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Torus break-down and bifurcations in coupled oscillators

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Abstract

To discover qualitative changes of solutions of differential equations, one has to study their bifurcations. We start with the well-known bifurcations of equilibria leading to periodic solutions, followed by bifurcations of periodic solutions (Neimark-Sacker and Hopf-Hopf) in a dissipative setting leading to quasi-periodic motion corresponding with tori. In their turn these families of quasi-periodic solutions may bifurcate to produce strange attractors. A number of examples illustrate the theory.

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Keywords: periodic solutions; bifurcations; quasi-periodic solutions; tori; chaos;

1. Introduction

The interest in bifurcations started with 18th and 19th century explorations of rotating fluid masses when the speed of rotation was varied. The first systematic bifurcation theory was given by Henri Poincaré in[10], vol. 1. Bifurcation phenomena in differential equations correspond with qualitative changes of the solutions, for instance changes of stability, emergence of new solutions or transitions like the Neimark-Sacker bifurcation that, starting with a periodic solution, produces quasi-periodic solutions. The theory has very important consequences for mechanics and the other natural sciences. We will briefly discuss bifurcations of equilibria and periodic solutions, followed by a treatment of the emergence of tori containing quasi-periodic solutions which can bifurcate again to produce chaos.

2. Bifurcations of equilibria

Consider the n-dimensional autonomous differential equation

$$\dot{x} = f(x,\mu), \; x \in \mathbb{R}^n, \tag{1}$$

with isolated equilibrium x_0, so $f(x_0) = 0$; μ is a real parameter. The theory of bifurcations of equilibria like Hopf, transcritical, saddle-node etc., are a well-documented area, see for instance[7,9,12]. All this started with Poincaré's very

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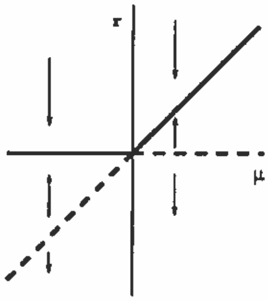

Fig. 1. Transcritical bifurcation of the amplitude of a periodic solution

detailed description[10] vol. 1, of what is now called the Hopf-bifurcation. This bifurcation was later discussed by Andronov and again later by Hopf. It arises if, when linearizing the vector field near x_0, for a certain value of the parameter $\mu = \mu_0$ two eigenvalues become purely imaginary. In this case a periodic solution can emerge as happens in the canonical example of the Van der Pol equation:

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}. \tag{2}$$

Some confusion may arise because of the terminology. Consider for instance the two-dimensional system in polar coordinates:

$$\begin{aligned} \dot{r} &= (\mu - r^2)r, \\ \dot{\theta} &= 1. \end{aligned} \tag{3}$$

The equation for the amplitude r shows the (supercritical) *pitchfork bifurcation* as for $\mu < 0$ we have stability of the origin, for $\mu > 0$ we have an attracting periodic solution (limit cycle). The bifurcation behaviour of the amplitude equation is then transferred to system (3) which is said to have a pitchfork bifurcation.

Another example is the system in polar coordinates:

$$\begin{aligned} \dot{r} &= \mu r - r^2, \\ \dot{\theta} &= 1. \end{aligned} \tag{4}$$

In the next section we will consider system (5) that contains the *transcritical bifurcation*, see Fig. 1, resulting in an unstable periodic solution for $\mu < 0$, a stable one for $\mu > 0$.

3. Bifurcations of periodic solutions producing tori

We will discuss two important scenarios for the emergence of quasi-periodic solutions corresponding geometrically with tori in phase-space. There exist other scenarios, for instance for Hamiltonian systems; see[12]. Combining analytic with geometric descriptions helps to visualize the dynamics and helps to present us with a global picture of the solutions.

3.1. The Neimark-Sacker bifurcation

Consider system (1) with dimension at least three, containing an isolated periodic solution. At a critical value of the parameter $\mu = \mu_0$, the periodic solution has two critical exponents with real part zero. Alternatively formulated: describing the phase-flow as a map, we have an isolated fixed point of the map with at $\mu = \mu_0$ two purely imaginary eigenvalues. Then:

1. A torus branches off the periodic solution;
2. The motion on the torus is quasi-periodic;
3. The corresponding fixed point is contained in an invariant circle.

Example
Consider the system from[3] containing damping, parametric excitation and nonlinear interaction:

$$\begin{aligned} &\ddot{x} + \varepsilon\kappa\dot{x} + (1 + \varepsilon\cos 2t)x + \varepsilon xy = 0, \\ &\ddot{y} + \varepsilon y + 4(1 + \varepsilon)y - \varepsilon x^2 \qquad = 0. \end{aligned} \tag{5}$$

Here and in the next examples the procedure will be as follows. The presence of a small parameter enables us to obtain analytic approximations of the solutions and some of the bifurcations. This information is supplemented by numerical bifurcation techniques to obtain more detailed or more global information. In the case of system (5) we find an isolated hyperbolic periodic solution for $0.546 < \kappa < 0.559$. At the value $\kappa = 0.546$, two real parts of the four eigenvalues vanish in a Neimark-Sacker bifurcation. A two-dimensional torus emerges containing two-frequency oscillations, one on timescale order 1 and one on timescale order $1/\varepsilon$.

