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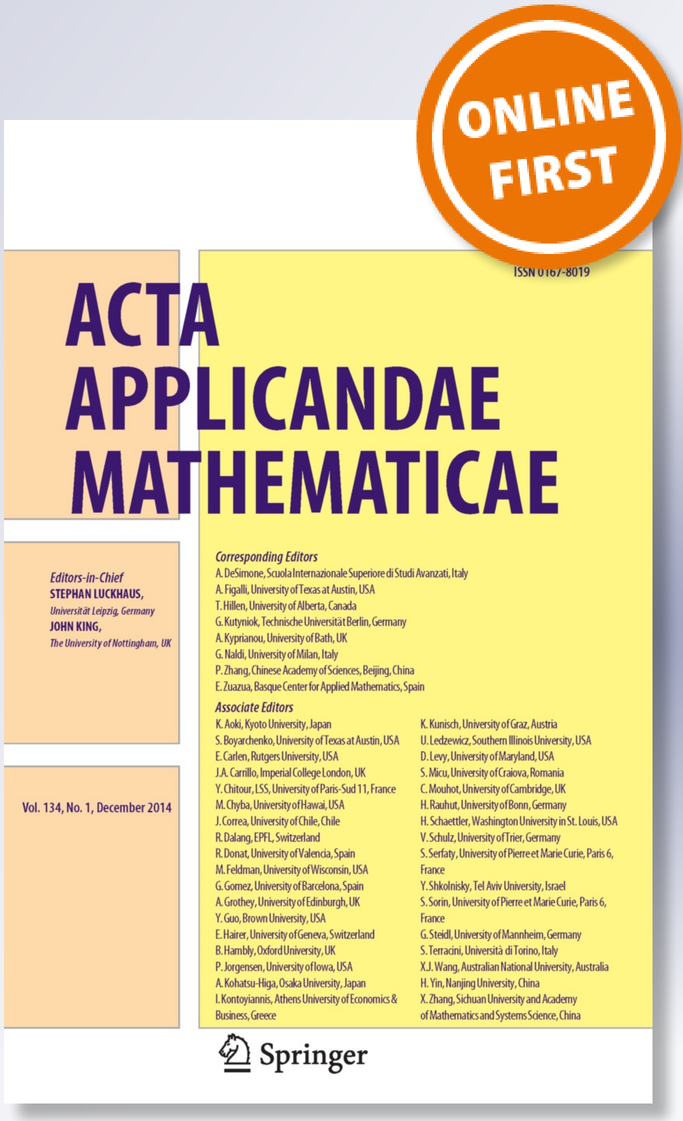

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Abstract This paper summarizes the present state of integrability of Hamiltonian normal forms and it aims at characterizing non-integrable behaviour in higher-dimensional systems. Non-generic behaviour in Hamiltonian systems can be a sign of integrability, but it is not a conclusive indication. We will discuss a few degenerations and briefly review the integrability of Hamiltonian normal forms in two and three degrees of freedom. In addition we discuss two integrable normal form Hamiltonian chains, FPU and $1:2:2:2:2:2$, and three non-integrable normal form chains, with emphasis on the $1:2:3:3:3:3$ resonance. To distinguish between various forms of non-integrability is a major issue; time-series and projections based on the presence of a universal quadratic integral of the normal forms can be a useful predictor.

Keywords Hamiltonian systems · Normal forms · Non-integrability · Hamiltonian chains · Time-series

1 Introduction

In the 'Méthodes Nouvelles de la Mécanique Célèste', vol. 1, [14] where Poincaré describes characteristic exponents and expansion of exponents in the presence of a small parameter, there is an intriguing discussion of characteristic exponents when first integrals exist. This volume of [14] also contains the famous proof that in general for time-independent Hamiltonian systems without additional assumptions, for instance regarding symmetry, no other first integrals exist besides the energy. Although integrability of a dynamical system is a non-generic feature, simplifying assumptions in the construction of models of mathematical physics quite often produce integrability. For more details and references regarding Poincaré's view see [23]. For references on results for Hamiltonian systems until 2007, in particular normal forms, see Chap. 10 of [17]. Although most Hamiltonian systems are non-integrable, remarkably enough a classification of non-integrable systems is lacking completely. It is a little bit like a situation where a biologist would distinguish all living creatures

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in for example birds and non-birds. In fact, it is even stronger than that as the set of non-birds is large but finite.

Normal forms turn out to present a useful approach for studying dynamical systems. We will review the present state of results for Hamiltonian normal forms and we will make a start with the analysis of the meaning of non-integrability. Note however, that even for integrable Hamiltonian systems of three or more degrees of freedom, general insight in the topology of phase-space is still very poor.

To study the stability of solutions of general dynamical systems of the form $\dot{x} = f(x)$, Poincaré came up with a more general concept than eigenvalues. These are the characteristic exponents introduced in chapter four of [14]. Assume that we know a particular solution $x = \phi(t)$, a non-trivial periodic solution, of the dynamical system. When studying neighbouring solutions of $\phi(t)$ to establish stability questions we put

$$x = \phi(t) + \xi.$$

The variational equations of $\phi(t)$ are obtained by substituting $x = \phi(t) + \xi$ into the differential equation and linearising for small ξ to obtain:

$$\dot{\xi} = \left.\frac{\partial f}{\partial x}\right|_{x=\phi(t)} \xi. \tag{1}$$

If $\phi(t)$ is periodic, the variational equations are linear equations with time-periodic coefficients (Floquet equations). The characteristic eigenvalue equation produces the characteristic exponents which are in the case of periodic $\phi(t)$, Floquet exponents. As $d\phi(t)/dt$ satisfies Eq. (1), in the case of perturbing around a periodic solution, one of the Floquet exponents will be zero.

From now on, we consider Hamiltonian systems derived from a time-independent, sufficiently differentiable function $H(p, q)$. We will have $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$ or alternatively $p \in \mathbb{R}^n$, $q \in \mathbb{T}^n$. The equations of motion are:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{2}$$

It can be more transparent to use another set of canonical variables: actions τ_i and angles ϕ_i. Near stable equilibrium $(p, q) = (0, 0)$, they are defined by:

$$p_i = \sqrt{2\tau_i}\cos\phi_i, \qquad q_i = \sqrt{2\tau_i}\sin\phi_i, \quad i = 1, 2, \ldots, n. \tag{3}$$

According to [14], the characteristic exponents of a particular solution of Eq. (2) are pairwise equal in size and have opposite sign. With $\phi(t)$ periodic and as $d\phi(t)/dt$ is a periodic solution of the variational equations, there will be at least two Floquet exponents zero. Poincaré proved in [14], vol. 1, that each additional time-independent integral besides the energy, yields two additional characteristic exponents zero for a periodic solution except in singular cases. A continuous family of periodic solutions on an energy manifold may indicate the existence of an extra integral, but on the other hand, an integrable Hamiltonian may still have isolated iso-energetic periodic solutions.

There is no conclusive relation between non-generic phenomena in Hamiltonian systems like a system at a specific bifurcation value and integrability. However, non-genericity can be an indication, a smoking gun, for integrability. One of the features we will look at, is degeneration of the variational equations, also continuous families of periodic solutions on an energy manifold and the presence of symmetries that show up in the normal form but not in the original system.

In Sect. 2, we summarize and discuss known results for two degrees of freedom systems while adding some remarks on neglected topics. In Sect. 3 we summarize results for three degrees of freedom systems including a description of techniques to study integrability. In Sect. 4 we present new results. We will consider chains of oscillators with integrable and non-integrable normal forms. The time series $H_2(t)$ will be one of the tools to characterize the dynamics. We show this for systems where the question of integrability has been settled, but we expect that this tool can also be used in experimental mathematics.

Interestingly, various degenerations and symmetries may also lead to superintegrable systems, but this will not be a topic in this paper.

2 Two Degrees of Freedom

This is a well-researched topic but with some basic questions left. We consider briefly a few classical examples.

2.1 Braun's Parameter Family

An extensively studied Hamiltonian with a discrete symmetric potential is:

$$H(p,q) = \frac{1}{2}\left(p_1^2 + p_2^2 + q_1^2 + \omega^2 q_2^2\right) - \frac{a_1}{3} q_1^3 - a_2 q_1 q_2^2. \tag{4}$$

This problem has been analyzed by many authors with regards to integrability and existence of periodic solutions. The original Hénon–Heiles problem, one of the interest-arousing papers of dynamical systems theory, corresponds with the case $\omega = 1$, $a_1 = 1$, $a_2 = -1$, [12] and [3]. The polynomial Hamiltonian case (4) is often referred to as Braun's parameter family; see also [1] and [21].

The equations of motion can be rewritten as:

$$\ddot{q}_1 + q_1 = a_1 q_1^2 + a_2 q_2^2, \qquad \ddot{q}_2 + \omega^2 q_2 = 2a_2 q_1 q_2. \tag{5}$$

If $a_2 = 0$, the system becomes trivially decoupled and is integrable. The bifurcation corresponding with a_2 passing through zero has been analyzed in [11]. The integrability at $a_2 = 0$ corresponds with a bifurcation of a degenerate nature.

