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Averaging normal forms for PDEs with applications to perturbed wave equations

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Summary. We start with a tutorial description of averaging-normalization for parabolic and hyperbolic PDEs on bounded domains. As an example of the parabolic case, we review results on advection-diffusion problems. New results are presented for linear and nonlinear wave equations with parametric excitation. Surprisingly enough, the normalization produces reduction to two-dimensional almost-invariant manifolds and no modal interaction at leading order.

1.1 Introduction

Normalization and normal forms play an important part in mathematical analysis and algebra. For instance $n \times n$-matrices can be put in Jordan normal form. Such an example also makes clear that normalization is not a unique procedure as the choice of normalization of matrices depends on its purpose. In the case of matrices there is a vast literature with many possibilities, but in all special cases and in other mathematical problems as well, the general aim of normalization is a simplification of the object by transformation.

In the case of ODEs of the form

$$\dot{x} = \varepsilon f(t, x),$$

with ε a small positive parameter, averaging normalization can be summarized as follows. Assume that the limit

$$f^0(z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(z, s) ds$$

exists. Introduce the averaging normalization transformation

$$x(t) = z(t) + \varepsilon \int_0^t (f(z, s) - f^0(z)) ds.$$

With a few assumptions and using elementary calculations one finds for z the equation

$$\dot{z} = \varepsilon f^0(z) + \varepsilon^2 f^1(t, z, \varepsilon).$$

The equation has been normalized to $O(\varepsilon)$, the simplification is the removal of the variable t and so-called nonresonant terms from the equation to $O(\varepsilon)$. With additional assumptions one can extend the normalization to $O(\varepsilon^2)$ and higher order.

This procedure for ODEs is well-known, for a description and references see [SVM]. The aim of the present paper is to describe in a tutorial way the normalization procedure for a number of PDEs (sections 2-4) and to discuss a few new examples. Averaging normalization for PDEs is of more recent date and the theory is far from complete. Additional material on this topic can be found in [Verhulst2005].

1.2 Normal forms for parabolic equations

A typical problem formulation is to consider an equation of the form

$$u_t + Lu = \varepsilon f(u),\ t \geq 0, \tag{1.1}$$

with given initial and boundary values, L a linear operator, u an element of a suitable function space, and $f(u)$ representing the linear and nonlinear perturbation terms.

The first step is to solve the 'unperturbed' problem

$$\frac{\partial u_0}{\partial t} + Lu_0 = 0,\ t \geq 0, \tag{1.2}$$

with the given initial and boundary values. If the domain has a simple geometrical shape like a circle or a rectangle, this may not present difficulties. In real-life problems, the domain is more complicated and one has to resort to numerical methods.

One may well ask: if we have to use numerical methods for the unperturbed problem, why would I not use these methods directly for the perturbed problem. The answer is that in evolution equations, long time numerical integrations may present a big obstacle. Averaging weeds out the short-periodic or short-oscillatory terms and this improves the interval of validity of the computations enormously. So, even if we have to perform numerical integration of the unperturbed and the normalized equation(s), this may still be an effective procedure.

1.2.1 Advection

To focus the discussion we consider a problem from [Krol91]. In this case, the domain is two-dimensional, the unperturbed equation is

$$\frac{\partial C_0}{\partial t} + \nabla(v_0.C_0) = 0,\ t \geq 0. \tag{1.3}$$

The equation describes advection for transport problems. We will consider the application to tidal basins like the North-Sea. In this case, the two-dimensional vector $v_0 = v_0(x, y, t)$ is the basic periodic flow due to tidal currents that is supposed to be known. The transportation of material, sediment or chemicals, is represented by the concentration C_0, the term $\nabla(v_0.C_0)$ represents the advection with the flow.

In the application to tidal basins one often considers the basic flow to be divergence free, so

$$\nabla.v_0 = 0.$$

The unperturbed equation becomes

$$\frac{\partial C_0}{\partial t} + v_0.\nabla C_0 = 0,\ t \geq 0. \tag{1.4}$$

Eq. (1.4) is a first order equation which can be integrated along the characteristics $P(t)(x, y)$, in this case also called streamlines. Due to the uniqueness of the solutions of eq. (1.4), $P(t)(x, y)$ is an invertible map with inverse $Q(t)(x, y)$.

