Systems with fast limit cycles and slow interaction

Ferdinand Verhulst[1]

We will review the theory of slow-fast systems that started with papers by Tykhonov, Pontryagin, Levinson, Anosov, Fenichel and other scientists. After this review we focus on systems with limit cycles. The Pontryagin-Rodygin theorem for slow-fast systems has an ingenious proof; also it has as advantage that it can be applied if the slow manifolds of the slow-fast system are all unstable. A serious disadvantage is that for application we have to know the fast solutions explicitly with the slow part in the form of parameters. Another disadvantage is the relatively short timescale where the results are valid. In practice there are very few cases where the theorem applies. However, the Pontryagin-Rodygin idea can be used again on assuming that the fast limit cycle arises in higher order approximation; this allows an approximation approach to study the slow motion. At this point we have still a restricted timescale but extension is then possible by looking for continuation on stable, in particular slow manifolds. We will demonstrate this extension of the theory by studying various types of self-excited, coupled slow and fast Van der Pol oscillators.

1. Introduction

We will be concerned with slow-fast systems of the form:

$$\varepsilon \frac{dy}{dt} = f(t, x, y) + \varepsilon R_1,\ \frac{dx}{dt} = g(t, x, y) + \varepsilon R_2. \tag{1}$$

$x \in \mathbb{R}^n, y \in \mathbb{R}^m, \varepsilon$ is a small, positive parameter.

With excuses to people left out we mention the following founding scientists:

1. Tikhonov (1952)
2. Flatto and Levinson (1955)
3. Anosov (1960)
4. Pontryagin - Rodygin (1960)
5. Tikhonov expansions, Vasil'eva (1963), O'Malley (1968)

---

[1]Proc. Conference: DYNAMICAL SYSTEMS – THEORY AND APPLICATIONS T2, Applicable solutions in non-linear dynamical systems pp. 511-422 (Jan Awrejcewicz, Marek Kaźmierczak, Pawel Olejnik, eds.) Łodz, Poland, Dec. 2019.

6. Fenichel (1971 - 1979)

7. Jones and Kopell (1994)

The early developments in the period 1952-1970 were concerned with asymptotic approximations and periodic solutions. In the second period new developments were stimulated by invariant manifold theory and new dynamics results.

2. Early results

We start with an example of (1): a prey-predator system with unequal interaction.
The population N preyed upon is abundant with respect to the predators P; the prey-population grows fast (we will return to the example later on):

$$\varepsilon \dot{N} = r(t)N\left(1 - \frac{N}{K(t)}\right) - \varepsilon NP, \; \dot{P} = cNP - dP - P^2. \tag{2}$$

The growth rate $r(t)$ and carrying capacity $K(t)$ are positive for $t \geq 0$ and T-periodic, c and d are positive constants; we have $N, P \geq 0$. Think for $r(t), K(t)$ of seasonal variations.
Tikhonov [13] studied system (1) by putting $\varepsilon = 0$ and supposing that $y = \phi(t,x)$ is an isolated, asymptotically stable root of the equation $f(t,x,y) = 0$ (with t, x as parameters). He proved then that $y(t)$ jumps fast to the 'slow' solution of the equation:

$$\frac{dx}{dt} = g(x, \phi(t,x)). \tag{3}$$

The resulting approximation is valid on a time-interval $O(1)$ for the slow motion of x. Examples show that without further assumptions this result is optimal. Vasil'eva [14] and O'Malley [10] improved Tikhonov's theorem in a practical way by assuming in addition that the stability of the root $y = \phi(t,x)$ is exponential:

$$ReSp \frac{\partial f(t,x,y)}{\partial y} |(y = \phi(t,x)) < 0. \tag{4}$$

With this assumption the jumps take time $O(\varepsilon)$ and we can obtain asymptotic expansions in ε valid on time intervals $O(1)$. Vasil'eva [14] uses matched asymptotic expansions, O'Malley [10] introduces multiple timescale expansions for this singular perturbation problem.

Qualitative results regarding the existence of periodic solutions of system (1) were obtained by Flatto and Levinson [8] and by Anosov [1]. Suppose that the slow equation (3) contains an isolated T_0-periodic solution, with only one multiplier 1, then the original system contains a T_ε-periodic solution with $T_\varepsilon \to T_0$ as $\varepsilon \to 0$. The theorems in [8, 1] show minor differences in formulation, the important step is that they involve structural stability of the asymptotic phenomena, they anticipate in a sense Fenichel's results.

