Apply 2.7 (including the remark) to $\left(\begin{smallmatrix} g & 0 \\ 0 & I_{j+1}\end{smallmatrix}\right)$, multiply from left and right, and rearrange to find $g'\in gE_{n+i}(A,B)$, $z\in M_{n+i,j+1}(B)$, $b\in E_{n+j}(A,B)$ with

$$\begin{pmatrix} g' & 0 \\ * & I_{j+1}\end{pmatrix}\begin{pmatrix} I_{n+i} & z \\ 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} I_{i+1} & 0 \\ 0 & b\end{pmatrix}=\begin{pmatrix} I_{n+i} & * \\ 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & I_{n-1} & 0 \\ * & 0 & I_{j+1}\end{pmatrix}.$$

Write the right-hand side as

$$\begin{pmatrix} I_{i+1} & 0 & u \\ 0 & I_{n-1} & w \\ 0 & 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & I_{n-1} & 0 \\ q & 0 & I_{j+1}\end{pmatrix}$$

and write g' as

$$\begin{pmatrix} I_{i+1}+uq & v \\ wq & I_{n-1}+M\end{pmatrix}.$$

Recall that

$$\begin{pmatrix} I_{i+1} & 0 & u \\ 0 & I_{n-1} & w \\ 0 & 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} I_{i+1} & 0 & 0 \\ 0 & I_{n-1} & 0 \\ q & 0 & I_{j+1}\end{pmatrix}\begin{pmatrix} I_{i+1} & v & -u \\ 0 & I_{n-1}+M & -w \\ 0 & -qv & I_{j+1}+qu\end{pmatrix}$$

$$=\begin{pmatrix} g' & 0 \\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ q & 0 & I\end{pmatrix}$$

(cf. [4, 2.14]) and use this to eliminate g'. One gets an equality

$$\begin{bmatrix} I_{i+1} & 0 & 0 \\ * & I_{n-1}+M & -w \\ * & -qv & I_{j+1}+qu \end{bmatrix} = \begin{bmatrix} I_{i+1} & * \\ 0 & b' \end{bmatrix} \quad \text{with } b' \in E_{n+j}(A,B).$$

Now use that

$$\begin{bmatrix} I_{n-1}+M & -w \\ -qv & I_{j+1}+qu \end{bmatrix}$$

is a conjugate of

$$\begin{bmatrix} I_{j+1}+qu & qv \\ w & I_{n-1}+M \end{bmatrix}.$$

Done.

3. Weak Mennicke symbols

3.1. In this section we look at the formalism of weak Mennicke symbols.

3.2. Notations. Let R be a commutative ring, $n \geq 2$. (We will make frequent use of SL_2 and therefore must restrict ourselves to commutative rings). If $v = (v_1, \ldots, v_n)$ is a row (always with entries in R), then $\langle v \rangle = \langle v_1, \ldots, v_n \rangle$ denotes the ideal generated by the v_i and $[v] = [v_1, \ldots, v_n]$ denotes the orbit of v under the action of $E_n(R)$. This orbit will only be considered when $v \in \mathrm{Um}_n(R)$, i.e. when $\langle v \rangle = R$. If J is an ideal in R, then $\mathrm{rad}(J)$ is the intersection of the maximal ideals that contain J. The 'transvection subgroup' $T_2(R)$ of $\mathrm{SL}_2(R)$ is generated by the matrices

$$\begin{bmatrix} 1-xay & -x^2a \\ y^2a & 1+xay \end{bmatrix} \quad \text{with } x, y, a \in R.$$

Observe that $T_2(R)$ is a normal subgroup of $\mathrm{GL}_2(R)$ with $E_2(R) \subset T_2(R) \subset E_3(R)$, and note that $T_2(R) \to T_2(S)$ is surjective if $R \to S$ is surjective (cf. [4, 3.2]). We denote the universal weak Mennicke symbol on $\mathrm{Um}_n(R)/E_n(R)$ by $\mathrm{wms} : \mathrm{Um}_n(R)/E_n(R) \to \mathrm{WMS}_n(R)$, $[v] \mapsto \mathrm{wms}(v)$. (See 1.3). The group $\mathrm{WMS}_n(R)$ may be described by the following presentation.

Generators are the $\mathrm{wms}(v)$ with $v \in \mathrm{Um}_n(R)$.

Relations are:

(1) $\mathrm{wms}(v) = \mathrm{wms}(vg)$ if $g \in E_n(R)$.

(2) If $(q, v_2, \ldots, v_n)$, $(1+q, v_2, \ldots, v_n)$ are unimodular and $r(1+q) \equiv q \bmod \langle v_2, \ldots, v_n \rangle$, then $\mathrm{wms}(q, v_2, \ldots, v_n) = \mathrm{wms}(r, v_2, \ldots, v_n)\mathrm{wms}(1+q, v_2, \ldots, v_n)$.

Here q, r, v_i are of course in R.

In cases where $\mathrm{WMS}_n(R)$ is known to be commutative, one may want to write its group law additively. We will use additive notation for the group structure on $\mathrm{Um}_n(R)/E_n(R)$ when wms has been shown bijective, for R with $\mathrm{sdim}(R) \le 2n-4$.

3.3. We now collect formulas and properties, most of which are well known in one form or another.

Lemma. *Let $n \ge 3$. Suppose $(a,b,v_3,\dots,v_n)$, $(c,d,v_3,\dots,v_n)$ are unimodular rows such that*

$$(\bar a,\bar b) \in (\bar c,\bar d)T_2(R/\mathrm{rad}\langle v_3,\dots,v_n\rangle).$$

Then $[a,b,v_3,\dots,v_n] = [c,d,v_3,\dots,v_n]$.

Proof. As $T_2(R) \to T_2(R/\mathrm{rad}\langle v_3,\dots,v_n\rangle)$ is surjective, and $T_2(R) \subset E_3(R)$, we may assume $(a,b) \equiv (c,d) \bmod \mathrm{rad}\,\langle v_3,\dots,v_n\rangle$.

Put $S = R/\langle v_3,\dots,v_n\rangle$. Then $(\bar a,\bar b) \in \mathrm{Um}_2(S)$ (observe our abuses of notation) and we may choose $e,f \in R$ so that if $\alpha = \binom{a\ b}{e\ f}$, then the determinant of α maps to 1 in S. We have

$$(c\ d)\begin{pmatrix} f & -b \\ -e & a \end{pmatrix} \equiv (1\ 0) \quad \mathrm{mod}\,\mathrm{rad}\,\langle v_3,\dots,v_n\rangle.$$

Thus $\overline{cf-de} \in \mathrm{GL}_1(S)$ and there is $\beta \in E_2(R)$ with

$$(c\ d)\begin{pmatrix} f & -b \\ -e & a \end{pmatrix}\beta \equiv (1\ 0) \quad \mathrm{mod}\,\mathrm{rad}\,\langle v_3,\dots,v_n\rangle.$$

Now $(\bar c\ \bar d)\bar\alpha^{-1}\bar\beta\,\bar\alpha = (\bar a\ \bar b)$ in $\mathrm{Um}_2(S)$ and $\bar\alpha^{-1}\bar\beta\,\bar\alpha \in T_2(S)$.

Use as above that $T_2(R) \to T_2(S)$ is surjective and finish the proof.

3.4. Lemma (Vaserstein's rule) (cf. [15, 5.2(a_2)]). *Let $n \ge 3$, let $(a,b,v_3,\dots,v_n)$, $(c,d,v_3,\dots,v_n)$ be unimodular and choose $e,f \in R$ so that the determinant of $\alpha = \binom{a\ b}{e\ f}$ has image 1 in $R/\langle v_3,\dots,v_n\rangle$. Put $(p\ q) = (c\ d)\alpha$ and $(r\ s) = (c\ d)\binom{f\ \ -b}{-e\ \ a}$. Assume that at least one of the pairs $(\bar a,\bar c)$, $(\bar a,\bar d)$, $(\bar b,\bar c)$, $(\bar b,\bar d)$ is unimodular over $R/\langle v_3,\dots,v_n\rangle$. Then*

$$\mathrm{wms}(c,d,v_3,\dots,v_n)\mathrm{wms}(a,b,v_3,\dots,v_n) = \mathrm{wms}(p,q,v_3,\dots,v_n)$$

and

$$\mathrm{wms}(c,d,v_3,\dots,v_n)(\mathrm{wms}(a,b,v_3,\dots,v_n))^{-1} = \mathrm{wms}(r,s,v_3,\dots,v_n).$$

Thus one computes with first rows in some SL_2, as is fitting for something called (weak) Mennicke symbol.

Remark. Considering $\mathrm{wms}(w\ g) - \mathrm{wms}(w)$ with w, g as in 7.9 below, one sees that the assumption on the pairs is relevant.

Proof of lemma. As $\binom{0\ -1}{1\ \ 0} \in E_2(R)$, we may assume $(\bar{b}, \bar{d})$ is unimodular over $S = R/\langle v_3, \dots, v_n \rangle$ (use 3.3). Adding a multiple of b to a and of c to d we may arrange $\bar{a} - \bar{c} = 1$ in S. Therefore we may assume $a - c = 1$ in R and thereafter we may add a multiple of a to b and of c to d to arrange $b = d$. (Of course we may lose unimodularity of $(\bar{b}, \bar{d})$. Also observe that by the previous lemma we did not cheat. Our modifications did not change $\mathrm{wms}(p, q, v_3, \dots, v_n)$ etc. and it suffices to prove the lemma for the case $a - c = 1, b = d$).

One of the defining relations for $\mathrm{WMS}_n(R)$ tells that

$$\begin{aligned}
&\mathrm{wms}(c, d, v_3, \dots, v_n)(\mathrm{wms}(a, b, v_3, \dots, v_n))^{-1} \\
&\quad = \mathrm{wms}(c, b, v_3, \dots, v_n)(\mathrm{wms}(1 + c, b, v_3, \dots, v_n))^{-1} \\
&\quad = \mathrm{wms}(fc, b, v_3, \dots, v_n).
\end{aligned}$$

To see that this equals $\mathrm{wms}(r, s, v_3, \dots, v_n)$, it suffices to check that $[\bar{r}, \bar{s}] = [\bar{f}\bar{c}, \bar{b}]$ in $\mathrm{Um}_2(S)/E_2(S)$. This is easy:

$$[\bar{r}, \bar{s}] = [\bar{c}, \bar{b}] \begin{pmatrix} \bar{f} & -\bar{b} \\ -\bar{e} & 1 + \bar{c} \end{pmatrix} = [\overline{fc - be}, \bar{b}] = [\bar{f}\bar{c}, \bar{b}].$$

We still have to prove

$$\mathrm{wms}(c, d, v_3, \dots, v_n) = \mathrm{wms}(p, q, v_3, \dots, v_n)(\mathrm{wms}(a, b, v_3, \dots, v_n))^{-1},$$

with $a - c = 1$, $b = d$. Now observe that $(\bar{p}, \bar{b})$ is unimodular over S, so that we may apply the part of the lemma that has been proved to conclude that the right-hand side equals $\mathrm{wms}((p, q)\binom{\ f\ \ -b}{-e\ \ \ a}, v_3, \dots, v_n)$ which of course equals the left-hand side, as $\binom{\ \bar{f}\ \ -\bar{b}}{-\bar{e}\ \ \ \bar{a}} = (\bar{\alpha})^{-1}$ in $\mathrm{SL}_2(S)$.

