A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices.

V. B. Mehta   Wilberd van der Kallen

1 Introduction

We exhibit a nice Frobenius splitting σ on $G \times^B \mathfrak{b}$ where $\mathfrak{b}$ is the Lie algebra of the Borel group B of upper triangular matrices in the general linear group $G = Gl_n$. What is nice about it, is that it descends along familiar maps and specializes to familiar subvarieties in a manner that is useful for the study of the singularities of closures of conjugacy classes of nilpotent n by n matrices. In particular, we show that these closures are simultaneously Frobenius split, are normal, and have rational singularities. The result on rational singularities is derived from a general vanishing theorem that will be proved in our paper [15]. Note that normality has already been proved by Donkin in [3]. His method uses a lot of representation theory and employs resolutions of the closures of conjugacy classes invented by Kraft and Procesi.

An alternative approach to these singularities has been given by G. Lusztig. In [11] he showed that the same singularities occur in Schubert varieties for Kac-Moody groups of affine Weyl groups. Now Schubert varieties for such infinite dimensional groups are mastered in Mathieu's book [12], where Mathieu shows they are normal and have rational singularities.

In contrast with this, our work remains in finite dimensions. It relies on explicit formulas. Indeed the formula for our splitting σ is given by a product of principal minors and the specialization of the splitting to subvarieties is based on an inspection of what happens to the determinants. To descend σ to the Lie algebra $\mathfrak{g}$ of G, (along the natural map $G \times^B \mathfrak{b} \to \mathfrak{g}$, cf. Grothendieck's "simultaneous resolution" [2]), we use a Galois theoretic argument. We find that above the generic point of $\mathfrak{g}$ the action of the Weyl group on σ is trivial. As preparation for that computation we first spell out trivializations of the canonical bundles of $G \times^B \mathfrak{b}$ and $G \times^T \mathfrak{t}$.

The geometry of conjugacy classes is of course simplest for the general linear groups. It may be of interest to try and extend our method to other semisimple groups, but there are some obstructions to this. For instance, for the symplectic group of rank 2 in characteristic 2 there is no function analogous to the product of principal minors. (No function that yields a splitting.)

The first author wishes to thank Shrawan Kumar for useful and stimulating discussions. The second author wishes to thank Steve Donkin for pointing out Lusztig's paper [11] and T. A. Springer for showing the importance of the Weyl group action.

2 Orientations.

We will need the relation between trivializations of the canonical bundle ω on three different spaces.

2.1 Notations. We work over an algebraically closed field k of characteristic p, $p > 0$. Let G be the group of n by n invertible matrices, B its subgroup of upper triangular matrices, $B = TU$ the usual decomposition. The unipotent radical of the Borel group B^- opposite to B we call U^-. Thus U^- consists of unipotent lower triangular matrices. The Lie algebra of G is called $\mathfrak{g}$ and is identified with the vector space—viewed as a variety—of n by n matrices. Similarly $\mathfrak{b}$ is the Lie algebra of B, $\mathfrak{u}^-$ is the Lie algebra of U^-, $\mathfrak{t}$ is that of T. The Weyl group is W. All this is viewed as being defined over the prime field in the usual way.

2.2 Volume forms. A nowhere vanishing global section of the canonical bundle on a variety is called a volume form. They exist only if the canonical bundle is trivial and then they are unique up to global units. In our examples the only global units are constants. We wish to choose volume forms on the three varieties $G \times^B \mathfrak{b}$, $G \times^T \mathfrak{t}$ and $\mathfrak{g}$. On $\mathfrak{g}$ this is very easy; one just chooses an ordered basis of $\mathfrak{g}$ and gets a generator of the top exterior power $\bigwedge^{top} \mathfrak{g}$ and thus a global generator $v[\mathfrak{g}]$ of the canonical bundle $\omega[\mathfrak{g}]$. On $G \times^B \mathfrak{b}$ we proceed as follows. A point p on this variety may be represented by a pair (g, X) with $g \in G$ and $X \in \mathfrak{b}$. Given such a pair we map the variety $U^- \times \mathfrak{b}$ into $G \times^B \mathfrak{b}$ by the local isomorphism $\tau_{g,X} : (x, Y) \mapsto (gx, X + Y)$. We fix a volume form $v[U^- \times \mathfrak{b}]$ of $U^- \times \mathfrak{b}$ and take its image in the canonical bundle of $G \times^B \mathfrak{b}$ to get a local section around p of the canonical bundle $\omega[G \times^B \mathfrak{b}]$.