An interesting quantitative result was produced by Pontryagin and Rodygin [11]. Consider again the slow-fast system (1) in autonomous form and assume:
1. For x fixed the fast equation for y contains an exponentially stable limit cycle of the form $y^*(\tau, x)$ with period $T(x)$ (x a parameter, $\tau = t/\varepsilon$).
2. For fixed positive T_1, T_2 we have $T_1 \leq T(x) \leq T_2$.
3. Replace t by $\varepsilon\tau$ and average $g(x, y)$ over the limit cycle.
Then with corresponding initial values we have with $\bar{x}(\tau)$ from the averaged equation:

$$y(\tau, \varepsilon) - y^*(\tau, \bar{x}) = O(\varepsilon), x(\tau, \varepsilon) - \bar{x}(\tau) = O(\varepsilon), \; \delta(\varepsilon) \leq \tau \leq L/\varepsilon. \tag{5}$$

For the Pontryagin-Rodygin theorem, weak and strong points are:

- The timescale of validity is restricted to $O(1/\varepsilon)$ in τ, so $O(1)$ in t.
- To apply the theorem we have to know $x^*(\tau, y)$ with period $T(y)$ explicitly; this will be rarely the case.
- Strong point: the result is also useful if the system contains an unstable slow manifold.

So in practice the Pontryagin-Rodygin theorem is useless but, as we shall see, it can be an inspiration for a related approach. First an example to show the weak points:

$$\begin{cases} \varepsilon\dot{x}_1 &= x_2 + y(x_1 - \frac{1}{3}x_1^3), \\ \varepsilon\dot{x}_2 &= -x_1 - yx_1^2, \\ \dot{y} &= 1 - y, \; 0 \leq y \leq 1. \end{cases} \tag{6}$$

There are two roots of the fast equation, called slow manifolds, both are unstable. Differentiation of the 1st equation with $x_1 = x$ produces:

$$\frac{d^2x}{d\tau^2} + x + yx^2 = yx'(1 - x^2) + \varepsilon(1 - y)(x - \frac{1}{3}x^3). \tag{7}$$

Doelman and F.V. [3] showed that at parameter value $y = 1/\sqrt{7}$ the limit cycle vanishes; see for an illustration fig. 1.

3. Modern results

Fenichel [4, 5, 6, 7] formulated geometric singular perturbation theory with many consequences for the existence of slow manifolds leading to a reformulation of quantitative results. The theory was introduced for autonomous systems where the geometric interpretation is easier, but it can be generalised to systems with periodic coefficients in time. Consider system (1) with the vector fields time-independent. Put $\varepsilon = 0$ and suppose as in Tykhonov's

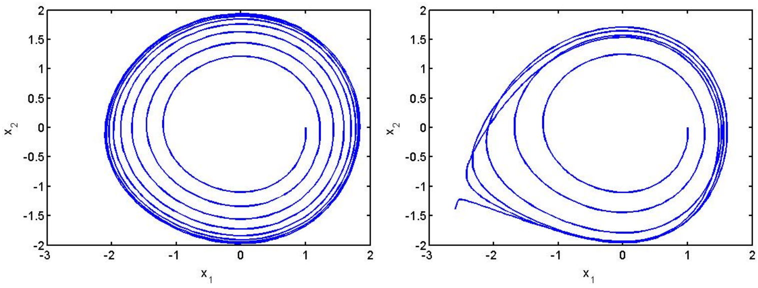

Figure 1. Left eq. (7) if y is constant, right varying y.

theorem that $y = \phi(x)$ is an isolated root of the (fast) y-equation with x as parameter. Fenichel: If $y = \phi(x)$ defines a compact manifold M_0 and

$$Re\, Sp \frac{\partial f(x,y)}{\partial y} |(y = \phi(x)) \neq 0$$

then system (1) contains an invariant manifold M_ε ε-close to M_0.
The dynamics on the slow manifold M_ε is approximated by the flow of the slow equation $dx/dt = g(x, y(x))$.
The theorem does not require asymptotic stability of the root, it has to be structurally stable and so be stable or unstable. Note that the existence and approximation of periodic solutions in a slow manifold is much easier than in the theorems of Flatto-Levinson and Anosov as we "got rid of" the fast dynamics and can restrict our attention to the slow manifold equation. One drawback is that the overall condition of compactness and the spectral assumption is sometimes not met in applications.

