Resonant periodic solutions in regularized impact oscillator

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Abstract

We use a perturbation approach to prove the occurrence of asymptotically stable periodic solutions in a bilinear oscillator whose one spring has nearly infinite stiffness. This leads to a singularly perturbed problem where the classical theory does not apply.

Keywords: impact oscillator, asymptotic stability, perturbation theory, resonant periodic solution, regularization
2010 MSC: 34A37, 34C29, 70K65, 70F35, 34D10, 34C25

1. Introduction

Since the observation by Glover, Lazer and McKenna [13] that a simple bilinear oscillator (i.e. harmonic oscillator with an elastic obstacle) can be used in the explanation of the failure of the Tacoma bridge, the study of periodic oscillations in bilinear oscillators received a lot of attention (see [2, 6, 8, 16, 21, 25, 30, 33] among others). Most recently, the analysis of the dynamics of bilinear oscillators helped to understand the loss of image quality in atomic force microscopy, see [26, 31]. More applications that involve the bilinear oscillator as a component can be found in survey [20].

The analysis in this paper concerns the following prototypic version of the bilinear oscillator

$$\begin{aligned} &\ddot{x} + x = \varepsilon f(t, x, \dot{x}, \varepsilon), && x > 0, \\ &\ddot{x} + \frac{1}{\varepsilon^2(\omega_\varepsilon)^2} x = g(t, x, \dot{x}, \varepsilon), && x \le 0, \end{aligned} \tag{1}$$

which can be viewed as a perturbed regularization of the impact oscillator

$$\begin{aligned} &\ddot{x} + x = 0, && x > 0, \\ &\dot{x}(t-0) = -\dot{x}(t+0), && x(t) = 0, \end{aligned} \tag{2}$$

in the terminology of [28]. Here f and g are smooth scalar functions, $\varepsilon > 0$ is a small parameter, and $\omega_\varepsilon \to \omega_0 \in \mathbb{R}$ as $\varepsilon \to 0$. The second equation of (1) can be considered as a smooth approximation of the impact law of (2), see [10, Section 1.2.4]. An energy dissipation approach comparing the dynamics of systems (1) and (2) is proposed in [4]. A comprehensive bifurcation analysis of a version of (1) is carried out in [25], in which paper the stroboscopic map (period map) for (1) is taken as a regularization of the stroboscopic map of (2). A direct proof of the existence of periodic solutions for general versions of (1) is accomplished in [8], but stability of periodic solutions was not analyzed. At the same time, the knowledge about asymptotic stability of periodic oscillations of (1) is of crucial importance in such applications of bilinear oscillator as e.g. regularization of non-deformable contacts in rigid-body mechanics ([1, 15, 16, 23]) or in electro-mechanical

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systems ([27]). Because system (2) doesn't admit an isolated periodic solution, the standard methods of singularly perturbation theory (see e.g. [32]) don't apply. To establish stability of periodic solutions to (1), the present paper develops an appropriate averaging principle, which approach has been earlier used in the context of impact oscillator in [3, 7, 11].

The main assumption of our approach is π-periodicity of the right-hand-sides of (1) in time. This doesn't mean that the period of the excitation coincides with the period of self-oscillations in (1) because a part of functions f and g can be viewed as small detuning of self-oscillations away from period π (the role of this detuning is played by the term εax in the example of Section 3). However, π-periodicity of the right-hand-sides of (1) implies that the period of excitation in (1) is close to the period of self-oscillations of (1), i.e. this paper deals with so-called resonant periodic solutions.

Linear system (2) can be viewed as an approximation of a nonlinear impact system of center type in the case where the dependence of the period-function on the coordinate is negligible compared to perturbation. When the dependence of the periods of the cycles of unperturbed system (2) on the initial condition is essential and cannot be neglected, the properties of the above-mentioned period-function have to be taken into account when constructing the averaging function, see [13]. As shown in [13] for the case of a piecewise smooth oscillator with bounded (in ε) terms, non-vanishing derivative of the period-function helps to reduce the dimension of the averaging function to 1. We expect that the same reduction is possible in the case of oscillator (1) with the unbounded entry $1/\varepsilon^2$, but we don't pursue this analysis in the present paper. Similar reduction of the dimension of the averaging function is expected when the obstacle of the oscillator is located at some $x > 0$ as opposed to $x = 0$ that we assume in (1). Indeed, placing the obstacle at $x > 0$ implies that the periods of the cycles of (2) depend on the initial condition in the same way how periods of the cycles of (2) depend on the initial condition when (2) bounces at $x = 0$, but nonlinear. Another potentially interesting extension that lies outside of the scope of the paper is the case where the cycles of system (2) share the same period which is incommensurable with the periods of excitations f and g. As shown in [25] various chaotic regimes are possible here. At the same time, following the classical result by Bogolyubov (see [5, Section 29]), one can also investigate the occurrence of asymptotically stable almost periodic solutions, but the construction of the averaging function will need to involve integration over an infinite interval. In the case of discontinuous system (1) computation of such an integral will lead to summation over a countable number of switchings. A useful source in this regard can be the papers [9, 12] which compute improper integrals (Melnikov integrals) for discontinuous systems in the context of homoclinic solutions.