Lemma 2.3 *This procedure defines a volume form on $G \times^B \mathfrak{b}$.*

Proof. Think of the canonical bundle as a variety and think of the desired volume form as a morphism of varieties. We need to check that the results patch when one varies (g, X). For this the key is to show that when (g_1, X_1) and (g_2, X_2) represent the same point p of $G \times^B \mathfrak{b}$, the resulting local sections v_1 and v_2 respectively of $\omega[G \times^B \mathfrak{b}]$ agree at p. Put $b = {g_1}^{-1} g_2$. Thus X_1 equals $\mathrm{Ad}(b)X_2$. Identify U^- in the obvious way with a neighborhood of the 'origin' B of G/B. Thus we get an identification of the tangent space of $U^- \times \mathfrak{b}$ at its origin $(1, 0)$ with $\mathfrak{g}/\mathfrak{b} \oplus \mathfrak{b}$. Consider the automorphism of $G/B \times \mathfrak{b}$ given by $(x, Y) \mapsto (bxb^{-1}, \mathrm{Ad}(b)Y)$. View it as a birational automorphism ψ of $U^- \times \mathfrak{b}$. We have $\tau_{g_1,X_1}\psi = \tau_{g_2,X_2}$, so we only need to check that the tangent map to ψ at the origin $(B, 0)$ has determinant 1. This determinant is the same as the determinant of $\mathrm{Ad}(b)$ restricted to $\mathfrak{b}$ times the determinant of the map induced by $\mathrm{Ad}(b)$ on $\mathfrak{g}/\mathfrak{b}$, so it is simply the determinant of $\mathrm{Ad}(b)$. Now the adjoint action of G is by linear transformations of determinant 1. (Recall that G is generated by its center together with its commutator subgroup.)

2.4 Orientation on $G \times^T \mathfrak{t}$. The reasoning is similar to the case of $G \times^B \mathfrak{b}$. Given (g, X) with $g \in G$ and $X \in \mathfrak{t}$ we map the variety $U^- \times U \times \mathfrak{t}$ into $G \times^T \mathfrak{t}$ by the local isomorphism $(x, y, Y) \mapsto (gxy, X + Y)$. We fix a volume form $v[U^- \times U \times \mathfrak{t}]$ of $U^- \times U \times \mathfrak{t}$ and take its image in the canonical bundle of $G \times^T \mathfrak{t}$ to get a local section of $\omega[G \times^B \mathfrak{b}]$. Identify the tangent space at $(1, 1)$ of $U^- \times U$ in the obvious way with the tangent space $\mathfrak{g}/\mathfrak{t}$ of G/T at its 'origin' T. This gives an identification of the tangent space of $U^- \times U \times \mathfrak{t}$ at its origin $(1, 1, 0)$ with $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$. The analogue of lemma 2.3 holds with a similar proof and we get a volume form $v[G \times^T \mathfrak{t}]$ on $G \times^T \mathfrak{t}$.

2.5 Comparison of volume forms. There is a natural map from $G \times^T \mathfrak{t}$ to $G \times^B \mathfrak{b}$, sending the class $[(g, X)]$ of (g, X) to the class $[(g, X)]$ of (g, X). There is also a natural map from $G \times^T \mathfrak{t}$ to $\mathfrak{g}$ sending $[(g, X)]$ to $\mathrm{Ad}(g)X$. We wish to know what happens to the volume forms under these maps. More specifically, the pull back of a volume form is a function times the volume form on the source, and we care about the divisor of that function. This is an exercise in computing determinants. We study the map from $G \times^T \mathfrak{t}$ to $\mathfrak{g}$ at $[(g, X)]$ by composing with the map from $U^- \times U \times \mathfrak{t}$ to $G \times^T \mathfrak{t}$, which was used for constructing the volume form $v[G \times^T \mathfrak{t}]$. Recall we identify the tangent space of $U^- \times U \times \mathfrak{t}$ at its origin $(1, 1, 0)$ with $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$ which in its turn may be identified with $\mathfrak{g}$ itself. The upshot is that we have to compute the determinant of the map from $\mathfrak{g}$ to $\mathfrak{g}$ which sends $Y + Z$ to $\mathrm{Ad}(g)([Y, X] + Z)$ for $Y \in \mathfrak{u}^- \oplus \mathfrak{u}$ and $Z \in \mathfrak{t}$. That determinant does not depend on g but only on X. One may view it as the product of the roots applied to X. The divisor of the function on $G \times^T \mathfrak{t}$ which arises as the coefficient of the volume form is thus twice the divisor of the reduced subvariety $G \times^T \mathfrak{t}_{\mathrm{irr}}$, where $\mathfrak{t}_{\mathrm{irr}}$ is the subvariety consisting of the elements having fewer than n distinct eigenvalues. So this is the answer when we pull back from $\mathfrak{g}$. Next let us pull back from $G \times^B \mathfrak{b}$. The determinant to consider is now the determinant of the endomorphism of $\mathfrak{u}^- \oplus \mathfrak{u} \oplus \mathfrak{t} = \mathfrak{u}^- \oplus \mathfrak{b}$ sending (Y^-, Y^+, Z) to $(Y^-, [Y^+, X] + Z)$. Therefore now the divisor is just once $G \times^T \mathfrak{t}_{\mathrm{irr}}$.