For an example of obtaining existence and approximation of a periodic solution we return to the time-periodic case of system (2), a prey-predator system with unequal growth of prey N. The growth rate $r(t)$ and carrying capacity $K(t)$ are T-periodic. The slow manifolds SM_1, SM_2 are described by: $SM_1 : N = 0$, $SM_2 : N = K(t)$. SM_1 is unstable, SM_2 is stable. $N = K(t)$ is a first order periodic approximation of the prey population in SM_2. We will find co-existence of prey and predator from the next approximation (omitted here).

Discussion of the Tikhonov theorem versus Fenichel.
The Tikhonov theorem can describe transient motion. Consider the Van der Pol equation with $\mu \gg 1$:

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}, \tag{8}$$

with Liénard transformation producing:

$$\frac{1}{\mu}\dot{x} = z + x - \frac{1}{3}x^3, \; \dot{z} = -\frac{1}{\mu}x. \tag{9}$$

With $1/\mu = \varepsilon$ we identify root $z = -x + \frac{1}{3}x^3$. Along the stable parts of the cubic curve

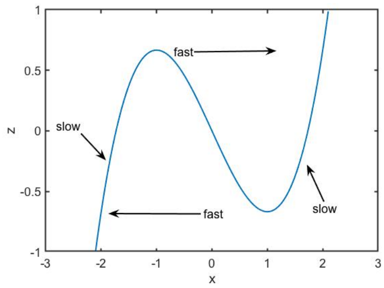

Figure 2. Slow manifolds and fast motion for Van der Pol relaxation.

we have slow motion, during a cycle 2 fast jumps take place. We obtain the well-known relaxation oscillation with slow motions periodically followed by fast transitions from one stable manifold to the other one; $\mu = 20$. The jumps are described by Tikhonov's theorem, choosing compact parts of the stable cubic curves we can apply Fenichel; the complete picture arises by patching parts together using singular perturbation theory.

4. Slow-fast limit cycles at higher order

Consider again the system of ODEs (1). Assume that leaving out the $\varepsilon R_1, R_2$ and $O(\varepsilon^2)$ terms, the system does not contain a limit cycle but suppose that, adding these perturbations, one or more limit cycles emerge by a bifurcation. A well-known example is the Hopf bifurcation arising in the Van der Pol-equation. Suppose we can solve the system when omitting the perturbation terms to construct the so-called variational equations in $\tau = t/\varepsilon$:

$$\begin{cases} r'_f &= \varepsilon F_1(r_f, \phi_f, r_s, \phi_s) + O(\varepsilon^2), r'_s = \varepsilon^2 G_1(r_f, \phi_f, r_s, \phi_s) + O(\varepsilon^3), \\ \phi'_f &= \Omega_1 + \varepsilon F_2(r_f, \phi_f, r_s, \phi_s) + O(\varepsilon^2), \phi'_s = \varepsilon\Omega_2 + \varepsilon^2 G_2(r_f, \phi_f, r_s, \phi_s) + O(\varepsilon^3), \end{cases} \tag{10}$$

where $'$ represents differentiation with respect to τ, the index f indicates a fast variable, s indicates slow. The dimensions of r_f, ϕ_f, r_s, ϕ_s depend on system (1). In this way we have reduced the system to a quasi-periodic system where averaging over angles is possible, see [12] ch. 5. If the limit cycle of the fast equation is asymptotically stable it is natural to average over the fast limit cycle which means averaging over ϕ_f. Averaging over angles involves the analysis of resonance manifolds. This theory has many aspects that are difficult to capture in one theorem. We summarize the procedure as it runs in general.

1. Average over the fast angles ϕ_f with attention to the possible presence of resonance manifolds. We obtain a system without angle ϕ_f.

2. Consider the resonance manifolds separately, see [12] ch. 5.

3. We can rescale $\tau \to t$ or alternatively use the slow angles ϕ_s as a timelike variable. The resulting slow-fast system may contain a slow manifold M.

4. Consider the dynamics in the slow manifold M by eliminating r_f. We can average over the slow angle ϕ_s in the slow manifold. Critical points will correspond with periodic solutions or tori producing interesting phenomena in the original slow-fast system (10).

5. This procedure enables us to extend the description of the dynamics to 3 timescales: $t/\varepsilon, t$ and εt, expressed in the corresponding angles. Some aspects of this analysis corresponds with the treatment of slow-fast systems in [2].

In actual applications we will meet problems of normal hyperbolicity of slow manifolds and certain degeneracies. This is not uncommon in applications as in practice symmetries and specific parameter values may destroy aspects of the general mathematical theory.

