Quenching instability associated with Whitney's umbrella

Ferdinand Verhulst
Mathematisch Instituut,
University of Utrecht

Prague, September 2011

Contents of the talk

- A basic paradox
- Whitney's umbrella
- Bottema's analysis
- Brouwer's rotating vessel
- Stabilization by modulating the rotation
- Nonlinear effects

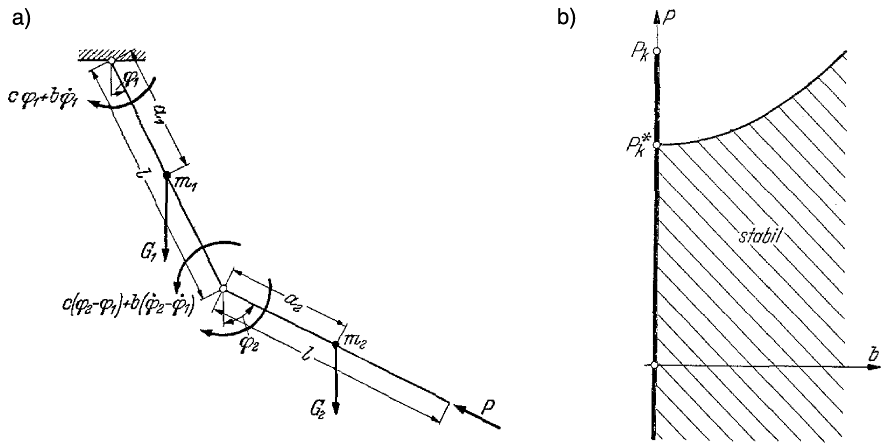
A basic paradox

General ‘knowledge’: dissipation stabilizes.
Example: mathematical pendulum without friction swings forever.
With friction: return to equilibrium position.

However for more than one degrees of freedom the situation is more complicated.

Example: the solar system. Tidal friction is hardly observable on a timescale of centuries, on longer timescales it may destabilize.
Examples: coupled pendula, rotor systems

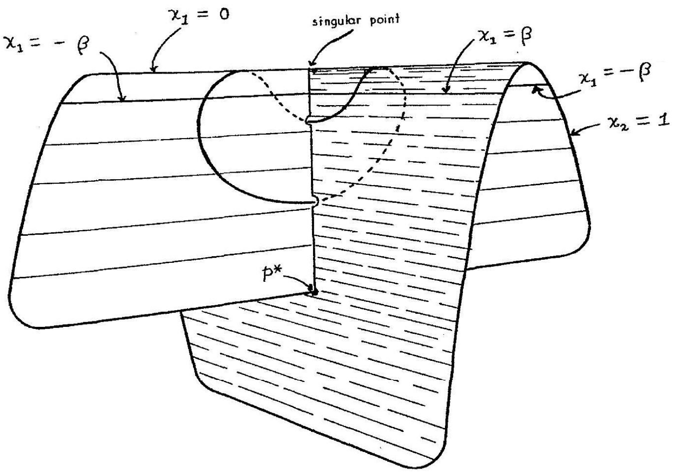
Ziegler's paradox ,1952



Original drawings from the Ziegler's work of 1952:
(a) double linked pendulum with constant follower load p,
(b) stability interval of the undamped and damped pendulum (damping parameter b).

The phenomenon was known much earlier: Thomson and Tait, Darwin, Poincaré.

Whitney's umbrella (1943)



Whitney's original sketch of the umbrella. Consider the C^k map $f : E^2 \mapsto E^3$ After transformation we have near the origin

$$y_1 = x_1^2,\ y_2 = x_2,\ y_3 = x_1 x_2, \qquad (1)$$

so that $y_1 \geq 0$ and on eliminating x_1 and x_2:

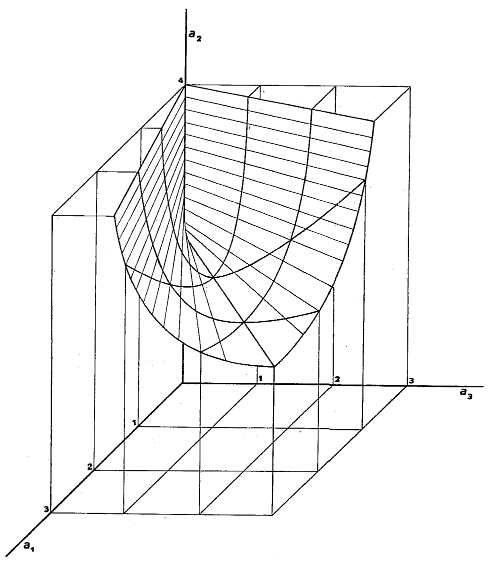
$$y_1 y_2^2 - y_3^2 = 0.$$

Linear ODEs with constant coefficients may produce (eigenvalue) characteristic equations with such singularities.