2.6 The action of W. An element w of the Weyl group acts on $G \times^T \mathfrak{t}$ through $[(g, X)] \mapsto [(gw^{-1}, \mathrm{Ad}(w)X)]$, with a slight abuse of notation.

Lemma 2.7 *The Weyl group acts through the sign representation on the linear span of the pull back to $G \times^T \mathfrak{t}$ of the volume form of $G \times^B \mathfrak{b}$.*

Proof. The map from $G \times^T \mathfrak{t}$ to $\mathfrak{g}$ is equivariant for W, when W acts trivially on $\mathfrak{g}$, so the pull back of the volume form of $\mathfrak{g}$ is invariant. We have to divide this pull back by the function defining $G \times^T \mathfrak{t}_{\mathrm{irr}}$, on which W indeed acts through the sign representation.

3 Frobenius splittings.

3.1 A partial order. In order to describe our computations, we need to single out a particular class of B invariant ideals of $\mathfrak{b}$. To this end we put a partial order on the set $\mathcal{I} = [1, n] \times [1, n]$ which indexes the coordinates on $\mathfrak{g}$. We declare that

$$(i, j) \leq (r, s) \Longleftrightarrow ((i \geq r \text{ and } j \leq s) \text{ or } j \leq s - r)$$

If S is an ideal for this partial order, i.e. if $(i, j) \leq (r, s)$ and $(r, s) \in S$ imply $(i, j) \in S$, then we define $\mathfrak{b}[S]$ to be the subspace of $\mathfrak{b}$ consisting of the matrices X with $X_{ij} = 0$ for $(i, j) \in S$. One easily sees that such a subspace is a B invariant ideal. Let us agree to use the notation $\mathfrak{b}[S]$ only when S is an ideal for the partial order. We will find a Frobenius splitting for all $G \times^B \mathfrak{b}[S]$ simultaneously.

3.2 Subdeterminants. If $M \in \mathfrak{g}$ is a matrix, let us indicate by a notation like $M_{\leq r, \geq s}$ the submatrix consisting of the entries whose row number is at most r and whose column number is at least s.

Lemma 3.3 *Let g be a unipotent lower triangular matrix and let $M \in \mathfrak{g}$ be such that $M_{\leq r, \leq n-r} = 0$ for some integer r between 1 and n. Then*

$$\det((gMg^{-1})_{\leq r, \leq r}) = \det(M_{\leq r, > n-r}) \det((g^{-1})_{> n-r, \leq r})$$

Proof. Exercise.

3.4 The choice of σ. We choose a very particular section σ of the anti-canonical bundle of $G \times^B \mathfrak{b}$. Namely, if we multiply σ by our volume form $v[G \times^B \mathfrak{b}]$, which we take to be defined over the prime field, then we require the resulting function to be the product over all integers r between 1 and n of the pull back from $\mathfrak{g}$ of the subdeterminant function $\det(X_{\leq r, \leq r})$. The power σ^{p-1} of σ defines by [13] a twisted linear endomorphism ϕ_σ of the structure sheaf of $G \times^B \mathfrak{b}$. Here twisted linear means that it is a morphism of sheafs of abelian groups satisfying the rule

$$\phi_\sigma(f^p g) = f \phi_\sigma(g).$$

If this endomorphism preserves the constant function 1, then it is in fact a Frobenius splitting. (This is indeed what will happen.)