3.5. Lemma (Useful formulas). *Let $n \geq 3$ and let $(a, v_2, \dots, v_n)$, $(b, v_2, \dots, v_n)$, (a, r) be unimodular. Choose $p \in R$ such that $ap \equiv 1 \bmod \langle v_2, \dots, v_n \rangle$. Then*

(i) $\mathrm{wms}(b, v_2, \dots, v_n)\mathrm{wms}(a, v_2, \dots, v_n) = \mathrm{wms}(a(b + p) - 1, (b + p)v_2, v_3, \dots, v_n)$.

(ii) $\mathrm{wms}(b, v_2, \dots, v_n)(\mathrm{wms}(a, v_2, \dots, v_n))^{-1} = \mathrm{wms}(1 - (a - b)p, (a - b)v_2, v_3, \dots, v_n)$.

(iii) $\mathrm{wms}(a, v_2, \dots, v_n)^{-1} = \mathrm{wms}(-p, v_2, \dots, v_n)$.

(iv) $[a, v_2, \dots, v_n] = [a, r^2 v_2, v_3, \dots, v_n] = [a, rv_2, rv_3, v_4, \dots, v_n]$.

(v) $\mathrm{wms}(a, v_2, \dots, v_n)\mathrm{wms}(b^2, v_2, \dots, v_n) = \mathrm{wms}(ab^2, v_2, \dots, v_n)$.

(vi) $[a^2, v_2, \dots, v_n] = [a, v_2^2, v_3, \dots, v_n]$.

(vii) $\mathrm{wms}(a, v_2, \dots, v_n)\mathrm{wms}(b, v_2, \dots, v_n) = \mathrm{wms}(b, v_2, \dots, v_n)\mathrm{wms}(a, v_2, \dots, v_n)$.

Proof. (i) and (ii) are easy consequences of Vaserstein's rule 3.4 (cf. [4, 3.7]). Taking $b = -p$ in (i) yields (iii). Part (vi) is due to Suslin [12, Lemma 2.10]. To prove (iv), choose s with $rs \equiv 1 \bmod \langle a \rangle$. Then

$$[a, v_2, \dots, v_n] = \left[a, (1, 0) \begin{pmatrix} v_2 & v_3 \\ * & * \end{pmatrix}, v_4, \dots, v_n \right]$$

$$= \left[a, (1,0)\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} v_2 & v_3 \\ * & * \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, v_4, \ldots, v_n\right]$$

$$= [a, r^2v_2, v_3, v_4, \ldots, v_n] = \left[a, (r^2v_2, r_3)\begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix}, v_4, \ldots, v_n\right]$$

$$= [a, rv_2, rv_3, v_4, \ldots, v_n]$$

where $\left(\begin{smallmatrix} v_2 & v_3 \\ * & * \end{smallmatrix}\right) \in \mathrm{SL}_2(S)$ with $S = R/\langle a, v_4, \ldots, v_n\rangle$, $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right) \in E_2(S) \subset T_2(S)$ and 3.3 was applied with the coordinates permuted. (This is of course allowed, as $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in E_2(R)$.)

Part (v) now follows as in [4, proof of 3.16(iii)].

It remains to prove (vii), which seems to be new. The idea is to try and show that taking inverses is a homomorphism. As in the proof of 3.4, we may assume $b = a - 1$. For simplicity of notation we take $n = 4$. Observe that $\langle a(1-a), v_2, v_3, v_4\rangle = R$ (inspect modulo arbitrary maximal ideals, as always). Say $a(1-a)d + p_2v_2 + p_3v_3 + p_4v_4 = 1$. Then

$$\mathrm{wms}(a-1, v_2, v_3, v_4)(\mathrm{wms}(a, v_2, v_3, v_4))^{-1} = \mathrm{wms}(1-(1-a)d, v_2, v_3, v_4)$$

$$= \mathrm{wms}(-(a-1)^2d, v_2, v_3, v_4)$$

(use $1-(1-a)d \equiv -(a-1)^2d \bmod \langle v_2, v_3, v_4\rangle$)

$$= \mathrm{wms}(-(a-1)^2d, v_2, v_3, -(1+a(1-a)d)p_4)^{-1}$$

(use $v_4(1+a(1-a)d)p_4 \equiv 1 \bmod \langle (a-1)^2d, v_2, v_3\rangle$)

$$= \mathrm{wms}((a-1)^2, v_2, v_3, -(1+a(1-a)d)p_4)^{-1}\mathrm{wms}(-d, v_2, v_3, -(1+a(1-a)d)p_4)^{-1}$$

$$= \mathrm{wms}(a-1, v_2^2, v_3, -(1+a(1-a)d)p_4)^{-1}\mathrm{wms}(-d, v_2, v_3, -p_4)^{-1}$$

$$= \mathrm{wms}(a-1, v_2^2, v_3, -p_4)^{-1}\mathrm{wms}(-d, v_2, v_3, -p_4)^{-1}$$

$$= \mathrm{wms}((a-1)^2, v_2, v_3, -p_4)^{-1}\mathrm{wms}(-d, v_2, v_3, -p_4)^{-1}$$

$$= \mathrm{wms}(-(a-1)^2d, v_2, v_3, -p_4)^{-1} = \mathrm{wms}(a, v_2, v_3, -p_4)(\mathrm{wms}(a-1, v_2, v_3, -p_4))^{-1}$$

(substitute $-p_4$ for v_4 and $-v_4$ for p_4 in the beginning of our computation)

$$= \mathrm{wms}(a, v_2, v_3, v_4)^{-1}\mathrm{wms}(a-1, v_2, v_3, v_4).$$

3.6. Theorem. *Let* $n \geq 3$.

(i) *There is a map* $\tau : \mathrm{WMS}_n(R) \to \mathrm{WMS}_n(R)$ *with* $\tau(ab) = \tau(b)\tau(a)$, $\tau(\mathrm{wms}(v)) = \mathrm{wms}(v)$ *for* $a, b \in \mathrm{WMS}_n(R)$, $v \in \mathrm{Um}_n(R)$.

(ii) *If* $\mathrm{wms} : \mathrm{Um}_n(R)/E_n(R) \to \mathrm{WMS}_n(R)$ *is surjective, then* $\mathrm{WMS}_n(R)$ *is abelian.*

Proof. View the involution τ as a homomorphism from $\mathrm{WMS}_n(R)$ to the opposite group. The existence of τ follows from 3.5(vii) and the universal property of $\mathrm{WMS}_n(R)$. Part (ii) follows from (i) because in the situation of (ii) the involution is the identity.

4. A group structure

4.1. Our aim in this section is

Theorem. *Let R be commutative, $n \geq 3$, $\operatorname{sdim}(R) \leq 2n-4$. (Recall $\operatorname{sdim}(R) = \operatorname{sr}(R)-1$). Then the universal weak Mennicke symbol $\operatorname{wms}: \operatorname{Um}_n(R)/E_n(R) \to \operatorname{WMS}_n(R)$ is bijective, so that (by 3.6) $\operatorname{Um}_n(R)/E_n(R)$ has the structure of an abelian group.*

Remark. We refrain from developing the theory relative to an ideal of R. As is amply illustrated in [4, §3], such an extension should present no difficulty because of excision [4, 3.21]. Thus the theorem generalizes [4, (3.6)].

4.2. The auxiliary results that are needed to prove the theorem are all immediate consequences of the theorem. Therefore the reader may want to move directly to the exercises at the end of Section 4. Just as in [4, §3] the group structure will be constructed in many steps. The bijectivity of wms will be obvious from the construction of the group structure. The way we compute the product or quotient of two orbits is such that it is far from obvious that operations are well-defined. That issue will be dealt with by a patient study of the effect of minor variations in the choices. For clarity we gradually expand the domain of our operations (so that ambiguity never looks too great) and discuss only the case $n=4$. This case displays all relevant phenomena. Thus $n=4$, $\operatorname{sdim}(R) \leq 4$, until 4.28.

4.3. Definition. Let $(a, b, v_3, v_4), (c, d, v_3, v_4)$ be unimodular and suppose that at least one of the pairs $(\bar{a}, \bar{c})$, $(\bar{a}, \bar{d})$, $(\bar{b}, \bar{c})$, $(\bar{b}, \bar{d})$ is unimodular over $R/\langle v_3, v_4 \rangle$. Choose e, f, r, s as in 3.4 and put $(a, b, v_3, v_4)/(c, d, v_3, v_4) = [r, s, v_3, v_4]$. It follows from 3.3 that this is well defined, i.e. independent of the choice of (e, f). Observe that we start with two rows and get an orbit.

4.4. Lemma. *If a, b, p, v_i are as in 3.5, then*

$$(b, v_2, v_3, v_4)/(a, v_2, v_3, v_4) = [1-(a-b)p, (a-b)v_2, v_3, v_4].$$

The right-hand side simplifies as in 3.2 when $a-b=1$.

Proof. As $(\bar{b}, \bar{v}_2)$ is unimodular over $R/\langle v_3, v_4 \rangle$, the left-hand side is defined and the equality is easily derived, cf. 3.5. Now if $a-b=1$, then $(1-(a-b)p)(1+b) \equiv b \bmod \langle v_2, v_3, v_4 \rangle$, which fits in 3.2.

4.5. Definition. Let $(a-1, v_2, v_3, v_4), (a, w_2, w_3, w_4)$ be unimodular. Put

$$\begin{aligned}
&(a-1, v_2, v_3, v_4)/(a, w_2, w_3, w_4) \\
&\quad = (a-1, v_2+(a-1)(v_2-w_2), v_3+(a-1)(v_3-w_3), v_4+(a-1)(v_4-w_4)) \\
&\qquad /(a, w_2+a(v_2-w_2), w_3+a(v_3-w_3), w_4+a(v_4-w_4)).
\end{aligned}$$

Observe that the right-hand side is defined because $v_i+(a-1)(v_i-w_i)=w_i+a(v_i-w_i)$ so that 4.3 and 4.4 apply. Also observe that we did not leave the orbits $[a-1,v_2,v_3,v_4]$, $[a,w_2,w_3,w_4]$ respectively, when changing 'numerator' and 'denominator'. (We are not allowed to leave orbits, as we are trying to define $[v]/[w]$ as v/w).

4.6. Lemma. *Let $v=(v_1,\dots,v_4)$, $w=(w_1,\dots,w_4)$ be unimodular, with $w_1-v_1=1$, cf. 4.5. The subgroup of $E_4(R)$ generated by the y^{ij} with $j\geq 2$ ($y\in R$, $i\neq j$, $1\leq i\leq 4$, $2\leq j\leq 4$) contains only elements with $(vg)/w=v/(wg)=v/w$.*

Proof. It is easy to see that $v/w=(vg)/(wg)$ for $g=y^{ij}$ with $i\geq 2$, $j\geq 2$. Therefore we may treat the second, third and fourth coordinate in the same manner. Now $(vy^{12})/w=v/(wy^{12})=v/w$ follows easily from 3.3 and 'Vaserstein's rule' 4.3 (cf. 3.4). For y^{32} the reasoning is similar. (Use that the z^{12} have already been treated). After the cases $g=y^{ij}$, treat the general case.

4.7. By way of exercise we treat

Lemma. *Let v,w be unimodular with $w_1-v_1=w_2-v_2=1$. Then*

$$v/w=(v_2,-v_1,v_3,v_4)/(w_2,-w_1,w_3,w_4).$$

Proof. By 4.6 we may assume $v_3=w_3$, $v_4=w_4$ (look at 4.5) and then this is just a case of 3.3 again.