The solution C_0 is constant along the characteristics, so on adding initial condition

$$C_0(x, y, 0) = \gamma(x, y),$$

we find the solution

$$C_0(P(t)(x, y), t) = \gamma(x, y),$$

so that

$$C_0(x, y, t) = \gamma(Q(t)(x, y)). \tag{1.5}$$

1.2.2 Advection-diffusion

Several types of perturbations of advection are possible. For the application in [Krol91] one considers the fact that tidal basins are open. This results in a small rest stream so that the tidal current is perturbed:

$$v(x, y, t) = v_0(x, y, t) + \varepsilon v_1(x, y).$$

The rest stream is assumed to be divergence free: $\nabla.v_1 = 0$.

A second perturbation arises from diffusion in the basin, expressed by the term $\varepsilon\Delta C$. The equation to be studied is then

$$\frac{\partial C}{\partial t} + v_0.\nabla C + \varepsilon v_1.\nabla C = \varepsilon\Delta C,\ t \geq 0. \tag{1.6}$$

with given initial condition $C(x, y, 0) = \gamma(x, y)$. This is still a linear problem. One should note that the tidal current has a period of nearly 12 hours, the effect of small diffusion entails a timescale of $6 - 12$ months.

1.2.3 The standard form for averaging

Using variation of constants we obtains a slowly varying system. The transformation is

$$C(x, y, t) = F(Q(t)(x, y), t).$$

If $\varepsilon = 0$, we have $C = C_0, F = \gamma$ and C_0 is constant on the characteristics. If $\varepsilon > 0$ and small, this results in a slowly varying F. By differentiation we obtain an equation of the form

$$\frac{\partial F}{\partial t} = \varepsilon L(t) F$$

with initial condition $F(x, y, 0) = \gamma(x, y)$. The linear operator $L(t)$ is computed using the perturbation terms and the unperturbed solution (from P and Q). In this problem $L(t)$ is uniformly elliptic and $T-$periodic in t. Averaging over t produces the approximating system

$$\frac{\partial \bar{F}}{\partial t} = \varepsilon L^0 \bar{F}$$

with initial value $\bar{F}(x, y, 0) = \gamma(x, y)$ and

$$L^0 = \frac{1}{T} \int_0^T L(t) dt.$$

In [Krol91] it is proved that $||F - \bar{F}||_\infty = O(\varepsilon)$ on the long timescale $1/\varepsilon$. For the corresponding approximation $\bar{C}$ of C, we have the same estimate. In [Krol91] also a number of extensions of the theory are indicated.

1.2.4 Reactions and sources

An extension with interesting aspects is to consider reactions of chemicals or sediment using a reaction term $f(C)$. Secondly it is natural to include localized sources indicated by $B(x, y, t)$ which in the case of tidal basins can be interpreted as periodic dumping of chemicals or sediment in the basin. Following [HKV] the equation becomes

$$\frac{\partial C}{\partial t} + v_0.\nabla C + \varepsilon v_1.\nabla C = \varepsilon \Delta C + \varepsilon f(C) + \varepsilon B(x, y, t),\ t \geq 0. \tag{1.7}$$

with given initial condition $C(x, y, 0) = \gamma(x, y)$. The reaction term will in general be nonlinear, for instance $f(C) = aC^2$ or $f(C) = aC^5$, depending on the type of reaction. $B(x, y, t)$ is periodic in t. Using again variation of constants, we obtain from eq. (1.7) a perturbation equation in the same way as shown above, but with a more complicated operator $L(t)$.

As the tidal period of $v_0(x, y, t)$ is near to 12 hours, it is natural to assume a common period T with the dumping process indicated by $B(x, y, t)$. Averaging produces an approximation $\bar{C}$ of the solution C of the initial value problem for eq. (1.7). Interestingly, the result is stronger than in the case without the source term. One can prove that $\bar{C}$ converges to the solution $\bar{C}_0$ of a time-independent boundary value problem, while C converges to a $T-$periodic solution which is $\varepsilon-$close to $\bar{C}_0$ *for all time*. The proof is based on a maximum principle and the use of suitable sub- and supersolutions of eq. (1.7). For details see [HKV].