3.5 Specializing to a subspace. Let S be an ideal of the partially ordered set $\mathcal{I}$ of 3.1, and (s, t) a maximal element of S, so that $S' = S - \{(s, t)\}$ is also an ideal. Assume $s \leq t$ so that the corresponding coordinate does not vanish on $\mathfrak{b}$. If $\sigma[S']$ is a global section of the anti-canonical bundle of $G \times^B \mathfrak{b}[S']$, which vanishes on $G \times^B \mathfrak{b}[S]$, we wish to define a *residue* $\operatorname{res} \sigma[S']$ of $\sigma[S']$ such that the twisted linear endomorphism $\phi_{\sigma[S']}$ of the structure sheaf $\mathcal{O}_{G \times^B \mathfrak{b}[S']}$ of $G \times^B \mathfrak{b}[S']$, defined by the $p-1$-st power of $\sigma[S']$, induces on the structure sheaf of the codimension 1 subspace $G \times^B \mathfrak{b}[S]$ the twisted linear endomorphism $\phi_{\sigma[S]}$ defined by the $p-1$-st power of $\sigma[S] := \operatorname{res}\sigma[S']$. To this end we consider on G/B an open set V which is small enough to ensure that the line bundle $G \times^B (\mathfrak{b}[S']/\mathfrak{b}[S])$ trivializes over V. Say f is a nowhere vanishing section over V of the dual line bundle. We choose $\operatorname{res}\sigma[S]$ so that

$$\operatorname{res}\sigma[S](\eta) = \sigma(\eta df/f)$$

for a local section η of the canonical bundle of $G \times^B \mathfrak{b}[S]$, with hopefully self explanatory notation. One checks that this is independent of the precise choice of f, and therefore patches as we vary V. One also checks the desired correspondence with twisted endomorphisms, using the explicit local formulas of [13].

Remark 3.6 In 3.5 it is actually not essential that f is a section of a line bundle. The "residue" in 3.5 is dual to a Poincaré residue (cf. [6]) restricted to the subvariety, and thus exists in greater generality.

3.7 Formulas for the specializations. If S is as above, and r is an integer between 1 and n, let $\delta_r[S]$ denote the matrix with a one at each entry $(i, j) \in S$ with $i + n = j + r$ and zeroes elsewhere. The open subset $U^- \times \mathfrak{b}[S]$ of $G \times^B \mathfrak{b}[S]$ is isomorphic to a linear affine space, so it has—up to a constant multiple—a natural choice of volume form. (The full space $G \times^B \mathfrak{b}[S]$ usually has no volume form.) Multiplying $\sigma[S]$ by such a volume form we claim to get a function sending $(g, X) \in U^- \times \mathfrak{b}[S]$ to a constant times the product over all integers r between 1 and n of the subdeterminants $\det((g(X + \delta_r[S])g^{-1})_{\leq r, \leq r})$.

Theorem 3.8 *For each ideal S of $\mathcal{I}$ there is a sequence of specializations (cf. 3.5) starting with σ of 3.4 and ending with $\sigma[S]$ as in 3.7. The $p-1$-st power of $\sigma[S]$ defines a Frobenius splitting on $G \times^B \mathfrak{b}[S]$. This splitting is also induced by the splitting of ϕ_σ of $G \times^B \mathfrak{b}$.*

3.9 Start of proof. We argue by induction on the size of S to show that specialization leads to the formulas indicated in 3.7, but we will go in the other direction to prove that one has Frobenius splittings. The formula for $\sigma[S]$ is by definition correct when $\mathfrak{b}[S]$ equals $\mathfrak{b}$. (Note that in this case $i > j$

for $(i,j) \in S$ so that $\delta_r[S]$ vanishes for $r \leq n$.) Therefore let us now assume S contains a maximal element (s,t) with $s \leq t$. We assume the formulas true for $S' = S - \{(s,t)\}$. Put $r = s + n - t$. For $(g, X) \in U^- \times \mathfrak{b}[S']$ the hypotheses of Lemma 3.3 apply with $M = X + \delta_r[S']$. Moreover $M_{\leq r, >n-r}$ is a block matrix

$$\begin{pmatrix} \alpha & * & * \\ 0 & X_{st} & * \\ 0 & 0 & \beta \end{pmatrix}$$

with determinant $\det(\alpha) X_{st} \det(\beta)$. We may use X_{st} as the f of 3.5, at least over the open subset U^- of G/B. As $U^- \times \mathfrak{b}[S]$ is dense in $G \times^B \mathfrak{b}[S]$, the hypotheses for the residue construction are satisfied and we only need to check that it replaces the factor $\det((X + \delta_r[S'])_{\leq r, >n-r})$ in the product for $\sigma[S']$ by the factor $\det((X + \delta_r[S])_{\leq r, >n-r})$. Indeed one must put X_{st} equal to zero in the regular function $\det(M_{\leq r, >n-r})/X_{st} = \det(\alpha)\det(\beta)$. And this gives the same as putting X_{st} equal to zero in $\det((X + \delta_r[S])_{\leq r, >n-r})$.

