Chaotic Modeling and Simulation (CMSIM) 4: 419–432, 2017

Hamiltonian and dissipative passage through resonance

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Abstract. This paper compares passage through resonance in the case of a conservative (Hamiltonian) system and in the case of a dissipative system. In the conservative case we will have the Poincaré recurrence theorem on a compact energy manifold; the implication is that if passage through resonance takes place, this will be repeated an infinite number of times with slightly modified positions and momenta. The form the recurrence takes will provide information about the internal structure of the resonance zone. We illustrate this for the $1:2:7$ resonance. One of the conclusions is that apart from actions, the angles (or phases) are important to characterize the dynamics.

For our dissipative system we have chosen a toy problem and a gyroscopic system that displays many phenomena. In these cases we have apart from passage the possibility of being caught into resonance. The gyroscopic example has been discussed in [12] and was summarized in [10] ch. 7.5.3, but we add a number of aspects. We will argue that our examples are typical for passage through resonance zones. Slow manifold theory adds insight to the phenomena.
Keywords: resonance, nonlinear, flywheel, recurrence.

MSC classification: 70H, 34E, 37J

1 The Hamiltonian system

In applications one often has to consider many degrees-of-freedom (dof) systems.Usually, the procedure is then to truncate the system to the resonant part, leaving out the non-resonant modes. One of the motivations for the first sections is to consider the validity of this procedure.
Consider the three dof Hamiltonian $H(p,q) = H_2(p,q) + H_3(p,q) + \dots$ with

$$H_2 = \frac{1}{2}\omega_1(p_1^2 + q_1^2) + \frac{1}{2}\omega_2(p_2^2 + q_2^2) + \frac{1}{2}\omega_3(p_3^2 + q_3^2). \tag{1}$$

The frequencies $\omega_1, \omega_2, \omega_3$ are chosen positive; we can approximate them by rational numbers as the rationals are dense in the set of real numbers. The

Received: 10 May 2017 / Accepted: 10 October 2017

ISSN 2241-0503

Hamiltonian terms $H_j(p, q), j = 3, 4, \dots$ are homogeneous polynomials in p, q of degree j. We assume that two of the frequencies, say ω_1 and ω_2, are close to the first order $1:2$ resonance of two dof. Often we rescale the frequencies to obtain $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = l$, $l \in \mathbb{Q}$; detuning effects to allow for small frequency perturbations can be added in applications. For an exhaustive list of first and second order resonances of three dof Hamiltonians see [10], tables 10.3-4. The three frequencies are not in first or second order resonance of three dof, so for the frequency l we exclude neighborhoods of the natural numbers $1, 2, \dots, 6$.

A powerful theorem on the stability of Hamiltonian systems in the sense of exponentially-long time invariance of the actions was formulated and proved by Nekhoroshev [6]. This theorem presupposes the absence of first or second order resonances in the system but in many applications a natural combination of low and higher order resonances takes place. This motivates to explore in detail combined low and higher order resonance to analyze the variation of the non-resonant actions (or amplitudes). The tools will be normal form theory and the use of the Poincaré recurrence theorem to characterize the dynamics in resonance zones.

It is convenient to scale the coordinates near the stable origin of the system by putting $p, q \to \varepsilon p, \varepsilon q$ and dividing by ε^2. This leads to the Hamiltonian

$$H(p, q) = \frac{1}{2}(p_1^2 + q_1^2) + (p_2^2 + q_2^2) + \frac{1}{2}l(p_3^2 + q_3^2) + \varepsilon H_3(p, q) + \varepsilon^2 \dots \quad (2)$$

So ε^2 is a measure for the energy with respect to stable equilibrium at the origin. We introduce action-angle coordinates I, ϕ by the transformation:

$$q_i = \sqrt{2I_i} \sin \phi_i, \; p_i = \sqrt{2I_i} \cos \phi_i, \; i = 1, 2, 3, \quad (3)$$

leading with (2) to

$$H = I_1 + 2I_2 + lI_3 + \varepsilon H_3 + \varepsilon^2 \dots \text{ and } \dot{I} = -\frac{\partial H}{\partial \phi}, \; \dot{\phi} = \frac{\partial H}{\partial I}.$$

We will also use amplitude-phase coordinates r, ψ with transformations:

$$q_i = r_i \cos(\omega_i t + \psi_i), \; \dot{q}_i = -\omega_i r_i \sin(\omega_i t + \psi_i). \quad (4)$$

1.1 Birkhoff-Gustavson or Born-approximation

After Birkhoff and Gustavson (earlier formulated by Born [1]) we will introduce a symplectic near-identity transformation producing normal forms (see [10] for theory and literature). Prominent terms in the normal forms are produced by the resonances induced by the frequencies $1, 2, l$. Introducing annihilation vectors $\mathbf{a} \in \mathbb{Z}^3$ with the property $a_1 + 2a_2 + a_3 l = 0$, the normal form will contain corresponding combination angles of the form:

$$\chi = a_1\phi_1 + a_2\phi_2 + a_3\phi_3.$$

We will call $\|a\| = |a_1| + |a_2| + |a_3|$ the norm of the annihilator. The Birkhoff-Gustavson normal form to H_4 in action-angle coordinates involves the angles ϕ_1, ϕ_2; it will be:

$$\begin{cases} H_{12} = & I_1 + 2I_2 + lI_3 + \varepsilon b_1 I_1 \sqrt{2I_2} \cos(\chi_1 - b_2) + \varepsilon^2 A(I_1, I_2, I_3), \\ & \chi_1 = 2\phi_1 - \phi_2. \end{cases} \tag{5}$$

We used the annihilator $(2, -1, 0)$. The constants b_1, b_2 are real (for potential problems $b_2 = 0$), A is a homogeneous, quadratic polynomial in I_1, I_2, I_3 (quartic in the amplitudes or positions and momenta).