Bottema solved the paradox, 1956

Consider a linear system with two degrees of freedom, constant coefficients, near $x = y = 0$:

$$\begin{aligned}\ddot{x} + a_{11}x + a_{12}y + b_{11}\dot{x} + b_{12}\dot{y} &= 0,\\ \ddot{y} + a_{21}x + a_{22}y + b_{21}\dot{x} + b_{22}\dot{y} &= 0.\end{aligned}$$



The ruled surface is the bifurcation manifold. There is a reduction of the 8 parameters to 3 for the bifurcation manifold

$$a_1 a_2 a_3 = a_1^2 + a_3^2.$$

A particle on a rotating vessel

Brouwer (1918) considered the equilibrium of a point mass moving under gravity on a surface S that is rotating with uniform angular velocity ω around a vertical axis l.

Consider O: point of S that is equilibrium.

Linearized equations near O with damping:

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} + c_1\dot{x} + (gk_1 - \omega^2)x &= 0, \\ \ddot{y} + 2\omega\dot{x} + c_2\dot{y} + (gk_2 - \omega^2)y &= 0. \end{aligned}$$

k_1 and k_2 are the curvatures in O, $k_2 \leq k_1$.

$-2\omega\dot{y}, 2\omega\dot{x}$ represents Coriolis force

gk_1x, gk_2y represents gravity

$-\omega^2x, -\omega^2y$ represents centrifugal force.

A particle on a rotating vessel

No damping $c_1 = c_2 = 0$

If $k_2 \leq k_1 < 0$ (bump): instability

Two other cases:

- $0 < k_2 < k_1$ (indentation of S).
  Stability iff $0 < \omega^2 < gk_2$ (slow rotation) or $\omega^2 > gk_1$ (fast rotation).

- $k_2 < 0$ and $k_1 > 0$, $k_1 > -k_2$ (saddle).
  Stability iff $\omega^2 > gk_1$.

Adding internal damping

Two cases (Bottema, 1976):

- $0 < k_2 < k_1$ (indentation of S).
  Stability iff $0 < \omega^2 < gk_2$.
  The fast rotation branch $\omega^2 > gk_1$ has vanished.

- A geometric saddle is always unstable with damping small or large.

Quenching the unstable saddle motion

In the engineering context, quenching of instabilities by a practical physical mechanism is important. Intuitively it is not clear what to propose but based on earlier studies we choose:

modulation of the vessel rotation.

Consider the rotation of a saddle, unstable by dissipation for any rotational velocity ω and put (to reduce the number of parameters)

$$k_2 = k > 0 \text{ and } \mathrm{k}_1 = -\mathrm{k}$$

Assume

$$\omega^2 > gk.$$

For the friction coefficient put $c_1 = c_2 = c$. Moreover

$$\omega^2 := \omega^2 + 2a\varepsilon \cos \nu t$$

with again ω a constant.

Quenching continued

The equations of motion become with small dissipation and small ω-modulation:

$$
\begin{aligned}
\ddot{x} - 2\omega \dot{y} - \varepsilon \frac{2a}{\omega} \cos \nu t \dot{y} + \varepsilon c \dot{x} - \beta^2 x &= 0, \\
\ddot{y} + 2\omega \dot{x} + \varepsilon \frac{2a}{\omega} \cos \nu t \dot{x} + \varepsilon c \dot{y} - \alpha^2 y &= 0.
\end{aligned}
$$

Two basic frequencies:

$$\alpha^2 = \omega^2 + gk, \ \beta^2 = \omega^2 - gk.$$

To perform averaging-normalization we transform (variation of constants) to a slowly varying system.

Using the transformation in the eqs of motion we find with MATHEMATICA (Theo Ruijgrok)

$$
\begin{aligned}
\dot{A} &= -\varepsilon\frac{\alpha\sin\alpha t}{\alpha^2-\omega^2}F_1+\varepsilon\frac{\omega\cos\alpha t}{\beta^2-\omega^2}F_2,\\
\dot{B} &= \varepsilon\frac{\alpha\cos\alpha t}{\alpha^2-\omega^2}F_1+\varepsilon\frac{\omega\sin\alpha t}{\beta^2-\omega^2}F_2,\\
\dot{C} &= \varepsilon\frac{\omega^2\sin\beta t}{\beta(\alpha^2-\omega^2)}F_1+\varepsilon\frac{\omega\cos\beta t}{\omega^2-\beta^2}F_2,\\
\dot{D} &= -\varepsilon\frac{\omega^2\cos\beta t}{\beta(\alpha^2-\omega^2)}F_1+\varepsilon\frac{\omega\sin\beta t}{\omega^2-\beta^2}F_2.
\end{aligned}
$$

We have in our quenching model

$$F_1=-c\dot{x}+\frac{2}{\omega}\dot{y}\cos\nu t+2ax\cos\nu t,$$

$$F_2=-c\dot{y}-\frac{2}{\omega}\dot{x}\cos\nu t+2ay\cos\nu t.$$

Applying averaging-normalization we have to make assumptions about the frequencies α, β, ν. We have the following 5 resonances:

- $2\alpha = \nu$. This is a Mathieu resonance that does not contribute to stabilization.