3.2. The Hopf-Hopf bifurcation

In two degrees of freedom systems we may find an equilibrium with two pairs of purely imaginary eigenvalues. To illustrate this, we slightly modify a four-dimensional example of Hale[8] with positive coefficients ω, a, b:

$$\begin{aligned} &\ddot{x} + x \quad\; = \varepsilon(1 - x^2 - ay^2)\dot{x}, \\ &\ddot{y} + \omega^2 y = \varepsilon(1 - y^2 - bx^2)\dot{y}. \end{aligned} \tag{6}$$

In the coordinate planes $x = \dot{x} = 0$ and $y = \dot{y} = 0$ we find isolated periodic solutions from the Van der Pol-equation. It is shown in[12] section 12.3.1 by averaging that if ω □ 1, and excluding $a = b = 1/2$ we have quasi-periodic motion outside the coordinate planes if $a, b > 1/2$ and $a, b < 1/2$, corresponding with a torus containing two-frequency oscillations, both on a timescale of order 1. As the torus in the averaged system is normally hyperbolic, it persists in the original system (6).
A remarkable bifurcation takes place if $a = b = 1/2$. Passing from $a, b > 1/2$ to $a, b < 1/2$ the torus goes from unstable to stable. At $a = b = 1/2$ the torus deforms to a 3-sphere in 4-dimensional phase-space covered with quasi-periodic solutions. The sphere is expected to be structurally unstable in the sense that higher order terms will destroy it.

4. Torus break-up

Most of the literature providing insight in the breakdown of tori has been obtained by studying explicitly given maps. The reason is that phenomena governed by differential equations are implicitly defined and much more difficult to demonstrate. Note that the breakdown of tori in Hamiltonian context is generic because of non-integrability and chaos, in a dissipative setting it asks for a more specific dynamics.

Suppose we have a dynamical system containing quasi-periodic solutions corresponding with a torus. The presence of stable and unstable periodic solutions in p/q-resonance on a torus may produce such a break-up; it can be triggered by heteroclinic tangencies of stable and unstable manifolds of periodic solutions, arising when a parameter is varied.

An early paper is[2] where the analysis discusses a two-dimensional map containing a smooth invariant circle with a pair of periodic orbits, one stable and one unstable. When the invariant manifolds of periodic orbits start crossing in homoclinic or heteroclinic tangencies the invariant circle looses differentiability and chaotic dynamics takes over.

More details of torus break-up can be found in[1] where it is established that the breakdown of the torus starts with *non-smoothness of the torus* The authors give three possibilities:

- The stable and unstable periodic orbits vanish through a bifurcation.
- Stable and unstable manifolds of the unstable periodic orbit intersect tangentially to form a homoclinic orbit.
- The stable periodic orbit looses stability.

Detailed studies of dissipative families of maps of an annulus into itself can be found in[6] and[13]. The torus corresponds with an invariant circle in the plane. A perturbation yields loss of smoothness of the invariant circle followed

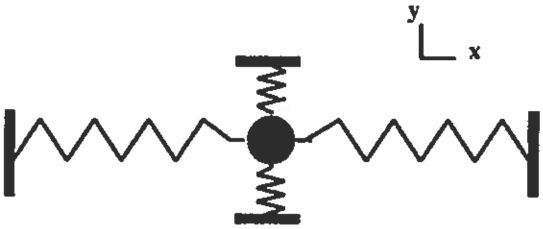

Fig. 2. Two coupled oscillators

by its destruction. In addition this involves homoclinic bifurcations, cascades of bifurcations, homoclinic and heteroclinic tangencies, 'small' strange attractors and 'large' strange attractors. In[13] the bifurcations display a dense occurrence of resonances. This causes a Cantorization of the bifurcation sets, making the system highly complex.

These map studies give a lot of insight, but we are left then with the task to identify the phenomena in models of real mechanical systems. We shall note that explicit mechanical systems will also show new phenomena.