Preliminary conclusions on the Birkhoff-Gustavson normal forms (5)

1. The normal form (5) is clearly integrable with integrals H_{12}, $I_1 + 2I_2$ and I_3.
2. Near stable equilibrium the energy manifolds are S^5 and are foliated in tori as in the case of two dof but with additional parametrization by the integral I_3.
3. On the energy manifold we find families of short-periodic solutions parametrized by I_3 which is a degeneration in the sense of Poincaré ([7] vol. 1).

The treatment of higher order resonance in [9] is quite general for two dof, the theory will be used with extensions from [11], see also [10].

2 Higher order interactions

We will be interested in the influence of higher order resonances on the long-term behaviour of the action I_3 and the general flow involving passage through resonance. To avoid too many parameters we will consider potential problems, for the qualitative aspects of the phenomena this is a certain restriction. The case of the 1 : 2 resonance for the first two modes is the simplest case as the resonance acts on the cubic part of the Hamiltonian.

2.1 The case $b_1 \neq 0$, $\|a\| \geq 5$

Note that the condition $b_1 = 0$ involves symmetry assumptions. If $b_1 \neq 0$ the system for $k = 2$ is not discrete symmetric in p_1, q_1. The Birkhoff-Gustavson normal form in action-angle coordinates will be of the form:

$$\begin{cases} H = & I_1 + 2I_2 + lI_3 + \varepsilon b_1 I_1 \sqrt{2I_2} \cos \chi_1 + \varepsilon^2 A(I) + \\ & \varepsilon^3 [D_1(I) + D_2(I) \cos \chi_2 + D_3(I) \cos \chi_3 + \varepsilon^4 \ldots] \end{cases} \tag{6}$$

$A(I)$ $(I = I_1, I_2, I_3)$ is a homogeneous quadratic polynomial in its arguments; $\chi_1 = 2\phi_1 - \phi_2$, $\chi_2, \chi_3, \ldots$ are combination angles that arise at higher order terms starting with ε^3. The equations are with a neighborhood of the normal modes excluded and omitting $O(\varepsilon^4)$ terms:

$$\begin{cases} \dot{I}_1 = & 2\varepsilon b_1 I_1 \sqrt{2I_2} \sin\chi_1 + \varepsilon^3[D_2(I)c_1 \sin\chi_2 + \ldots] + \varepsilon^4 \ldots, \\ \dot{\phi}_1 = & 1 + \varepsilon b_1 \sqrt{2I_2} \cos\chi_1 + \varepsilon^2 \frac{\partial A}{\partial I_1} + \varepsilon^3[\frac{\partial D_1}{\partial I_1} + \frac{\partial D_2}{\partial I_1} \cos\chi_2 + \ldots] + \varepsilon^4 \ldots \\ \dot{I}_2 = & -\varepsilon b_1 I_1 \sqrt{2I_2} \sin\chi_1 + \varepsilon^3[D_2(I)c_2 \sin\chi_2 + \ldots] + \varepsilon^4 \\ \dot{\phi}_2 = & 2 + \varepsilon b_1 \frac{I_1}{\sqrt{2I_2}} \cos\chi_1 + \varepsilon^2 \frac{\partial A}{\partial I_2} + \varepsilon^3[\frac{\partial D_1}{\partial I_2} + \frac{\partial D_2}{\partial I_2} \cos\chi_2 + \ldots] + \varepsilon^4 \ldots \\ \dot{I}_3 = & -\varepsilon^3[D_2(I)c_3 \sin\chi_2 + \ldots] + \varepsilon^4 \ldots \\ \dot{\phi}_3 = & l + \varepsilon^2 \frac{\partial A}{\partial I_3} + \varepsilon^3[\frac{\partial D_1}{\partial I_3} + \frac{\partial D_2}{\partial I_3} \cos\chi_2 + \ldots] + \varepsilon^4 \ldots \end{cases} \tag{7}$$

Starting with ε^3 the dots stand for terms dependent on the actions and possibly other resonant combination angles. It follows from the equations and the compactness of the energy manifold that $H_2 = I_1 + 2I_2 + lI_3$ is conserved to $O(\varepsilon)$ for all time. In addition we have

$$\frac{d}{dt}(I_1 + 2I_2) = \varepsilon^3[(c_1 + 2c_2)D_2(I) \sin\chi_2 + \ldots] + \varepsilon^4 \ldots.$$

We conclude that $I_1 + 2I_2$ and I_3 are conserved to $O(\varepsilon)$ on the timescale $1/\varepsilon^2$. If χ_2 and the other combination angles at ε^3 are timelike ($\dot{\chi}_2$ etc. sign-definite), we can remove the $O(\varepsilon^3)$ terms by normalization and improve the estimate.