1.3 Two basic normal form theorems

Consider the semilinear initial value problem

$$\frac{dw}{dt} + \mathcal{A}w = \varepsilon f(w, t, \varepsilon), \quad w(0) = w_0, \tag{1.8}$$

where $-\mathcal{A}$ generates a uniformly bounded C_0-group $G(t), -\infty < t < +\infty$, on the Banach space X. We assumed the presence of a group instead of a semi-group as our attention will now be turned at hyperbolic problems.

We assume the usual regularity conditions:

- f is continuously differentiable and uniformly bounded on $\bar{D} \times [0, \infty) \times [0, \varepsilon_0]$, where D is an open, bounded set in X.
- f can be expanded with respect to ε in a Taylor series, at least to some order.

The group $G(t)$ generates a generalized solution of eq. (1.8) as a solution of the integral equation:

$$w(t) = G(t)w_0 + \varepsilon \int_0^t G(t-s) f(w(s), s, \varepsilon) ds.$$

Using the variation of constants transformation $w(t) = G(t)z(t)$ for eq. (1.8), we find the so-called standard form (see [SVM] or [Verhulst2005])

$$\frac{dz}{dt} = \varepsilon F(z, s, \varepsilon), \;\; F(z, s, \varepsilon) = G(-s) f(G(s)z, s, \varepsilon). \tag{1.9}$$

In what follows we assume that $F(z, s, \varepsilon)$ is an almost-periodic function in a Banach space, satisfying Bochner's criterion, see for instance [Verhulst2005]. The average F^0 is defined by:

$$F^0(z) = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(z, s, 0) ds. \tag{1.10}$$

Applying normalization by the averaging transformation

$$z(t) = v(t) + \varepsilon \int_0^t \left(F(v, s, 0) - F^0(v) \right) ds, \;\; v(0) = w_0, \tag{1.11}$$

produces the normal form equation

$$\frac{dv}{dt} = \varepsilon F^0(v) + O(\varepsilon^2)$$

with $O(\varepsilon^2)$-term still time-dependent. There are at least two problems here: the generalized Fourier spectrum of the almost-periodic function F contains an infinite number of frequencies and the integral in eq. (1.11) may not be bounded for all time as is the case for periodic functions.

1.3.1 Averaging theorem

The averaging approximation $\bar{z}(t)$ of $z(t)$ is obtained by omitting the $O(\varepsilon^2)$-terms:

$$\frac{d\bar{z}}{dt} = \varepsilon F^0(\bar{z}),\ \bar{z}(0) = w_0. \tag{1.12}$$

Under these rather general conditions, [Buit93] (or [Verhulst2005]) provides the following theorem:

Theorem 1. *(general averaging)*
Consider eq. (1.8) and the corresponding $z(t)$, $\bar{z}(t)$ given by eqs. (1.9) and (1.12) under the basic conditions stated above. If $G(t)\bar{z}(t)$ exists in an interior subset of D on the timescale $1/\varepsilon$, we have $v(t) - \bar{z}(t) = o(1)$ and

$$z(t) - \bar{z}(t) = o(1)\,\text{as}\,\varepsilon \to 0$$

on the timescale $1/\varepsilon$. If $F(z, t, 0)$ is periodic in t, the error is $O(\varepsilon)$.

1.3.2 Approximations for all time

In the case of attraction, averaging-normalization leads to stronger approximation results. The results can be described as follows. Consider the initial value problem in a Banach space

$$\dot{x} = \varepsilon f(x, t),\ x(0) = x_0.$$

Suppose that we can average the vector field:

$$f^0(z) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(z, s)ds$$

and thus can consider the averaged equation

$$\dot{z} = \varepsilon f^0(z),\ z(0) = x_0.$$

We have the following result by Sanchez-Palencia ([SP1] and [SP2]):

Theorem 2. *Suppose that the vector fields f and f^0 are continuously differentiable and that $z = a$ is an asymptotically stable critical point (in linear approximation) of the averaged equation. If x_0 lies within the domain of attraction of a, we have*

$$x(t) - z(t) = o(1) \,\text{as}\, \varepsilon \to 0$$

for $t \geq 0$. If the vector field f is periodic in t, the error is $O(\varepsilon)$ for all time.