4.8. Lemma. *Let v,w be as in* 4.6, $y\in R$, $2\leq i\leq 4$, $2\leq j\leq 4$. *Then*

$$v/w=(v_1+yv_iw_j,v_2,v_3,v_4)/(w_1+yv_iw_j,w_2,w_3,w_4).$$

Proof. Observe that the right-hand side is defined by 4.5. By 4.6 we may assume $i=j=2$. Argue as in 4.7.

4.9. Lemma. *With the same v,w, let $p_i,q_i\in R$ and put*

$$v_1'=v_1+p_2v_2+p_3v_3+p_4v_4,\qquad w_1'=w_1+q_2w_2+q_3w_3+q_4w_4.$$

Assume $w_1'-v_1'$ also equals 1 and assume $(v_2,v_3,v_4,w_2,w_3,w_4)$ is unimodular.
Then $v/w=(v_1',v_2,v_3,v_4)/(w_1',w_2,w_3,w_4)$.

Proof. The row $(p_2,p_3,p_4,q_2,q_3,q_4)$ satisfies $p_2v_2+p_3v_3+p_4v_4=q_2w_2+q_3w_3+q_4w_4$. The solutions of that equation are spanned by the trivial ones, such as $(0,w_2,0,v_3,0,0)$. (To see this, go local and observe that locally at least one of v_2,v_3,v_4,w_2,w_3,w_4 is a unit.) Therefore the lemma follows from the previous one.

4.10. Definition. Let $v,w\in \mathrm{Um}_4(R)$ be such that $(v_2,v_3,v_4,w_2,w_3,w_4)$ is unimodular. Choose $a\in(1+v_1+\langle v_2,v_3,v_4\rangle)\cap(w_1+\langle w_2,w_3,w_4\rangle)$ (this is possible) and put $v/w=(a-1,v_2,v_3,v_4)/(a,w_2,w_3,w_4)$.

Lemma 4.9 tells that this is well defined. Also, if both 4.10 and 4.5 apply then they agree (same for '4.10 and 4.3' and for '4.5 and 4.3').

4.11. As always we embed $E_2(R)$ in the upper left corner of $E_4(R)$.

4.12. Lemma. *Let $v, w \in \mathrm{Um}_4(R)$ be such that $(v_3, v_4, w_2, w_3, w_4)$ is unimodular. For $g \in E_2(R)$ one has $v/w = (vg)/w$.*

Proof. The case $g = y^{12}$ is easy: You do not need v_2 when constructing a, and if you take $a \in (1 + v_1 + \langle v_3, v_4\rangle) \cap (w_1 + \langle w_2, w_3, w_4\rangle)$, then the problem boils down to yet another application of 3.3 or 4.6. The case $g = y^{21}$ is still easier: You may not need v_2, but it is not forbidden to use it. The two cases suffice.

4.13. Lemma. *Let $v, w \in \mathrm{Um}_4(R)$ be such that $(v_2, v_3, v_4, w_2, w_3, w_4)$ is unimodular. For every g in the subgroup of $E_4(R)$ generated by the y^{ij} with $i \geq 2$, one has $v/w = (vg)/w = v/(wg)$.*

Proof. The case $g = y^{i1}$, $i \geq 2$ is clear from Definition 4.10 and if $i \geq 2$, $j \geq 2$, then one reduces to the case $w_1 - v_1 = 1$ where 4.6 applies.

4.14. Lemma. *With the same v, w one does not change v/w when adding a multiple of $w_1 v_2$ to v_3, w_2, w_3, w_4 or v_4.*

Proof. Adding it to v_3 or v_4 is treated by 4.13. Also by 4.13 we only need to look at adding $w_1 v_2$ to w_2. We have to compare v/w with $v/(w_1, w_2 + y w_1 v_2, w_3, w_4)$ where $y \in R$. Observe that $(v_2, v_3, v_4, w_2 + y w_1 v_2, w_3, w_4)$ is unimodular, so that both 'fractions' are defined. Neither fraction changes if we add an element of $\langle v_2, v_3, v_4\rangle$ to v_1 or an element of $\langle w_3, w_4\rangle$ to w_1 (observe that w_1 occurs in several places). Therefore we may assume that $w_1 - v_1 - 1 = z w_2$ for some $z \in R$. We have

$$v/w = v/(w_1 - z w_2, w_2, w_3, w_4)$$

and

$$v/(w_1, w_2 + y w_1 v_2, w_3, w_4)$$

$$= (v_1 - yz w_1 v_2, v_2, v_3, v_4)/(w_1 - z w_2 - yz w_1 v_2, w_2 + y w_1 v_2, w_3, w_4).$$

Both right-hand sides are such that 4.6 applies, so we may add elements of $\langle v_1, v_2\rangle = \langle v_1 - yz w_1 v_2, v_2\rangle$ to v_3 and v_4 and elements of $\langle w_1 - z w_2, w_2\rangle = \langle w_1 - z w_2 - yz w_1 v_2, w_2 + y w_1 v_2\rangle$ to w_3 and w_4, so as to reduce to the situation of 4.3. Then it is easy as usual.

4.15. Lemma. *Let $v, w \in \mathrm{Um}_4(R)$ be such that $(v_2, v_3, v_4, w_2, w_3, w_4)$ and $(v_1, v_3, v_4, w_2, w_3, w_4)$ are also unimodular. Then $v/w = (v_2, -v_1, v_3, v_4)/w$.*

Proof. By 4.14 we may add multiples of $w_1 v_1 v_2$ to v_3, v_4, w_2, w_3, w_4. As $\mathrm{sdim}(R) \leq 4$, it thus follows from the remark below that we may assume that 4.12 applies.

4.16. Remark. We need in the proof of 4.15 that $(v_3, v_4, w_2, w_3, w_4, w_1 v_1 v_2)$ is unimodular. For such rows, unimodularity is most easily checked by factoring out an arbitrary maximal ideal of R. We are then looking at a field in which $\bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{w}_2, \bar{w}_3, \bar{w}_4$ are not all zero, $\bar{v}_1, \bar{v}_4, \bar{v}_3, \bar{w}_2, \bar{w}_3, \bar{w}_4$ are not all zero and $\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4$ are not all zero. The result is thus obvious. We will need this argument for many similar situations. We will also assume the reader knows the equivalent forms of the stable range condition in [16].

4.17. Lemma. *Let v, w be as in 4.13. Let $p_i, q_i \in R$ be such that*

$$(v_2, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1, w_2 + q_2 v_1 w_1, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1)$$

is unimodular. Then

$$\begin{aligned} v/w = {} & (v_1, v_2, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1) \\ & /(w_1, w_2 + q_2 v_1 w_1, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1) \end{aligned}$$

Proof. Adding multiples of $w_1 v_2$ to v_3, v_4, w_2, w_3, w_4 we may arrange that $(v_3, v_4, w_2, w_3, w_4)$ is also unimodular. But then

$$(v_1, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1, w_2 + q_2 v_1 w_1, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1)$$

is unimodular too, so that 4.15 allows us to replace (v_1, v_2) by $(v_2, -v_1)$ on both sides of the desired equality. Now make the appropriate substitutions in 4.14.

4.18. Remark. We may switch numerator and denominator on both sides and get a true statement again. This is because our reasoning has a qualitative nature and depends more on 4.3 than on 4.4, say. (Of course v/w will just be the inverse of w/v, but this we did not prove yet.) We will leave it to the reader to derive 'inverted' versions of 4.12, 4.14, 4.15, 4.17 where the rôles of numerator and denominator have been interchanged. One may also prove at this stage that w/v is the inverse of v/w (see 4.26 for the definition of inverses).

4.19. Definition. For $v, w \in \mathrm{Um}_4(R)$ choose $p_i, q_i \in R$ so that

$$(v_2, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1, w_2 + q_2 v_1 w_1, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1)$$

is unimodular and define $v /\!/ w$ to be

$$(v_1, v_2, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1)/(w_1, w_2 + q_2 v_1 w_1, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1).$$

(Compare 4.16, 4.10.) By 4.17 the orbit $v /\!/ w$ is independent of the choices. We write $v /\!/ w$ instead of v/w because we do not want to claim yet that the present definition agrees with all the earlier ones. Of course it will turn out that $v /\!/ w$ equals v/w when both are defined. Also 4.19 agrees with 4.10 (take $p_i = q_i = 0$).

4.20. Lemma. *Let $v/\!/w \in \mathrm{Um}_4(R)$. Then $v/\!/w$ does not change if we add multiples of $v_2 w_1$ to v_3, v_4, w_2, w_3, w_4.*

Proof. Use 4.14.

4.21. Lemma. *Let $v, w \in \mathrm{Um}_4(R)$. Then $v/\!/w$ does not change if we add a multiple of $v_1 w_1$ to v_2.*

Proof. Let $y \in R$. We ask if $v/\!/w = (v_1, v_2 + yv_1 w_1, v_3, v_4)/\!/w$. Neither side changes if we add a multiple of $v_1 w_1$ to v_3, v_4, w_2, w_3, w_4. (See 4.19.) Combining this with 4.20 we see that $v/\!/w$ does not change if we add elements of $\langle v_1 w_1, v_2 w_1\rangle$ to v_3, v_4, w_2, w_3, w_4. Similarly $(v_1, v_2 + yv_1 w_1, v_3, v_4)/\!/w$ does not change if we add elements of $\langle v_1 w_1, (v_2 + yv_1 w_1) w_1\rangle = \langle v_1 w_1, v_2 w_1\rangle$ to v_3, v_4, w_2, w_3, w_4. Therefore (see 4.16) we may assume that $(v_3, v_4, w_2, w_3, w_4)$ is unimodular. Apply 4.12.

4.22. Definition. For $v, w \in \mathrm{Um}_4(R)$ choose $p_i, q_i \in R$ so that

$$(v_2 + p_2 v_1 w_1, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1, w_2, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1)$$

is unimodular and put $v/\!/w$ equal to

$$(v_1, v_2 + p_2 v_1 w_1, v_3 + p_3 v_1 w_1, v_4 + p_4 v_1 w_1)/(w_1, w_2, w_3 + q_3 v_1 w_1, w_4 + q_4 v_1 w_1).$$

This is the inverted form of 4.19, cf. 4.18. To see that it agrees with 4.19 observe that $v/\!/w$ in the sense of 4.19 does not change if we add a multiple of $v_1 w_1$ to v_2 (see 4.21) or to v_3, v_4, w_2, w_3, w_4. (See 4.19.) We may therefore invoke the compatibility of 4.19 with 4.10.

4.23. Lemma. *Let $v, w \in \mathrm{Um}_4(R)$ and let $g \in E_4(R)$. Then $v/\!/w = (vg)/\!/w = v/\!/(wg)$.*

Proof. First let $g = y^{ij}$ with $i \geq 2$, $j \geq 2$. As we may add multiples of $v_1 w_1$ to $v_2, v_3, v_4, w_2, w_3, w_4$, we may assume that $(v_2, v_3, v_4, w_2, w_3, w_4)$ is unimodular, so that 4.13 applies. Next let $g \in E_2(R)$. Let us ask if $v/\!/w$ equals $(vg)/\!/w$. (The 'inverted' question whether $v/\!/w$ equals $v/\!/(wg)$ is treated similarly, cf. 4.22, 4.18.)

As in 4.21 we may add elements of $\langle w_1 v_1, w_1 v_2\rangle$ to v_3, v_4, w_2, w_3, w_4 and make that 4.12 applies. The y^{ij} with $i \geq 2$, $j \geq 2$ generate, together with $E_2(R)$, all of $E_4(R)$, whence the lemma.