Example 1

Interaction of slow-fast Van der Pol limit cycles.
We assume $(x_1, x_2) \in \Gamma_1, (y_1, y_2) \in \Gamma_2$ with $\Gamma_1, \Gamma_2 \subset \mathbb{R}^2$, compact subsets containing the origin. This produces a relatively simple interaction problem as we have only one fast angle (or time). Consider the system with parameters a_1, a_2, positive frequency ω, parameter $\mu > 0$:

$$\begin{cases} x_1' &= x_2 + \varepsilon(x_1 - \frac{1}{3}x_1^3),\ x_2' = -x_1 - \varepsilon a_1 y_1^2 x_1', \\ \dot{y}_1 &= y_2 + \mu(y_1 - \frac{1}{3}y_1^3),\ \dot{y}_2 = -\omega^2 y_1 - \mu a_2 x_1^2 \dot{y}_1. \end{cases} \tag{11}$$

Differentiation is respectively with respect to $\tau = t/\varepsilon$ and t. The slow manifold of system (11) is given by the plane $x_1 = x_2 = 0$. If $\varepsilon = 0$ the slow manifold is Lyapunov (neutrally) stable. It contains an asymptotically stable slow limit cycle corresponding with the Van der Pol-oscillator in (y_1, y_2) coordinates. Another invariant manifold is given by the plane $y_1 = y_2 = 0$ where a fast Van der Pol-oscillator is found in (x_1, x_2) coordinates. The question of interest is the interaction of the two oscillators outside the 2 coordinate planes.

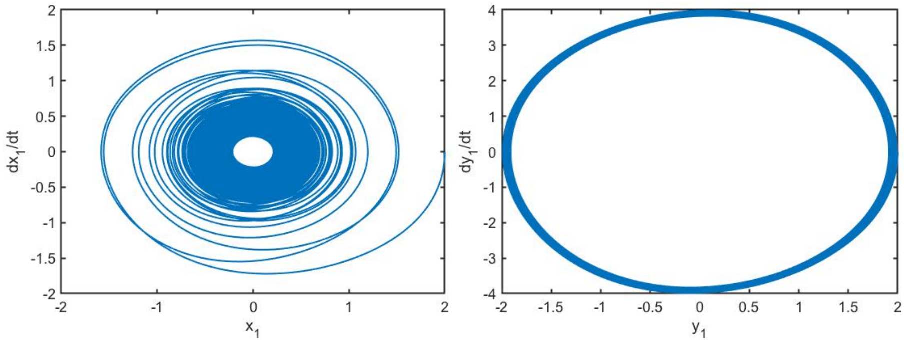

Figure 3. Numerical approximation of a stable torus of system (12). Left the projection of $x_1(t), \dot{x}_1(t)$ (fast), right the projection of $y_1(t), \dot{y}_1(t)$ (slow) with $x_1(0) = 2, \dot{x}_1(0) = 0, y_1(0) = 2, \dot{y}_1(0) = 0$, $\varepsilon = \mu = 0.1, a_1 = 0.5, a_2 = 0.3, \omega = 2$. The choice of parameters corresponds with the analysis of the slow manifold (16) leading to a stable torus.

The more familiar (equivalent) scalar equations in respectively τ and t are:

$$\begin{cases} \frac{d^2x_1}{d\tau^2} + x_1 &= \varepsilon(1 - x_1^2 - a_1 y_1^2)\frac{dx_1}{d\tau}, \\ \frac{d^2y_1}{dt^2} + \omega^2 y_1 &= \mu(1 - y_1^2 - a_2 x_1^2)\frac{dy_1}{dt}. \end{cases} \tag{12}$$

We transform to amplitude-angle variables: $x_1 = r_1 \sin\phi_1, x_1' = r_1\cos\phi_1, y_1 = r_2 \sin\phi_2, \dot{y}_1 = r_2\omega\cos\phi_2$. The equations from system (12) produce with differentiation with respect to τ:

$$\begin{cases} r_1' &= \varepsilon\cos^2\phi_1[1 - r_1^2\sin^2\phi_1 - a_1 r_2^2 \sin^2\phi_2], \\ \phi_1' &= 1 + \varepsilon\sin\phi_1\cos\phi_1[1 - r_1^2\sin^2\phi_1 - a_1 r_2^2\sin^2\phi_2], \\ r_2' &= \varepsilon\mu r_2\cos^2\phi_2[1 - r_2^2\sin^2\phi_2 - a_2 r_1^2\sin^2\phi_1], \\ \phi_2' &= \varepsilon\omega - \varepsilon\mu\sin\phi_2\cos\phi_2[1 - r_2^2\sin^2\phi_2 - a_2 r_1^2\sin^2\phi_1]. \end{cases} \tag{13}$$