1.4 Normal forms for hyperbolic equations

A straightforward application is to consider semilinear initial value problems of hyperbolic type,

$$u_{tt} + Au = \varepsilon f(u, u_t, t, \varepsilon), \quad u(0) = u_0, u_t(0) = v_0, \tag{1.13}$$

where A is a positive, self-adjoint linear operator on a separable Hilbert space and f satisfies the basic conditions. In our applications later on, we will be concerned with the case that we have one space dimension and that for $\varepsilon = 0$ we have a linear, dispersive wave equation by choosing:

$$Au = -u_{xx} + u.$$

To make the relation with Eq. (1.8) explicit, one writes $u_1 = u, u_2 = u_t$ and

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= u_2, \\ \frac{\partial u_2}{\partial t} &= -Au_1 + \varepsilon f(u_1, u_2, t, \varepsilon). \end{aligned}$$

One uses the operator (with eigenvalues and eigenfunctions) associated with this system.

In particular and to focus ideas, consider the case of the boundary conditions $u(0, t) = u(\pi, t) = 0$.

In this case, a suitable domain for the eigenfunctions is $\{u \in W^{1,2}(0, \pi) : u(0) = u(\pi) = 0\}$. Here $W^{1,2}(0, \pi)$ is the Sobolev space consisting of functions $u \in L_2(0, \pi)$ that have first-order generalised derivatives in $L_2(0, \pi)$. The eigenvalues are $\lambda_n = \sqrt{n^2 + 1}, n = 1, 2, \cdots$ and the spectrum is nonresonant. The implication is that $F(z, s, 0)$ in expression (1.10) is almost-periodic.

Assume now for Eq. (1.13) homogeneous Dirichlet conditions or homogeneous Neumann conditions. The denumerable eigenvalues in this case are $\lambda_n = \omega_n^2$ and the corresponding eigenfunctions $v_n(x)$. Substitution of the expansion

$$u(x, t) = \sum u_n(t) v_n(x) \tag{1.14}$$

into eq. (1.13) and taking inner products with the eigenfunctions $v_n(x)$, produces the infinite set of coupled second-order equations

$$\ddot{u}_n + \omega_n^2 u_n = \varepsilon F(\mathbf{u}, t, \varepsilon), \tag{1.15}$$

with $\mathbf{u}$ representing the infinite set $u_n, \dot{u}_n$ with $n = 1, 2, 3, \ldots$ in the Dirichlet case, $n = 0, 1, 2, \ldots$ in the Neumann case.

We shall discuss the procedure for a few examples. The variation of constants transformation, introduced in the preceding sections, takes in the case of the infinite-dimensional system (1.15) the following form. The standard transformation $u_n, \dot{u}_n \to y_{n_1}, y_{n_2}$ of the form

$$\begin{aligned} u_n &= y_{n_1} \cos \omega_n t + \frac{y_{n_2}}{\omega_n} \sin \omega_n t, \\ \dot{u}_n &= -\omega_n y_{n_1} \sin \omega_n t + y_{n_2} \cos \omega_n t, \end{aligned}$$

is introduced in system (1.15), followed by averaging. An alternative transformation to the standard form, $u_n, \dot{u}_n \to r_n, \psi_n$, employs amplitude-phase coordinates:

$$u_n = r_n \cos(\omega_n t + \psi_n), \ \dot{u}_n = -r_n \omega_n \sin(\omega_n t + \psi_n). \tag{1.16}$$

In general, averaging leaves us with an infinite-dimensional system that may still be difficult to analyze. In principle however, it is simpler and will admit analysis.

In our analysis of hyperbolic PDEs, we will be interested in the case that we have a resonance between a finite number of modes k and that the infinite number of other, non-resonant modes are attracted to a stationary solution. To fix ideas, assume that these stationary states correspond with the trivial solutions of the modes as will be the case in our examples. The attraction is produced by dissipation.

With these assumptions, we shall split system (1.15) into two subsystems, a finite-dimensional resonant system and an infinite-dimensional non-resonant system.