3.10 The splitting of G/B. We take $S = \mathcal{I}$ and investigate $\sigma[S]$. It is a section of the anti-canonical bundle of $G \times^B 0 = G/B$. Its $p-1$-st power defines a twisted endomorphism $\phi_{\sigma[S]}$ of the structure sheaf, which is a Frobenius splitting if $\phi_{\sigma[S]}(1) = 1$. As G/B is complete, we may try the criterion in terms of local coordinates around a special point, given in Proposition 6 of [13]. As special point we take the origin B, and we restrict to its neighborhood $U^- = U^- B/B$. But for the coordinates of $g \in U^-$ we take the matrix coefficients Y_{ij} with $i > j$ of the *inverse* matrix. The product of $\sigma[S]$ with the volume form of U^- is given by the product of the subdeterminants $\det((g^{-1})_{>n-r, \leq r})$. For $r = 1$ this gives $Y_{n,1}$. Putting that coordinate equal to zero one gets $Y_{n-1,1}Y_{n,2}$ as subdeterminant for $r = 2$. Putting those two coordinates equal to zero too, one gets $Y_{n-2,1}Y_{n-1,2}Y_{n,3}$ as the subdeterminant for $r = 3$. Proceeding in this manner, one sees that the $p-1$-st power of the product of the subdeterminants has 1 as the coefficient of the monomial $\prod_{i>j} Y_{ij}^{p-1}$. This shows that we have a splitting. (As it happens, the normalizing constant is automatically correct because we started with something defined over the prime field and then took a $p-1$-st power. Otherwise we would have had to rescale ϕ_σ.)

3.11 End of proof of 3.8. We now know that ϕ_σ induces a splitting on $G \times^B 0$, so $\phi_\sigma(1)$ restricts to 1 on $G \times^B 0$. It remains to show that it is 1 on all of $G \times^B \mathfrak{b}$. Now if c is a nonzero constant, we get an automorphism h_c of $G \times^B \mathfrak{b}$ given by $[(g, X)] \mapsto [(g, cX)]$. Under this automorphism the section σ goes to a nonzero constant multiple of itself, because determinants are multilinear. This implies that the zero set of $\phi_\sigma(1)$ is invariant under h_c, for all nonzero c. As $\phi_\sigma(1)$ is 1 on the zero section of the vector bundle $G \times^B \mathfrak{b}$ over G/B, the result follows. Alternatively, one may show that σ extends to a complete variety, by embedding $\mathfrak{b}$ as an open subset of the projective space $P(\mathfrak{b} \oplus k)\ldots$ □

4 Conjugacy classes of nilpotent matrices.

4.1 Partitions. Given the conjugacy class $C(N)$ of a nilpotent element N of $\mathfrak{g}$, one may associate to it two partitions of n. The first, say $\pi[N] = (\pi[N]_1, \ldots, \pi[N]_r)$, consists of the sizes of the Jordan blocks, in descending order. The second is the dual partition $\pi'[N]$. It may also be read off the dimensions of the kernels of the powers of N, in an easy way. Let F be the partial flag in k^n whose i-th part is spanned by the first $\pi'[N]_1 + \cdots + \pi'[N]_i$ standard basis vectors, $(0 \le i \le \pi[N]_1)$, and let $P[N]$ be the stabilizer of the flag F. Then $P[N]$ is a parabolic subgroup and the sizes of the blocks in its Levi subgroup are exactly given by $\pi'[N]$. Let $\mathfrak{r}[N]$ be the Lie algebra of the unipotent radical $R(P[N])$ of $P[N]$. It has an open orbit, called $\mathfrak{r}[N]_{\mathrm{reg}}$, under the action of $P[N]$. This orbit is the open dense subset consisting of the elements whose powers have maximal rank, i. e. such that the i-th power has rank $\pi'[N]_{i+1} + \cdots + \pi'[N]_s$, where $s = \pi[N]_1$ is also the number of simple factors of the Levi group. The regular orbit is also the intersection of the G orbit of N with $\mathfrak{r}[N]$. An element X of the Lie algebra $\mathfrak{r}[N]$ does not just preserve the flag; it actually satisfies $X(F_i) \subset F_{i-1}$ and thus induces maps $F_i/F_{i-1} \to F_{i-1}/F_{i-2}$. One checks that it belongs to the regular orbit if and only if all these maps are injective.

