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The Schur Multipliers of $SL(3, \mathbb{Z})$ and $SL(4, \mathbb{Z})$

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Introduction

In his book Introduction to Algebraic K-Theory Milnor proves the following two facts

(i) for $n \geqq 3$ the group $K_2(n, \mathbb{Z})$ has order 2 (Theorem 10.1),

(ii) for $n \geqq 5$ the Schur multiplier of $SL(n, \mathbb{Z})$ equals $K_2(n, \mathbb{Z})$ (Theorem 5.10, Remark).

This suggests the following question: What are the Schur multipliers of $SL(3, \mathbb{Z})$ and $SL(4, \mathbb{Z})$? In both cases we will see that the Schur multiplier is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Notations

If G is a perfect group, then we will denote its universal central extension by $\tilde{G} \to G$. The kernel of $\tilde{G} \to G$ is the Schur multiplier $M(G)$ of G. Let $n > 2$. In $\widetilde{SL}(n, \mathbb{Z})$ we define elements x_{ij} by the requirements:

(i) The image of x_{ij} under $\widetilde{SL}(n, \mathbb{Z}) \to SL(n, \mathbb{Z})$ is the elementary matrix whose ij-th entry is 1.

(ii) If $i \neq j$ and k is minimal among the numbers $1, 2, \ldots, n$, distinct from i, j, then $(x_{ik}, x_{kj}) = x_{ij}$.

(The symbol (,) is used for commutators. The reader is assumed to be familiar with "Steinberg's trick", i.e. the use of the fact that (a, b) only depends on the classes mod $\ker(\tilde{G} \to G)$ to which x and y belong.)

We put $w_{ij} = x_{ij} x_{ji}^{-1} x_{ij}$, as usual. Let $ST(n, \mathbb{Z})$ be defined as in [2]. For $n \geqq 3$ there is a natural homomorphism $\widetilde{SL}(n, \mathbb{Z}) \to ST(n, \mathbb{Z})$.

Theorem. (i) *$M(SL(3, \mathbb{Z}))$ is generated by two elements of order $2: w_{12}^4$ and $w_{12} w_{21}$.*

They are distinct.

(ii) *$M(SL(4, \mathbb{Z}))$ is generated by two elements of order $2: w_{12}^4$ and (x_{12}, x_{34}).*

They are distinct.

Remark. $w_{12} w_{21} = 1$ in $M(SL(4, \mathbb{Z}))$.

Corollary. $M(SL(3, \mathbb{Z})) \simeq M(SL(4, \mathbb{Z})) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

Proof of the Theorem. (a) Let $n = 3$ or 4. First we show that $M(SL(n, \mathbb{Z}))$ has order bigger than 2. Consider the homomorphism $\alpha: ST(n, \mathbb{Z}) \to ST(n, \mathbb{F}_2)$.

It induces a homomorphism $\tilde{\alpha}: \widetilde{ST}(n, \mathbb{Z}) \to \widetilde{ST}(n, \mathbb{F}_2)$. (See [3], § 7.) Suppose $M(ST(n, \mathbb{Z})) = 0$. Then the Steinberg relations hold in $\widetilde{ST}(n, \mathbb{Z})$, and $\tilde{\alpha}(x_{ij}^2) = \tilde{\alpha}((x_{ik}^2, x_{kj})) = (\tilde{\alpha}(x_{ik}^2), *) = 1$, because $\tilde{\alpha}(x_{ik}^2)$ is central. (The numbers i, j, k or i, j, k, l will always be meant to be distinct.) Therefore the relations for $ST(n, \mathbb{F}_2)$ hold in the image of $\tilde{\alpha}$, and $\widetilde{ST}(n, \mathbb{F}_2) \simeq ST(n, \mathbb{F}_2)$. But it is known that $M(ST(n, \mathbb{F}_2)) \neq 0$. (See [2], p. 48.) This yields a contradiction. We therefore know that $M(ST(n, \mathbb{Z})) \neq 0$. Then $|M(SL(n, \mathbb{Z}))| = |M(ST(n, \mathbb{Z}))| \cdot |K_2(n, \mathbb{Z})| > |K_2(n, \mathbb{Z})| = 2$.