1.5 Linear waves with parametric excitation

Consider the linear wave equation

$$u_{tt} - c^2 u_{xx} + \varepsilon^k \beta u_t + (\omega_0^2 + \varepsilon \gamma \phi(t)) u = 0, \ t \geq 0, 0 < x < \pi, \tag{1.17}$$

with boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$, small, periodic or almost-periodic parametric excitation $\varepsilon \gamma \phi(t)$ and small damping ($\beta > 0$); also $\omega_0 > 0$. The positive parameter $k \in \mathbb{N}$ indicates the size of the damping. For $\varepsilon = 0$ the model reduces to the dispersive wave equation of section 1.4. In [Rand] the experimental motivation for this model is discussed, for instance a line of coupled pendula with vertical (parametric) forcing or the linearized behavior of water waves in a vertically forced channel. Related mechanical problems can be found in [SeM].

1.5.1 Modal expansion

Using the eigenfunctions for the Neumann problem $v_n(x) = \cos nx$, and eigenvalues $\omega_n^2 = \omega_0^2 + n^2c^2,\ n = 0, 1, 2, \cdots$, we expand the solution as

$$u(x,t) = \sum_{0}^{\infty} u_n(t) \cos nx.$$

Taking L_2-inner products with $v_n(x)$ produces the infinite dimensional system

$$\ddot{u}_n + \omega_n^2 u_n = -\varepsilon^k \beta \dot{u}_n - \varepsilon\gamma u_n \phi(t),\ n = 0, 1, 2, \cdots, \tag{1.18}$$

with suitable initial conditions. System (1.18) is fully equivalent with eq. (1.17). Note that the normal mode solutions do satisfy system (1.18), enabling the existence of an infinite number of finite- and infinite-dimensional invariant manifolds of eq. (1.17). A question that remains is about the overall dynamics and another about the dynamics within the invariant manifolds. We will consider a number of cases to illustrate the subtleties involved.

1.5.2 The Mathieu case $\phi(t) = \cos 2t$, no resonance

We will show that if no basic frequency of the unperturbed modes, determined by the eigenvalues ω_n^2, resonates with the parametric frequency, all solutions will decay to zero if ε is small enough. The explicit condition for non-resonance is that for $n = 0, 1, 2, \cdots$

$$\omega_n^2 (= \omega_0^2 + n^2c^2) \neq m^2,\ m = 0, 1, 2, \cdots .$$

Assume $k = 1$.
In the case of non-resonance we have, after introducing variation of constants as in section 1.4 by $u_n, \dot{u}_n \to y_{n_1}, y_{n_2}$, the averaged normal form

$$\dot{y}_{n_1} = -\frac{1}{2}\varepsilon\beta y_{n_1} + O(\varepsilon^2),\ \dot{y}_{n_2} = -\frac{1}{2}\varepsilon\beta y_{n_2} + O(\varepsilon^2),\ n = 0, 1, 2, \cdots .$$

The solutions decay to first order to the trivial solution. Omitting the $O(\varepsilon^2)$-terms we obtain approximations for the solutions that are, according to theorem 2, valid for all time. We have explicitly

$$\begin{aligned} u_n(t) &= e^{-\frac{1}{2}\varepsilon\beta t}(u_n(0)\cos\omega_n t + \frac{\dot{u}_n(0)}{\omega_n}\sin\omega_n t) + o(1), \\ \dot{u}_n(t) &= e^{-\frac{1}{2}\varepsilon\beta t}(-u_n(0)\omega_n \sin\omega_n t + \dot{u}_n(0)\cos\omega_n t) + o(1), \end{aligned}$$

$n = 0, 1, 2, \cdots$, with the estimates $o(1)$ as $\varepsilon \to 0$ and *validity of the estimates for all positive time* ($t \geq 0$). For the energy of the modes of the system we have

$$E_n(t) = \frac{1}{2}(\dot{u}_n^2(t) + \omega_n^2 u_n^2(t)) = E_n(0)e^{-\varepsilon\beta t} + o(1)$$

for all time. This agrees with the standard theory for Mathieu equations.

What happens if the damping is smaller, $k > 1$? In this case we have to perform higher order averaging, to $O(\varepsilon^k)$. The results are qualitatively the same, but the attraction takes place on a longer timescale.