From now on we assume $a_2 \neq 0$. As we shall see, one can reduce the system to one-parameter dependence by introducing as in [1]:

$$\lambda = \frac{a_1}{3a_2}.$$

If $a_1 = a_2 = a$, $\omega = 1$, a second independent integral exists; we can write this integral I as:

$$I(p,q) = \frac{1}{2}(p_1 + p_2)^2 + \frac{1}{2}(q_1 + q_2)^2 - \frac{a}{3}(q_1 + q_2)^3. \tag{6}$$

The Normal Mode Symmetries of the form $f(x,y) = f(-x,y)$ are called discrete symmetry in x, also reflection symmetry or mirror symmetry. The discrete symmetry in Braun's parameter family produces immediately one periodic solution: the normal mode $q_2 = \dot{q}_2 = 0$. In a neighbourhood of the origin of 4-dimensional phase-space, the energy manifold, parametrized by putting $H(p,q) = E_0$ and so putting E_0 sufficiently small, is topologically the 3-sphere. The normal mode corresponds with an elliptic function $\phi(t)$ described by the equation:

$$\ddot{q}_1 + q_1 = a_1 q_1^2,$$

which is periodic for $0 < E_0 < 1/(6a_1^2)$. For this range of the energy, we have a continuous family of periodic solutions. If $a_1 = 0$, the elliptic functions become harmonic and they are periodic and synchronous for all values of the energy. At the value $E_0 = 1/(6a_1^2)$, $a_1 \neq 0$, the energy manifold bifurcates and becomes non-compact.

Putting $q_1 = \phi(t) + \xi(t)$, $q_2(t) = \eta(t)$ as above, we obtain the variational equations:

$$\ddot{\xi} + \xi = 2a_1\phi(t)\xi, \qquad \ddot{\eta} + \omega^2\eta = 2a_2\phi(t)\eta. \tag{7}$$

As we have seen before, if $a_1 = 0$, $\phi(t)$ is harmonic, the variational equation for η is a Mathieu-equation. It is well-known that in this case there exist open sets in a_1, a_2, ω parameter-space where the solutions of the variational solutions are either stable or unstable; see also [3] and [4].

Periodic Solutions if $\omega = 1$ If $a_1 = a_2, \omega = 1$, an integrable case, the variational equations (7) for ξ and η become identical which can be seen as a degeneration.

Following [21], we introduce $q_1 = \mu q_2$ into the equations of motion; we find easily two periodic solutions if the righthand side of

$$\mu^2 = \frac{a_2}{2a_2 - a_1} = \frac{1}{2 - 3\lambda}$$

is positive. So we have to assume $\lambda < \frac{2}{3}$. The equations for q_1, q_2 are:

$$\ddot{q}_2 + q_2 = 2a_2\mu q_2^2, \qquad q_1 = \mu q_2.$$

The equations produce periodic elliptic functions if the energy E_0 is not too large. Note that for a fixed value of the energy, we find not more than two isolated solutions, even in the integrable case $a_1 = a_2 = a$. The variational equations become:

$$\ddot{\eta} + \eta = 4a_2\mu\phi(t)\eta, \qquad \xi(t) = \mu\eta(t). \tag{8}$$

In the integrable case $a_1 = a_2 = a$, we have $\mu = \pm 1$; if $\mu = +1$, the variational equations for ξ and η become identical as above for the normal mode. Again, as in [2], monodromy arguments apply, leading to integrability.

2.2 Hamiltonian Normal Forms for the 1 : 1-Resonance

The normal forms of two degrees-of-freedom, time-independent Hamiltonian systems near equilibrium are always integrable, so these cases are in this respect less interesting but they present testcases. It may give insight when degeneracies are encountered. In studying normal forms, the existence of periodic orbits can be deduced from the existence of non-degenerate critical points of an averaged (normal form) Hamiltonian on a reduced space. The normalization yields a reduction as the angles emerge in the normal form as combination angles. This reduction corresponds with the existence of an integral of the normal form which enables us to project periodic orbits on relative equilibria of the reduced Hamiltonian. For the general theory see [17] and [4], for the specific problems considered here [21] and [3]; note that there are many other studies of these examples.

Consider Braun's parameter family for the 1 : 1-resonance ($\omega = 1$). According to [21], the normal form to quartic terms has two, four or six periodic solutions for each value of the (small) energy with the exception of the cases:

$$\lambda = \frac{a_1}{3a_2} = \pm\frac{1}{3}.$$

For $\lambda = -1/3$ which corresponds with the Hénon–Heiles problem, we have for the normal form up to quartic terms four isolated periodic solutions on the energy manifold and two continuous families of periodic solutions. The degeneracy is resolved by normalizing to the Hamiltonian terms of degree six: the two families split into four isolated periodic solutions, see [9].

$\lambda = 1/3$ corresponds with the integrable case discussed above. The degeneracy corresponds with integrability but the correspondence is weak as the full system, before normalization, has isolated periodic solutions only. A small perturbation of the continuous families or normalization to higher order will destroy the continuous families and/or produce isolated periodic solutions.

2.3 Discussion

Although we have extensive knowledge about two degrees of freedom systems, this knowledge is mainly restricted to normal form results that are by definition local, i.e. near equilibria or known periodic solutions, and to incidental cases of integrability. Near stable equilibria and near stable periodic solutions, KAM tori exist. A basic question is whether open sets of KAM tori with positive measure exist in phase-space away from the known stable solutions.

A second, more general question concerns the global picture of phase-space. Consider for instance Braun's parameter family of Hamiltonians. Near stable equilibrium, the energy manifold is compact; increasing the energy, the manifold bifurcates at the saddle equilibria. Some of the families of periodic solutions connect stable and unstable equilibrium, whereas other families emerge from stable equilibrium but vanish when increasing the energy. A global picture of bifurcations involving the changes of topology of the phase-flow, for instance for Braun's parameter family, would be very instructive.

3 Three Degrees of Freedom

Consider the Hamiltonian with three degrees of freedom expanded in a neighbourhood of stable equilibrium:

$$H(p,q) = H_2 + H_3 + H_4 + \cdots \tag{9}$$

with

$$H_2 = \frac{1}{2}\left(p_1^2 + q_1^2\right) + \left(p_2^2 + q_2^2\right) + \frac{n}{2}\left(p_3^2 + q_3^2\right).$$

In action variables:

$$H_2 = \tau_1 + 2\tau_2 + n\tau_3.$$

The $H_i, i = 3, 4, \ldots$ are homogeneous polynomials of p and q, the index gives the degree. The general cubic part H_3 has 56 terms, the quartic part H_4 126 terms, etc. The normal form is indicated by a bar. We will consider the first order resonances $n = 1, 2, 3, 4$ [17] or:

$$1:2:1, \qquad 1:2:2, \qquad 1:2:3, \qquad 1:2:4.$$

As we shall see, first order normalization of the Hamiltonian (9) (normalization to cubic terms) produces a drastic reduction of 56 parameters to four or six. We will indicate the Hamiltonian normal form by $\bar{H}$; normalization will take place to a certain degree, cubic terms, quartic terms etc. The normalization to a certain degree will be the final normal form if the qualitative results obtained from this normal form are persistent with respect to higher

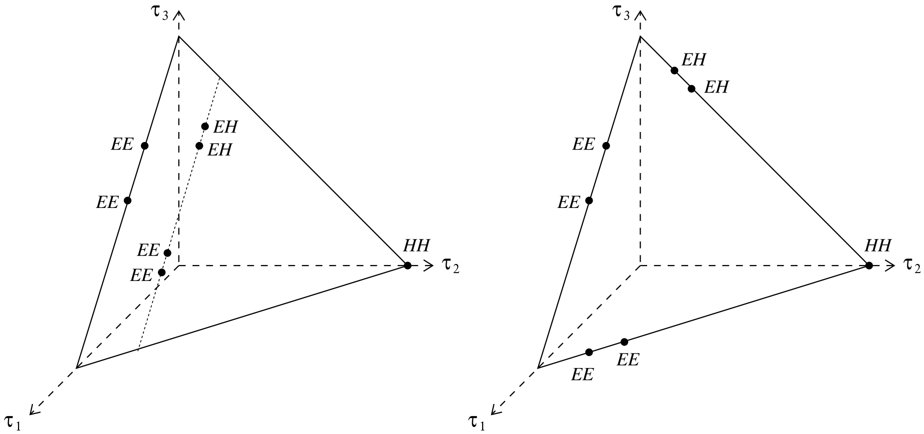

Fig. 1 Left, the energy simplex for the normal form (10) of the $1:2:1$ resonance; there are seven families of periodic solutions. Right, the case $a_3 = 0$ which involves discrete symmetry in the first or the third degree of freedom; the families of periodic solutions have moved to the normal mode hyperplanes

order normalization or if the required quantitative precision is sufficient. We will indicate symmetries leading to integrability as summarized in [17], Chap. 10. Spherical or axial symmetries are not considered as these lead directly to the existence of integrals. We will consider discrete (or reflection) symmetries that are natural in physical models; think of the vertical plane in the mathematical pendulum model or the motion of stars with respect to the galactic plane of a rotating galaxy.

As we will see, the use of a so-called energy simplex with the three actions on the axes can be helpful. Periodic solutions are indicated by dots, their stability is characterized by the letters O (two zero eigenvalues), H (two real eigenvalues), E (two purely imaginary eigenvalues) and C (four complex eigenvalues); see for instance Fig. 1.

Integrability of Normal Forms In the case of two degrees of freedom, the Hamiltonian normal forms are all integrable with integrals H_2 and $\bar{H} = H_2 + \bar{H}_3 + \bar{H}_4 + \cdots$. This holds for any algebraic Hamiltonian normal form from which the wide-spread notion derives that non-integrability effects like chaotic dynamics for two degrees of freedom near stable equilibrium has exponentially small measure; see for instance [17] or [22].