- $2\beta = \nu$, also a Mathieu resonance making matters worse.

- The sum-resonance $\alpha = \beta + \nu$.

- The sum-resonance $\alpha + \beta = \nu$.

- Special resonance $\alpha = 3\beta = \beta + \nu$.

Eigenvalue calculation

- Mathieu resonances: unstable.

- Special resonance $\alpha = 3\beta = \beta + \nu$: unstable equilibrium.

- Sum-resonances $\alpha = \beta + \nu$ and $\alpha + \beta = \nu$. Stability if

$$c < a \frac{K}{\omega\sqrt{4 - K^2}} \text{ with } \mathrm{K} = \frac{\mathrm{gk}}{\omega^2}.$$

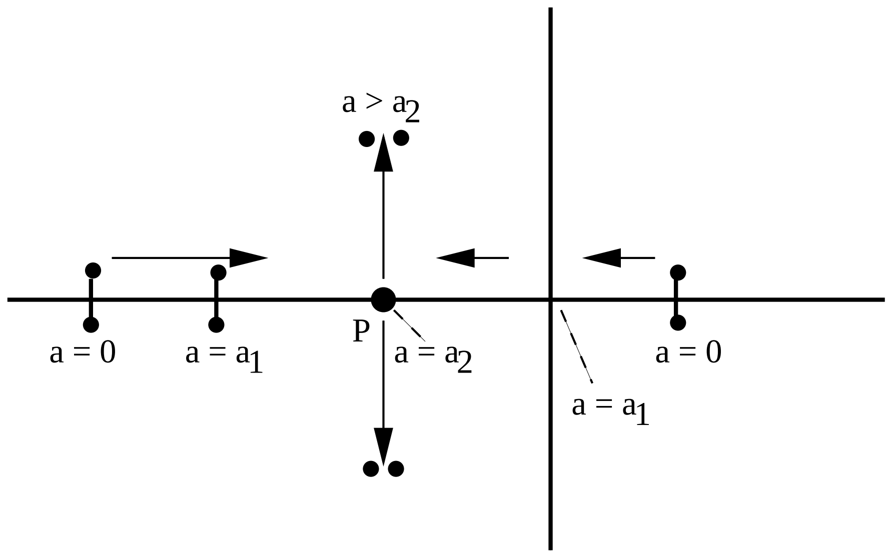
  The sum-resonances correspond with a synchronization of the basic vessel frequencies and the modulation of the vessel rotation.

The single-well in the unstable case

Basic frequencies:

$$\omega \pm \sqrt{gk}.$$

Stabilizing combination resonance $2\sqrt{gk} = \nu$.



Movement of the eigenvalues near equilibrium at the single-well with dissipation and small modulation of the rotation ω.

Nonlinear extensions

Damping

$$\varepsilon(c\dot{x} - d\dot{x}^3),\ \varepsilon(c\dot{y} - d\dot{y}^3).$$

$d > 0$ softening damping, $d < 0$ hardening damping.

The unstable case was $\omega^2 > gk$. Will nonlinear damping increase the instability or will there be stabilization at some distance of the origin?

It turns out nonlinearities do not stabilize but there are new phenomena, like quasi-periodic motion.

Conclusions

- Mathematically, the bifurcational behavior is described by the Whitney umbrella as indicated. In mechanical terms, the enlarging of the instability-domains is caused by the coupling between the two degrees of freedom which arises in the presence of damping.

- Bottema's solution in 1956 was ignored. Google Scholar gives no citations of the paper in the period 1956-2008.

- The generality of Bottema's results enable us to discuss the part played by asymmetric and symmetric damping.

- In the context of dissipation-induced instability, the influence of asymmetric and symmetric damping was studied extensively by Kirillov (2005b, 2007), Kirillov and Seyranian (2005a). In these papers (see Kirillov and Verhulst, ZAMM 2010) Bottema's results were also generalized to higher (more than 4) dimensions.

- The phenomena described here are basically linear. Further away from equilibrium and in some critical cases, Krein-collision or small real parts near the umbrella surface, nonlinear terms may come into play.

- It is remarkable that one can stabilize equilibrium as shown in Brouwer's rotating vessel.