We can average over the fast angle ϕ_1 to obtain the approximating system:

$$\begin{cases} \tilde{r}_1' &= \frac{1}{2}\varepsilon[1 - \frac{1}{4}\tilde{r}_1^2 - a_1\tilde{r}_2^2\sin^2\tilde{\phi}_2], \\ \tilde{r}_2' &= \varepsilon\mu\tilde{r}_2\cos^2\tilde{\phi}_2[1 - \tilde{r}_2^2\sin^2\tilde{\phi}_2 - \frac{a_2}{2}\tilde{r}_1^2], \\ \tilde{\phi}_2' &= \varepsilon\omega - \varepsilon\mu\sin\phi_2\cos\phi_2[1 - r_2^2\sin^2\phi_2 - \frac{a_2}{2}\tilde{r}_1^2]. \end{cases} \tag{14}$$

Starting at the same initial values as r_1, r_2, ϕ_2, the approximations $\tilde{r}_1, \tilde{r}_2, \tilde{\phi}_2$ have validity $O(\varepsilon)$ on the timescale $1/\varepsilon$ in τ (as in the Pontryagin-Rodygin theorem). We conclude from system (14) that the only possibility to quench the fast oscillator *completely* is if $a_1 > 0$. We will consider the case $\mu = O(\varepsilon)$.

Put $\mu = \varepsilon\mu_0$ with μ_0 a positive constant independent of ε. In this case $\tilde{\phi}_2$ is timelike with respect to $\tilde{r}_2$ and we can reformulate system (14) as:

$$\begin{cases} \frac{d\tilde{r}_1}{d\tilde{\phi}_2} &= \frac{1}{2\omega}[1 - \frac{1}{4}\tilde{r}_1^2 - a_1\tilde{r}_2^2\sin^2\tilde{\phi}_2] + O(\varepsilon), \\ \frac{d\tilde{r}_2}{d\tilde{\phi}_2} &= \varepsilon\frac{\mu_0}{\omega}\tilde{r}_2\cos^2\tilde{\phi}_2[1 - \tilde{r}_2^2\sin^2\tilde{\phi}_2 - \frac{a_2}{2}\tilde{r}_1^2] + O(\varepsilon^2). \end{cases} \tag{15}$$

System (15) has a slow manifold given by

$$1 - \frac{1}{4}\tilde{r}_1^2 - a_1\tilde{r}_2^2\sin^2\tilde{\phi}_2 = 0. \tag{16}$$

The slow manifold is stable if we find a positive solution for $\tilde{r}_1$. Eliminating $\tilde{r}_1$ with (16) we find for the dynamics in the slow manifold:

$$\frac{d\tilde{r}_2}{d\tilde{\phi}_2} = \varepsilon\frac{\mu_0}{\omega}\tilde{r}_2\cos^2\tilde{\phi}_2[1 - 2a_2 - \tilde{r}_2^2(1 - 4a_1a_2)\sin^2\tilde{\phi}_2] + O(\varepsilon^2). \tag{17}$$

There is no obstruction to average again, this time over $\tilde{\phi}_2$. We find in the slow manifold the equation:

$$\frac{d\tilde{\tilde{r}}_2}{d\tilde{\phi}_2} = \varepsilon\frac{\mu_0}{2\omega}\tilde{\tilde{r}}_2[1 - 2a_2 - \frac{1}{4}(1 - 4a_1a_2)\tilde{\tilde{r}}_2^2]. \tag{18}$$

The solutions of eq. (18) are valid to $O(\varepsilon)$ on an $1/\varepsilon$ timescale in $\tilde{\phi}_2$, which means a long timescale in t. A periodic solution in $\tilde{\phi}_2$ arises if

$$\tilde{\tilde{r}}_2^2 = 4\frac{1 - 2a_2}{1 - 4a_1a_2} \tag{19}$$

with positive righthand side. If in addition both $1 - 2a_2$ and $1 - 4a_1a_2$ are positive, the periodic solution $\tilde{\tilde{r}}_2(\tilde{\phi}_2)$ is stable. A corresponding approximation in the slow manifold for $\tilde{r}_1$ can be found from eq. (16). The approximations for r_1 and r_2 represent a torus in 4-space; a finite-dimensional torus will contain quasi-periodic solutions. See fig. 3 for an illustration.

The analysis of coupled slow-fast Van der Pol-oscillators is more complicated if we have more than 2 oscillators. To avoid discussing too many cases we reduce the number of free parameters.

Example 2

The case of one fast and two slow Van der Pol-oscillators.