In the case of three degrees of freedom and first order resonance, the normal form $H_2 + \bar{H}_3$ near stable equilibrium is in general not integrable, the $1:2:2$ resonance is an exception. The methods used to establish non-integrability are varying quite a lot and also the normal form dynamics is different for the various first order resonances. We will give a discussion for each of these four resonances. The following methods have been used:

- Ingenious inspection of the normal form or obvious signs of integrability, see [20].
- Extension into the complex domain and analysis of singularities, see for instance [6].
- Applying Shilnikov–Devaney theory to establish the existence of a transverse homoclinic orbit on the energy manifold, see [13]. This method does not only prove non-integrability but also demonstrates and localizes chaotic behaviour.
- Using Ziglin–Morales–Ramis theory where the idea is to study the monodromy group of a particular nontrivial solution; in a number of cases this study leads to non-integrability. This involves the variational equation and the characteristic exponents in the spirit of Poincaré's chapter four in [14]. In an extension one introduces the differential Galois

group which has to be Abelian if the Hamiltonian system has as many meromorphic independent integrals as degrees of freedom. If on the other hand, the Galois group associated with a particular solution is non-commutative, the system is non-integrable. See for references [2].

Note that Shilnikov–Devaney theory is the only method discussed here, that can be used to prove non-integrability while at the same time giving explicit results on the dynamics.

3.1 The 1 : 2 : 1-Resonance

Calculating the normal form (see [19] or [17]) to cubic terms, we find:

$$\begin{aligned}\bar{H}(\tau,\phi) &= \tau_1 + 2\tau_2 + \tau_3 + 2\sqrt{2\tau_2}\big[a_1\tau_1\cos(2\phi_1 - \phi_2 - a_2) \\ &\quad + a_3\sqrt{\tau_1\tau_3}\cos(\phi_1 - \phi_2 + \phi_3 - a_4) + a_5\tau_3\cos(2\phi_3 - \phi_2 - a_6)\big],\end{aligned} \tag{10}$$

where a_i, $i = 1\dots 6$ are real constants. The energy simplex of the normal form $H_2 + \bar{H}_3$ is displayed in Fig. 1 where the three actions are shown, the angles are ignored in this picture. There are seven families of periodic solutions on the energy manifold. The normal form (10) has two independent first integrals, H_2 and $\bar{H}$. In [6], non-integrability of the normal form is shown by extending into the complex domain and showing that a certain sub-class of periodic solutions involves infinite branching of the manifolds. This excludes the presence of a third analytic first integral of (10). The existence of non-analytic first integrals remains an open question that got little or no attention in the literature.

Indications of Integrability Discrete symmetry in the first or third degree of freedom affects the positions of the periodic solutions but the system (10) remains non-integrable, the same reasoning as above applies. Discrete symmetry in the second degree of freedom produces a huge degeneration: $\bar{H} = H_2$, the normal form $\bar{H}_3$ vanishes. To cubic degree, the normal form is integrable (see also [17]). The spherical-spring pendulum with spring frequency two and swing frequencies one is a mechanical example with discrete symmetry in the first and third mode. As the force field is also identical for these modes, it is not surprising that in this case the normal form is integrable; see [7].

Hamiltonian (14), to be discussed later, does not possess these symmetries.

In [2] the general 1 : 2 : 1 resonance has been considered again. A particular solution is selected with corresponding variational equation. It turns out that the monodromy group is not Abelian from which can be concluded that this resonance is not integrable by meromorphic integrals.

3.2 The 1 : 2 : 2-Resonance

For the normal form to cubic terms, we find:

$$\begin{aligned}\bar{H}(\tau,\phi) &= \tau_1 + 2\tau_2 + 2\tau_3 + 2\tau_1\big[a_1\sqrt{2\tau_2}\cos(2\phi_1 - \phi_2 - a_2) \\ &\quad + a_3\sqrt{2\tau_3}\cos(2\phi_1 - \phi_3 - a_4)\big],\end{aligned} \tag{11}$$

where a_i, $i = 1\dots 4$ are real constants. Surprisingly enough the normal form (11) is integrable with independent integrals $H_2, \bar{H}$ and an additional quadratic integral (see [20]). The energy simplex is shown in Fig. 2.

A strong indication for integrability is that the solutions in the (τ_2, τ_3) hyperplane described by $\tau_1 = 0$ are forming a continuous set of periodic solutions which are harmonic

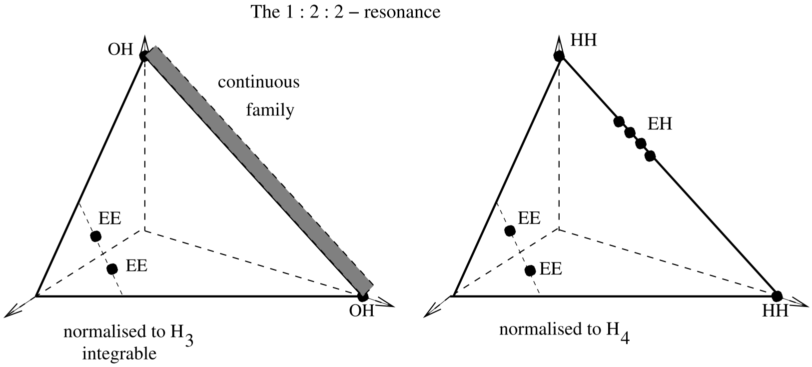

Fig. 2 Left, the energy simplex for the normal form (11) of the 1 : 2 : 2 resonance; there are two families of periodic solutions and one continuous set on the submanifold $p_1 = q_1 = 0$ embedded in the energy manifold. Right, the energy simplex for the Hamiltonian normalized to H_4 (expression not displayed in the text); the continuous family has splitted into six periodic solutions

functions. The iso-energetic periodic solutions in the (τ_2, τ_3) hyperplane are characterized by two extra zero eigenvalues (or characteristic exponents zero). So we have, following Poincaré [14], a candidate for the existence of a second integral besides the energy. Analyzing the normal form equations, see [19] or [20], we find a reduction in the equations for the actions which produces the second integral.

In [20], a typical H_4 perturbation is added to the Hamiltonian to test for persistence. The continuous family in the hyperplane $\tau_1 = 0$ breaks up into six families of periodic solutions. The non-integrability of the normal form to H_4 is still an open question, numerical calculations suggest non-integrability.

An integrable normal form can often be used to apply the KAM theorem by considering the original system (before normalization) as a perturbation of the normal form. The Hamiltonian normal form has to satisfy certain non-degeneracy requirements. The degeneracy of the $(p_1, q_1) = (0, 0)$ submanifold is in itself not an obstruction but the fact that all the periodic solutions are harmonic is one. If the normal form $H_2 + \bar{H}_3 + \bar{H}_4$ might be integrable, the application of the KAM theorem would be feasible.

Remarkably enough, the integrability to cubic terms also holds for an arbitrary number of degrees of freedom in $1:2:\ldots:2$ resonance [20]; see Sect. 4.

3.3 The 1 : 2 : 3-Resonance

The normal form (see [19] or [17]) to cubic terms is:

$$\bar{H}(\tau, \phi) = \tau_1 + 2\tau_2 + 3\tau_3 + 2\sqrt{2\tau_1\tau_2}\big[a_1\sqrt{\tau_3}\cos(\phi_1 + \phi_2 - \phi_3 - a_2) + a_3\sqrt{\tau_1}\cos(2\phi_1 - \phi_2 - a_4)\big], \tag{12}$$

where a_i, $i = 1 \ldots 4$ are real constants. The energy simplex of the normal form $H_2 + \bar{H}_3$ is displayed in Fig. 3. There are seven families of periodic solutions on the energy manifold and there are no obvious degeneracies. Chaotic behaviour has been demonstrated in the $1:2:3$ Hamiltonian normal form in [13] in the case of the presence of a periodic solution with complex eigenvalues. In this case, one has to include the calculation of $\bar{H}_4$ as the dynamics induced by $H_2 + \bar{H}_3$ contains a continuous family of homoclinic solutions. The family breaks

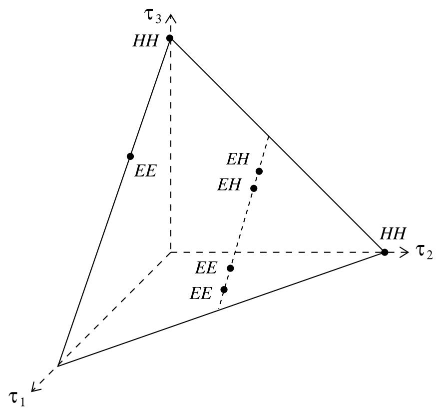

Fig. 3 The energy simplex for the normal form (12) of the 1 : 2 : 3 resonance; there are seven families of periodic solutions. The normal mode for $\tau_2(p_2, q_2)$ has either two positive and two negative eigenvalues (*HH* as indicated in the figure) or complex eigenvalues (C), dependent on the parameters of the normal form $\bar{H}_3$

up to a transverse homoclinic solution when adding $\bar{H}_4$ leading to Shilnikov–Devaney bifurcation, chaos and non-integrability. At that stage, the integrability of the normal form $H_2 + \bar{H}_3$ was still an open question.

Interestingly, it is shown in [13] that an invariant manifold N exists for the normal form $H_2 + \bar{H}_3$, defined by $H_2 = E_0$, $\bar{H}_3 = 0$ with E_0 a positive constant. A two-dimensional ellipsoid is embedded in N, another embedded manifold contains homoclinic and heteroclinic solutions.