4.24. Definition. For $[v], [w] \in \mathrm{Um}_4(R)/E_4(R)$ put $[v]/[w] = v/\!/w$. It follows from 4.23 that this is well defined. It should also be clear that $[v]/[w] = v/w$ whenever the right-hand side is defined. We leave this as an exercise.

Let $[e]$ denote the orbit of $(1, 0, 0, 0)$ and put $[v]^{-1} = [e]/[v]$ for $v \in \mathrm{Um}_4(R)$. Also put $[v][w] = [v]/([w]^{-1})$.

4.25. Lemma. *In the situation of* 3.4 *we have*

$$[c,d,v_3,v_4][a,b,v_3,v_4]=[p,q,v_3,v_4] \quad \textit{and} \quad [a,b,v_3,v_4]^{-1}=[-f,b,v_3,v_4].$$

Proof. To get the last equality, apply 4.3 to compute $(-1,0,v_3,v_4)/v$. To get the first equality, observe that we may assume, as in 3.4 that $(\bar{b},\bar{d})$ is unimodular over $R/\langle v_3,v_4\rangle$. Then $[c,d,v_3,v_4][a,b,v_3,v_4]=(c,d,v_3,v_4)/(-f,b,v_3,v_4)$ may be computed with 4.3, so that we are computing in some $\mathrm{SL}_2(S)/T_2(S)$ again.

4.26. Remark. The fact that $[a,b,v_3,v_4]\mapsto[-f,b,v_3,v_4]$ defines a map $\mathrm{Um}_4(R)/E_4(R)\to\mathrm{Um}_4(R)/E_4(R)$ also follows from [15, 13.1], where one uses only that R is commutative (no dimension hypotheses are involved). Thus inverses may be defined by the formula of 3.5(iii) before defining other fractions.

4.27. Lemma. *For* $v,w,z\in\mathrm{Um}_4(R)$ *one has* $([v][w])[z]=[v]([w][z])$.

Proof. We start modifying the representatives of the orbits. Adding multiples of v_1w_1 to v_2,v_3,v_4,w_2,w_3,w_4 as in 4.19 (cf. 4.16) we make $(v_2,v_3,v_4,w_2,w_3,w_4)$ unimodular. Next we make w_1-v_1 equal to 1 by adding an element of $\langle v_2,v_3,v_4\rangle$ to v_1 and one of $\langle w_2,w_3,w_4\rangle$ to w_1, just like in 4.10. Next make that $v_2=w_2$, $v_3=w_3$, $v_4=w_4$, just like in 4.5.

Observe that this probably spoils unimodularity of $(v_2,v_3,v_4,w_2,w_3,w_4)$. We will usually leave it to the reader to guess what properties to preserve and what properties to give up when making modifications. We now have $v=(v_1,v_2,v_3,v_4)$, $w=(w_1,v_2,v_3,v_4)$. (By this notation we suggest that if in the sequel we modify v_2, it should be modified both within v and within w). Adding multiples of $v_1w_1z_1$ to v_2,v_3,v_4,z_2,z_3,z_4 we make $(v_2,v_3,v_4,z_2,z_3,z_4)$ unimodular. Next we make z_1-v_1 equal to 1 by adding an element of $\langle v_2,v_3,v_4\rangle$ to v_1 and one of $\langle z_2,z_3,z_4\rangle$ to z_1. Next we make that $z_1-v_1w_1$ equals 1 by adding an element of $\langle v_2,v_3,v_4\rangle$ to w_1 and an element of $\langle v_1z_2,v_1z_3,v_1z_4\rangle$ to z_1. Finally we obtain that $z_2=v_2$, $z_3=v_3$, $z_4=v_4$ by adding suitable multiples of z_1 to z_2,z_3,z_4 and of v_1w_1 to v_2,v_3,v_4. Our representatives of the orbits have now taken the form $v=(v_1,v_2,v_3,v_4)$, $w=(w_1,v_2,v_3,v_4)$, $z=(z_1,v_2,v_3,v_4)$. (The fact that $z_1-v_1w_1$ equals 1 is not relevant now, but it is good to know for later.) We may compute $[u]=[v][w]$ by Vaserstein's rule (4.25, 3.4), which yields $u=(w_1(v_1+p)-1,(v_1+p)v_2,v_3,v_4)$ with $pw_1\equiv 1 \bmod \langle v_2,v_3,v_4\rangle$ (compare 3.5(i)). In particular, as $u_1=w_1(v_1+p)-1\equiv w_1v_1 \bmod \langle v_2,v_3,v_4\rangle$, the row (u_1,v_2,v_3,v_4) is unimodular and $[u][z]$ may also be computed by Vaserstein's rule. Thus with these representatives $([v][w])[z]$ may be entirely computed by Vaserstein's rule, i.e. by a matrix multiplication modulo $\langle v_3,v_4\rangle$. Similarly $[v]([w][z])$ may be computed by applying Vaserstein's rule twice, and the lemma thus follows from associativity of matrix multiplication.

4.28. *End of proof of Theorem* 4.1. We now have an associate composition law on $\mathrm{Um}_4(R)/E_4(R)$ and one easily checks that our inverse (see 4.24) is a group inverse, so that we have a group structure. From 4.4 and the universal property of $\mathrm{wms}: \mathrm{Um}_4(R)/E_4(R) \to \mathrm{WMS}_4(R)$ it follows that wms has an inverse. Done.

4.29. Exercise. Let R be commutative, $n \geq 3$, $\mathrm{sdim}(R) \leq 2n-3$. Show that $\mathrm{WMS}_n(R)$ is abelian. (Use 3.6 and read the proof of 4.27.)

4.30. Exercise. With the same R, n, let finitely many orbits under $E_n(R)$ be given in $\mathrm{Um}_n(R)$. Show that one may choose representatives in such a way that for any two orbits, the chosen representatives differ at most in their first coordinate. Hint: Use induction on the number of orbits and look at the proof of 4.27 again.

5. A module structure

5.1. Let R be commutative, $n \geq 3$, $\mathrm{sdim}(R) \leq 2n-4$, such that Theorem 4.1 applies. In this section we study how the natural action of $\mathrm{GL}_n(R)$ on $\mathrm{Um}_n(R)/E_n(R)$ interacts with the group structure on $\mathrm{Um}_n(R)/E_n(R)$.

5.2. Notations. We write the group structure on $\mathrm{Um}_n(R)/E_n(R)$ additively, so that $[v][w]$ of 4.24 is now written as $[v]+[w]$. For $g \in \mathrm{GL}_n(R)$ the orbit of its first row is denoted $[g]$. Thus $[g]=0$ for $g \in E_n(R)$. Put

$$[v]g = [vg] - [g]$$

for $v \in \mathrm{Um}_n(R)$, $g \in \mathrm{GL}_n(R)$. Observe that the right-hand side depends only on the orbit of v (and on g), because $E_n(R)$ is normal in $\mathrm{GL}_n(R)$. The notation $[v]g$ will be justified by the following theorem:

5.3. Theorem (Module structure). *Let R be a commutative ring, $n \geq 3$, $\mathrm{sdim}(R) \leq 2n-4$.*

(i) *The rule $([v], g) \mapsto [v]g$ makes $\mathrm{Um}(R)/E_n(R)$ into a right $\mathrm{GL}_n(R)$-module.*

(ii) *If $\mathrm{sdim}(R) = n-1$ and $g \in \mathrm{SL}_n(R)$, then $[vg] = [v] + [g]$ for $v \in \mathrm{Um}_n(R)$. (cf. [13, Proposition 1.3], [4, (3.16)(iv)]).*

(iii) *If n is odd and $g \in \mathrm{GL}_n(R) \cap E_{n+1}(R)$, then $[vg] = [v]g = [v]$ for $v \in \mathrm{Um}_n(R)$. (See also 5.4, [4, 4.2].)*

(iv) *If $\mathrm{sdim}(R) \leq 2n-5$ and $g \in \mathrm{GL}_n(R) \cap E_{n+1}(R)$, then $[vg] = [v] + [g]$ for $v \in \mathrm{Um}_n(R)$.*

5.4. In connection with part (iii) it is good to recall a theorem of Vaserstein, valid without any conditions on $\mathrm{sdim}(R)$. It reads

Theorem (Vaserstein [20, §6, Theorem 10]). *Let R be a commutative ring, n an odd integer, $n \geq 3$. If $g \in \mathrm{GL}_n(R) \cap E_{n+1}(R)$ and $h \in \mathrm{GL}_n(R)$, then $[hg] = [h]$.*

5.5. In the same vein, one has

Lemma. *With R, n as in* 5.4, *let* $t \in \mathrm{GL}_1(R)$, $h \in \mathrm{GL}_n(R)$. *Then* $[ht] = [h]$.

Proof. $[ht] = [h\; t\; t^{-1}\; h^{-1}\; t\; h]$ by 5.4 and

$$[th] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & I \end{pmatrix} h\right] = [h].$$

5.6. In the other direction, we know that for $v \in \mathrm{Um}_n(R)$, $t \in \mathrm{GL}_1(R)$, one need not have $[vt] = [v]$, even if n is odd, $\mathrm{sdim}(R) = n-1$. To get a counterexample one takes $t = -1$ and considers the degree of a self map of a sphere. (See [4, (3.37)].)

There is a difference between the behaviour of orbits of arbitrary unimodular rows and orbits of completable rows. The completable rows may be more relevant, cf. [20], but we find it difficult to single them out. Observe also the difference between odd n and even n. Theorem 5.4 must fail for even n, as 5.5 fails for even n (see [4, 4.13(i)] and compare it with the first lines of 5.6). There is some connection with the special role of squares in 3.5(iv), 3.5(v) (compare [1]).

5.7. We start with the proof of Theorem 5.3.

Proposition. *Let R, n be as in* 5.3. *Let $v, w \in \mathrm{Um}_n(R)$ and $g \in \mathrm{GL}_n(R)$. Then* $([w] - [v])g = [w]g - [v]g$.

Proof. We first seek convenient representatives v, w, z of $[v]$, $[w]$, $[g^{-1}]$ respectively. As in 4.27 we may arrange that $v = (1+q, v_2, \ldots, v_n)$ $w = (q, v_2, \ldots, v_n)$. If $g = \mathrm{diag}(1, \ldots, 1, t)$, then the proposition follows from 3.5. We may therefore assume $\det(g) = 1$. First choose $z = (z_1, \ldots, z_n)$ in $[g^{-1}]$. Add elements of $\langle q(1+q)z_1, v_2 z_1 \rangle$ to $v_3, \ldots, v_n, z_2, \ldots, z_n$ to make $(v_3, \ldots, v_n, z_2, \ldots, z_n)$ unimodular. (cf. 4.16). Then add an element of $\langle v_3, \ldots, v_n \rangle$ to v_2 and one of $\langle z_2, \ldots, z_n \rangle$ to z_1 to make $z_1 - v_2 = 1$. Use z_1, v_2 to make that $z_2 = q$, $z_3 = v_3, \ldots,$ $z_n = v_n$. Add $q(z_1+1)$ to z_1 and observe that this makes $(z_1, q(q+1), z_3, \ldots, z_n)$ unimodular. (The new z_1 is the old $z_1 + q(z_1+1)$.) Then add a multiple of $q(q+1)$ to v_2 and an element of $\langle z_1, z_3, \ldots, z_n \rangle$ to z_2 to achieve $v_2 = z_2$. Write $a = z_1$. We now have $[v] = [1+q, v_2, \ldots, v_n]$, $[w] = [q, v_2, \ldots, v_n]$, $[g^{-1}] = [a, v_2, \ldots, v_n]$. As we may just as well prove the proposition with g replaced by any other representative of $gE_n(R)$, we further assume there are b, m_i in R with

$$g^{-1} = \begin{pmatrix} a & v_2 & v_3 & \cdots & v_n \\ b & m_2 & m_3 & \cdots & m_n \\ & & * & & \end{pmatrix}.$$

Next we want moreover that $(v_2, \ldots, v_n, m_3, \ldots, m_n)$ is unimodular. We claim that this can be achieved by adding multiples of the first two columns of g^{-1} to the other ones and a multiple of the first column to the second. To see this, use that the image of $\binom{a\ \ v_2}{b\ \ m_2}: R^2 \to R^2$ contains all multiples of $r = q(q+1)a \det\binom{a\ \ v_2}{b\ \ m_2}$ and use that $(r, v_2, \dots, v_n, m_3, \dots, m_n)$ is unimodular. (Make sure to preserve the connection between v, w, z, a, q, g). After all this preparation, write

$$g^{-1} = \begin{pmatrix} a & v_2 \cdots v_n \\ * & M \end{pmatrix}$$

where M is an $n-1$ by $n-1$ matrix.