1.5.3 The Mathieu case $\phi(t) = \cos 2t$, one Floquet resonance

A nontrivial case arises if one of the eigenvalues equals 1 or is ε-close to it (this is called the first Floquet resonance), and there are no other accidental resonances. Suppose that $\omega_m^2 = 1 + \varepsilon d$, $m \neq 0$ and $k = 1$. The parameter d indicates the detuning from the resonance. Using averaging-normalization in amplitude-phase variables (1.16), we find after averaging, with some abuse of notation using the same r_n, ψ_n for the variables,

$$\begin{aligned}
\dot{r}_n &= -\varepsilon\frac{\beta}{2}r_n + O(\varepsilon^2),\ \ n \neq m,\\
\dot{\psi}_n &= O(\varepsilon^2),\ \ n \neq m,\\
\dot{r}_m &= \frac{1}{2}\varepsilon r_m(-\beta + \frac{\gamma}{2}\sin 2\psi_m) + O(\varepsilon^2),\\
\dot{\psi}_m &= \frac{1}{2}\varepsilon(d + \frac{\gamma}{2}\cos 2\psi_m) + O(\varepsilon^2)\ \ (m \neq 0).
\end{aligned}$$

The solution decays to the trivial solution if $\beta > |\gamma|/2$ (damping exceeds excitation). Suppose now that $2\beta/|\gamma| < 1$ with two solutions for ψ_m from

$$\sin 2\psi_m = \frac{2\beta}{\gamma}.$$

This value of ψ_m corresponds with a periodic solution if also

$$d + \frac{\gamma}{2}\cos 2\psi_m = 0.$$

This produces the condition

$$\beta^2 + d^2 = \frac{\gamma^2}{4},$$

representing the first order approximation to the well-known Foquet instability tongue in parameter space.

1.5.4 The case of quasi-periodic resonance

As we have started with an infinite-dimensional system, there is no end to the complications that may arise. Take for instance the case of a spectrum

containing the first Floquet resonance $\omega_m = 1$ and a detuned higher order resonance, for instance $\omega_j = 4 + \delta(\varepsilon)d$. There are no other resonances.
In this case, all except two modes decay to a neighborhood of the trivial solution. The two remaining modes are described by

$$\ddot{u}_m + \omega_m^2 u_m = -\varepsilon^k \beta \dot{u}_m - \varepsilon\gamma u_m \phi(t),$$
$$\ddot{u}_j + \omega_j^2 u_j = -\varepsilon^k \beta \dot{u}_j - \varepsilon\gamma u_j \phi(t).$$

The analysis follows again finite-dimensional Floquet theory and this decoupling is in fact typical for the linear parametric wave equation. For a survey of perturbation methods for such parametric resonance problems see [Verhulst2009].

1.6 Nonlinear waves with parametric excitation

Consider the wave equation

$$u_{tt} - c^2 u_{xx} + \varepsilon\beta u_t + (\omega_0^2 + \varepsilon\gamma\cos 2t)u = \varepsilon(au^2 + bu^3),\ t \geq 0, 0 < x < \pi, \quad (1.19)$$

with boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$, small, periodic parametric excitation $\varepsilon\gamma\cos 2t$ and small damping ($\beta > 0$); also $\omega_0 > 0$. For $\varepsilon = 0$ the model reduces again to the dispersive wave equation of section 1.4.
In contrast to the case of a linear PDE, we expect now modal interactions. It turns out, surprisingly enough, that this is in general not the case.

1.6.1 Modal expansion

Using as before the eigenfunctions for the Neumann problem $v_n(x) = \cos nx$, and eigenvalues $\omega_n^2 = \omega_0^2 + n^2c^2,\ n = 0, 1, 2, \cdots$, we expand the solution as

$$u(x,t) = \sum_0^\infty u_n(t)\cos nx.$$

Taking L_2-inner products with $v_n(x)$ produces the infinite dimensional system

$$\ddot{u}_n + \omega_n^2 u_n = -\varepsilon\beta\dot{u}_n - \varepsilon\gamma u_n \cos 2t + \varepsilon f_n(\mathbf{u}),\ n = 0, 1, 2, \cdots, \quad (1.20)$$

with suitable initial conditions; $\mathbf{u} = (u_0, u_1, u_2, \cdots)$. The nonlinear terms are quadratic and cubic with constant coefficients.