In [2], Ziglin–Morales–Ramis theory was used to study the normal form $H_2 + \bar{H}_3$ of this resonance again. After selecting a particular solution and deriving the variational equation one finds that the monodromy group generates the Galois group which is not Abelian except in some special cases of the parameters: $a_1 = 0$, $a_3 = 0$ and $a_1 = a_3$. So apart from these cases we have non-integrability. The case $a_1 = a_3$ is technically more complicated but it is shown in [2] that also in this case for the normal form $H_2 + \bar{H}_3$ not more than two meromorphic integrals exist.

Indications of Integrability Discrete symmetry in the first or third degree of freedom (or both) produces integrability of the normal form. Discrete symmetry in the second degree of freedom leads to integrability of both the normal forms $H_2 + \bar{H}_3$ and $H_2 + \bar{H}_3 + \bar{H}_4$.

3.4 The 1 : 2 : 4-Resonance

The normal form (see [19] or [17]) to cubic terms is:

$$\begin{aligned}\bar{H}(\tau, \phi) = \tau_1 + 2\tau_2 + 4\tau_3 + 2\big[&a_1\tau_1\sqrt{2\tau_2}\cos(2\phi_1 - \phi_2 - a_2)\\ &+ a_3\tau_2\sqrt{2\tau_3}\cos(2\phi_2 - \phi_3 - a_4)\big],\end{aligned} \tag{13}$$

where a_i, $i = 1 \ldots 4$ are real constants.

As in other first resonances, a particular solution can be selected in the $p_1 = q_1$ submanifold. It is shown in [2] that the corresponding monodromy group is not Abelian apart from special parameter choices. The conclusion is again that a third meromorphic integral does in general not exist.

Indications of Integrability Discrete symmetry in (p_1, q_1) does not make the normal form integrable, but discrete symmetry in (p_2, q_2) or (p_3, q_3) does. If we have discrete symmetry in both the second and third degree of freedom, the normal form degenerates to $\bar{H}_3 = 0$, we have to compute higher order normal forms.

4 Chains of Oscillators

There are not so many results on normal forms of non-integrable chains of Hamiltonian oscillators. We will discuss a few cases. We mention here the thesis [5] that handles coupled pendula in 1 : 1 resonance. The emphasis in this work is not on integrability questions but on universal unfoldings of the 1 : 1 : ... : 1 resonance and on the calculation of the generators of normal forms.

4.1 The $H_2(t)$ Time Series as a Predictor

Showing that a Hamiltonian system or its normal form is non-integrable is important but only in low-dimensional cases, the dynamics is clear and can easily be visualized. Well-known are the Poincaré maps of two degrees of systems that were started off by the famous paper [12]. In the case of dissipative systems we have the possibility of exploring time series to find strange attractors and possibly apply reconstruction theory, see for instance the survey [18].

What to do in the case of a time series associated with a symplectic map or a time series directly derived from a Hamiltonian system? The case of an explicit description of the dynamics of the 1 : 2 : 3 resonance (Sect. 3) is unfortunately rare.

Here we propose to exploit the fact that near stable equilibrium, the quadratic part of the Hamiltonian, $H_2(p,q)$, is always an integral of the normal form and is an $O(\varepsilon)$ approximation of the flow induced by the original Hamiltonian system (before normalization) valid for all time. The small parameter ε arises from the scaling $p \to \varepsilon p$, $q \to \varepsilon q$ in a neighbourhood of equilibrium. For the Hamiltonian, this results in

$$H(p,q) = H_2(p,q) + \varepsilon H_3(p,q) + \varepsilon^2 \dots$$

Consider the time series $H_2(t)$ of the Hamiltonian $H(p,q)$. We have that H_2 is an integral of $\bar{H}$ with the estimate

$$H_2(t) = H_2(0) + O(\varepsilon)$$

which is valid for all time. We can supplement this time series with the behaviour of H_2 with respect to one of the actions (or angles). We omit the exceptional case that H_2 is not only an integral of the normal form $\bar{H}$ but also of H. In the sequel we will meet the following cases:

1. $H(p,q)$ is integrable, the solutions near stable equilibrium will generally show quasi-periodic motion on tori and other manifolds. This results in a regular pattern for the time series $H_2(t)$ around $H_2(0)$. Also a H_2, τ_i diagram (with τ_i a specific, suitable action) will show a smooth projection of the Hamiltonian flow.
2. $H(p,q)$ is non-integrable, its normal form $\bar{H}$ is integrable. Chaotic motion is restricted to a narrow range measured in ε around the integrals of the normal form (for details see [17]). The time series $H_2(t)$ is accordingly expected to show a regular pattern within the given error estimates. For the H_2, τ_i diagram the result may look different. Some regions in phase-space show more chaos than other ones. Initial conditions near an unstable periodic solution will generally generate more chaos. If we have chaotic motion of a certain restricted measure, the chaotic jumping will result in a less smooth diagram. In the corresponding MATLAB plot, points will be connected by straight or nearly straight lines. In regions of phase-space where chaos is not effective, the H_2, τ_i diagram will look smooth. We will meet both cases for the 1 : 2 : 2 : 2 : 2 : 2 resonance.

3. In general, both $H(p,q)$ and its normal form are non-integrable. We expect an irregular time series $H_2(t)$ with jumpy (non-smooth) H_2, τ_i diagrams.

Note, that also in the chaotic case, the orbits are smooth, but their jumpy character is reflected in the projections. Doubling the precision of integration, the apparent non-smoothness is replaced by sharp turnings and direction changes of the orbits.

4.2 The Periodic Fermi–Pasta–Ulam Chain

A famous example is the FPU chain, reported in [8]. It is a model for point masses moving on a circle with nearest-neighbour interaction. In the 1950s, numerical integrations for this model were surprising as the model showed recurrence phenomena instead of so-called 'thermalization'. In the case of recurrence, energy is continually exchanged by the individual modes; in the case of thermalization, the energy spreads out equally over the modes.

For not too large values of the energy, the explanation lies in the application of the KAM theorem, but for this application the system has to be near-integrable and non-degenerate in the sense of KAM. This is highly non-trivial as the chain has many different resonances. In [15] it was shown that in a number of prominent cases, the corresponding normal form of the FPU chain is integrable which explains the recurrence phenomena.

A version of the periodic FPU chain with 6 particles, [16], contains already typical phenomena. In this case, there are two $1:1$ resonances appearing in $\bar{H}_4$, and the $1:2:1$ resonance which acts on H_3. However, this three degrees of freedom resonance is degenerate for the FPU chain, i.e. $\bar{H}_3 = 0$. The resulting normal form $H_2 + \bar{H}_4$ has 6 independent integrals which Poisson commute. This normal form serves as the integrable system that is used to apply the KAM theorem.

4.3 The $1:2:2:\ldots:2$ Resonance

It has been proved in [20] that the normal form $H_2 + \bar{H}_3$ of this n degrees of freedom system is integrable. The implication is that for not too large values of the energy, we may encounter recurrence phenomena as found for the FPU chain. We will demonstrate this for Hamiltonian (14). The phenomena will be most striking when choosing initial values near an unstable periodic solution. The (p_2, q_2) normal mode is such a solution. In Fig. 4 we present a numerical calculation of the second action, τ_2, in the cases of 3, 4, 5 and 6 degrees of freedom (modes) during 100 periods. The Hamiltonian to be used for $\omega = 2$ is:

$$H(p,q) = \frac{1}{2}\left(p_1^2+q_1^2\right) + \left(p_2^2+q_2^2\right) + \frac{\omega}{2}\sum_{i=3}^{6}\left(p_i^2+q_i^2\right) - \varepsilon q_1^2 \sum_{i=2}^{6} a_i q_i - \varepsilon q_2^2\left(c_1 q_1 + \sum_{i=3}^{6} c_i q_i\right) - \varepsilon q_1 q_2 \sum_{i=3}^{6} b_i q_i. \quad (14)$$

For $\omega = 1, 3, 4$, Hamiltonian (14) has not a discrete symmetry that leads to integrability of the normal form $H_2 + \bar{H}_3$, see Sect. 3; the terms with b_i coefficients are breaking the symmetry.

Figure 4 shows that when adding degrees of freedom (modes), the energy decrease of the unstable (p_2, q_2) normal mode becomes less; the coefficients a_i, b_i, c_i were chosen identical to avoid deviations caused by them. The phenomenon of energy fluctuations with increasing number of modes is tied in with the integrability of the normal form of this resonance. According to [20], we have that, apart from H_2 and $H_2 + \bar{H}_3$ in the case of n degrees of

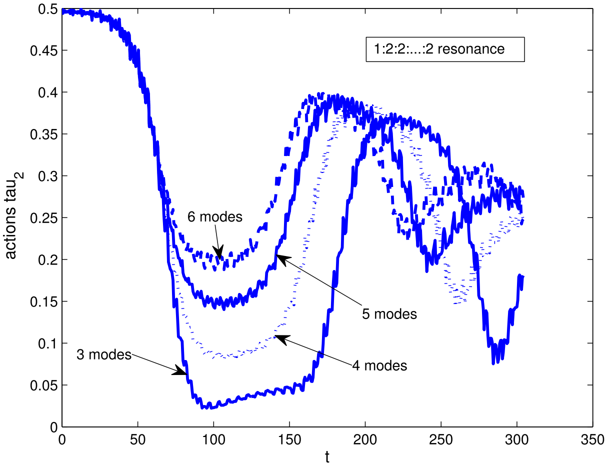

Fig. 4 The actions of the (p_2, q_2) normal mode with time during 100 periods in the case of the $1:2:2:\ldots:2$ resonance of Hamiltonian (14). The cases of 3, 4, 5 and 6 degrees of freedom (modes) are indicated; $a_2 = \cdots = a_6 = 1$, $b_3 = \cdots = b_6 = 0.5$, $c_3 = \cdots = c_6 = 0.5 = c_1$, $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = \cdots = q_6(0) = 0.01$, $p_i(0) = 0$ $(i = 1, \ldots, 6)$