(b) We now start computing in $\widetilde{SL}(n, \mathbb{Z})$. Put $A = (x_{12}, x_{13})$, $B = (x_{21}, x_{31})$. Then A is central, so $A = {}^{w}A$ for all $w \in \widetilde{SL}(n, \mathbb{Z})$. Taking a product of elements of type w_{ij} for w, one sees that $(x_{ij}, x_{ik}) = A$ for all i, j, k (dinstinct, by convention). So $A^2 = (x_{12}, x_{13})(x_{13}, x_{12}) = 1$. One proves by double induction (in both directions, starting from 1) that $(x_{ij}^m, x_{ik}^n) = A^{mn}$. Similarly, $B^2 = 1$ and $(x_{ji}^m, x_{ki}^n) = B^{mn}$ for all $m, n \in \mathbb{Z}$. One has $(x_{ik}^m, x_{kj}^n) = (x_{ik}^{m-1}, (x_{ik}, x_{kj}^n))(x_{ik}, x_{kj}^n)(x_{ik}^{m-1}, x_{kj}^n) = A^{mn-n}(x_{ik}, x_{kj}^n)(x_{ik}^{m-1}, x_{kj}^n)$ and similarly: $(x_{ik}, x_{kj}^n) = (x_{ik}, x_{kj}^{n-1})(x_{ik}, x_{kj}) B^{n-1}$. From this it is easy to see (by double induction again) that $(x_{ik}^m, x_{kj}^n) = (x_{ik}, x_{kj})^{mn} A^{m(m-1)n/2} \cdot B^{n(n-1)m/2}$ for $m, n \in \mathbb{Z}$. Applying this, we see ${}^{w_{12}}x_{12} = {}^{w_{12}}(x_{13}, x_{32}) = (x_{23}^{-1}, x_{31}) = A(x_{23}, x_{31})^{-1} = Ax_{21}^{-1}$ and similarly ${}^{w_{12}}x_{21} = x_{12}^{-1}B$. So $w_{12} = {}^{w_{12}}w_{12} = Bx_{21}^{-1}x_{12}x_{21}^{-1} = Bw_{21}^{-1}$, or $B = w_{12}w_{21}$. But also ${}^{w_{12}}w_{21} = Ax_{12}^{-1}x_{21}x_{12}^{-1} = Aw_{12}^{-1}$, or $A = w_{12}w_{21}$. So $A = B$.

(c) If we divide $\widetilde{SL}(3, \mathbb{Z})$ by the group generated by B, then it follows from (b) that the quotient satisfies the defining relations for $ST(3, \mathbb{Z})$. So B generates $M(ST(3, \mathbb{Z}))$ and $M(SL(3, \mathbb{Z}))$ is generated by B, w_{12}^4. [Recall that w_{12}^4 generates $K_2(3, \mathbb{Z})$.] As w_{12}^4 is central, it is easy to see that $w_{12}^8 = w_{12}^4 w_{21}^4 = B^4 = 1$. This finishes the proof of Part (i).

(d) Next let $n = 4$. Put $C = (x_{12}, x_{34})$. Then $A = (x_{12}, (x_{13}, x_{34})) = (Ax_{13}, Cx_{34})(x_{13}, x_{34})^{-1} = 1$. So $A = B = 1$. Furthermore $C = {}^{w_{31}w_{24}}C = (x_{34}, x_{12})$, so $C^2 = 1$. One shows as above that $(x_{ij}^m, x_{kl}^n) = C^{mn}$ for $m, n \in \mathbb{Z}$. If $k < l$ then $1 = {}^{w_{lk}}(x_{ij}^{-1}(x_{ik}, x_{kj})) = Cx_{ij}^{-1}(x_{il}, x_{lj})$, so $(x_{il}, x_{lj}) = Cx_{ij}$. The proof is finished as in (c).

Corollary. *Let $n = 3$ or 4, $m \in \mathbb{Z}$.*
Then $M(SL(n, \mathbb{Z}/m)) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ *if* $m \in 4\mathbb{Z}$,
$\simeq 1$ *if m is odd,*
$\simeq \mathbb{Z}/2$ *in other cases.*

Proof.

(e) If m is odd then $B = B^{m-1}B = (x_{21}^m, x_{31})$ is mapped to $1 \in M(ST(3, \mathbb{Z}/m))$. If m is even however, the image of B is non-trivial because $M(ST(3, \mathbb{Z}/2)) \neq 1$. [Compare with (a).] Similarly C is killed in $M(ST(4, \mathbb{Z}/m))$ if and only if m is odd.

(f) It follows from the work of Dennis and Stein that $K_2(n, \mathbb{Z}/m) \simeq K_2(\mathbb{Z}/m)$ is 2-cyclic if $m \in 4\mathbb{Z}$ and trivial otherwise. (See [4], Section 12, cf. [2], 10.8.) The proof is completed as in Parts (c) and (d) above.

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