We will be interested in $[(f, v_2, \dots, v_n)g]$ for several values of f with $(f, v_2, \dots, v_n)$ unimodular.

Following [4, proof of 3.16(iv)] we let N be the adjoint (= matrix of minors) of M, such that $MN = NM = \det(M)I$. Write $d = \det(M)$. The first row of g then equals $(d, u_2, \dots, u_n)$ with $(u_2, \dots, u_n) = -(v_2, \dots, v_n)N$, by Cramer's rule. (To avoid part of the computation with Cramer's rule one may observe that

$$(1, 0, \dots, 0) = (d, u_2, \dots, u_n)g^{-1} = (1, d(v_2, \dots, v_n) + (u_2, \dots, u_n)M)$$

implies

$$d(v_2, \dots, v_n)N = -(u_2, \dots, u_n)MN = -d(u_2, \dots, u_n).$$

Now cancel d, which is allowed because it suffices to check the formula for a domain, or more specifically for a localisation of a polynomial ring over $\mathbb{Z}$.) We get

$$\begin{aligned}[(f, v_2, \dots, v_n)g] &= [(f - a + a, v_2, \dots, v_n)g] \\ &= [1 + (f-a)d, -(f-a)(v_2, \dots, v_n)N].\end{aligned}$$

Choose $s \in \langle m_3, \dots, m_n\rangle \cap (1 + \langle v_2, \dots, v_n\rangle)$. (Use unimodularity of $(m_3, \dots, m_n, v_2, \dots, v_n)$.) Then

$$[(f, v_2, \dots, v_n)g] = [1 + (f-a)ds, -(f-a)(v_2, \dots, v_n)N]$$

(Use $NM = dI$ or use that $\langle u_2, \dots, u_n\rangle = \langle v_2, \dots, v_n\rangle$.) By 3.5 we may write

$$\begin{aligned}[(f, v_2, \dots, v_n)g] &= [1 + (f-a)ds, -(f-a)d^2s^2(v_2, \dots, v_n)N] \\ &= [1 + (f-a)ds, ds(v_2, \dots, v_n)N].\end{aligned}$$

Modulo $1 + (f-a)ds$ the element ds is invertible. Therefore we consider $N \in \mathrm{GL}_{n-1}(R[1/(ds)])$. Its inverse is $(1/d)M$. Now $(m_3, \dots, m_n)$ is unimodular over $R[1/s]$, so $(1/d)M \in \mathrm{GL}_{n-2}(R[1/(ds)])\, E_{n-1}(R[1/(ds)])$ and its inverse N lies in that same group. We get $(ds)^{2k+1}N = \binom{1\ \ 0}{0\ \ T}E$ for some $k \geq 0$, where $E \in E_{n-1}(R[1/(ds)])$, T is a matrix over R with $\det(T)$ equal to a power of d times a power of s. (Choose k large to get rid of denominators of T and to get rid of zero-divisor trouble). Arguing as in [4, p. 381] we find $[(f, v_2, \dots, v_n)g] = [1 + (f-a)ds, v_2, x]$ where x stands for $(v_3, \dots, v_n)T$. Now $\mathrm{rad}\langle x\rangle$ contains $\mathrm{rad}\langle \det(T)(v_3, \dots, v_n)\rangle = \mathrm{rad}\langle ds(v_3, \dots, v_n)\rangle$. We will be computing modulo $\langle ds(v_3, \dots, v_n)\rangle$, when applying 3.3. Put $S = R/\langle dsv_3, \dots, dsv_n\rangle$ and choose $z \in R$ with $(\bar{a}\bar{d}\bar{s} + \bar{v}_2\bar{z})\bar{d}\bar{s} = \bar{d}\bar{s}$ in S. (Recall that $s \equiv 1 \bmod \langle v_2, \dots, v_n\rangle$ and note that $ad \equiv 1 \bmod \langle v_2, \dots, v_n\rangle$ because $g^{-1}g = I$.)

Also choose $c, e, y \in R$ with $y(1+q) \equiv q \bmod \langle v_2, \ldots, v_n \rangle$, $cy \equiv 1 \bmod \langle v_2, \ldots, v_n \rangle$, $(\bar{c}\bar{y} + \bar{e}\bar{v}_2)\bar{d}\bar{s} = \bar{d}\bar{s}$ in S. Put $B = a(e - cz(1-y))$ and check that $(1-(1-y)ads) \times (1+(1-y)acds) + dsv_2B(1-y)$ has image 1 in S.

Also check that $(1 - ads + a^2ds)(1 + (1-a)ds)$ and $(1 - ads + a^2ds - a^2dsy) \times (1 + (1+q-a)ds)$ have image 1 in $S/\langle \bar{v}_2 \rangle$. We compute as in Section 3, using 3.2, 3.3, 3.5 to see that

$$\begin{aligned}
&([w]-[v])g \\
&\quad = [y, v_2, \ldots, v_n]g = [(y, v_2, \ldots, v_n)g] - [(1, v_2, \ldots, v_n)g] \\
&\quad = [1+(y-a)ds, v_2, \boldsymbol{x}] - [1+(1-a)ds, v_2, \boldsymbol{x}] \quad \text{(see above)} \\
&\quad = [1-(1-y)ds(1-ads+a^2ds), (1-y)dsv_2, \boldsymbol{x}] \\
&\quad = [1-(1-y)ads, (1-y)dsv_2, \boldsymbol{x}]
\end{aligned}$$

and

$$\begin{aligned}
&([w]g)-([v]g) \\
&\quad = [wg]-[vg] = [1+(q-a)ds, v_2, \boldsymbol{x}] - [1+(1+q-a)ds, v_2, \boldsymbol{x}] \\
&\quad = [1-ds(1-ads+a^2ds-a^2dsy), dsv_2, \boldsymbol{x}] = [1-(1-y)ads, dsv_2, \boldsymbol{x}]
\end{aligned}$$

so that

$$\begin{aligned}
&([w]g)-([v]g)-([w]-[v])g \\
&\quad = [1-(1-y)ads, dsv_2(1-y)ads, \boldsymbol{x}] - [1-(1-y)ads, dsv_2(1-y), \boldsymbol{x}] \\
&\quad = [(1-(1-y)ads)dsv_2(1-y)(1-ads), 1-dsv_2(1-y)(1-ads)B, \boldsymbol{x}] \\
&\quad = [(1-(1-y)ads)(1-y)dsv_2^2z, 1-dsv_2^2z(1-y)B, \boldsymbol{x}] \\
&\quad = [(1-(1-y)ads)(1-y)dsz, \\
&\qquad\quad 1-v_2z(1-(1-(1-y)ads)(1+(1-y)acds)), \boldsymbol{x}] \\
&\quad = [(1-(1-y)ads)(1-y)dsz, 1-v_2z(1-y)ads, \boldsymbol{x}] \\
&\quad = [(1-y)dszv_2^2-(1-y)^2ad^2s^2zv_2^2, 1-v_2z(1-y)ads, \boldsymbol{x}] \\
&\quad = [(1-y)dszv_2^2-(1-y)dsv_2, 1-v_2z(1-y)ads, \boldsymbol{x}] \\
&\quad = [-(1-y)ad^2s^2v_2, 1-v_2z(1-y)ads, \boldsymbol{x}] \\
&\quad = [-(1-y)av_2, 1-v_2z(1-y)ads, \boldsymbol{x}] \\
&\quad = [-(1-y)av_2, 1, \boldsymbol{x}] = 0.
\end{aligned}$$

5.8. Proof of 5.3(i). If A is an abelian group, write T_a for the translation by a ($a \in A$), and write $T(A) = \{T_a \colon a \in A\}$ for the subgroup of the permutation group which consists of translations. ($T_a(z) = a + z$). Let $\operatorname{Aff}(A)$ be the semi-direct product, within the permutation group on the set A, of $T(A)$ and the automorphism group $\operatorname{Aut}(A)$. Then Proposition 5.7 tells us that for $g \in \mathrm{GL}_n(R)$ the map $[v] \mapsto [vg]$ is an element of $\operatorname{Aff}(\mathrm{Um}_n(R)/E_n(R))$. Projecting $\operatorname{Aff}(\mathrm{Um}_n(R)/E_n(R))$ onto $\operatorname{Aut}(\mathrm{Um}_n(R)/E_n(R))$ sends the element $[v] \mapsto [vg]$ to the element $[v] \mapsto [v]g$. By composition we get an anti-homomorphism $\mathrm{GL}_n(R) \to \operatorname{Aut}(\mathrm{Um}_n(R)/E_n(R))$ sending g to $[v] \mapsto [v]g$. In other words, we have a right module for $\mathrm{GL}_n(R)$.

5.9. Part (ii) of 5.3 will follow from

Theorem. (cf. [4, 3.25(iii)]). *Let R be commutative, $n \geq 3$, $\mathrm{sdim}(R) = n-1$. Let $1 \leq k \leq n$. Let $(v_1, \dots, v_n) \in \mathrm{Um}_n(R)$ and let T be a k by k matrix over R with first row $(u_1, \dots, u_k)$ such that $\overline{\det(T)}$ is a square of a unit in $R/\langle v_{k+1}, \dots, v_n\rangle$. Then*

$$[(v_1, \dots, v_k)T, v_{k+1}, \dots, v_n] = [v_1, \dots, v_n] + [u_1, \dots, u_k, v_{k+1}, \dots, v_n].$$

Proof. By induction on k. The case $k=1$ follows from 3.5(iv). Let $k \geq 2$. By 3.5 we may multiply T by $\mathrm{diag}(1, \dots, 1, \det(T))$ so that we may assume $\det(T) = d^2$, where $d \in R$. Let S be the adjoint (= matrix of minors) of T, so that $ST = d^2 I$. We claim

$$[(v_1, \dots, v_k)S, v_{k+1}, \dots, v_n] = [v_1, \dots, v_n] - [u_1, \dots, u_k, v_{k+1}, \dots, v_n].$$