Consider the case that apart from χ_1 we have at H_5 one more combination angle given by $\chi_2 = a_1\phi_1 + a_2\phi_2 + a_3\phi_3$. For the combination angles χ_1, χ_2 we have from system (7) the equations:

$$\begin{cases} \dot{\chi}_1 = & \varepsilon b_1 (2\sqrt{2I_2} - \frac{I_1}{\sqrt{2I_2}}) \cos\chi_1 + \varepsilon^2 (2\frac{\partial A}{\partial I_1} - \frac{\partial A}{\partial I_2}) + \\ & \varepsilon^3 (F_3(I) + F_4(I) \cos\chi_2) + \varepsilon^4 \ldots \\ \dot{\chi}_2 = & \varepsilon b_1 \left(a_1 \sqrt{2I_2} + a_2 \frac{I_1}{\sqrt{2I_2}}\right) \cos\chi_1 + \varepsilon^2 \left(a_1 \frac{\partial A}{\partial I_1} + a_2 \frac{\partial A}{\partial I_2} + a_3 \frac{\partial A}{\partial I_3}\right) + \\ & \varepsilon^3 \left(F_5(I) + F_6(I) \cos\chi_2\right) + \varepsilon^4 \ldots. \end{cases} \tag{8}$$

To $O(\varepsilon^2)$ the equations for χ_1, χ_2 depend on χ_1 and I only, the terms $O(\varepsilon^3)$ depend on I and $\cos\chi_2$. As in the case of two dof, the presence of a resonance manifold (or zone) produces small variation of the actions and local oscillations of χ_2. The actions I_1, I_2 will be nearly constant in a neighborhood of a stable $1:2$ short-periodic solution. In the Birkhoff-Gustavson normal form (7) to $O(\varepsilon)$ these are obtained by putting $\sin\chi_1 = 0$ and $\dot{\chi}_1 = 0$ leading to $\chi_1 = 0, \pi$ and solutions $I_1 = I_{10}, I_2 = I_{20}$ of:

$$2\sqrt{2I_2} - \frac{I_1}{\sqrt{2I_2}} = 0,\ I_1 + 2I_2 = E_0 (= I_1(0) + 2I_2(0)).$$

This yields:

$$I_{10} = \frac{2}{3}E_0,\ I_{20} = \frac{1}{6}E_0, \tag{9}$$

with I_3 still undetermined. A 5-dimensional neighborhood of these values of I_1, I_2, χ_1 with I_3 still free defines *the resonance zone* M of the system. So on a given energy manifold, the two I_1, I_2 periodic solutions are parametrized by I_3

and correspond with two manifolds M_1, M_2 embedded in the resonance zone M of the energy manifold. The dynamics on M is described with system (7) by:

$$\begin{cases} \dot{I}_j = & \varepsilon^3 C_1 \sin\chi_2 + \varepsilon^4 \ldots,\ j = 1,2,3, \\ \dot{\chi}_1 = & \varepsilon^2 A_0 + \varepsilon^3 (A_1 + A_2 \cos\chi_2) + \varepsilon^4 \ldots, \\ \dot{\chi}_2 = & \pm\varepsilon B_0 + \varepsilon^2 B_1 + \varepsilon^3 (B_2 + B_3 \cos\chi_2) + \varepsilon^4 \ldots \end{cases} \tag{10}$$

with $A_0, A_1, A_2, B_1, B_2, B_3, , C_j, j = 1,2,3$ depending on the solutions I_{10}, I_{20} of (9) and I_3; B_0 is a constant that in general will not vanish. We conclude that in general χ_2 is timelike as long as the orbits remain in the resonance zone M. For orbits in M we can average over χ_2 to obtain $O(\varepsilon)$ invariance of I_3 on the timescale $1/\varepsilon^3$.
Generically we have the following result:

Corollary 21 *1. From the normal form* (7) *we know that I_3 is limited to $O(\varepsilon)$ variations on the timescale $1/\varepsilon^2$. Estimates like this for I_3 are valid on a long polynomial timescale, but not as long as in Nekhoroshev estimates, see [6].*

2. **The orbits outside the resonance zone** *M.*
The strongest changes in the actions arise for I_1, I_2 if χ_1 is not constant, the orbits are approximately moving on toroidal manifolds with significant exchange of energy with small modulations caused by the third dof; an example is given in the next section.

3. **Passage through the resonance zone** *M.*
In the resonance zone M where the manifolds M_1, M_2 are embedded the recurrence of the Hamiltonian flow is expected to be delayed by quasi-trapping. See for an illustration fig. 1 to be discussed below.

3 The 1 : 2 : 7 resonance

To fix ideas and as an explicit example we consider the 1 : 2 resonance in combination with higher order resonance for $l = 7$. The combination angles that may arise correspond with resonances acting on $H_3, \ldots, H_6$:

$$H_3(\|a\| = 5):\ \chi_1 = 2\phi_1 - \phi_2,\ H_5:\ \chi_2 = \phi_1 + 3\phi_2 - \phi_3,$$
$$H_6(\|a\| = 6):\ \chi_3 = 3\phi_1 + 2\phi_2 - \phi_3,\ \chi_4 = \phi_1 - 4\phi_2 + \phi_3, \chi_5 = 4\phi_1 - 2\phi_2.$$

From (8) we have for $j = 2, \ldots, 5$:

$$\dot{\chi}_j = \varepsilon b_1 \left(a_1\sqrt{2I_2} + a_2 \frac{I_1}{\sqrt{2I_2}} \right) \cos\chi_1 + \varepsilon^2 \left(a_1 \frac{\partial A}{\partial I_1} + a_2 \frac{\partial A}{\partial I_2} + a_3 \frac{\partial A}{\partial I_3} \right) + \varepsilon^3 \ldots$$

In the resonance zone M we have with (9):

$$a_1\sqrt{2I_2} + a_2 \frac{I_2}{\sqrt{2I_2}} = \sqrt{\frac{E_0}{3}}(a_1 + 2a_2).$$