We will present an abbreviated account with an illustration. Consider the system of 3 scalar equations with parameters $a, b > 0$:

$$\begin{cases} x'' + x &= \varepsilon(1 - x^2 - by^2 - bz^2)x', \\ \ddot{y} + y &= \varepsilon(1 - ax^2 - y^2 - z^2)\dot{y}, \\ \ddot{z} + z &= \varepsilon(1 - ax^2 - y^2 - z^2)\dot{z}, \end{cases} \tag{20}$$

where again a prime indicates differentiation with respect to $\tau = t/\varepsilon$, a dot indicates differentiation with respect to time t. The coordinate planes correspond with invariant manifolds, we consider the dynamics outside the coordinate planes.

Using amplitude-angle variables and fast time τ also for the 2 slow equations we have the equivalent system:

$$\begin{cases} r_1' &= \varepsilon r_1 \cos^2\phi_1(1 - r_1^2\sin^2\phi_1 - br_2^2\sin^2\phi_2 - br_3^2\sin^2\phi_3), \\ \phi_1' &= 1 - \frac{\varepsilon}{2}\sin 2\phi_1(1 - r_1^2\sin^2\phi_1 - br_2^2\sin^2\phi_2 - br_3^2\sin^2\phi_3), \\ r_2' &= \varepsilon^2 r_2\cos^2\phi_2(1 - ar_1^2\sin^2\phi_1 - r_2^2\sin^2\phi_2 - r_3^2\sin^2\phi_3), \\ \phi_2' &= \varepsilon - \frac{\varepsilon^2}{2}\sin 2\phi_2(1 - ar_1^2\sin^2\phi_1 - r_2^2\sin^2\phi_2 - r_3^2\sin^2\phi_3), \\ r_3' &= \varepsilon^2 r_3\cos^2\phi_3(1 - ar_1^2\sin^2\phi_1 - r_2^2\sin^2\phi_2 - r_3^2\sin^2\phi_3), \\ \phi_3' &= \varepsilon - \frac{\varepsilon^2}{2}\sin 2\phi_3(1 - ar_1^2\sin^2\phi_1 - r_2^2\sin^2\phi_2 - r_3^2\sin^2\phi_3). \end{cases} \tag{21}$$

As before we can average over the fast angle ϕ_1 to find:

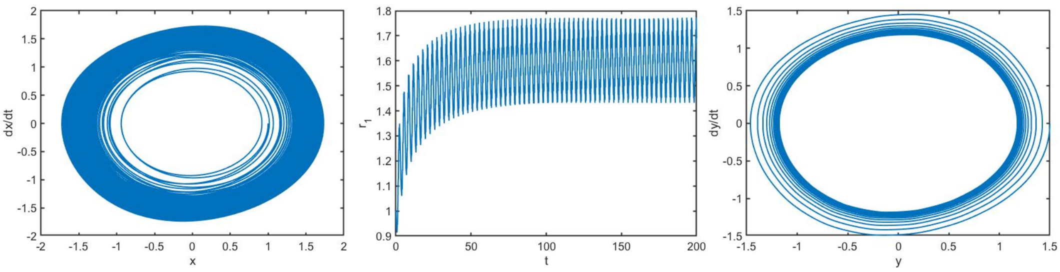

Figure 4. Numerical approximation of a stable torus of system (20). Left the projection of the phaseplane $x_1(t), \dot{x}_1(t)$ (fast), middle $r_1(t) = \sqrt{x^2(t) + \dot{x}^2(t)}$, right the projection of the phaseplane $y_1(t), \dot{y}_1(t)$ (slow) with $x_1(0) = 1, \dot{x}_1(0) = 0, y_1(0) = 1.5, \dot{y}_1(0) = 0, z(0) = 1.3, \dot{z}(0) = 0$, $\varepsilon = 0.1, a = b = 0.3$. The choice of parameters corresponds with the analysis of the slow manifold (27) leading to a stable torus.