4.2 Resolution. Let ρ denote the map from $G \times^B \mathfrak{b}$ to $\mathfrak{g}$ given by $[(g, X)] \mapsto \mathrm{Ad}(g)(X)$. We want to take a "direct image" of the splitting ϕ_σ along ρ.

Theorem 4.3 *The twisted endomorphism of $\rho_*(\mathcal{O}_{G\times^B\mathfrak{b}})$ induced by ϕ_σ leaves invariant the subsheaf $\mathcal{O}_\mathfrak{g}$, and thus yields a Frobenius splitting ϕ of $\mathfrak{g}$. For each nilpotent matrix N in $\mathfrak{g}$ this splitting is compatible with the closure of the conjugacy class of N.*

Proof. Let f be a regular function, defined on some open subset V of $\mathfrak{g}$. As ρ is proper, $\Gamma(\rho^{-1}(V),\mathcal{O}_{G\times^B\mathfrak{b}})$ is finite over $\Gamma(V,\mathcal{O}_\mathfrak{g})$. Moreover, $\mathfrak{g}$ is a normal variety, so to prove that $\phi_\sigma(f)\in\Gamma(V,\mathcal{O}_\mathfrak{g})$, it suffices to show that it is in the function field $k(\mathfrak{g})$ of $\mathfrak{g}$. Now the function field of $G\times^B\mathfrak{b}$ is the same as the function field of $G\times^B\mathfrak{b}_{\mathrm{reg}}=G\times^T\mathfrak{t}_{\mathrm{reg}}$, and the latter function field is a Galois extension with group W of $k(\mathfrak{g})$, see [2]. Thus what we need to show is that $\phi_\sigma(f)$ is W invariant. As f is W invariant, this will follow if the restriction of the Frobenius splitting ϕ_σ to $G\times^T\mathfrak{t}_{\mathrm{reg}}$ is invariant. It is indeed invariant because of Lemma 2.7 and the construction of ϕ_σ. (The $p-1$-st power of the sign representation is the trivial representation, and functions that are pulled back from $\mathfrak{g}$ are invariant.) To prove the last sentence of the theorem, we use that $\mathfrak{r}[N]$ is one of the $\mathfrak{b}[S]$ of 3.1, because the parts of $\pi'[N]$ are ordered by descending size. Thus by theorem 3.8 ϕ_σ leaves invariant the ideal sheaf of $G\times^B\mathfrak{r}[N]$ and thus ϕ leaves invariant the ideal sheaf of $\rho(G\times^B\mathfrak{r}[N])$, which is indeed the closure of the conjugacy class of N.

Notation 4.4 If $N\in\mathfrak{g}$ is a nilpotent element and S is such that $\mathfrak{r}[N]=\mathfrak{b}[S]$, we write $\sigma[N]$ for $\sigma[S]$, $\mathfrak{b}[N]$ for $\mathfrak{b}[S]$, $\delta_r[N]$ for $\delta_r[S]$. By $\rho[N]$ we denote the restriction of ρ to $G\times^B\mathfrak{r}[N]$ with as target the closure of the conjugacy class of N.

Proposition 4.5 *If $N\in\mathfrak{g}$ is nilpotent, there is a principal effective divisor D which contains the exceptional locus of $\rho[N]:G\times^B\mathfrak{r}[N]\to\overline{C(N)}$ and on which $\sigma[N]$ vanishes.*

Proof. Let us show first that $\sigma[N]$ vanishes on the exceptional locus. This locus is the complement of $G\times^B\mathfrak{r}[N]_{\mathrm{reg}}$. As its intersection with the open set $U^-\times\mathfrak{r}[N]$ is dense, we may restrict attention to that open set. Let us consider $(g,X)\in U^-\times\mathfrak{r}[N]$ such that $\sigma[N]$ does not vanish at $[(g,X)]$. We have to show that $X\in\mathfrak{r}[N]_{\mathrm{reg}}$. The map $F_i/F_{i-1}\to F_{i-1}/F_{i-2}$ induced by X is given by a submatrix α of X with $\pi'[N]_i$ columns and $\pi'[N]_{i-1}$ rows. $(2\le i\le\pi[N]_1)$. Let β be the submatrix of α consisting of the bottom $\pi'[N]_i$ rows. Choose r such that β is one of the blocks along the diagonal in $(X + \delta_r[S])_{\leq r, >n-r}$. Then the hypotheses of Lemma 3.3 apply with $M = X + \delta_r[S]$, so $\det(M_{\leq r, >n-r})$ does not vanish. But $M_{\leq r, >n-r}$ is a block matrix of the form