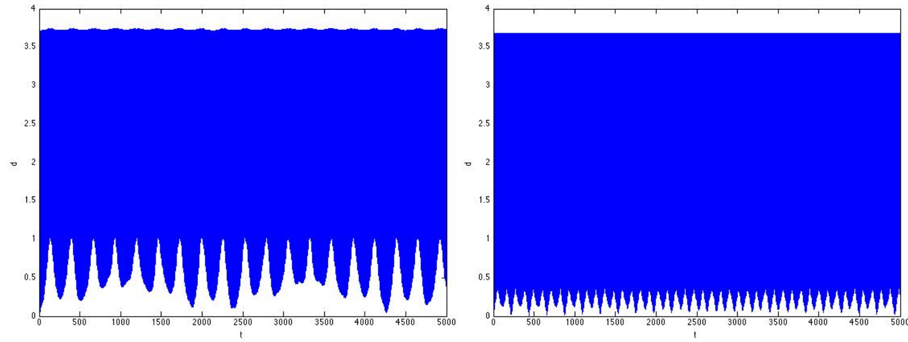

Fig. 1. The Euclidean distance d to the initial values of orbits generated by the $1:2:7$ resonant Hamiltonian (11), $\varepsilon = 0.1$ and 5000 time-steps. Left we started outside the resonance zone M and will pass repeatedly through the zone with $q_1(0) = 0.1, q_2(0) = 0.5, q_3(0) = 0.5$, velocities zero. Right we started in the resonance zone M at $q_1(0) = 0.5, q_2(0) = 0.17678, q_3(0) = 0.5$, velocities zero resulting in stronger recurrence. In M the actions I_1, I_2 show variations of order 10^{-3}, the action I_3 associated with frequency 7 shows variations of order 10^{-4}.

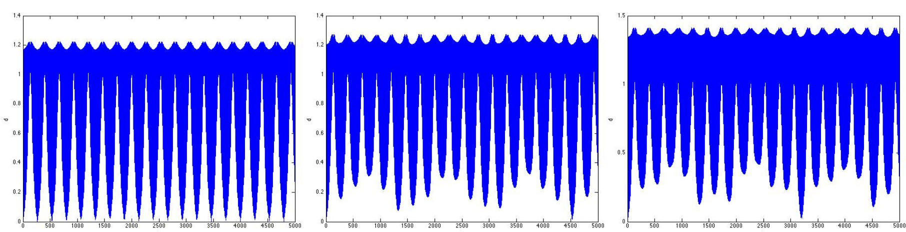

Fig. 2. The Euclidean distance d to the initial values of orbits generated by the $1:2:7$ resonant Hamiltonian (11), $\varepsilon = 0.1$ and 5000 time-steps. Starting outside the resonance zone M we have repeatedly passage of M. We took $q_1(0) = 0.1, q_2(0) = 0.5$, velocities zero, and to show the influence of the non-resonant third mode from left to right $q_3(0) = 0.0, 0.05, 0.1$. The recurrence depends strongly on the third mode.

With the discussion given earlier we conclude that χ_2, χ_3, χ_4 are timelike. This holds generally for a combination angle with a_1, a_2 of the annihilation vector if $a_1/a_2 \neq -2$. We have $\chi_5 = 2\chi_1$, it interesting to consider the influence of the resonances connected with χ_2 (timelike) and χ_1, χ_5 (not timelike). We consider a Hamiltonian with non-resonant terms omitted (an *intermediate* normal form) to consider the dynamics of the $1:2:7$ Hamiltonian resonance:

$$\begin{cases} H = & \frac{1}{2}(\dot{q}_1^2 + q_1^2) + \frac{1}{2}(\dot{q}_2^2 + 4q_2^2) + \frac{1}{2}(\dot{q}_3^2 + 49q_3^2) - \varepsilon q_1^2 q_2 + \\ & \frac{\varepsilon^2}{4}(q_1^4 + 2q_2^4 + q_3^4 + 8q_2^2 q_3^2) + \varepsilon^3 q_1 q_2^3 q_3 + \varepsilon^4 q_1^4 q_2^2. \end{cases} \tag{11}$$

The equations of motion can be written as:

$$\begin{cases} \ddot{q}_1 + q_1 = & 2\varepsilon q_1 q_2 - \varepsilon^2 q_1^3 - \varepsilon^3 q_2^3 q_3 - \varepsilon^4 4 q_1^3 q_2^2, \\ \ddot{q}_2 + 4q_2 = & \varepsilon q_1^2 - \varepsilon^2 (2q_2^3 + 4q_2 q_3^2) - \varepsilon^3 3 q_1 q_2^2 q_3 - \varepsilon^4 2 q_1^4 q_2, \\ \ddot{q}_3 + 49 q_3 = & -\varepsilon^2 (q_3^3 + 4q_2^2 q_3) - \varepsilon^3 q_1 q_2^3. \end{cases} \tag{12}$$

The second and third normal modes are solutions of the equations of motion (12). It is convenient to use for the normal form polar coordinates: $q = r\cos(\omega t + \psi)$, $\dot{q} = -r\omega \sin(\omega t + \psi)$. Using the corresponding combination angles in ψ we find by normalization (excluding the normal modes):