5. A simple mechanical system

A system of two coupled oscillators with positive damping δ, κ, see Fig. 2, and nonlinear interaction was considered in[4]:

$$\begin{aligned} \ddot{x} + \delta x^2 \dot{x} + x + \gamma x^3 + xy &= 0, \\ \ddot{y} + \kappa \dot{y} + 4y + bx^2 &= 0. \end{aligned} \tag{7}$$

As the damping coefficients δ, κ are positive it is easy to see that the phase-flow is exponentially contracting; if $b > 0$ the origin is the only stable equilibrium (and limit set), so we choose $b < 0$ which introduces a new element in the dynamics. The nonlinear damping of the x oscillator is not essential (it implies a small modification), we might as well take linear damping. It turns out that γ □ 0 is crucial as it produces asynchronous oscillations. Surprisingly enough, we will see that in the case $b < 0$ tori, resonances and chaos are produced.

5.1. Periodic and quasi-periodic solutions

For system (7) various scalings of the variables are possible corresponding with different localizations in phase-space. A suitable scaling near the origin is $x \to \sqrt{\varepsilon}x,\ y \to \varepsilon y$, with ε a small positive parameter; we put $\kappa \to \varepsilon\kappa$, $b \to \varepsilon b$. After this rescaling, we find for $\varepsilon = 0$ 'unperturbed' harmonic equations. The analytical technique used is again averaging-normalization with results:

- We find a hyperbolic critical point (equilibrium) of the averaged equations.
- The critical point corresponds with an asymptotically stable periodic solution (relative equilibrium)
- Decreasing γ, the periodic solution leaves the neighbourhood of the origin

Using the analytical results as a start we apply numerical bifurcation techniques, i.e. Auto and Matcont. We start at $\gamma = 0.2$ and decrease γ to find the torus displayed in fig. 3. As the system is 4-dimensional, a Poincaré-map will still be 3-dimensional, so for its display we prefer to use a projection on the $y, \dot{y}$-plane.

One can find different dynamical aspects in other domains of phase-space and by other methods. Another suitable scaling is $x \to \varepsilon^{1/4}x$ with y not rescaled; this concerns a larger domain than the size considered above both for x and y. In this case we find after the rescaling for $\varepsilon = 0$ the 'unperturbed' equations (8):

$$\ddot{x} + (1 + y)x = 0,\ \ \ddot{y} + 4y = 0, \tag{8}$$

resulting in Mathieu-functions as solutions for 'unperturbed' $x(t)$. One can use the Poincaré-Lindstedt method using the known features of Mathieu-functions to determine the existence (by the implicit function theorem) and approximation of two π-periodic solutions. The Mathieu-function has a periodic branch and a branch with unbounded solutions.

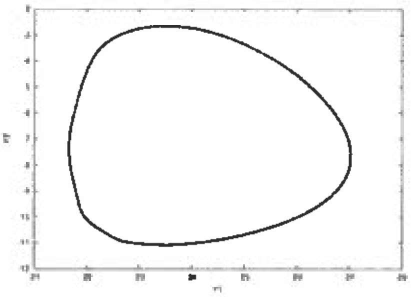

Fig. 3. Torus in projection of Poincaré-section $(y, \dot{y})$ at $\gamma = 0.092$, $b = -0.5$, $\delta = 0.4$, $\kappa = 0.1$, horizontal $y \in [21, 28]$, vertical $\dot{y} \in [-12, -2]$; from[4]

For application of the Poincaré-Lindstedt method we have to use the periodic branch; for details see[4]. Also in this region further away from the origin, we find bifurcations resulting in tori.

5.2. Break-up of the torus

Considering the presence of Arnold tongues, we are confronted with a very complicated structure that can be characterized by Cantor sets. So we will look for a prominent tongue within this structure. Using Auto and Matcont when decreasing γ in the torus regime, we find a bifurcation diagram with a large 1 : 6 resonance tongue bounded by saddle-node bifurcation curves. At the saddle-node curves in the diagram, the stable and unstable period 6 solutions that exist for the parameter values within the tongue, vanish. An example of the 1 : 6 stable (node) and unstable (saddle) periodic solutions on the torus at $\gamma = 0.09184$ is given in Fig. 4. We keep on decreasing γ. The 1 : 6 stable

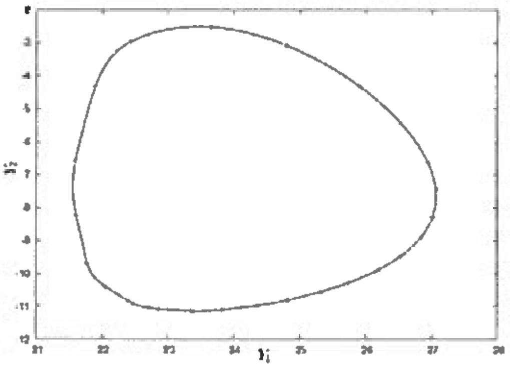

Fig. 4. Stable and unstable 1 : 6 periodic solutions on the torus in projection of Poincaré-section $(y, \dot{y})$ at $\gamma = 0.09184$, horizontal $y \in [21, 28]$, vertical $\dot{y} \in [-12, -2]$; from[4]

(node) and unstable (saddle) periodic solutions on the torus at $\gamma = 0.0903$ have a heteroclinic connection resulting in a cascade of period doublings, the torus becomes non-smooth. After the cascade of period doublings the torus has vanished. A strange attractor emerges with Kaplan-Yorke dimension 2.32..., see Fig. 5.