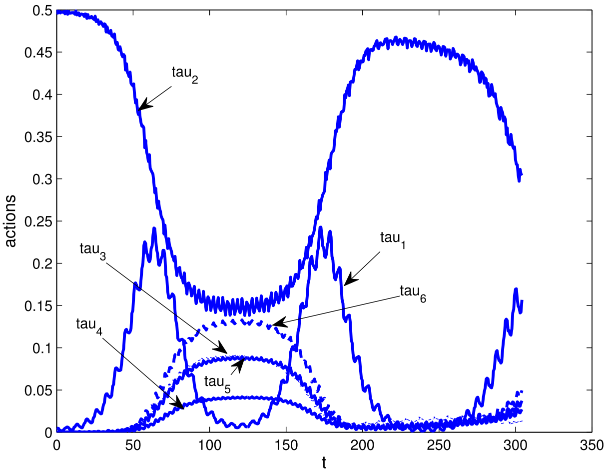

Fig. 5 The different actions of the $1:2:2:2:2:2$ resonance of Hamiltonian (14) starting near the (p_2, q_2) normal mode; time runs during 100 periods. We have $a_2 = a_3 = 1$, $a_4 = 0.6$, $a_5 = 0.9$, $a_6 = 1.1$, $b_3 = 0.5$, $b_4 = -0.5$, $b_5 = -0.6$, $b_6 = -0.7$, $c_1 = c_3 = 0.5$, $c_4 = 0.6$, $c_5 = 0.7$, $c_6 = 0.8$, $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = \cdots = q_6(0) = 0.01$, $p_i(0) = 0$ $(i = 1, \ldots, 6)$

freedom in $1:2:2:\ldots:2$ resonance, $(n-2)$ independent quadratic integrals exist describing $(n-2)$ 2-tori, linking two by two the modes with frequency 2. The implication is that the Hamiltonian (14) in this resonance case is near-integrable in asymptotic sense (for estimates see [17]). Adding more modes, the energy of the unstable (p_2, q_2) mode is spread out over the tori with a recurrence that decreases with the number of modes. This is clear from the picture if the coefficients of the Hamiltonian are well-balanced (of the same order of magnitude for the different degrees of freedom) as corresponds with our choice in Fig. 4. If the coefficients of the Hamiltonian are not well-balanced, certain interactions may be suppressed.

Figure 5 shows the actions with time in the case of six modes. Here the coefficients a_i, b_i, c_i have been chosen differently to be able to identify the modes with frequency 2. On this interval of time, the modes 3, 4, 5 and 6 are not much excited. This might be caused by the fact that only two of the characteristic exponents of the unstable (p_2, q_2) normal mode are generated by $\bar{H}_3$, for the other two exponents one needs $\bar{H}_4$ (see Fig. 2).

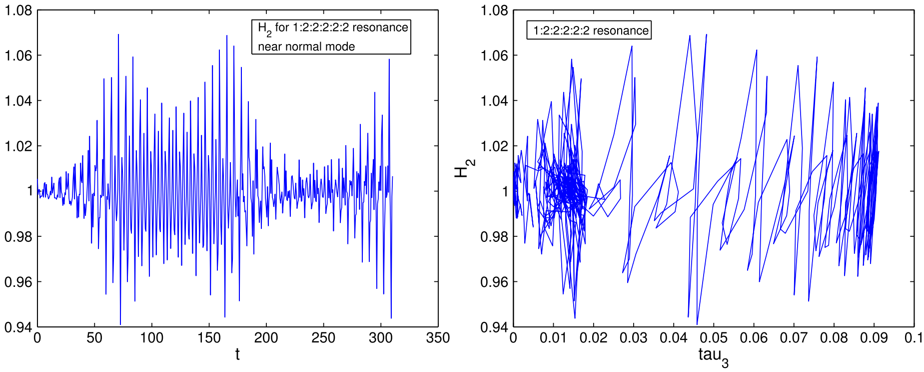

Fig. 6 H_2 for the $1:2:2:2:2:2$ resonance of Hamiltonian (14) starting near the unstable (p_2, q_2) normal mode, $H_2(0) = 1.005$; time runs during 100 periods. We have $a_2 = a_3 = 1$, $a_4 = 0.6$, $a_5 = 0.9$, $a_6 = 1.1$, $b_3 = 0.5$, $b_4 = -0.5$, $b_5 = -0.6$, $b_6 = -0.7$, $c_1 = c_3 = 0.5$, $c_4 = 0.6$, $c_5 = 0.7$, $c_6 = 0.8$, $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = \cdots = q_6(0) = 0.01$, $p_i(0) = 0$ $(i = 1, \ldots, 6)$. There is some energy pumping into the higher order modes but still $O(\varepsilon)$, $\varepsilon = 0.1$. Right, the projection H_2, τ_3 that shows chaotic jumps

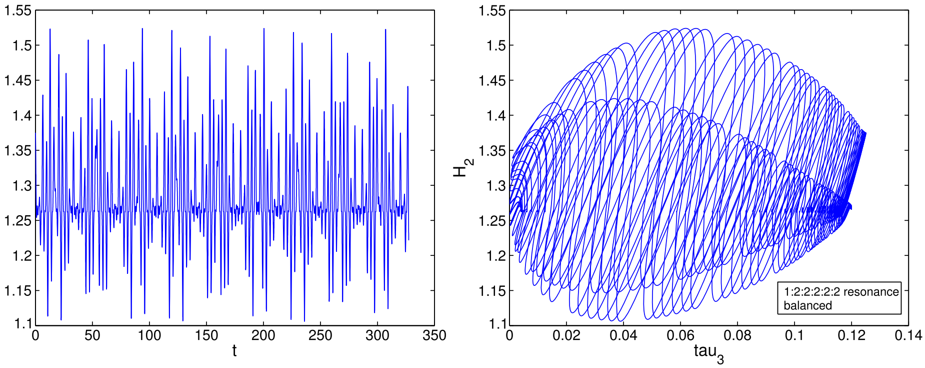

Fig. 7 H_2 for the $1:2:2:2:2:2$ resonance of Hamiltonian (14), $H_2(0) = 1.375$; time runs during 100 periods. We have $a_i = b_i = 1$, $c_i = 0$, $i = 1$, $q_i(0) = 0.5$, $p_i(0) = 0$, $i = 1, \ldots, 6$. There is more energy pumping into the higher order modes than in Fig. 6 but still $O(\varepsilon)$, $\varepsilon = 0.1$. Right is the projection H_2, τ_3 that looks rather smooth

As H_2 is an integral of the normal form, it is conserved to $O(\varepsilon)$ for the original Hamiltonian (14). In Fig. 6 we show the behaviour of this approximate integral for the data corresponding with Fig. 5. This choice means that we expect energy pumping into the modes 1 and $3, \ldots, 6$ as the normal mode (p_2, q_2) is unstable. The H_2, τ_3 diagram shows the small scale chaos.

It is interesting to compare the behaviour of H_2 in the case of a more balanced distribution of initial values, further away from the unstable normal mode. The result is depicted in Fig. 7, the deviation of the normal form integral H_2 are larger but still $O(\varepsilon)$. Both in Figs. 6 and 7, the fluctuations around $H_2(0)$ are quite regular. This is according to expectation as, although the numerical integration is for the full non-integrable system, its normal form is integrable.

Consider now a projection of the dynamics instead of the time series for H_2. We have chosen H_2 as a function of τ_3. In this resonance case, chaotic jumps take place indicated by

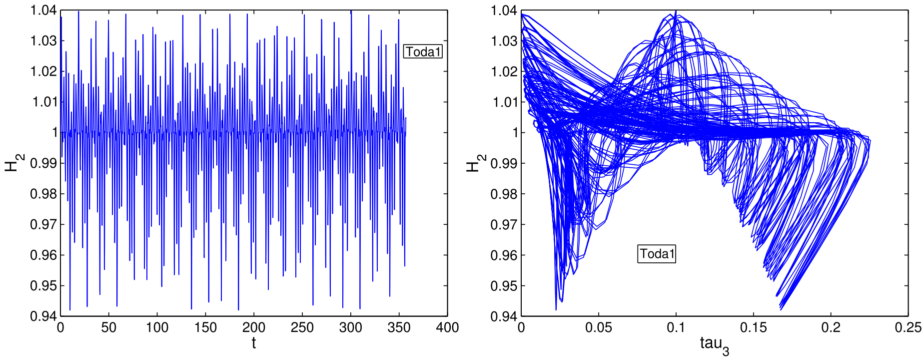

Fig. 8 The periodic Toda lattice with six particles, constants $m = a = 1$, $v = 2$. Initial values: $q_i(0) = 0.3$, $i = 1, \dots, 6$, $p_1(0) = p_2(0) = 0.5$, $p_5(0) = p_6(0) = -0.5$, $p_3(0) = 0.1$, $p_4(0) = -0.1$; H_2 starts at $H_2(0) = 1$ and displays regular behaviour, comparable with the behaviour of $H_2(t)$ in the case of the $1:2:2:2:2:2$ resonance. However (on the right), the H_2, τ_3 projection is smooth, suggesting the projection of a quadric which agrees with the integrability of the Toda chain

straight lines; these lines arise as MATLAB connects subsequent points of the time series, see again Fig. 6. This shows that the time series of H_2 gives a good indication of the near-integrability of this resonant chain, while the $H_2 - \tau_3$ projection reveals the underlying non-integrability. For the initial values chosen in Fig. 7, the chaotic motion is clearly less effective.