$$\begin{pmatrix} * & * & * \\ 0 & \beta & * \\ 0 & 0 & * \end{pmatrix},$$

so β has full rank and the map $F_i/F_{i-1} \to F_{i-1}/F_{i-2}$ is injective. It follows that $X \in \mathfrak{r}[N]_{\mathrm{reg}}$. To finish, check that the map $(g, X) \mapsto \det(\beta)$ defines a regular function on $G \times^B \mathfrak{r}[N]$.

Theorem 4.6 *If $N \in \mathfrak{g}$ is nilpotent, then $\overline{C(N)}$ is normal and has rational singularities.*

4.7 Start of proof. By 4.3, $\overline{C(N)}$ is Frobenius split, so normality will follow from [14] if we find any normal variety mapping onto $\overline{C(N)}$ with connected fibres. One may use a map from [10], but we prefer to use the following theorem.

Theorem 4.8 (Spaltenstein) *The fibres of $\rho[N] : G \times^B \mathfrak{r}[N] \to \overline{C(N)}$ are connected.*

Proof. Let $M \in \mathfrak{r}[N]$ and let F be the partial flag corresponding with $P[N]$. Note that $X \in \mathfrak{r}[N]$ if and only if $X(F_i) \subset F_{i-1}$ for all i. The conjugacy classes of nilpotents that intersect $\mathfrak{r}[N]$ are those that are contained in the image $\overline{C(N)}$ of $\rho[N]$. By section 1 of [4] the criterion for X to belong to such a class is that $\dim(\ker X^i) \geq \dim(\ker N^i)$ for all $i \geq 1$. In other words, the condition is that $\pi[X] \leq \pi[N]$ in the "closure ordering" of partitions, called "dominance" order in [5]. The fibre $\rho[N]^{-1}(M)$ of M is parametrized by

$$\{\, g \in G \mid \mathrm{Ad}(g^{-1})(M)(F_i) \subset F_{i-1} \text{ for all } i \,\}/B,$$

which maps onto a set of partial flags

$$\mathcal{F} = \{\, F' \mid \dim(F'_i) = \dim(F_i), \qquad M(F'_i) \subset F'_{i-1} \,\}$$

through a map $g \mapsto g(F)$, which is a proper map with connected fibres. To prove the theorem it thus suffices to show that $\mathcal{F}$ is connected. Now $\mathcal{F}$

maps onto a set $\mathcal{V}$ of linear subspaces of $\ker(M)$ by the map f which assigns to a partial flag its first part. By an induction hypothesis we may assume the fibres of f to be connected, as they are of the same nature as $\mathcal{F}$, but for smaller n. It thus remains to understand why $\mathcal{V}$ is connected. By the remarks above $\mathcal{V}$ consists of the $\pi'[N]_1$ dimensional subspaces L of $\ker(M)$ for which the map $X = M_L$ induced by M on k^n/L satisfies $\dim(\ker X^i) \geq \dim(\ker N^i) - \dim(L)$ for $i \geq 1$. Now one can be quite explicit about the way the Jordan type of M_L—or the Young/Ferrers diagram of its partition—depends on the choice of L. (We may now forget about N.) Let $e_{i,j}$ be a Jordan basis of k^n for M, with

$$e_{ij} \mapsto e_{i,j-1} \mapsto \cdots e_{i1} \mapsto 0.$$

One may think of this basis as indexed by the boxes of the Young diagram of $\pi = \pi[M]$. For a subspace L of $\ker M = \operatorname{span}(e_{11}, \ldots, e_{1s})$ we define a *pivot* to be an integer i such that L intersects $e_{1i} + \operatorname{span}(e_{11}, \ldots, e_{1,i-1})$. Taking vectors that realize the respective pivots gives a basis of L and the Jordan type of M_L is obtained from that of M by making the i-th block one smaller if i is a pivot. (Exercise.) In terms of $\pi[M]$, one should subtract 1 from $\pi[M]_i$ when i is a pivot, and then reorder the parts again by size, if necessary. If L has i as a pivot, but not $i-1$, then one may modify the corresponding basis vector, and thus also L, to lower the pivot by one. Doing this, the other pivots remain the same and $\dim(\ker(M_L)^i)$ can only increase, as one sees by looking at the partitions. So one does not leave $\mathcal{V}$ this way. It is easy to realize such a lowering of a pivot in a one parameter family of subspaces in which the general element has the original set of pivots and the special element has the "lower" set. This family thus lies inside $\mathcal{V}$. The process may be repeated until the set of pivots is $\{1, \ldots, \dim(L)\}$, in which case $L = \operatorname{span}(e_{11}, \ldots, e_{1,\dim(L)})$. All of $\mathcal{V}$ is thus in the same connected component as this particular subspace.