$$\begin{cases} \dot{r}_1 = & -\frac{\varepsilon}{2} r_1 r_2 \sin\chi_1 + \frac{\varepsilon^3}{16} r_2^3 r_3 \sin\chi_2 + \frac{\varepsilon^4}{8} r_1^3 r_2^2 \sin 2\chi_1, \\ \dot{\psi}_1 = & -\frac{\varepsilon}{2} r_2 \cos\chi_1 + \varepsilon^2 (\frac{3}{8} r_1^2 - \frac{9}{64} r_1^2 - \frac{1}{16} r_2^2) + \\ & \frac{1}{16} \varepsilon^3 \frac{r_2^3 r_3}{r_1} \cos\chi_2 + \frac{\varepsilon^4}{8} r_1^2 r_2^2 (6 + \cos 2\chi_1), \\ \dot{r}_2 = & \frac{\varepsilon}{8} r_1^2 \sin\chi_1 + \frac{3}{32} \varepsilon^3 r_1 r_2^2 r_3 \sin\chi_2 - \frac{\varepsilon^4}{32} r_1^4 r_2 \sin 2\chi_1, \\ \dot{\psi}_2 = & -\frac{\varepsilon}{8} \frac{r_1^2}{r_2} \cos\chi_1 + \varepsilon^2 (\frac{3}{8} r_2^2 + \frac{1}{2} r_3^2 - \frac{1}{32} r_1^2) + \varepsilon^3 \frac{3}{32} r_1 r_2 r_3 \cos\chi_2 + \\ & \frac{\varepsilon^4}{32} r_1^4 (6 + \cos 2\chi_1), \\ \dot{r}_3 = & -\frac{\varepsilon^3}{112} r_1 r_2^3 \sin\chi_2, \\ \dot{\psi}_3 = & \frac{\varepsilon^2}{56} 3 r_3^2 + \frac{\varepsilon^2}{7} r_2^2 + \frac{\varepsilon^3}{112} \frac{r_1 r_2^3}{r_3} \cos\chi_2. \end{cases} \tag{13}$$

It follows from system (13) that

$$\frac{d}{dt}(r_1^2 + 4r_2^2) = O(\varepsilon^3),\ \frac{d}{dt}(r_1^2 + 4r_2^2 + 49 r_3^2) = O(\varepsilon^5).$$

Because of the compactness of the energy manifold we have $r_1^2 + 4r_2^2 + 49r_3^2 - 2E = O(\varepsilon)$ for all time (E a constant). From the estimate for the action I_3 we have that $r_1^2 + 4r_2^2 - 2E_0 = O(\varepsilon)$ on the timescale $1/\varepsilon^2$ (E_0 a constant). We find for the combination angles:

$$\begin{cases} \dot{\chi}_1 = & -\frac{\varepsilon}{r_2} (r_2^2 - \frac{1}{8} r_1^2) \cos\chi_1 + \frac{1}{2} \varepsilon^2 (r_1^2 - r_2^2 - r_3^2) + \frac{\varepsilon^3}{8} \frac{r_2 r_3}{r_1} (r_2^2 - \frac{3}{4} r_1^2) \cos\chi_2 + \\ & \frac{\varepsilon^4}{4} r_1^2 (6 + \cos 2\chi_1)(r_2^2 - \frac{r_1^2}{8}), \\ \dot{\chi}_2 = & -\frac{\varepsilon}{2r_2} (r_2^2 + \frac{3}{4} r_1^2) \cos\chi_1 + \frac{\varepsilon^2}{4} (\frac{9}{16} r_1^2 + \frac{103}{28} r_2^2 - \frac{39}{14} r_3^2) + \\ & \frac{\varepsilon^3}{16} (\frac{r_2^3}{r_1} r_3 + \frac{9}{2} r_1 r_2 r_3 - \frac{1}{7} \frac{r_1^2 r_2^3}{r_3}) \cos\chi_2, \\ \dot{\chi}_5 = & 2\dot{\chi}_1. \end{cases} \tag{14}$$

The resonance zone M is for fixed energy E a neighbourhood of $\sin\chi_1 = 0, \pi, r_1^2 = 8r_2^2$. The angle χ_2 is timelike to first order in the resonance zone M, so we do not expect corresponding higher order resonance manifolds; this also holds for the angles χ_3 and χ_4. Putting $\sin\chi_1 = 0, r_1^2 = 8r_2^2$ in M we have timelike $O(\varepsilon^3)$ perturbations decreasing with r_3:

$$\dot{r}_1 = \frac{\varepsilon^3}{16} r_2^3 r_3 \sin\chi_2,\ \dot{r}_2 = \frac{3\sqrt{2}}{16} \varepsilon^3 r_2^3 r_3 \sin\chi_2.$$

To illustrate the part played by resonance zone M we consider recurrence starting outside M and in M, see fig. 1. Inside M the variations of the actions are

small as expected. When passing through the resonance zone (fig. 1 left), we observe delay of recurrence because of quasi-trapping in the resonance zone.
It is interesting to observe the influence of small values of $q_3(0)$. In fig. 2 we show that for $q_3(0) = 0$ we have relatively strong recurrence and little effect of passage through M. This picture changes immediately for small nonzero values of $q_3(0)$.The presence of r_3 in the denominator of the $O(\varepsilon^3)$ term in the system (14) suggests that resonance for small positive values of r_3 might arise but the polar coordinate transformation is not valid for $q_3 \to 0$; note that starting at $q_3(0) = 0.5$, see fig. 1, the recurrence is stronger than for small $q_3(0)$. An indication of the influence of $q_3(0)$ is that the variation of $\dot{\chi}_1$ to $O(\varepsilon^2)$ is maximal if $q_3(0) = 0$, minimal if $r_3^2(0) = 7r_1^2(0)$.

Remark
The discussion for the $1 : 2 : 7$ resonance is typical for the $1 : 2 : \omega$ resonance where ω is chosen such the system is not in first or second order three dof resonance. An approximate higher order resonance with annihilation vector (a_1, a_2, a_3) will result in equations for the combination angles of the form (14) leading to a timelike combination angle χ in the resonance zone. The conclusion is that we have quite generally that for such systems outside the normal mode planes the $1 : 2$ resonance dominates but quasi-trapping in M can be effective and recurrence will be delayed for a large set of initial values.