4.4 Comparison with the Toda Lattice

It is instructive to compare the analysis of the near-integrable $1:2:2:2:2:2$ resonance with the dynamics of the periodic Toda lattice which is integrable. The Hamiltonian describes a lattice with nearest-neighbour interaction and is of the form:

$$H(p,q) = \frac{1}{2m}\sum_{1}^{n} p_i^2 + v\sum_{1}^{(n)}\left(e^{\frac{q_i - q_{i+1}}{a}} - 1\right),$$

m, v and a are constants; we omit the ε-scaling. We identify $q_{i+1} = q_1$, see for instance [10]. The Hamiltonian is completely integrable with momentum integral $\sum p_i^2$. Without restriction of generality we choose zero for this momentum integral. The normal form of the Toda lattice has again H_2 as an integral; for 6 particles we produce the H_2 time series in Fig. 8. On the right we present the H_2, τ_3 projection that is smooth as expected.

4.5 The $1:2:3:\dots:3$ Resonance

It is interesting to compare the energy fluctuations of the unstable (p_2, q_2) normal mode of Hamiltonian (14) for the $1:2:2:\dots:2$ resonance with one of the non-integrable cases. We will consider in some detail $\omega = 3$, the case in which we have a fairly complete picture of the non-integrable dynamics in [13]. The action τ_2 is displayed in Fig. 9 for the cases of 3, 4, 5 and 6 modes with frequency 3, but for these parameter values the (p_2, q_2) normal mode is unstable with real eigenvalues (HH). The variation of the action of the (p_2, q_2) normal mode does not follow the same pattern as in the case $\omega = 2$. Recurrence phenomena however, are present as before.

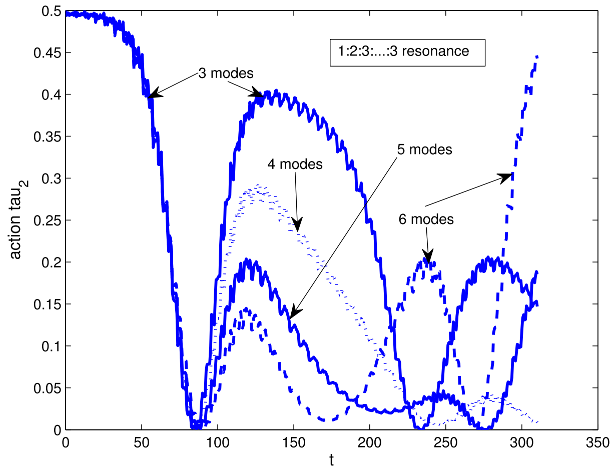

Fig. 9 The action of the (p_2, q_2) normal mode with time during 100 periods in the case of the $1:2:3:\ldots:3$ resonance of Hamiltonian (14). The periodic normal mode solution (p_2, q_2) is unstable and has real eigenvalues (HH); $\varepsilon = 0.1$. The cases of 3, 4, 5 and 6 modes are indicated; $a_2 = \cdots = a_6 = 1$, $b_3 = \cdots = b_6 = 0.5$, $c_3 = \cdots = c_6 = 0.5 = c_1$, $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = \cdots = q_6(0) = 0.01$, $p_i(0) = 0$ $(i = 1, \ldots, 6)$

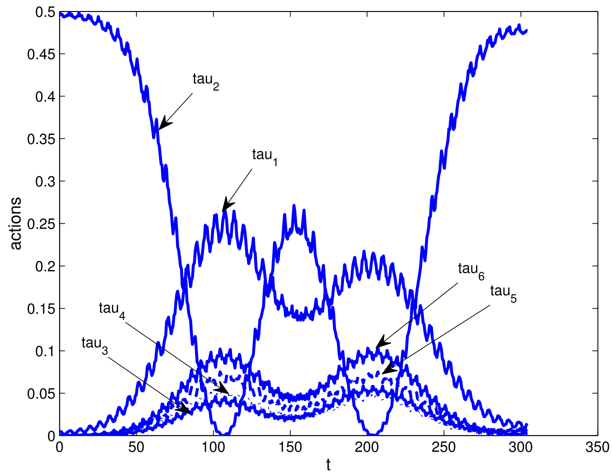

Fig. 10 The different actions of the $1:2:3:3:3:3$ resonance of Hamiltonian (14) starting near the (p_2, q_2) normal mode; time runs during 100 periods. The periodic normal mode solution (p_2, q_2) is unstable and has real eigenvalues (HH); $\varepsilon = 0.1$. We have $a_2 = a_3 = 1$, $a_4 = 0.6$, $a_5 = 0.9$, $a_6 = 1.1$, $b_3 = 0.5$, $b_4 = -0.5$, $b_5 = -0.6$, $b_6 = -0.7$, $c_1 = c_3 = 0.5$, $c_4 = 0.6$, $c_5 = 0.7$, $c_6 = 0.8$, $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = \cdots = q_6(0) = 0.01$, $p_i(0) = 0$ $(i = 1, \ldots, 6)$

The equations of motion corresponding with the normalized Hamiltonian $H_2 + \bar{H}_3$ of (14) will contain only the parameters a_2 and b_i, $i = 3, \ldots, 6$; compare also the normal form in [13]. The parameters $a_3, \ldots, a_6$, and c_i play no part at this level of approximation.

Analogous to the $1:2:3$ resonance, we find five normal modes of the normal form. The (p_1, q_1) normal mode does not exist, except in degenerate cases. The normal mode (p_2, q_2) is unstable (HH or C). The homoclinic and ellipsoidal manifolds discovered in [13] will be higher-dimensional and will act as an organizing center for the chaotic orbits.

The analysis of [13] is focused on the case where the second normal mode is complex unstable. This produces a horseshoe map in the dynamics with strong consequences. First we consider the case of the unstable second normal mode with real eigenvalues (HH).

Figure 10 shows the actions with time in the case of six modes. As above, the coefficients a_i, b_i, c_i have been chosen differently to be able to identify the modes with equal frequency 3, but for this choice there are no complex unstable periodic solutions. The modes $3, \ldots, 6$ do not show much excitation on this interval of time. This agrees with the normal form calculation which shows that for these modes the interaction runs mainly through the

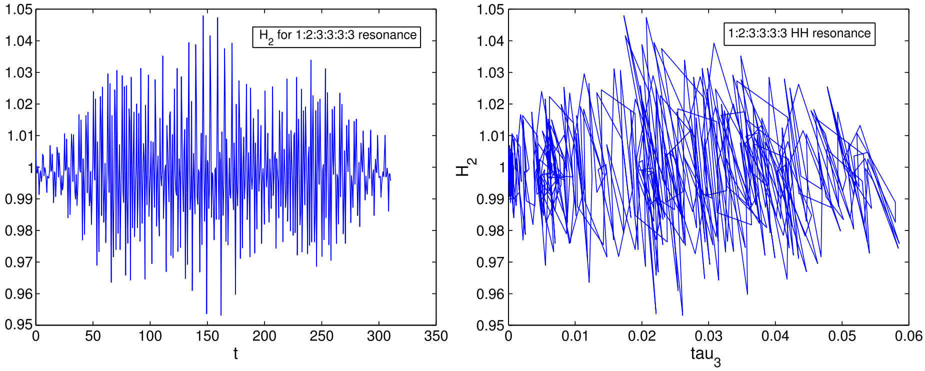

Fig. 11 H_2 for the $1:2:3:3:3:3$ resonance of Hamiltonian (14) starting near the unstable (p_2, q_2) normal mode (HH), $H_2(0) = 1.005$; time runs during 100 periods. We have the same data as in Fig. 10. There is some energy pumping into the higher order modes but still $O(\varepsilon)$, $\varepsilon = 0.1$. On the right the H_2, τ_3 projection

combination angles $\phi_1 + \phi_2 - \phi_i$, $i = 3, \dots, 6$. This relative quiet behaviour is reflected in the deviations of the normal form integral H_2 in Fig. 11. However, the H_2, τ_3 projection on the right displays the presence of chaos as predicted in [2].

The Case of Complex Instability of the Second Normal Mode The analysis in [13] is for the $1:2:3$ resonance, but it can easily be extended to our case of six modes. Of interest is the invariant submanifold N imbedded in the energy manifold. N is obtained from the normal form $H_2 + \bar{H}_3$ by putting $H_2 = E_0$ (E_0 a positive constant) and $\bar{H}_3 = 0$. Note that H_2 and $\bar{H}_3$ are integrals of the normal form. According to [13], the invariant manifold N contains periodic orbits like the second normal mode, homoclinic and heteroclinic sorbits and other interesting phenomena. Imbedded again in N is an ellipsoid, which contains a large part of the interesting flow. The integrals $H_2 = E_0$ and $\bar{H}_3 = 0$ represent approximations of the original Hamiltonian flow that is valid for all time, so the corresponding manifolds do shadow part of the flow of the original Hamiltonian.

We start with a picture of the actions outside N ($\bar{H}_3 \neq 0$) in Fig. 12, left. Both the actions τ_1 and τ_2 display violent exchanges. The actions $\tau_3, \dots, \tau_6$ act at a relatively low level (τ_6 is typical), but they have mutual exchanges that are again significant, see Fig. 12, right.