System (1.20) is fully equivalent with eq. (1.19). Note that the normal mode solutions *do not* satisfy system (1.20), so we have not apriori normal mode invariant manifolds of eq. (1.19). We will distinguish between the following cases:

- Wave speed and dispersion parameter c and ω_0 are $O(1)$ quantities with respect to ε.

- The wave speed c is $O(\varepsilon)$. In this case we have, assuming that ω_0 is an $O(1)$ quantity, for a finite number of modes the $1 : 1 : 1 : \cdots$-resonance. This case has been discussed in [BMV].
- The dispersion is small: $\omega_0 = O(\varepsilon)$. In this case the system (1.20) is fully resonant. This problem is unsolved, see for instance the discussion in [Verhulst2005].

1.6.2 Averaging-normalization

Assuming that c and ω_0 are $O(1)$ quantities with respect to ε, we will carry out the averaging process. The fact that the spatial dimension is 1 means that all eigenvalues are single; this simplifies the averaging-normalization.

1.6.3 One Floquet resonance

Assume that one of the eigenvalues is near-resonant with respect to parametric excitation, for instance

$$\omega_0^2 = 1 + \varepsilon d,$$

with d the detuning. The equations of motion become for $n = 0, 1, 2, \cdots$

$$\ddot{u}_n + (1 + n^2c^2)u_n = -\varepsilon(du_n + \beta\dot{u}_n + \gamma u_n \cos 2t) + \varepsilon f_n(\mathbf{u}). \tag{1.21}$$

Assume that there are no other resonances between the frequencies ω_n. Introducing again amplitude-phase variables (1.16), we find after averaging, with some abuse of notation using the same r_n, ψ_n for the variables:

$$\begin{aligned}
\dot{r}_0 &= \frac{1}{2}\varepsilon r_0(-\beta + \frac{1}{2}\gamma \sin 2\psi_0),\\
\dot{\psi}_0 &= \frac{1}{2}\varepsilon(d + \frac{1}{2}\gamma\cos 2\psi_0 - \frac{3}{4}br_0^2 - \frac{3}{4}b\sum_{k=1}^{\infty} r_k^2),\\
\dot{r}_n &= -\frac{1}{2}\varepsilon\beta r_n,\ n = 1, 2, \cdots,\\
\dot{\psi}_n &= \varepsilon b h_n(\mathbf{u}).
\end{aligned}$$

The righthand sides h_n are quadratic in $u_0, u_1, \cdots$. The modes $n = 1, 2, \cdots$ are exponentially decreasing, nontrivial behavior can take place in mode 0 governed by

$$\begin{aligned}
\dot{r}_0 &= \frac{1}{2}\varepsilon r_0(-\beta + \frac{1}{2}\gamma \sin 2\psi_0),\\
\dot{\psi}_0 &= \frac{1}{2}\varepsilon(d + \frac{1}{2}\gamma\cos 2\psi_0 - \frac{3}{4}br_0^2).
\end{aligned}$$

For a critical point to exist, we have the condition (as in subsection 1.5.3)

$$2\beta/|\gamma| < 1.$$

The solution decays to the trivial solution if $\beta > |\gamma|/2$ (damping exceeds excitation). Suppose now that we have solutions for ψ_m from

$$\sin 2\psi_m = \frac{2\beta}{\gamma}.$$

This critical value of ψ_m corresponds with a periodic solution if also

$$d + \frac{1}{2}\gamma \cos 2\psi_0 - \frac{3}{4} b r_0^2 = 0.$$

This is a different situation from the linear case discussed earlier, as this condition also determines r_0. Suppose we find a positive solution for r_0^2. For the eigenvalues of the critical point we find

$$\lambda_{1,2} = -\beta \pm \sqrt{5\beta^2 - \gamma^2 - 2d\gamma \cos 2\psi_0}.$$

From the existence condition we have $\gamma^2 > 4\beta^2$, so at exact Floquet resonance ($d = 0$), we have stability of the periodic solution. If $4\beta^2 < \gamma^2 < 5\beta^2$, the critical point is a node, if $\gamma^2 > 5\beta^2$, the critical point is a focus and around the stable periodic solution the solutions are spiralling in.
The picture changes if $d \neq 0$ and large enough.