To prove this claim we first imitate the proof of 4.27 and add multiples of $u_1 v_1$ to $u_2, \dots, u_k, v_2, \dots, v_n$ to make $(u_2, \dots, u_k, v_2, \dots, v_n)$ unimodular. (Change S, T accordingly). Then we add elements of $\langle u_2, \dots, u_k\rangle$ to u_1 and of $\langle v_2, \dots, v_n\rangle$ to v_1 to achieve $u_1 - v_1 = 1$ (change S, T again). After that we add multiples of v_1 to $v_2, \dots, v_k$ and of u_1 to $u_2, \dots, u_k$ to make that $u_i = v_i$ for $i = 2, \dots, k$ (change S, T for the last time). Say

$$T = \begin{pmatrix} u_1 & v_2 \cdots v_k \\ * & M \end{pmatrix}$$

where M is $k-1$ by $k-1$ with adjoint N. As in the proof of 5.7 the first row of S is $(\det(M), -(v_2, \dots, v_k)N)$, so that

$$\begin{aligned}[(v_1, \dots, v_k)S, v_{k+1}, \dots, v_n] &= [(u_1 - 1, v_2, \dots, v_k)S, v_{k+1}, \dots, v_n] \\ &= [d^2 - \det(M), (v_2, \dots, v_k)N, v_{k+1}, \dots, v_n].\end{aligned}$$

Now the determinant of N is a square modulo $d^2 - \det(M)$, and the inductive hypothesis tells us

$$\begin{aligned}&[d^2 - \det(M), (v_2, \dots, v_k)N, v_{k+1}, \dots, v_n] \\ &\quad = [d^2 - \det(M), v_2, \dots, v_n] + [d^2 - \det(M), (1, 0, \dots, 0)N, v_{k+1}, \dots, v_n].\end{aligned}$$

As $NM = \det(M)I$, we get

$$\begin{aligned}&[d^2 - \det(M), (1, 0, \dots, 0)N, v_{k+1}, \dots, v_n] \\ &\quad = [d^2, (1, 0, \dots, 0)N, v_{k+1}, \dots, v_n].\end{aligned}$$

But this vanishes because d^2 is a unit $\mathrm{mod}\,\langle v_{k+1}, \dots, v_n\rangle$. Further $u_1 = v_1 + 1$ yields $[v_1, \dots, v_n] - [u_1, v_2, \dots, v_n] = [p_1, v_2, \dots, v_n]$ where p_1 is such that $p_1 u_1 \equiv u_1 - 1$ $\mathrm{mod}\,\langle v_2, \dots, v_n\rangle$. Now $(d^2 - \det(M))u_1 \equiv d^2(u_1 - 1) \bmod \langle v_2, \dots, v_n\rangle$, so

$$[p_1, v_2, \dots, v_n] = [d^2 p_1, v_2, \dots, v_n] = [d^2 - \det(M), v_2, \dots, v_n].$$

Our claim follows. Substitute $(v_1,\dots,v_k)T$ for $(v_1,\dots,v_k)$ and you get

$$\begin{aligned}&[(v_1,\dots,v_k)T, v_{k+1},\dots,v_n]-[u_1,\dots,v_k,v_{k+1},\dots,v_n]\\&\quad=[(v_1,\dots,v_k)TS,v_{k+1},\dots,v_n]=[d^2(v_1,\dots,v_k),v_{k+1},\dots,v_n]\\&\quad=[v_1,\dots,v_n].\end{aligned}$$

5.10. Lemma. *Let R be commutative, $n\geq 3$, n odd, $\operatorname{sdim}(R)\leq 2n-3$. Let $v\in \mathrm{Um}_n(R)$ and let $a,t\in R$, $u\in M_{1,n-1}(R)$, $w\in M_{n-1,1}(R)$, $M\in M_{n-1,n-1}(R)$ be such that $g=\left(\begin{smallmatrix}1+at & u\\ wt & M\end{smallmatrix}\right)$ is invertible.*

Put $h=\left(\begin{smallmatrix}1+at & tu\\ w & M\end{smallmatrix}\right)$, which is thus also invertible. (cf. (2.3)).

Then $[vg]=[vh]$.

Proof. We may add multiples of the first column of g to the other columns and adapt h accordingly. Similarly we may add multiples of the other columns of h to the first column and adapt g accordingly. Put $(z_1,\dots,z_n)=z=vg$. We have to show that $[zg^{-1}h]=[z]$. Imitating 4.30 we make $(u_2,\dots,u_n,z_2,\dots,z_n)$ unimodular, next make z_1 equal to a, next make that $(z_2,\dots,z_n)=u$. After that we get

$$[zg^{-1}]=[(1+at+z_1-1-at,u)g^{-1}]=[1+(z_1-1-at)d,-(z_1-1-at)uN]$$

with $d=\det(M)$ and N the adjoint of M (compare proof of 5.7). As n is odd, repeated application of 3.5(iv) yields $[zg^{-1}]=[1+(z_1-1-at)d,-uN]$ and

$$[zg^{-1}h]=[(1+at)(1+(z_1-1-at)d)-uNw,(t+(z_1-1-at)dt-d)u],$$

which equals $[(1+at)(1+(z_1-1-at)d)-uNw,u]$ by some more applications of 3.5(iv). Now check that $(1+at)(1+(z_1-1-at)d)-uNw\equiv z_1 \bmod \langle u\rangle$.

5.11. *Proof and sharpening of* 5.3(iii). Clearly 5.3(iii) follows from 5.10 and 2.2. In fact one sees that the equality $[vg]=[v]$ is valid for n odd, $n\geq 3$, $\operatorname{sdim}(R)\leq 2n-3$, $g\in \mathrm{GL}_n(R)\cap E_{n+1}(R)$. But we need $\operatorname{sdim}(R)\leq 2n-4$ if we want to talk of $[v]g$, as this requires 4.1.

5.12. Proof of 5.3(iv). Because of 5.3(iii) we may assume n is even. We may also assume there is $h\in E_n(R)$ with

$$h=\begin{pmatrix}p & st & u\\ q & 1+at & w\\ r & zt & M\end{pmatrix}\quad\text{and}\quad g=\begin{pmatrix}p & s & u\\ qt & 1+at & tw\\ r & z & M\end{pmatrix}$$

where $u=(u_3,\dots,u_n)$, $w=(w_3,\dots,w_n)$, $p,q,s,t,a\in R$, etc. (apply 2.2 to the transpose of g and conjugate). Put $x=vg$. As in 5.10 we may add elements of $\langle x_1p, x_1st\rangle$ to $x_2,\dots,x_n,u_3,\dots,u_n$ and make $(x_2,\dots,x_n,u_3,\dots,u_n)$ unimodular (adapt w etc.). Next we may make that $p-x_1=1$, next that $(x_3,\dots,x_n)=u$, $s=x_2$, $p-x_1=1$. Let T be the adjoint of $\left(\begin{smallmatrix}1+at & tw\\ z & M\end{smallmatrix}\right)$ and let d be the determinant of the latter. As in the proof of 5.7 we get $[v]=[xg^{-1}]=[1-d,(s,u)T]$. But $\bar{T}\in E_n(R/\langle 1-d\rangle)\cap \mathrm{GL}_{n-1}(R/\langle 1-d\rangle)$ because $h\in E_n(R)$ (operate on $\bar{h}$, using $\det(\bar{T})=1$, and use the Whitehead Lemma 2.3). Therefore 5.11 shows $[v]=[1-d,s,u]$. We have to compare this with $[x]-[g]$, which is easy.

6. Higher Mennicke symbols

6.1. We now turn to the higher Mennicke symbols of Suslin [13]

Theorem. *Let R be commutative, $n\geq 3$.*

(i) *If* $\mathrm{sdim}(R)\leq 2n-4$, *then the kernel of the universal Mennicke symbol* $\mathrm{ms}:\mathrm{Um}_n(R)/E_n(R)\to \mathrm{MS}_n(R)$ *is a submodule (for the module structure of* 5.3).

(ii) *If* $\mathrm{sdim}(R)\leq 2n-5$ *and n is even, then the universal Mennicke symbol induces a bijection* $\mathrm{Um}_n(R)/(\mathrm{GL}_n(R)\cap E_{n+1}(R))\to \mathrm{MS}_n(R)$.

Remark. For odd n, see 5.3(iii) and 4.1, cf [4, §4].

Proof of Theorem. (i) The map $\mathrm{ms}:\mathrm{Um}_n(R)/E_n(R)$ is of course obtained as the composite homomorphism $\mathrm{Um}_n(R)/E_n(R)\xrightarrow{\cong}\mathrm{WMS}_n(R)\longrightarrow \mathrm{MS}_n(R)$ and its kernel is generated by elements of the form $[pq,v_2,\ldots,v_n]-[p,v_2,\ldots,v_n]-[q,v_2,\ldots,v_n]$, by the definition of $\mathrm{MS}_n(R)$. Let $g\in \mathrm{GL}_n(R)$ with $g^{-1}=\left(\begin{smallmatrix} a & u\\ * & M\end{smallmatrix}\right)$, where $u=(u_2,\ldots,u_n)$, M is an $n-1$ by $n-1$ matrix with determinant d and adjoint N, cf. proof of 5.7. By the usual argument we can make $(v_2,\ldots,v_n,u_3,\ldots,u_n)$ unimodular (add elements of $\langle apq,u_2pq\rangle$ to its entries), next make a equal to $1+q$, u_2 equal to $1+p$, next make a equal to $1-pq$, next make $(v_2,\ldots,v_n)$ equal to u. As in 5.7 we find

$$\mathrm{ms}[(p,u)g]=\mathrm{ms}[1+(p-a)d,-(p-a)uN]=\mathrm{ms}[1+(p-a)d,uN].$$

Now $ad\equiv 1 \bmod \langle uN\rangle$ (look at gg^{-1}), so $\mathrm{ms}[(p,u)g]=\mathrm{ms}[pd,uN]$. Similarly,

$$\begin{aligned}&\mathrm{ms}(([pq,u]-[p,u]-[q,u])g)\\ &\quad=\mathrm{ms}[(pq,u)g]-\mathrm{ms}[(p,u)g]-\mathrm{ms}[(q,u)g]+\mathrm{ms}[(1,u)g]\\ &\quad=\mathrm{ms}[pqd,uN]-\mathrm{ms}[pd,uN]-\mathrm{ms}[qd,uN]+\mathrm{ms}[d,uN]=0.\end{aligned}$$

This proves (i).

(ii) Now let $v\in \mathrm{Um}_n(R)$, $g,h\in \mathrm{GL}_n(R)$ with g and h related as in 5.10. The proof of 5.10 shows that $\mathrm{ms}(vg)=\mathrm{ms}(vh)$. It follows from this and from 2.2 that $\mathrm{ms}(vk)=\mathrm{ms}(v)$ for $k\in \mathrm{GL}_n(R)\cap E_{n+1}(R)$ (apply 2.2 to the transpose of k). Therefore there is a map $\mathrm{Um}_n(R)/(\mathrm{GL}_n(R)\cap E_{n+1}(R))\to \mathrm{MS}_n(R)$. This map is of course surjective (see 4.1). To prove injectivity we consider the kernel of $\mathrm{ms}:\mathrm{Um}_n(R)/E_n(R)\to \mathrm{MS}_n(R)$. From Lemma 3.5(ii) it follows that this kernel is generated by elements of the form $[1+at,tv_2,v_3,\ldots,v_n]-[1+at,v_2,v_3,\ldots,v_n]$. As n is even we may follow Suslin as in [4, proof of 4.2] to find $k\in \mathrm{GL}_n(R)\cap E_{n+1}(R)$ with

$(1+at, tv_2, v_3, \ldots, v_n)k = (1+at, v_2, \ldots, v_n)$. So the kernel of $\mathrm{ms}: \mathrm{Um}_n(R)/E_n(R) \to \mathrm{MS}_n(R)$ coincides with the subgroup generated by the elements $[vk]-[v]$ with $v \in \mathrm{Um}_n(R)$, $k \in \mathrm{GL}_n(R) \cap E_{n+1}(R)$. By 5.3(iv) this is the same as the subgroup generated by the $[k]$ with $k \in \mathrm{GL}_n(R) \cap E_{n+1}(R)$. Thus if $\mathrm{ms}(v) = \mathrm{ms}(w)$ for some $v, w \in \mathrm{Um}_n(R)$, then there is $k \in \mathrm{GL}_n(R) \cap E_{n+1}(R)$ with $[v]-[w]=[k]$. Again by 5.3(iv) this implies $[v]=[wk]$, whence $v \in w\ (\mathrm{GL}_n(R) \cap E_{n+1}(R))$.