3.1 Nekhoroshev's theorem for discrete symmetry, $b_1 = 0$

In mathematical applications, for instance pendulum models, it is natural to have symmetries. Assuming discrete symmetry in the first dof q_1, v_1 we have $b_1 = 0$ in Hamiltonian H_{12} of system (5). In this case we can often apply Nekhoroshev's [6] theorem to obtain for the three actions only variations $O(\delta(\varepsilon))$ with $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$ on an exponentially long timescale. This application involves the proof of the steepness of the Hamiltonian $H(I, \phi) = H_{12} + \varepsilon^3 \ldots$ with $b_1 = 0$: $H_{12} = I_1 + 2I_2 + lI_3 + \varepsilon^2 A(I_1, I_2, I_3)$. A sufficient condition for steepness of the full Hamiltonian $H(I, \phi)$ is convexity of H_{12}, see [8]. This is the case if $A(I)$ is positive definite. In this case we find that for every orbit of H starting in the neighborhood of the origin we have for the action:

$$I(t) - I(0) = O(\varepsilon^a) \,\mathrm{for}\, t = O(e^{1/\varepsilon^b}) \,\mathrm{with\,constants}\, a, b > 0. \tag{15}$$

Note that in degenerate cases, for instance where A vanishes or $A = A(I_1, I_3)$, we can still apply the results of higher order resonance.

4 Resonance in the dissipative case

Resonances and resonance zones are abundant in mechanical systems, see for instance [3] or [2] for nice collections of engineering problems. An interesting classical example is the motion determined by gravitation and tidal friction in a binary star system or alternatively a planet with a moon. In this problem the energy of the two-body system changes monotonically while total angular momentum is conserved. The dynamics leads to either permanent disruption of the system or motion into resonance where the rotation of the two bodies is co-planar, circular and 1 : 1 (facing each other permanently); for details and references see [5]. The phenomenon of passage into resonance is a basic possibility in mechanical systems.

4.1 General notions

Consider the $n+m$ dimensional system

$$\dot{x} = \varepsilon X(\phi, x),\ \dot{\phi} = \Omega(x) + \varepsilon \Omega_1(x, \phi), \tag{16}$$

with $x \in \mathbb{R}^n$, $\phi \in \mathbb{T}^m$ where x is a slowly varying, n-dimensional state variable, ϕ represents m angles describing a m-dimensional torus. The vector function X is periodic in the angles, we have the multiple Fourier expansion:

$$X(\phi, x) = \sum_{k=0}^{k=\infty} (a_k(x)\cos(k_1\phi_1 + \ldots + k_m\phi_m) + b_k(x)\sin(k_1\phi_1 + \ldots + k_m\phi_m)),$$

with $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$. Resonance zones in $\mathbb{R}^n$ are determined as neighborhoods of

$$k_1\Omega_1(x) + \ldots + k_m\Omega_m(x) = 0.$$

The resonance zones may exist if the corresponding Fourier coefficient $c_k(x)$ does not vanish. Passage through a resonance zone will always take place if the system (16) is conservative; solutions can not be trapped. Outside the resonance zones we can average over the angles. This results to first order in an $O(\varepsilon)$ approximation of the state variable x on the timescale $1/\varepsilon$.
If system (16) is dissipative, a subset of the solutions entering a resonance zone can be trapped into resonance. When deriving the equations that are valid in a resonance zone one usually finds to first order a conservative system consisting of a pendulum equation or coupled pendulum equations; this is quite deceptive. To establish trapping in a resonance zone one usually has to compute at least a second order approximation.

Example 1. Consider the toy problem:

$$\begin{cases} \dot{x}_1 = & \varepsilon + 2\varepsilon\cos(\phi_1 - \phi_2) - \varepsilon x_1, \\ \dot{x}_2 = & \varepsilon \sin(\phi_1 - \phi_2), \\ \dot{\phi}_1 = & x_1 + x_2, \\ \dot{\phi}_2 = & x_2. \end{cases} \tag{17}$$

There is one combination angle in the system: $\chi = \phi_1 - \phi_2$. We find

$$\dot{\chi} = x_1,$$

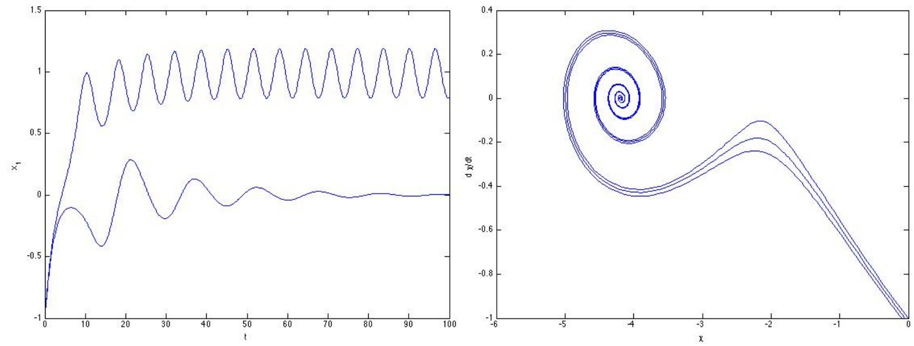

Fig. 3. Left the behaviour of $x_1(t)$ with time, $\varepsilon = 0.1$, based on system (17). When starting in $x_1(0)) = -1, \chi(0) = 0$ the solution moves into the resonance zone near $x_1 = 0$. Starting in $x_1(0)) = -1, \chi(0) = 0.2$ the solution passes through the resonance zone and ends up near $x_1 = 1$. Right passage into resonance in the $\chi, \dot{\chi}$ phase-plane for system (17) with $x_1(0)) = -1$ and starting values $\chi(0) = 0, -0.05, -0.1$

so that χ will be timelike when outside the region around $x_1 = 0$. A neighborhood of $x_1 = 0$ will be a resonance zone. Outside this zone we can average over the angles to obtain approximations by solving:

$$\dot{x}_1 = \varepsilon - \varepsilon x_1,\ \dot{x}_2 = 0.$$

As the averaging approximation of $x_1(t)$ tends towards 1, the orbits will enter the resonance zone when starting with $x_1(0) < 0$. To consider the behavior of the solutions in the resonance zone we rescale near $x_1 = 0$ by $x_1 = \sqrt{\varepsilon} y$. In the resonance zone we find for y and χ the equations:

$$\begin{cases} \dot{y} = \sqrt{\varepsilon} + 2\sqrt{\varepsilon}\cos\chi - \varepsilon y, \\ \ddot{\chi} - 2\varepsilon\cos\chi = \varepsilon - \varepsilon\sqrt{\varepsilon} y. \end{cases} \tag{18}$$

