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The Schur Multipliers of $SL(3, \mathbb{Z})$ and $SL(4, \mathbb{Z})$

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Introduction

In his book Introduction to Algebraic K -Theory Milnor proves the following two facts

- (i) for $n \geq 3$ the group $K_2(n, \mathbb{Z})$ has order 2 (Theorem 10.1),
- (ii) for $n \geq 5$ the Schur multiplier of $SL(n, \mathbb{Z})$ equals $K_2(n, \mathbb{Z})$ (Theorem 5.10, Remark).

This suggests the following question: What are the Schur multipliers of $SL(3, \mathbb{Z})$ and $SL(4, \mathbb{Z})$? In both cases we will see that the Schur multiplier is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Notations

If G is a perfect group, then we will denote its universal central extension by $\tilde{G} \rightarrow G$. The kernel of $\tilde{G} \rightarrow G$ is the Schur multiplier $M(G)$ of G . Let $n > 2$. In $\widetilde{SL}(n, \mathbb{Z})$ we define elements x_{ij} by the requirements:

- (i) The image of x_{ij} under $\widetilde{SL}(n, \mathbb{Z}) \rightarrow SL(n, \mathbb{Z})$ is the elementary matrix whose ij -th entry is 1.
- (ii) If $i \neq j$ and k is minimal among the numbers $1, 2, \dots, n$, distinct from i, j , then $(x_{ik}, x_{kj}) = x_{ij}$.

(The symbol $(,)$ is used for commutators. The reader is assumed to be familiar with "Steinberg's trick", i.e. the use of the fact that (a, b) only depends on the classes mod $\ker(\tilde{G} \rightarrow G)$ to which x and y belong.)

We put $w_{ij} = x_{ij} x_{ji}^{-1} x_{ij}$, as usual. Let $ST(n, \mathbb{Z})$ be defined as in [2]. For $n \geq 3$ there is a natural homomorphism $\widetilde{SL}(n, \mathbb{Z}) \rightarrow ST(n, \mathbb{Z})$.

Theorem. (i) $M(SL(3, \mathbb{Z}))$ is generated by two elements of order 2: w_{12}^4 and $w_{12} w_{21}$.

They are distinct.

(ii) $M(SL(4, \mathbb{Z}))$ is generated by two elements of order 2: w_{12}^4 and (x_{12}, x_{34}) .

They are distinct.

Remark. $w_{12} w_{21} = 1$ in $M(SL(4, \mathbb{Z}))$.

Corollary. $M(SL(3, \mathbb{Z})) \simeq M(SL(4, \mathbb{Z})) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

Proof of the Theorem. (a) Let $n = 3$ or 4. First we show that $M(SL(n, \mathbb{Z}))$ has order bigger than 2. Consider the homomorphism $\alpha: ST(n, \mathbb{Z}) \rightarrow ST(n, \mathbb{F}_2)$.

It induces a homomorphism $\tilde{\alpha} : \widetilde{ST}(n, \mathbf{Z}) \rightarrow \widetilde{ST}(n, \mathbf{F}_2)$. (See [3], § 7.) Suppose $M(ST(n, \mathbf{Z})) = 0$. Then the Steinberg relations hold in $\widetilde{ST}(n, \mathbf{Z})$, and $\tilde{\alpha}(x_{ij}^2) = \tilde{\alpha}((x_{ik}^2, x_{kj})) = (\tilde{\alpha}(x_{ik}^2), *) = 1$, because $\tilde{\alpha}(x_{ik}^2)$ is central. (The numbers i, j, k or i, j, k, l will always be meant to be distinct.) Therefore the relations for $ST(n, \mathbf{F}_2)$ hold in the image of $\tilde{\alpha}$, and $\widetilde{ST}(n, \mathbf{F}_2) \simeq ST(n, \mathbf{F}_2)$. But it is known that $M(ST(n, \mathbf{F}_2)) \neq 0$. (See [2], p. 48.) This yields a contradiction. We therefore know that $M(ST(n, \mathbf{Z})) \neq 0$. Then $|M(SL(n, \mathbf{Z}))| = |M(ST(n, \mathbf{Z}))| \cdot |K_2(n, \mathbf{Z})| > |K_2(n, \mathbf{Z})| = 2$.