We compare these results with the case of the same value of $H_2 = 3.615$ but $\bar{H}_3 = 0$, so the initial values are located on the invariant manifold N of the normal form. The mode corresponding with τ_1 dominates for a large part, see Fig. 13 (left), but at a lower energy level the exchanges between the four modes with frequency 3 are relatively strong; see Fig. 13 (right).

It is interesting to compare the deviations of the normal form integral H_2 in the case of complex instability of the normal mode (p_2, q_2) in Fig. 14 with the results of HH instability (Figs. 11 and 14). In the submanifold $H_3 = 0$, the deviations become larger after $1/\varepsilon$ periods, in the case $H_3 \neq 0$, the growth of deviations starts earlier. One expects the accuracy to improve when adding the H_4 normal form to describe the dynamics. The H_2, τ_3 projection on the right displays the presence of chaos suggested by the irregularity of the time series.

In Fig. 15 we show the $H_2(t)$ time series in the case that we start near to the invariant manifold N. The deviations from $H_2(0)$ are slightly smaller.

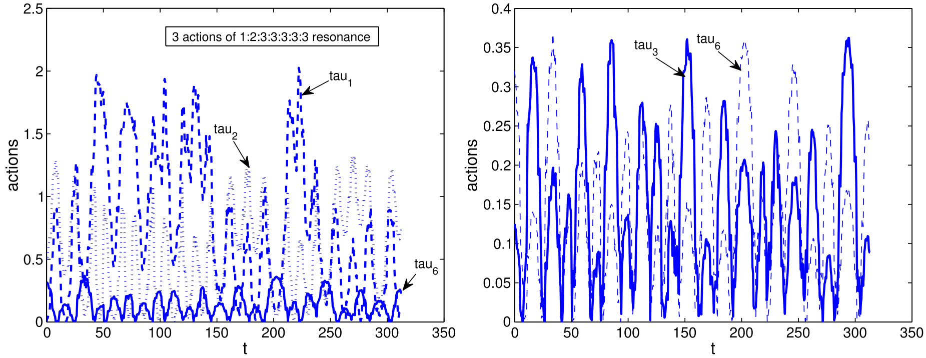

Fig. 12 Left, the actions τ_1, τ_2, τ_6 of the $1:2:3:3:3:3$ resonance of Hamiltonian (14); time runs during 100 periods; $\varepsilon = 0.1$. The periodic normal mode solution (p_2, q_2) is unstable and has complex eigenvalues (C). Right, the actions τ_5, τ_6. We have $a_i = 0.5$, $b_i = 3$, $c_i = 0$, $i = 1, \dots, 6$; initial values: $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = 0.5$, $q_4(0) = 0.6$, $q_5(0) = 0.7$, $q_6(0) = 0.8$, $p_i(0) = 0$ $(i = 1, \dots, 6)$; $H_2 = 3.615$, $\bar{H}_3 = -0.798\varepsilon$

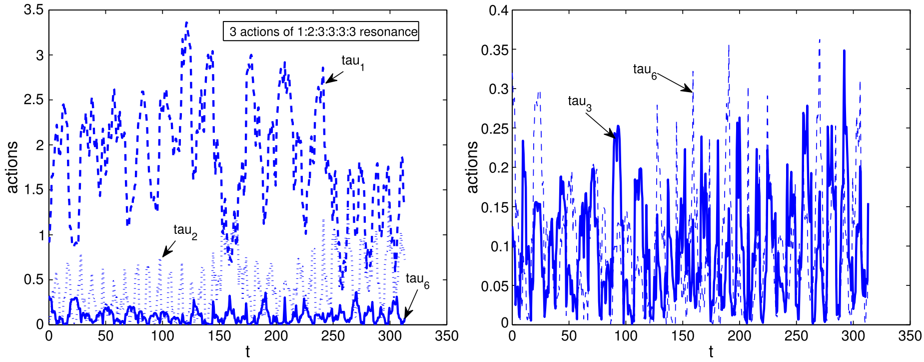

Fig. 13 Left, the actions τ_1, τ_2, τ_6 of the $1:2:3:3:3:3$ resonance of Hamiltonian (14) starting on the submanifold N, $H_2 = 3.615$, $\bar{H}_3 = 0$; $\varepsilon = 0.1$. The periodic normal mode solution (p_2, q_2) is unstable and has complex eigenvalues (C). We have $a_i = 0.5$, $b_i = 3$, $c_i = 0$, $i = 1, \dots, 6$; initial values: $q_1(0) = -1.372914$, $q_2(0) = 0.2509$, $q_3(0) = 0.5$, $q_4(0) = 0.6$, $q_5(0) = 0.7$, $q_6(0) = 0.8$, $p_i(0) = 0$ $(i = 1, \dots, 6)$. Right, the actions τ_3, τ_6

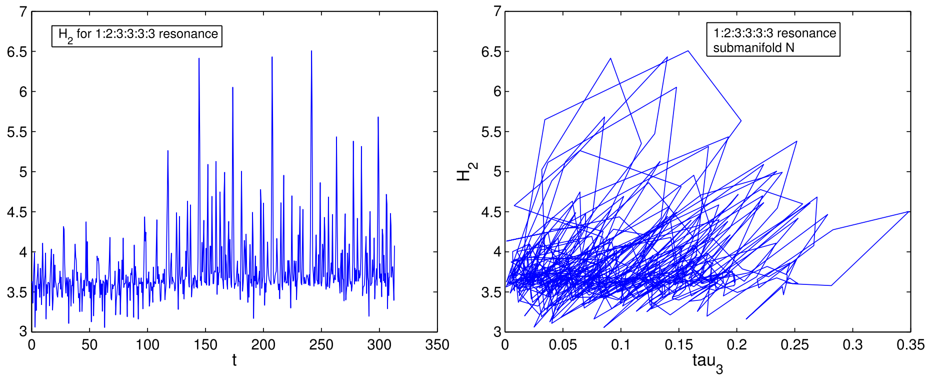

Fig. 14 The behaviour of H_2 in the case of the $1:2:3:3:3:3$ resonance, $\varepsilon = 0.1$ when starting on the submanifold N, $H_2 = 3.615$, $\bar{H}_3 = 0$ with data of Fig. 13. The periodic normal mode solution (p_2, q_2) is unstable and has complex eigenvalues (C). Right the H_2, τ_3 projection

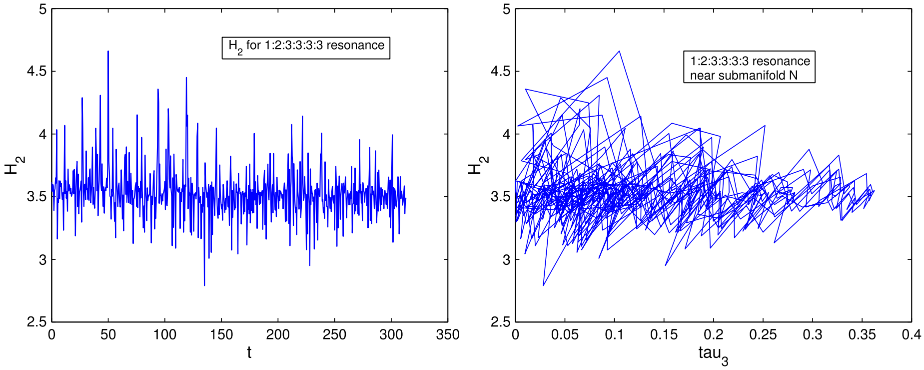

Fig. 15 The behaviour of H_2 for the $1:2:3:3:3:3$ resonance, $\varepsilon = 0.1$, when starting near the submanifold N, $H_2 = 3.615$, $\bar{H}_3 = -0.798\varepsilon$ corresponding with Fig. 12. The periodic normal mode solution (p_2, q_2) is unstable and has complex eigenvalues (C). Right, the H_2, τ_3 projection

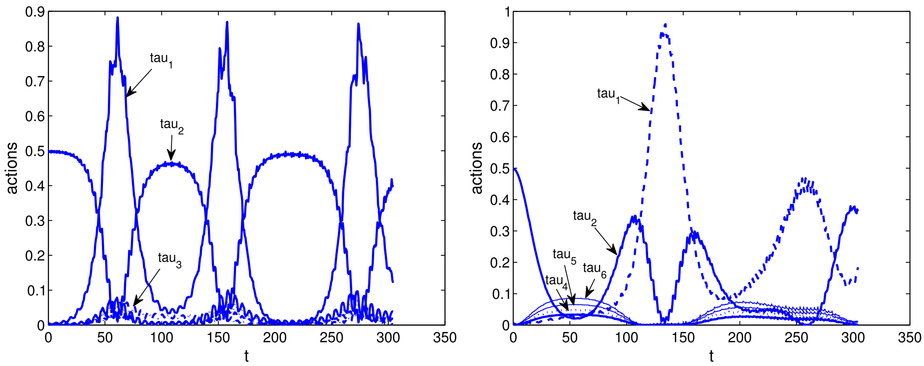

Fig. 16 Left, the different actions of the $1:2:1:\ldots:1$ resonance of Hamiltonian (14), right the actions of the $1:2:4:\ldots:4$ resonance. In both cases we started near the (p_2, q_2) normal mode; time runs during 100 periods; $\varepsilon = 0.1$. There is strong interaction between the first and second mode. The three higher modes have (on this interval of time) less energy than the third one. For both resonances we took $a_2 = a_3 = 1$, $a_4 = 0.6$, $a_5 = 0.9$, $a_6 = 1.1$, $b_3 = 0.5$, $b_4 = -0.5$, $b_5 = -0.6$, $b_6 = -0.7$, $c_1 = c_3 = 0.5$, $c_4 = 0.6$, $c_5 = 0.7$, $c_6 = 0.8$, $q_1(0) = 0.1$, $q_2(0) = 1$, $q_3(0) = \cdots = q_6(0) = 0.01$, $p_i(0) = 0$ $(i = 1, \ldots, 6)$

4.6 The $1:2:1:\ldots:1$ and $1:2:4:\ldots:4$ Resonances

We know little about the dynamics of the other two chains except that their normal forms are not integrable. For the sake of completeness and as an inspiration for future research, we produce the actions with time for the $1:2:1:\ldots:1$ and $1:2:4:\ldots:4$ resonances in Fig. 16. For these initial conditions, most of the energy remains in the first two modes. Both cases show lively recurrent behaviour, they merit further analysis.