6.2. Remark. As in [4, 4.2] we may also formulate (ii) as an exact sequence $\mathrm{GL}_n(R) \cap E_{n+1}(R) \to \mathrm{WMS}_n(R) \to \mathrm{MS}_n(R) \to 0$, valid under the conditions of 6.1(ii).

6.3. The module structure of Section 5 also makes sense for MS_n.

Definition. Let R be commutative, $n \geq 3$, $\mathrm{sdim}(R) \leq 2n-4$. For $g \in \mathrm{GL}_n(R)$, $v \in \mathrm{Um}_n(R)$ we put $(\mathrm{ms}[v])g = \mathrm{ms}[vg] - \mathrm{ms}[g]$. This makes $\mathrm{MS}_n(R)$ into a right $\mathrm{GL}_n(R)$-module, by 6.1(i). Observe that there also is an action of $\mathrm{GL}_n(R)$ on the set $\mathrm{MS}_n(R)$, given by $(\mathrm{ms}[v], g) \mapsto \mathrm{ms}[vg]$. (It is well defined because the previous action is well defined.) Compare 5.8.

7. Comparison with the topological case

7.1. We start with some 'algebraic geometry', then turn to algebraic topology for illustrations of the theory developed above.

7.2. It is clear that $\mathrm{Um}_n(R)$ may be viewed as the set of morphisms $\mathrm{Spec}(R) \to \mathbb{A}^n$ whose image avoids the origin. (Here $\mathbb{A}^n$ denotes affine n-space, i.e. $\mathrm{Spec}\,\mathbb{Z}[X_1, \ldots, X_n]$.) Now real affine n-space minus the origin is homotopy equivalent with the sphere S^{n-1}. Therefore it is reasonable to think of $\mathrm{Um}_n(R)/E_n(R)$ as being analogous to $[X, S^{n-1}]$, cf. 1.1, 1.2, This connection can be made more precise. See [4, 14, 20] and 7.7 below.

7.3. It is inconvenient that $\mathbb{A}^n$ minus the origin is not an affine scheme. We can remedy this as follows:

Notation. Let R be commutative, $n \geq 2$. Put

$$\mathrm{SUm}_n(R) = \{(v, w) \in \mathrm{Um}_n(R) \times \mathrm{Um}_n(R) \colon \textstyle\sum v_i w_i = 1\}.$$

Elements of $\mathrm{SUm}_n(R)$ are sometimes called split unimodular sequences, see [3]. The group $\mathrm{GL}_n(R)$ acts from the right on $\mathrm{SUm}_n(R)$ by the formula $(v, w)g = (vg, w^{\mathrm{T}}g^{-1})$, where ${}^{\mathrm{T}}g^{-1}$ denotes the transpose inverse of g.

7.4. Lemma. *Let R be commutative, $n \geq 3$. The projection $(v, w) \mapsto v$ induces a bijection* $\mathrm{SUm}_n(R)/E_n(R) \to \mathrm{Um}_n(R)/E_n(R)$.

Proof. Use [11, 1.2 and 1.3].

Remark. For $n=2$ one would need $T_2(R)$ (see 3.2) instead of $E_2(R)$.

7.5. For $r\geq 1$ put $f_r = 1-\sum_{i=1}^{r} X_i Y_i$ in the polynomial ring $\mathbb{Z}[X_1,\dots,X_r,Y_1,\dots,Y_r]$. (Compare [13, Theorem 2.3].) Clearly $\mathrm{SUm}_n(R)$ consists of morphisms $\operatorname{Spec} R \to \operatorname{Spec}\mathbb{Z}[X_1,\dots,X_n,Y_1,\dots,Y_n]/\langle f_n\rangle$.

Speculation. Let $n\geq 3$, $r\geq 1$, and let R be commutative with $\mathrm{sdim}(R)\leq 2n-4$ and with $\mathrm{sdim}(R[X_1,\dots,X_r,Y_1,\dots,Y_r])\leq 2n+2r-4$. Let $R\{r\}$ denote the fibred product of $R[X_1,\dots,X_r,Y_1,\dots,Y_r]$ and $\mathbb{Z}[X_1,\dots,X_r,Y_1,\dots,Y_r]/\langle f_r\rangle$ over $R[X_1,\dots,X_r,Y_1,\dots,Y_r]/\langle f_r\rangle$. One may show that $\mathrm{sdim}(R\{r\})\leq 2n+2r-4$. We define a homomorphism $\mathrm{WMS}_n(R)\to\mathrm{WMS}_{n+r}(R\{r\})$ by $[v_1,\dots,v_n]\mapsto[X_1,\dots,X_r,v_1f_r,\dots,v_nf_r]$. We leave it to the reader to interpret the notation and to show that one gets a homomorphism. Does this homomorphism behave like a suspension map $[X,S^{n-1}]\to[S^rX,S^{n+r-1}]$? (Compare [4, 4.12].)

Caution. Do not read v_1f_r as a product of two elements v_1 and f_r of $R\{r\}$. Usually v_1 does not exist in $R\{r\}$. This type of confusion could be avoided by introducing notations for weak Mennicke symbols relative to ideals. (Here one would be interested in symbols over $R[X_1,\dots,X_r,Y_1,\dots,Y_r]$ relative to its ideal $f_rR[X_1,\dots,X_r,Y_1,\dots,Y_r]$.)

7.6. We turn to algebraic topology.

Notations. Let X be a finite CW complex of dimension d and let $C(X)$ be the ring of real valued continuous functions (for variations on this theme see [14]). Let SX denote the suspension of X. (But SO_n denotes the special orthogonal group, acting from the right on $\mathbb{R}^n$ and thus on S^{n-1}). Say $n\geq 2$. Each $v\in\mathrm{Um}_n(C(X))$ determines a map $X\to\mathbb{R}^n-\{0\}$, which in its turn determines a map $\arg(v): X\to S^{n-1}$ by composition with the standard homotopy equivalence $\mathbb{R}^n-\{0\}\to S^{n-1}$. We thus get an element $[\arg(v)]$ of $[X,S^{n-1}]$. (As $n\geq 2$, we may ignore base points, cf. [4, (4.11)(1)].) Similarly an element of $\mathrm{SL}_n(C(X))$ determines one of $[X,\mathrm{SO}_n]$ (compare [8, §7]). Now suppose $d\leq 2n-4$, $n\geq 3$. Then $[X,S^{n-1}]$ is a stable cohomotopy group $\pi^{n-1}(X)$ and the reasoning in [4; 4.10, 4.11, 4.12] shows that $\mathrm{Um}_n(C(X))\to\pi^{n-1}(X)$ (given by $v\mapsto[\arg(v)]$) induces a group isomorphism $\mathrm{Um}_n(C(X))/E_n(C(X))\to\pi^{n-1}(X)$. Let us record this as

7.7. Theorem. *Let $n\geq 3$ and let X be a finite* CW *complex of dimension d with $d\leq 2n-4$. Then the natural map* $\mathrm{Um}_n(C(X))/E_n(C(X))\to\pi^{n-1}(X)$ *is bijective and it is a universal weak Mennicke symbol.*

Second proof. We now give a more illuminating proof, due to Ofer Gabber. First recall how Borsuk [2] describes the group structure on his cohomotopy groups

$\pi^{n-1}(X)$. Given $\alpha,\beta\in\pi^{n-1}(X)$ and $y_0\in S^{n-1}$ there are representatives f_1,f_2, of α,β respectively and disjoint open subsets G_1,G_2 of X such that $f_i(X-G_i)=\{y_0\}$ for $i=1,2$. Define $f:X\to S^{n-1}$ so that f coincides with f_1 on G_1, with f_2 on G_2, with both on $X-(G_1\cup G_2)$. Then f is continuous and it represents $\alpha+\beta$. Let us write N for the north pole $(1,0,\dots,0)$ of S^{n-1} and S for its south pole $(-1,0,\dots,0)$. Then an alternative description of Borsuk's addition goes as follows (compare with the addition in terms of framed cobordism classes for the case that X is a compact differentiable manifold [9]). One first chooses representatives f_1,f_2 of α,β respectively so that $f_1^{-1}(N)$ is disjoint from $f_2^{-1}(N)$ (here $f_i^{-1}(N)$ means $f_i^{-1}(\{N\})$). There is a continuous $f:X\to S^{n-1}$ such that $f^{-1}(N)=f_1^{-1}(N)\cup f_2^{-1}(N)$ and such that f coincides with f_i on a neighborhood of $f_i^{-1}(N)$ for $i=1,2$. Any such f represents $\alpha+\beta$. (To derive this description from the previous one, first observe that given any neighborhood U of $f_i^{-1}(N)$, there is $g_i:X\to S^{n-1}$ which is homotopic to f_i, coincides with f_i on a neighborhood of $f_i^{-1}(N)$, and which sends $X-U$ to the south pole. Also use that X is compact.) As it is well known that $\mathrm{Um}_n(C(X))/E_n(C(X))\to\pi^{n-1}(X)$ is bijective (use [20, Theorem 12] for a proof), we have to understand now why this map is a weak Mennicke symbol. First let $v\in\mathrm{Um}_n(C(X))$. Inspecting $\arg(|v_1|,v_2,\dots,v_n)^{-1}(V)$ for V a small neighborhood of N ones sees that

$$[\arg(v_1,\dots,v_n)]+[\arg(-v_1,v_2,\dots,v_n)]=[\arg(|v_1|,v_2,\dots,v_n)].$$

But $[\arg(|v_1|,v_2,\dots,v_n)]$ is of course trivial. It follows that $[\arg(-v_1,v_2,\dots,v_n)]=-[\arg(v_1,v_2,\dots,v_n)]$. Now if $(q,v_2,\dots,v_n),(1+q,v_2,\dots,v_n)$ are unimodular, and $r(1+q)\equiv q \bmod \langle v_2,\dots,v_n\rangle$, then we inspect in a similar way the inverse image of N under $f=\arg(r,v_2,\dots,v_n)$. Put $f_1=\arg(q,v_2,\dots,v_n)$, $f_2=\arg(-1-q,v_2,\dots,v_n)$. It is easy to see that $f^{-1}(N)$ is the disjoint union of the $f_i^{-1}(N)$. (Let $x\in f^{-1}(N)$. One has $v_2(x)=\cdots=v_n(x)=0$, so that $r(x)(1+q(x))=q(x)$. One also has $r(x)>0$, so that either $1+q(x)>q(x)>0$ or $-q(x)>-1-q(x)>0$.) Moreover, if V is a small neighborhood of N, then f agrees with f_i on $f_i^{-1}(V)$ up to perturbations along great circles. It is easy to find homotopies to counteract these perturbations, so that $[f]$ must be $[f_1]+[f_2]$. Thus

$$\begin{aligned}[\arg(r,v_2,\dots,v_n)]&=[\arg(q,v_2,\dots,v_n)]+[\arg(-1-q,v_2,\dots,v_n)]\\&=[\arg(q,v_2,\dots,v_n)]-[\arg(1+q,v_2,\dots,v_n)],\end{aligned}$$

as required. Universality of the symbol is easy (use 4.30 and 3.5(i)).