5.3. Conclusions for the simple mechanical system

- Averaging-normalization and the Poincaré-Lindstedt method are tools to find and locate periodic solutions and tori.
- The programs Auto, Content and Matcont provide the tools for subsequent numerical bifurcation results.
- We studied in this mechanical system the prominent resonance 1 : 6, many other resonances can be located producing interesting phenomena.
- We briefly mentioned domains that are further away from the origin and which also contain periodic solutions and strange attractors. Again the analysis is by rescaling and with mixed analytic-numeric tools.

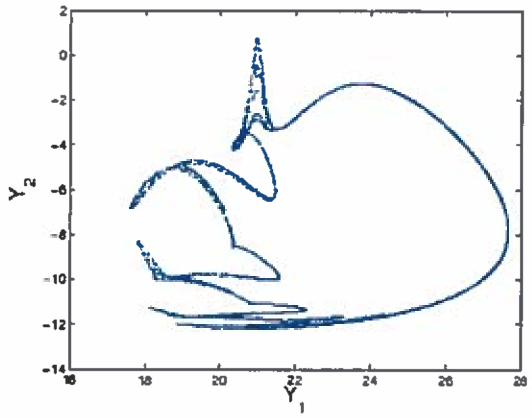

Fig. 5. A strange attractor at $\gamma = 0.0892$, horizontal y, vertical $\dot{y}$; from[4]

6. General conclusions

The systematic study of maps as represented by the references has been an inspiration to consider realistic dynamical systems corresponding with actual mechanical systems. Averaging-normalization and numerical bifurcation techniques can be very helpful. Remarkable enough, complicated phenomena as predicted by studies of maps can already be identified in our simple example from[4]. An interesting side-remark is that the phenomena described here, confirm a visionary paper by Ruelle and Takens[11] who suggested a new bifurcation scenario where a periodic solution produces subsequently a torus and then a strange attractor.

In[5] a (relatively simple) three degrees of freedom mechanical system is considered. Interestingly, it turns out that in this case new bifurcation phenomena arise. This makes a strong case for exploring torus break-up in more realistic mechanical models.

References

1. Afraimovich, V.S. and Shil'nikov, L.P., Invariant two-dimensional tori, their breakdown and stochasticity, *Amer. Math. Soc, Transl.* 149, pp. 201-212 (1991).
2. Aronson, D.G., Chory, M.A., Hall, G.R., McGehee, R.P., Bifurcations from an invariant circle for two-parameter families of maps of the plane: a computer-assisted study, *Commun. Math. Phys.* 83, pp. 303-354 (1982).
3. Bakri, T, Nabergoj, R., Tondl, A. and Verhulst, F, Parametric excitation in nonlinear dynamics, *Int. J. Nonlinear Mech.* 39, pp. 311-329 (2004).
4. Bakri, T., Verhulst, F.: Bifurcations of quasi-periodic dynamics: torus breakdown. *Z. Angew. Math. Phys.* 65, pp. 1053-1076 (2014).
5. Bakri, T., Kuznetsov, Yu.A., Verhulst, F., Torus Bifurcations in a Mechanical System, *J. Dyn. Diff Equat.*, DOI: 10.1007/s10884-013-9339-9 2013.
6. Broer, H.W., Simó, C., Tatjer, J.C., Towards global models near homoclinic tangencies of dissipative diffeomorphisms, *Nonlinearity* 11, pp. 667-770 (1998).
7. Guckenheimer, J. and Holmes, P., Nonlinear oscillations, dynamical systems and bifurcations of vector fields, *Appl. Math. Sciences 42*, 5th printing, Springer (1997).
8. Hale, J.K., Ordinary differential equations, J. Wiley (1969).
9. Kuznetsov, Yu.A., Elements of applied bifurcation theory, 2nd ed. Springer (2005).
10. Poincaré, Henri, Méthodes Nouvelles de la Mécanique Célèste, 3 vols., Gauthier-Villars, Paris, (1892, 1893, 1899).
11. Ruelle, D. and Takens, F., On the nature of turbulence, *Commun. Math. Phys.* 20, pp. 167-192 (1971).
12. Verhulst,F., Methods and applications of singular perturbations, Springer (2005).
13. Vitolo, R., H.W. Broer, C. Simó, Quasi-periodic bifurcations of invariant circles etc., *Regular and Chaotic Dynamics* 16, pp. 154-184 (2011).