4.9 End of proof of 4.6. The map $\rho[N]$ factors through the birational map $G \times^{P[N]} \mathfrak{r}[N] \to \overline{C(N)}$ and the higher direct images of $\mathcal{O}_{G\times^B \mathfrak{r}[N]}$ in $G \times^{P[N]} \mathfrak{r}[N]$ vanish because $G \times^B \mathfrak{r}[N] \to G \times^{P[N]} \mathfrak{r}[N]$ is a fibration with fibre $P[N]/B$ to which Kempf's vanishing theorem applies. Further the splitting of $G \times^B \mathfrak{r}[N]$ is compatible with the zero set of $\sigma[N]$ and its subdivisor D from 4.5 (see [13] Remark on page 34; observe that the scheme $\sigma[N] = 0$

can not contain a divisor with multiplicity > 1 because that would make the splitting wrong in local coordinates). So by [15] it follows from 4.5 that the higher direct images in $\overline{C(N)}$ of the structure sheaf of $\overline{G \times^B \mathfrak{r}}[N]$ vanish. By the Leray spectral sequence the higher direct images in $\overline{C(N)}$ of the structure sheaf of $G \times^{P[N]} \mathfrak{r}[N]$ also vanish. And that structure sheaf is also isomorphic with the canonical bundle. We have thus checked the conditions stated by Kempf on page 51 of [8]. □

References

[1] W. Borho und H. Kraft, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen, Comm. Math. Helv. 54(1979), 61–104.

[2] E. Brieskorn, Singular elements of semi-simple algebraic groups, Actes du Congrès International des Mathématiciens 1970, tome 2, 279–284 Gauthiers-Villars, Paris (1971).

[3] S. Donkin, The normality of closures of conjugacy classes of matrices, Invent. Math. 101 (1990), 717–736.

[4] M. Gerstenhaber, On nilagebras and linear varieties of nilpotent matrices, III, Annals of Math. Vol. 70(1959), 167–205.

[5] M. Gerstenhaber, On dominance and varieties of commuting matrices, Annals of Math. 73 (1961), 324–348.

[6] Griffiths and Harris, Principles of algebraic geometry, Wiley&Sons, New York (1978).

[7] W. Hesselink, The normality of closures of orbits in a Lie algebra, Comm. Math. Helv. 54, (1979), 105–110.

[8] G. Kempf et al., Toroidal embeddings, Lecture Notes in Math. 339, Springer 1973.

[9] H. Kraft, Parametrisierung von Konjugationsklassen in $\mathfrak{sl}_n$, Math. Annalen 234 (1978), 209–220.

[10] H. Kraft and C. Procesi, Closures of conjugacy classes are normal, Inv. Math. 53 (1979), 227–247.

[11] G. Lusztig, Green polynomials and singularities of unipotent classes, Advances in Math. 42 (1981), 169–178.

[12] O. Mathieu, Formules de caractères pour les algèbres de Kac-Moody générales, Astérisque 159-160 (1988).

[13] V. B. Mehta and A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties, Annals of Math. 122 (1985), 27–40.

[14] V. B. Mehta and V. Srinivas, Normality of Schubert varieties, American Journal of Math. 109 (1987), 987–989.

[15] V. B. Mehta and W. van der Kallen, On a Grauert-Riemenschneider vanishing theorem for Frobenius split varieties in characteristic p, to appear in Inventiones Math.

[16] A. Ramanathan, Equations defining Schubert varieties and Frobenius splitting of diagonals, Publ. Math. IHES 65 (1987), 61–90.

Addresses

V. B. Mehta, School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400005
INDIA

Wilberd van der Kallen, Mathematisch Instituut, Budapestlaan 6,
P. O. Box 80.010, 3508 TA Utrecht
The Netherlands
electronic mail: vdkallen@math.ruu.nl