7.8. Theorem. *For X,n as in* 7.7 *the group $\pi^{n-1}(X)$ is a right $[X,\mathrm{SO}_n]$-module as described in* 1.1.

Proof. This is just Theorem 5.3(i), because of 7.7. As in 7.7 we now give a second proof. We have to explain the formula

$$[vg]+[wg]=[zg]+[g]\quad\text{when}\quad[v]+[w]=[z],$$

where $v, w, z \in \mathrm{Um}_n(C(X))$, $g \in \mathrm{SO}_n(C(X)) \subseteq \mathrm{GL}_n(C(X))$. One may choose representatives v, w, g of the respective homotopy classes in such a way that there are disjoint open subsets G_1, G_2, G_3 of X such that $\arg(v)(X-G_1)=S$, $\arg(w)(X-G_2)=S$ and such that $g(x)$ stabilizes S for $x \in X-G_3$. (For a proof of this in the style of this paper see the exercise below.) Represent $[g]$ by $x \mapsto S \cdot g(x)$ (so $X-G_3$ goes to S) and represent $[z]$ by the map which agrees with $\arg(v)$ on G_1, with $\arg(w)$ on G_2, with both on $X-(G_1 \cup G_2)$. Then $[zg]$ gets represented by a map which sends $X-(G_1 \cup G_2 \cup G_3)$ to S, agrees with $x \mapsto \arg(v)(x) \cdot g(x)$ on G_1 with $x \mapsto \arg(w)(x) \cdot g(x)$ on G_2 and with $x \mapsto S \cdot g(x)$ on G_3. Similar descriptions hold for $[vg]$, $[wg]$, $[g]$ and the result follows easily.

Exercise. Show that v, w, g may be chosen so that $v=(v_1, v_2, \dots, v_n)$, $w=(-1-v_1, v_2, \dots, v_n)$, g has first row $(p_1, \dots, p_n)$ with $(v_2, \dots, v_n, p_2, \dots, p_n)$ unimodular. (Argue as in 4.27.) Next choose an ε-neighborhood V_ε of the north pole so that $\arg(v)^{-1}(V_\varepsilon)$, $\arg(w)^{-1}(V_\varepsilon)$, $\{x \in X\colon S \cdot g(x) \in V_\varepsilon\}$ are disjoint. Finally show that v, w, g may be chosen as claimed above (compare 7.7).

7.9. Main example. We take $X=\mathrm{SO}_2 \times S^r$ with $r \geq 3$, $n=r+1$. Observe that $d=r+1=n$ is just one higher than in [4]. Also observe that $\mathrm{sdim}(C(X))=d \leq 2n-4$, so that our theory applies. We now recall a result of G.W. Whitehead [21]. Embed SO_2 in the upper left-hand corner of SO_n, so that $\mathrm{SO}_2 \to \mathrm{SO}_n$ represents the non-trivial element of $\pi_1(\mathrm{SO}_n)$ (here we view SO_2 as a circle). Map X to S^r by the action of SO_2 on S^r, which is after all just a map $\mathrm{SO}_2 \times S^r \to S^r$. That gives an element $[v]$ of $[X, S^r] \simeq \mathrm{Um}_n(C(X))/E_n(C(X))$. Suspending it yields $S[v] \in [SX, S^{r+1}]$. Now Whitehead shows that one may realize SX by collapsing in the sphere

$$S^{r+2}=\{(x_1, \dots, x_{r+3})\colon \textstyle\sum x_i^2=1\}$$

both subspheres $\{(x_1, x_2, 0, \dots, 0)\colon x_1^2+x_2^2=1\}$ and $\{(0, 0, x_3, \dots, x_{r+3})\colon \sum x_i^2=1\}$ (this is not his language). This collapsing map $S^{r+2} \to SX$ induces a map

$$[SX, S^{r+1}] \to [S^{r+2}, S^{r+1}]$$

and Whitehead proves that the composite map

$$[\mathrm{SO}_2, \mathrm{SO}_n] \to [X, S^r] \to [SX, S^{r+1}] \to [S^{r+2}, S^{r+1}]$$

is a bijection, where the first map sends the non-trivial element to v and the trivial element to the class $[w]$ of the projection of $X=\mathrm{SO}_2 \times S^r$ onto its second factor. (All objects in the sequence are abelian groups, as the first object equals $\pi_1(\mathrm{SO}_n)$ and the other ones are stable. But the first map is not a homomorphism.) Thus $[v] \neq [w]$. Projecting X onto its first factor SO_2 yields $g \in \mathrm{SL}_2(C(X))$ and one has $[w]g=[v]$. We thus have found $g \in \mathrm{SL}_2(C(X))$, $w \in \mathrm{Um}_n(C(X))$, with $[w]g \neq [v]$. This shows that the module structure of Theorem 5.3 is not always trivial. (Recall it would be trivial for $d=n-1$, at least when restricted to $\mathrm{SL}_n(C(X))$, by 5.3(ii).) Now let r be odd, $r>4$. We claim that even $\mathrm{ms}(v)$ is not equal to $\mathrm{ms}(w)$ (notations of 6.1). Suppose they were equal. By Theorem 6.1 we would have $k \in \mathrm{SL}_n(C(X))$ such that $[vk]=[w]$ and such that k describes an element of $\ker([X,\mathrm{SO}_n] \to [X,\mathrm{SO}_{n+1}])$. Now the Puppe sequence $[SX,S^n] \to [X,\mathrm{SO}_n] \to [X,\mathrm{SO}_{n+1}]$ tells that k must be in the image of $[SX,S^n] \to [X,\mathrm{SO}_n]$. As $n=r+1$ we may rewrite $[SX,S^n]$ as $[X,S^r]$ and the map to $[X,\mathrm{SO}_n]$ then comes from the familiar map $S^r \to \mathrm{SO}_n$ whose composite with the first row map $\mathrm{SO}_n \to S^r$ has degree 2. The class $[k]$ of the first row of k is thus a multiple of 2 (recall that $[X,S^r]$ is stable). As $\pi_{r+2}(S^{r+1})$ has two elements and $[vk]=[v]+[k]$ by 5.3(iv), the image of $[vk]$ in $\pi_{r+2}(S^{r+1})$ under Whitehead's homomorphism $[X,S^r] \to [SX,S^{r+1}] \to \pi_{r+2}(S^{r+1})$ is equal to the image of $[v]$, not the image of $[w]$. This contradicts the bijectivity of $[\mathrm{SO}_2,\mathrm{SO}_n] \to \pi_{r+2}(S^{r+1})$ cited above. Now take $r=7$. As $\mathrm{SO}_8 \to S^7$ splits, there is $h \in \mathrm{SL}_8(C(X))$ whose first row yields $[w]$. Suppose $\mathrm{ms}[ghg^{-1}h^{-1}]$ vanishes. Then

$$0=(\mathrm{ms}[ghg^{-1}h^{-1}])hg=\mathrm{ms}[gh]-\mathrm{ms}[hg].$$

But $\mathrm{ms}[gh]$ is just $\mathrm{ms}[h]$ because $g \in \left(\begin{smallmatrix}1&0\\0&g\end{smallmatrix}\right)E_3(C(X))$. We would thus find $\mathrm{ms}[w]=\mathrm{ms}[gh]=\mathrm{ms}[hg]=\mathrm{ms}[v]$, in contradiction with the above. Similarly, if $r=3$ we find $h \in \mathrm{SL}_4(C(S))$ with $[ghg^{-1}h^{-1}] \neq 0$ in $\mathrm{WMS}_n(C(X))$. Let us record this as

7.10. Proposition. (i) *There is a 4-dimensional ring R and there are $g \in \mathrm{SL}_2(R)$, $h \in \mathrm{SL}_4(R)$ with $ghg^{-1}h^{-1} \notin \mathrm{GL}_3(R)E_4(R)$.*

(ii) *There is an 8-dimensional ring R and there are $g \in \mathrm{SL}_2(R)$, $h \in \mathrm{SL}_8(R)$ such that $ghg^{-1}h^{-1} \notin \mathrm{GL}_7(R)(\mathrm{GL}_8(R) \cap E_9(R))$.*

Proof. Take

$$R = \mathbb{R}[x_1,x_2,y_1,\ldots,y_{r+1}]/(x_1^2+x_2^2=1=y_1^2+\cdots+y_{r+1}^2)$$

with $r=3,7$ respectively, and note that our examples in 7.9 may be realized over this subring of $C(X)=C(\mathrm{SO}_2 \times S^r)$ (cf. [7, p.31]). Now use 4.1, 6.1.

Remark. Recall that in (ii) we have $ghg^{-1}h^{-1} \in E_{10}(R)$ by stability for K_1.

7.11. Example. Let X be as in 7.6, $n \geq 3$, n odd, $d \leq 2n-4$. We claim that the image under the first row map $[X,\mathrm{SO}_n] \to [X,S^{n-1}]$ is annihilated by 2. (We do not claim this map is a homomorphism, as we know better by 7.9.) To prove the claim, let $g \in \mathrm{SL}_n(C(X))$ with first row $(v_1,\ldots,v_n)$. Recall that $[-v_1,v_2,\ldots,v_n]=-[v_1,v_2,\ldots,v_n]$. (see proof of 7.7 or derive this from 3.5 using the fact that $\sum v_i^2$ is a square of a unit). Now apply 5.5.

Exercise. Give an ordinary proof, i.e. one that does not use 7.7.

7.12. Example. Let X be as in 7.6 again, $n \geq 3$, $d \leq 2n-4$. Assume also that X is a co-H-space (e.g. a sphere), so that the $[X,\mathrm{SO}_r]$ are abelian. Then the image of $[X,\mathrm{SO}_n]$ in $([X,\mathrm{SO}_{n+1}]/\mathrm{im}[X,\mathrm{SO}_{n-1}])$ is annihilated by 2. Again one could easily prove this with standard methods, but we argue as follows. We have an exact sequence $[X, \mathrm{SO}_{n-1}] \to [X, \mathrm{SO}_n] \to [X, S^{n-1}]$ and if n is odd, the result follows from 7.11. Let n be even and let $g \in \mathrm{SL}_n(C(X))$ represent an element of $[X, \mathrm{SO}_n]$. Say g has first row $(v_1, \ldots, v_n)$. Let $t = \mathrm{diag}(-1, 1, \ldots, 1)$. Then tgt and g have the same image in $[X, \mathrm{SO}_{n+1}]$, but by the computation in 7.11 we have $[tgt] = -[g]$ in $[X, S^{n-1}]$.

7.13. Example. Consider the Hopf map $S^3 \to S^2$. It is represented by the first row of an element of $\mathrm{SL}_3(C(S^3))$. Now look at the natural bijection $\mathrm{Um}_3(C(S^3))/E_3(C(S^3)) \to \pi_3(S^2)$. Suppose this bijection were a weak Mennicke symbol. We could then argue as in 7.11 and find that the class of the Hopf map is annihilated by 2 in $\pi_3(S^2)$. As this class has actually infinite order, it follows that in Theorem 4.1 it would not suffice to have $\mathrm{sdim}(R) \leq 2n-3$ when $n=3$.

7.14. Main question. How does $\mathrm{WMS}_n(R)$ relate with

$$(\mathrm{GL}_n(R)E(R))/(\mathrm{GL}_{n-1}(R)E(R))? \quad \text{(cf. [20])}.$$

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