In Fig. 17 we present the $H_2(t)$ time series for the $1:2:1:1:1:1$ resonance with the corresponding H_2, τ_3 projection that shows chaotic jumps. The analogous pictures for the $1:2:4:4:4:4$ resonance are presented in Fig. 18.

5 Discussion

– We did not discuss in extenso the approximate character of normal forms with respect to the original systems. A normalized Hamiltonian approximates the original Hamiltonian

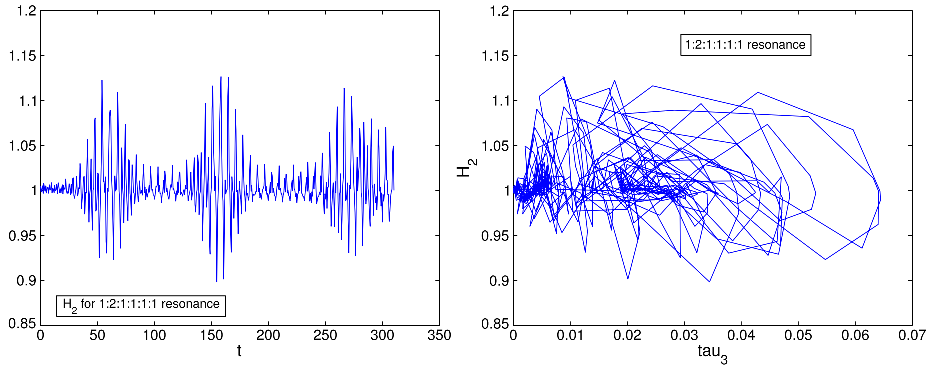

Fig. 17 Left, we present the time series $H_2(t)$ for the $1:2:1:1:1:1$ resonance of Hamiltonian (14) with parameter values as in Fig. 16, $H_2(0) = 1.005$. Although the system is non-integrable, the time series looks fairly regular. Right, the H_2, τ_3 projection that shows chaotic jumps

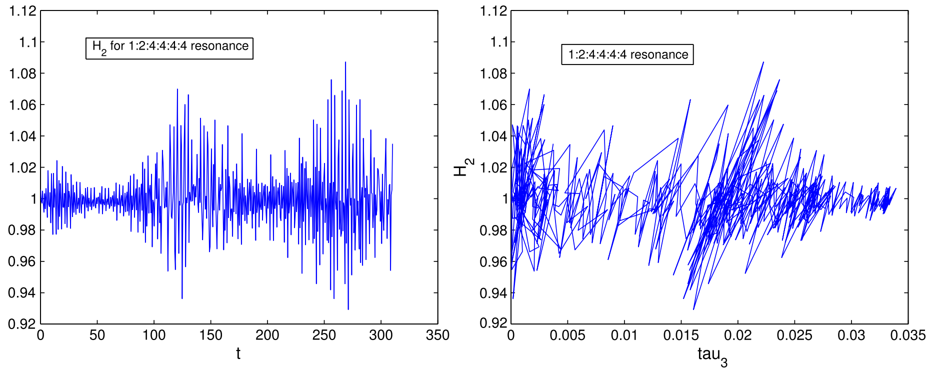

Fig. 18 Left, we present the time series $H_2(t)$ for the $1:2:4:4:4:4$ resonance of Hamiltonian (14) with parameter values as in Fig. 16, $H_2(0) = 1.005$. Right, the H_2, τ_3 projection that shows chaotic jumps

in a clear-defined sense, but this does not imply that the dynamics of the normal form approximates the dynamics of the original system in all respects. There are many delicate questions and interesting answers for this, it is one of the main topics of [17].

– In our numerical examples we chose 'well-balanced' Hamiltonians, i.e. monomials with not too different coefficients. More special choices might change the dynamics.
– In the case of the $1:2:2$ resonance, the integrability of the normal form $H_2 + \bar{H}_3 + \bar{H}_4$ is still an open question. Preliminary calculations suggest non-integrability.
– We have indicated, as in [21], that the introduction of discrete symmetries merits special attention. In some cases, these symmetries produce integrability of the normal form to a certain order, but certainly not always.
– Unfortunately, there is not a 'simple smoking gun' for (non-)integrability, in each resonance case we need different techniques. Qualitative methods, like analysis of singularities as in [6] or the use of the Abelian character of certain groups as in [2], are very important. They should be a starting point for additional dynamical characterization of the dynamics. The $H_2(t)$ time series near stable equilibrium seems to be a useful predictor.

– The chains of oscillators discussed in Sect. 4 merit further study as we have here cases of an integrable normal form and three non-integrable normal form chains. A more detailed comparison of the dynamics will improve our insight. In particular, we expect strong influence of the resonances produced by equal frequencies. Such an analysis requires normalization to H_4.

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References

1. Braun, M.: On the applicability of the third integral of motion. J. Differ. Equ. **13**, 300–318 (1973)
2. Christov, O.: Non-integrability of first order resonances in Hamiltonian systems in three degrees of freedom. Celest. Mech. Dyn. Astron. **112**, 149–167 (2012)
3. Churchill, R.C., Pecelli, G., Rod, D.L.: A survey of the Hénon–Heiles problem with applications to related examples. In: Casati, G., Ford, J. (eds.) Como Conf. Proc. Springer Lecture Notes in Physics, vol. 93, pp. 76–136 (1979)
4. Churchill, R.C., Kummer, M., Rod, D.L.: On averaging, reduction and symmetry in Hamiltonian systems. J. Differ. Equ. **49**, 359–414 (1983)
5. de Jong, H.: Quasiperiodic Breathers in Systems of Weakly Coupled Pendula. Thesis Rijksuniversiteit, Groningen (1999)
6. Duistermaat, J.J.: Non-integrability of the 1 : 2 : 1-resonance. Ergod. Theory Dyn. Syst. **4**, 553–568 (1984)
7. Dullin, H., Giacobbe, A., Cushman, R.: Monodromy in the resonant swing spring. Physica D **190**, 15–37 (2004)
8. Fermi, E., Pasta, J., Ulam, S.: Los Alamos report LA-1940. In: E. Fermi, Collected Papers, vol. 2, pp. 977–988 (1955)
9. Gustavson, F.G.: On constructing formal integrals of a Hamiltonian system near an equilibrium point. Astron. J. **71**, 670–686 (1966)
10. Gutzwiller, M.C.: Chaos in Classical and Quantum Mechanics. Springer, Berlin (1990)
11. Hanßmann, H., Sommer, B.: A degenerate bifurcation in the Hénon–Heiles family. Celest. Mech. Dyn. Astron. **81**, 249–261 (2001)
12. Hénon, M., Heiles, C.: The applicability of the third integral of motion, some numerical experiments. Astron. J. **69**, 73–79 (1964)
13. Hoveijn, I., Verhulst, F.: Chaos in the 1 : 2 : 3 Hamiltonian normal form. Physica D **44**, 397–406 (1990)
14. Poincaré, H.: Les Méthodes Nouvelles de la Mécanique Célèste, 3 vols. Gauthier-Villars, Paris (1892, 1893, 1899)
15. Rink, B.: Symmetry and resonance in periodic FPU-chains. Commun. Math. Phys. **218**, 665–685 (2001)
16. Rink, B., Verhulst, F.: Near-integrability of periodic FPU-chains. Physica A **285**, 467–482 (2000)
17. Sanders, J.A., Verhulst, F., Murdock, J.: Averaging Methods in Nonlinear Dynamical Systems, 2nd edn. Applied Mathematical Sciences, vol. 59. Springer, Berlin (2007)
18. Takens, F.: Reconstruction theory and nonlinear time series analysis. In: Broer, H.W., Hasselblatt, B., Takens, F. (eds.) Handbook of Dynamics, vol. 3. Elsevier, Amsterdam (2010)
19. Van der Aa, E.: First order resonances in three-degrees-of-freedom systems. Celest. Mech. **31**, 163–191 (1983)
20. Van der Aa, E., Verhulst, F.: Asymptotic integrability and periodic solutions of a Hamiltonian system in 1 : 2 : 2-resonance. SIAM J. Math. Anal. **15**, 890–911 (1984)
21. Verhulst, F.: Discrete symmetric dynamical systems at the main resonances with applications to axi-symmetric galaxies. Philos. Trans. R. Soc. Lond. A **290**, 435–465 (1979)
22. Verhulst, F.: Extension of Poincare's program for integrability, chaos and bifurcations. Chaotic Model. Simul. (CMSIM) **1**, 3–16 (2011)
23. Verhulst, F.: Henri Poincaré, Impatient Genius. Springer, Berlin (2012)