1.6.4 Additional low-order resonances

Assuming we have the $1 : 2$ parametric resonance in mode 0, the conditions for a combined low-order resonance in system (1.21) are

$$\frac{1}{1 + m^2 c^2} = \frac{1}{4}, \frac{1}{9},$$

for certain mode m. We find respectively $m^2 c^2 = 3$ and $m^2 c^2 = 8$. These choices produce a $1 : 2$- and a $1 : 3$-resonance respectively.

Analysis of the possibility of a first- or second-order resonance in three degrees of freedom according to the resonance classification in [SVM] produces no positive results, so we will consider two degrees of freedom only. It is no restriction to choose $m = 1$ and we will have three frequencies: ω_0, ω_1 and the frequency of parametric excitation 2.

1.6.5 Combined Floquet and 1 : 2-resonance

We assume

$$\omega_0^2 = 1 + \varepsilon d_1,\ c^2 = 3 + \varepsilon(d_2 - d_1),\ \omega_1^2 = 4 + \varepsilon d_2,$$

with d_1, d_2 indicating the detunings of the three frequencies. The equations of motion from system (1.21) which may show modal interaction become:

$$\ddot{u}_0 + \omega_0^2 u_0 = -\varepsilon\beta\dot{u}_0 - \varepsilon\gamma u_0 \cos 2t + \varepsilon a(u_0^2 + \frac{1}{2}u_1^2) + \varepsilon b u_0(u_0^2 + \frac{3}{2}u_1^2),$$
$$\ddot{u}_1 + \omega_1^2 u_1 = -\varepsilon\beta\dot{u}_1 - \varepsilon\gamma u_1 \cos 2t + \varepsilon a 2u_0 u_1 + \varepsilon b u_1(3u_0^2 + \frac{3}{4}u_1^2).$$

We find after averaging, using the same r_n, ψ_n for the variables:

$$\dot{r}_0 = \frac{1}{2}\varepsilon r_0(-\beta + \frac{1}{2}\gamma \sin 2\psi_0),$$
$$\dot{\psi}_0 = \frac{1}{2}\varepsilon(d_1 + \frac{1}{2}\gamma\cos 2\psi_0 - \frac{3}{4}br_0^2 - \frac{3}{4}br_1^2),$$
$$\dot{r}_1 = -\frac{1}{2}\varepsilon\beta r_1,$$
$$\dot{\psi}_1 = \varepsilon\frac{1}{4}(d_2 - \frac{1}{2}b(3r_0^2 + \frac{9}{8}r_1^2)).$$

We conclude that, because of symmetry in the equations of motion, the 1 : 2-resonance is degenerate in this case. This symmetry degeneration is described in detail in [TuV].

1.6.6 Combined Floquet and 1 : 3-resonance

We can repeat the analysis, assuming

$$\omega_0^2 = 1 + \varepsilon d_1,\ c^2 = 8 + \varepsilon(d_2 - d_1),\ \omega_1^2 = 9 + \varepsilon d_2.$$

As for the 1 : 2-resonance, we find that the 1 : 3-resonance in this case is degenerate because of symmetry. The only active resonance for system (1.21) takes place in mode 0.

1.7 Discussion

1. We conclude that after an interval of time, asymptotically larger than $1/\varepsilon$ (for instance $1/\varepsilon^2$), the righthand sides of the infinite-dimensional, non-resonant systems which we encountered in sections 1.5 and 1.6, become $o(1)$. Starting with $o(1)$ initial conditions, the non-resonant modes remain $o(1)$.
2. The manifold where the fast dynamics takes place is almost-invariant. We conjecture that very small fluctuations are possible for the higher order modes, arising from the presence of higher order resonance manifolds containing stable and unstable periodic solutions with corresponding intersecting stable and unstable manifolds. These resonance manifolds are of very small size and the analysis to describe them is subtle. For an analysis of such resonance manifolds in two degrees of freedom Hamiltonian systems, see [TuV].
   A related discussion, for a different PDE, can be found in [Wit].

3. The parametrically excited wave equation with dispersion and wave speed independent of ε, displays a remarkable reduction to low-dimensional (one mode) behavior. This becomes clear by averaging-normalization. The equation is also of practical interest; applications are cited in [Rand]. A number of the phenomena we found, periodic and quasi-periodic solutions, are stable and in this way open for experimental investigation.

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