The first order - $O(\sqrt{\varepsilon})$ - equation for χ is indeed a forced pendulum equation and so without trapping, the equilibria are a centre and a saddle for the solutions of $\cos\chi = -1/2$. For the trapping phenomenon we need the next order averaging approximation; see [12] and [4] for technical details.

4.2 The oscillating flywheel with progressive damping

Here we will consider a slightly extended version of an illustrative gyroscopic model described in [3] ch. 8.3 and [2] ch. 3.3 (see fig. 4); the basic analysis for linear damping ($b = 0$) and a hard spring ($a = 1$) was carried out in [12] . Consider a flywheel with mass m_2 and external energy source that can oscillate in the vertical x-direction, it has a small eccentric mass m_1; put $m = m_1 + m_2$. The eccentric mass makes an angle ϕ with the vertical. The equations of motion are with gravitational constant g:

$$\begin{cases} m\ddot{x} + \beta(\dot{x}) + cx + f(x) &= m_1 r(\dot{\phi}^2\cos\phi + \ddot{\phi}\sin\phi), \\ J\ddot{\phi} &= M(\dot{\phi}) - M_w(\dot{\phi}) + m_1 r\ddot{x}\sin\phi + m_1 gr\sin\phi. \end{cases} \tag{19}$$

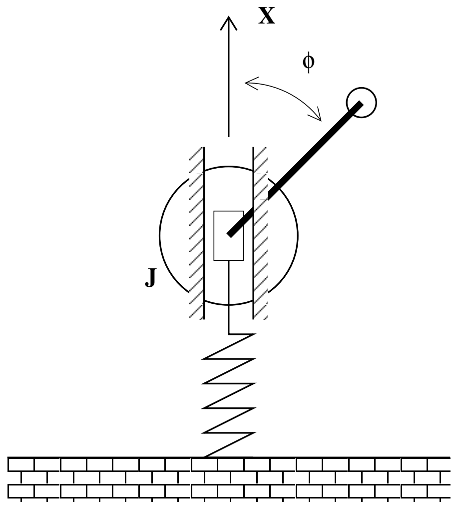

Fig. 4. The oscillating flywheel with excentric mass (**X** should be identified with x in the text).

We assume that m_1 and the nonlinearities are small ($O(\varepsilon)$), also the motor characteristic $M(\dot{\phi}) - M_w(\dot{\phi})$ divided by J. The motor characteristic has a negative slope, taken linear. We find after rescaling:

$$\begin{cases} \ddot{x} + x & = \varepsilon(-ax^3 - \dot{x} - b\dot{x}^3 + \dot{\phi}^2 \cos\phi) + \varepsilon^2 \ldots, \\ \ddot{\phi} & = \varepsilon(\frac{1}{4}(2 - \dot{\phi}) + (1 - x)\sin\phi) + \varepsilon^2 \ldots \end{cases} \tag{20}$$

For $a > 0$ we have a hard spring, for $a < 0$ a soft one; $b > 0$ produces progressive damping. Here we choose $a = 1, b \geq 0$.
We introduce the transformation

$$x, \dot{x} \to r, \phi_2 \, (r > 0), \;\; \phi, \dot{\phi} \to \phi_1, \Omega$$

by $x = r\sin\phi_2, \dot{x} = r\cos\phi_2, \phi = \phi_1, \dot{\phi}_1 = \Omega$. The transformed equations become:

$$\begin{cases} \dot{r} = & \varepsilon\cos\phi_2(-r^3\sin^3\phi_2 - r\cos\phi_2 - br^3\cos^3\phi_2 + \Omega^2\cos\phi_1) + \varepsilon^2 \ldots, \\ \dot{\Omega} = & \varepsilon(\frac{1}{4}(2 - \Omega) + \sin\phi_1 - r\sin\phi_2\sin\phi_1) + \varepsilon^2 \ldots \\ \dot{\phi}_1 = & \Omega, \\ \dot{\phi}_2 = & 1 + \varepsilon(r^2\sin^4\phi_2 + \frac{1}{2}\sin 2\phi_2 + br^2\sin\phi_2\cos^3\phi_2 - \frac{\Omega^2}{r}\cos\phi_1\sin\phi_2) + \\ & \varepsilon^2 \ldots \end{cases} \tag{21}$$

To $O(\varepsilon)$ the righthand sides of system (21) contains apart from ϕ_1, ϕ_2 the angle combination $\chi = \phi_1 - \phi_2$. Following [10] ch. 7 or [13] we have to distinguish between the behaviour in a neighborhood M of $\dot{\chi} = \dot{\phi}_1 - \dot{\phi}_2 = 0$ (the resonance zone) and outside the zone M.