(b) We now start computing in $\widetilde{SL}(n, \mathbf{Z})$. Put $A = (x_{12}, x_{13})$, $B = (x_{21}, x_{31})$. Then A is central, so $A = {}^w A$ for all $w \in \widetilde{SL}(n, \mathbf{Z})$. Taking a product of elements of type w_{ij} for w , one sees that $(x_{ij}, x_{ik}) = A$ for all i, j, k (distinct, by convention). So $A^2 = (x_{12}, x_{13})(x_{13}, x_{12}) = 1$. One proves by double induction (in both directions, starting from 1) that $(x_{ij}^m, x_{ik}^n) = A^{mn}$. Similarly, $B^2 = 1$ and $(x_{ji}^m, x_{ki}^n) = B^{mn}$ for all $m, n \in \mathbf{Z}$. One has $(x_{ik}^m, x_{kj}^n) = (x_{ik}^{m-1}, (x_{ik}, x_{kj}^n)) (x_{ik}, x_{kj}^n) (x_{ik}^{m-1}, x_{kj}^n) = A^{m(n-1)} (x_{ik}, x_{kj}^n) (x_{ik}^{m-1}, x_{kj}^n)$ and similarly: $(x_{ik}, x_{kj}^n) = (x_{ik}, x_{kj}^{n-1}) (x_{ik}, x_{kj}) B^{n-1}$. From this it is easy to see (by double induction again) that $(x_{ik}^m, x_{kj}^n) = (x_{ik}, x_{kj})^{mn} A^{m(m-1)n/2} \cdot B^{n(n-1)m/2}$ for $m, n \in \mathbf{Z}$. Applying this, we see ${}^{w_{12}} x_{12} = {}^{w_{12}}(x_{13}, x_{32}) = (x_{23}^{-1}, x_{31}) = A(x_{23}, x_{31})^{-1} = Ax_{21}^{-1}$ and similarly ${}^{w_{12}} x_{21} = x_{12}^{-1} B$. So $w_{12} = {}^{w_{12}} w_{12} = Bx_{21}^{-1} x_{12} x_{21}^{-1} = Bw_{21}^{-1}$, or $B = w_{12} w_{21}$. But also ${}^{w_{12}} w_{21} = Ax_{12}^{-1} x_{21} x_{12}^{-1} = Aw_{12}^{-1}$, or $A = w_{12} w_{21}$. So $A = B$.

(c) If we divide $\widetilde{SL}(3, \mathbf{Z})$ by the group generated by B , then it follows from (b) that the quotient satisfies the defining relations for $ST(3, \mathbf{Z})$. So B generates $M(ST(3, \mathbf{Z}))$ and $M(SL(3, \mathbf{Z}))$ is generated by B, w_{12}^4 . [Recall that w_{12}^4 generates $K_2(3, \mathbf{Z})$.] As w_{12}^4 is central, it is easy to see that $w_{12}^8 = w_{12}^4 w_{21}^4 = B^4 = 1$. This finishes the proof of Part (i).

(d) Next let $n = 4$. Put $C = (x_{12}, x_{34})$. Then $A = (x_{12}, (x_{13}, x_{34})) = (Ax_{13}, Cx_{34})(x_{13}, x_{34})^{-1} = 1$. So $A = B = 1$. Furthermore $C = {}^{w_{31} w_{24}} C = (x_{34}, x_{12})$, so $C^2 = 1$. One shows as above that $(x_{ij}^m, x_{kl}^n) = C^{mn}$ for $m, n \in \mathbf{Z}$. If $k < l$ then $1 = {}^{w_{ik}}(x_{ij}^{-1}(x_{ik}, x_{kl})) = Cx_{ij}^{-1}(x_{il}, x_{lj})$, so $(x_{il}, x_{lj}) = Cx_{ij}$. The proof is finished as in (c).

Corollary. Let $n = 3$ or 4 , $m \in \mathbf{Z}$.

Then $M(SL(n, \mathbf{Z}/m)) \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ if $m \in 4\mathbf{Z}$,
 $\simeq 1$ if m is odd,
 $\simeq \mathbf{Z}/2$ in other cases.

Proof.

(e) If m is odd then $B = B^{m-1} B = (x_{21}^m, x_{31})$ is mapped to $1 \in M(ST(3, \mathbf{Z}/m))$. If m is even however, the image of B is non-trivial because $M(ST(3, \mathbf{Z}/2)) \neq 1$. [Compare with (a).] Similarly C is killed in $M(ST(4, \mathbf{Z}/m))$ if and only if m is odd.

(f) It follows from the work of Dennis and Stein that $K_2(n, \mathbf{Z}/m) \simeq K_2(\mathbf{Z}/m)$ is 2-cyclic if $m \in 4\mathbf{Z}$ and trivial otherwise. (See [4], Section 12, cf. [2], 10.8.) The proof is completed as in Parts (c) and (d) above.

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