$$\begin{cases} \tilde{r}_1' &= \frac{\varepsilon}{2}\tilde{r}_1(1 - \frac{1}{4}\tilde{r}_1^2 - 2b\tilde{r}_2^2\sin^2\tilde{\phi}_2 - 2b\tilde{r}_3^2\sin^2\tilde{\phi}_3), \\ \tilde{r}_2' &= \varepsilon^2\tilde{r}_2\cos^2\tilde{\phi}_2(1 - \frac{a}{2}\tilde{r}_1^2 - \tilde{r}_2^2\sin^2\tilde{\phi}_2 - \tilde{r}_3^2\sin^2\tilde{\phi}_3), \\ \tilde{\phi}_2' &= \varepsilon - O(\varepsilon^2), \\ \tilde{r}_3' &= \varepsilon^2\tilde{r}_3\cos^2\tilde{\phi}_3(1 - \frac{a}{2}\tilde{r}_1^2 - \tilde{r}_2^2\sin^2\tilde{\phi}_2 - \tilde{r}_3^2\sin^2\tilde{\phi}_3), \\ \tilde{\phi}_3' &= \varepsilon - O(\varepsilon^2). \end{cases} \tag{22}$$

System (22) has in τ a relatively fast amplitude r_1, fast angles ϕ_2, ϕ_3 and slow amplitudes r_2, r_3. We will average over the fast angles outside the resonance domain determined by $\tilde{\phi}_2' = \tilde{\phi}_3'$. We find for the slow amplitudes:

$$\begin{cases} \tilde{r}_2' &= \frac{\varepsilon^2}{2}\tilde{r}_2(1 - \frac{a}{2}\tilde{r}_1^2 - \frac{1}{4}\tilde{r}_2^2 - \frac{1}{2}\tilde{r}_3^2), \\ \tilde{r}_3' &= \frac{\varepsilon^2}{2}\tilde{r}_3(1 - \frac{a}{2}\tilde{r}_1^2 - \frac{1}{2}\tilde{r}_2^2 - \frac{1}{4}\tilde{r}_3^2). \end{cases} \tag{23}$$

Interesting dynamics may happen if the following 2 ellipsoids intersect:

$$\frac{a}{2}\tilde{r}_1^2 + \frac{1}{4}\tilde{r}_2^2 + \frac{1}{2}\tilde{r}_3^2 = 1,\ \frac{a}{2}\tilde{r}_1^2 + \frac{1}{2}\tilde{r}_2^2 + \frac{1}{4}\tilde{r}_3^2 = 1.$$

This leads to

$$\tilde{r}_2 = \tilde{r}_3,\ \frac{a}{2}\tilde{r}_1^2 = 1 - \frac{3}{4}\tilde{r}_2^2. \tag{24}$$

In this case the dynamics of system (20) reduces to the case of 1 fast and 1 slow oscillator which we discussed before. Looking for solutions in system (20) of the form $y^2 = z^2$ gives a shortcut to the problem.

The resonance cases $\phi_2 - \phi_3 = 0, \pi$ lead to a first integral:

$$r_2 = \frac{r_2(0)}{r_3(0)} r_3. \tag{25}$$

Replacing ϕ_3 or $\phi_3 + \pi$ by ϕ_2 we can average over ϕ_2 to obtain in the 2 resonance cases:

$$\begin{cases} \tilde{\tilde{r}}_2' &= \frac{\varepsilon^2}{2}\tilde{\tilde{r}}_2(1 - \frac{a}{2}\tilde{\tilde{r}}_1^2 - \frac{1}{4}\tilde{\tilde{r}}_2^2 - \frac{1}{4}\tilde{\tilde{r}}_3^2), \\ \tilde{\tilde{r}}_3' &= \frac{\varepsilon^2}{2}\tilde{\tilde{r}}_3(1 - \frac{a}{2}\tilde{\tilde{r}}_1^2 - \frac{1}{4}\tilde{\tilde{r}}_2^2 - \frac{1}{4}\tilde{\tilde{r}}_3^2). \end{cases} \tag{26}$$

Another approach is to consider first the slow manifold obtained from system (22) for $\tilde{r}_1$. We find (leaving out the tildes)

$$r_1^2 = 4 - 8b(r_2^2 \sin\phi_2^2 + r_3^2 \sin\phi_3^2). \tag{27}$$

Eliminating $\tilde{r}_1$ from the equations for $\tilde{r}_2, \tilde{r}_3$ in system (22) we have after averaging over the angles for the dynamics in the slow manifold:

$$\begin{cases} \tilde{\tilde{r}}_2' &= \frac{\varepsilon^2}{2}\tilde{\tilde{r}}_2[1 - 2a - (\frac{1}{4} - ab)\tilde{\tilde{r}}_2^2 - 2(\frac{1}{4} - ab)\tilde{\tilde{r}}_3^2]. \\ \tilde{\tilde{r}}_3' &= \frac{\varepsilon^2}{2}\tilde{\tilde{r}}_3[1 - 2a - 2(\frac{1}{4} - ab)\tilde{\tilde{r}}_2^2 - (\frac{1}{4} - ab)\tilde{\tilde{r}}_3^2]. \end{cases} \tag{28}$$