Outside M the angles ϕ_1, ϕ_2 and χ are timelike and we an average over them to find $O(\varepsilon)$ approximations for r and Ω from:

$$\dot{r} = -\frac{\varepsilon}{2}r(1 + \frac{3}{4}br^2),\ \dot{\Omega} = \frac{\varepsilon}{4}(2 - \Omega). \tag{22}$$

It is clear that outside M the value of Ω tends tot 2; the amplitude r of the oscillating flywheel tends to zero if the damping is linear ($b = 0$) or progressive ($b > 0$) and after passage of M. There are two equilibria, $r = 0$ and $r^2 = -\frac{4}{3b}$ if $b < 0$, we do not consider this case. Starting at $\Omega(0) < 1 - O(\sqrt{\varepsilon})$ we will show that we will move into the resonance zone if $b \geq 0$.
In the resonance zone M (in a neighborhood of $\Omega = 1$) we rescale

$$\omega = \frac{\Omega - 1}{\sqrt{\varepsilon}}.$$

Motivation for the choice of $O(\sqrt{\varepsilon})$ for the size of the resonance zone can be found in [10] ch. 7 or [13] ch. 12. The resonance zone M corresponds with $\omega = 0$. Neglecting $O(\varepsilon\sqrt{\varepsilon})$ terms and averaging over timelike angles we find the equations:

$$\begin{cases} \dot{r} = & -\frac{\varepsilon}{2}r(1 + \frac{3}{4}br^2 - \cos\chi), \\ \dot{\omega} = & \frac{\sqrt{\varepsilon}}{2}(\frac{1}{2} - r\cos\chi) - \frac{\varepsilon}{4}\omega, \\ \dot{\chi} = & \sqrt{\varepsilon}\omega - \frac{\varepsilon}{2}(\frac{3}{4}r^2 + \frac{1}{r}\sin\chi). \end{cases} \tag{23}$$

In system (23) the variables ω, χ are relatively fast moving, r is slow. To $O(\sqrt{\varepsilon})$ the flow in M is described by

$$r(t) = r(0),\ \ddot{\chi} + \frac{\varepsilon}{2}r(0)\cos\chi = \frac{\varepsilon}{4}. \tag{24}$$

For χ we have a pendulum equation with constant forcing; the equilibria satisfy $r(0)\cos\chi = \frac{1}{2}$, a centre and a saddle. It is interesting to note that this conservative first approximation in the resonance zone is typical for dissipative systems as in example 1, see for more details [13]. We draw the following conclusions.

1. The saddle point remains unstable under higher order perturbations. It has been shown in [12] by a second order calculation (including the $O(\varepsilon)$ terms) for the case $a = 1, b = 0$, that the centre point turns into a stable focus. The method used was developed by Haberman [4] and employs the energy values associated with the stable and unstable manifolds of the saddle loop encircling the centre.
2. The implication of the analysis is that there exist initial conditions of system (20) such that the solutions are caught into the resonance zone. Other solutions will pass through the resonance zone and will approach the state of constant rotation of the flywheel and absence of vertical oscillations ($r = 0, \Omega = 2$).
3. Admitting progressive damping $b > 0$ will not change the first order conservative equation (24) or the location of the resonance zone, but it will affect

the stability region which is determined by the next order approximation of system (23). The asymptotic analysis is analogous to the treatment of our example 1 but of course more cumbersome. More information can be obtained from the divergence of the flow in the resonance zone. We find from system (23) for the divergence at the critical points:

$$-\frac{\varepsilon}{4}(3br^2 + 1 + \frac{2}{r}\cos\chi). \tag{25}$$

As expected, $b > 0$ enhances the attraction at $O(\varepsilon)$.

4. Slow manifold theory, see [13] for references, throws more light on a system like (23). The equations for ω, χ are called the fast subsystem of (23) (timescale $\sqrt{\varepsilon}t$), r varies relatively slowly (timescale εt). The slow manifold is defined by the zeros of the fast system, i.e.

$$r\cos\chi - \frac{1}{2}, \, \omega = 0.$$

The eigenvalue equation to $O(\sqrt{\varepsilon})$ for these zeros becomes:

$$\lambda^2 - \frac{r}{2}\sin\chi = 0.$$

To this order of approximation $r = r(0)$, the slow manifold exists if $\sin\chi > 0$. Near the centre point that we obtained earlier, we have to expand to $O(\varepsilon)$ to find the next order of the eigenvalue equation. Solving the equation to the next order produces a small negative perturbation of the purely imaginary solutions corresponding with an attracting region in the resonance zone.

5 Conclusions

We have discussed the case of a combined 1 : 2 and higher order resonance in the case of a family of three dof time-independent Hamiltonian systems. In these systems the presence of resonance zones is a natural phenomenon. In such zones stable and unstable periodic solutions can be found but when orbits enter a resonance zone, the Poincaré recurrence theorem guarantees the passage of the orbits through the zone. However, quasi-trapping in the resonance zone may delay the passage; the implication is that the recurrence properties of the orbits reflect the complex structure of the resonance zone. So recurrence may serve as a tool to explore resonance zones.

The situation is qualitatively different in the case of dissipative systems. Again, resonance zones may be present, but in this case orbits may be caught in resonance. In mechanical engineering this may cause undesirable phenomena. The general theory is well-documented , see [10] or [13] for references. We present two illustrative cases, example 1 which is a nice toy problem, and then quite typical passage through and into resonance of a gyroscope that is elastically mounted. In the second case the results are sensitive to the choice of parameters as expected. In the dissipative case we can formulate the equations in the resonance zone in terms of slow manifold theory. It is quite easy and natural to generalize this for the resonance zones of system (16). The procedure gives faster access to the problem of the existence of an attracting region in the resonance zone, but to determine the shape of this region one still has to perform the second order asymptotics in the spirit of [4]. To first order we have for an isolated resonance zone a two-dimensional problem, the second order involves $n + 2$ dimensions.

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