We can draw several conclusions from system (28). An important one is that we find in the slow manifold of system (22) a stable torus if

$$0 < a < \frac{1}{2},\ 0 < ab < \frac{1}{4}. \tag{29}$$

See fig. 4. We leave out the figures for other cases suggested by conditions (29); for instance changing in the data of fig. 4 b to $b = 1$, the slow solutions y, z tend to a stable periodic solution (if the y, z frequencies would be different we would obtain a stable 2-dimensional torus), the fast oscillator is quenched and tends to the $x, \dot{x}$ coordinate plane. Reversing the role of the parameters by putting $a = 1, b = 0.3$ the slow solutions y, z are quenched and tend to the 2 coordinate planes, the fast solution tends to a limit cycle in the $x, \dot{x}$ phaseplane.

The examples of slow-fast oscillators with Van der Pol self-excitation that we discussed have common features like averaging first over a fast angle, averaging after that over slow angles, the presence of slow manifolds and the possibility of local resonance manifolds. To facilitate the demonstrations we have in the examples only a few parameters. A consequence of this is more symmetry producing sometimes non-generic reductions in the analysis. It would be of interest to repeat studying the examples with more parameters. Many more bifurcations are to be expected.

5. Conclusions

1. Slow-fast systems arise naturally in applications involving interactions and quenching.

2. Limit cycles obtained in a perturbation framework can be used to study long time slow-fast interactions described by 3 timescales: $t/\varepsilon, t$ and εt.

Acknowledgement
The author declares that he has no conflict of interest.

References

[1] Anosov, D.V. *On limit cycles in systems of differential equations with a small parameter in the highest derivatives*, Mat Sb (1960) 50, pp. 299-334; transl. AMS Trans. Ser. 2, vol. 33, pp. 233-276.

[2] Blekhman, I.I. *Vibrational Mechanics-nonlinear dynamics effects, general approach, application*, (2000) Singapore, World Scientific.

[3] Doelman, A. and Verhulst, F. *Bifurcations of strongly non-linear self-excited oscillations*, (1994) Math. Methods Appl. Sciences 17, pp. 189-207.

[4] Fenichel, N. *Persistence and smoothness of invariant manifolds for flows*, (1971) Indiana Univ. Math. J. 21, pp. 193–225.

[5] Fenichel, N. *Asymptotic stability with rate conditions*, (1974) Indiana Univ. Math. J. 23, pp. 1109–1137.

[6] Fenichel, N. *Asymptotic stability with rate conditions, II*, (1977) Indiana Univ. Math. J. 26, pp. 81–93.

[7] Fenichel, N. *Geometric singular perturbations theory for ordinary differential equations*, (1979) J. Diff. Eq. 31, pp. 53–98.

[8] Flatto L.and Levinson, N. *Periodic solutions of singularly perturbed systems*, (1955) J. Rat. Mech. Analysis 4, pp. 943-950.

[9] Jones, C.K.R.T. *Geometric singular perturbation theory*, in *Dynamical Systems*, Montecatini Terme (1994) (Johnson, R., ed.) Lecture Notes in Mathematics 1609, pp. 44–118, Springer.

[10] O'Malley, Jr., R.E., *Topics in singular perturbations*, (1968) Adv. Math. 2, pp. 365–470.

[11] Pontryagin L.S. and Rodygin, L.V. *Approximate solution of a system of ordinary differential equations involving a small parameter in the derivatives*, (1960) Soviet Math. Doklady 1, pp. 237-240 (1960), orig. Russian: Dokl. Ak. Nauk SSSR 131, pp. 255-258.

[12] Sanders, J.A. Verhulst,F. and Murdock, J. *Averaging methods in nonlinear dynamical systems* 2nd ed., (2007) Appl. Math. Sciences 59, Springer, New York etc..

[13] Tikhonov, A.N. *Systems of differential equations containing a small parameter mutiplying the derivative*, (1952) Mat. Sb. 31, pp. 575-586.

[14] Vasil'eva, A.B., *Asymptotic behaviour of solutions to certain problems involving nonlinear differential equations containing a small parameter multiplying the highest derivatives*, (1963) Russ. Math. Surv. 18, pp. 13–84.

Ferdinand Verhulst, PhD: University of Utrecht, Mathematisch Instituut, PO Box 80.010, 3508 TA Utrecht, The Netherlands (*f.verhulst@uu.nl*).

The author gave a presentation of this paper during one of the conference sessions.