The paper is organized as follows. In the next section we prove Theorem 1, which is our main result. This theorem introduces an analogue of the subharmonic Melnikov function $\bar{P}(A, \theta)$ (also known as bifurcation function or averaging function, see [14]). Stable zeros of $\bar{P}(A, \theta)$ correspond to the cycles of the limiting system (2) that transform to asymptotically stable periodic solutions of (1) as ε crosses 0. The proof is based on the analysis of the associated period-map that comes by appropriately combining the slow part of the flows of system (1) on $x > 0$ and the fast part corresponding to $x < 0$. An example with particular choices of the right-hand-sides $f(t, x, \dot{x}, \varepsilon) = -ax - c_1\dot{x} + \mu_1\dot{x}(1 - \dot{x}^2) + \gamma \sin 2t$ and $g(t, x, \dot{x}, \varepsilon) = -(c_2 + \varepsilon c_1)\dot{x} + (\mu_2 + \varepsilon\mu_1)\dot{x}(1 - \dot{x}^2) + \varepsilon\gamma \sin 2t$ is presented in Section 3, where an explicit condition for the occurrence of π-periodic solutions in (1) is obtained in terms of the parameters a, c_1, μ_1, γ, c_2, μ_2. Same section features numeric simulations confirming our theoretic predictions. The respective Wolfram Mathematica Notebook (where we also computed the function $\bar{P}(A, \theta)$ of the example of Section 3) is uploaded as a supplemental material. A conclusions section concludes the paper.

2. Main result

The main achievement of the paper is the following bifurcation function that allows to judge about the occurrence of resonant periodic solutions in (1):

$$\bar{P}(A,\theta) = -\int_0^{\pi/2-\theta} \begin{pmatrix} \sin(\tau+\theta) \\ (1/A)\cos(\tau+\theta) \end{pmatrix} (f(\tau, A\cos(\tau+\theta), -A\sin(\tau+\theta), 0) + 2\omega_0 A\cos(\tau+\theta))\mathrm{d}\tau$$
$$-\int_{\pi/2-\theta}^{\pi} \begin{pmatrix} \sin(\tau+\theta+\pi) \\ (1/A)\cos(\tau+\theta+\pi) \end{pmatrix} (f(\tau, A\cos(\tau+\theta+\pi), -A\sin(\tau+\theta+\pi), 0) + 2\omega_0 A\cos(\tau+\theta+\pi))\mathrm{d}\tau$$
$$-\omega_0 \int_0^{\pi} \begin{pmatrix} \sin(s+\pi/2) \\ 0 \end{pmatrix} g(\pi/2-\theta, 0, -A\sin(s+\pi/2), 0)\,\mathrm{d}s.$$

Naturally, when $\omega_0 = 0$ and $\theta = \pi/2$, this function coincides with the averaging function of the forced linear oscillator, see e.g. [5, 29].

We prove the following theorem.

Theorem 1. *Let $f, g \in C^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$ be π-periodic with respect to the first variable. If equation (1) has a π-periodic solution x_ε for all $\varepsilon > 0$ sufficiently small and*

$$(x_\varepsilon(0), \dot{x}_\varepsilon(0)) \to (x_0, v_0) \quad as \quad \varepsilon \to 0, \quad with\ x_0 > 0, \tag{3}$$

then

$$\bar{P}(A_0, \theta_0) = 0 \tag{4}$$

for $(A_0, \theta_0) \in (0,\infty)\times(-\pi/2, \pi/2)$ given by

$$(x_0, v_0) = (A_0\cos\theta_0, -A_0\sin\theta_0). \tag{5}$$

Conversely, if (4) holds for some $(A_0, \theta_0) \in (0,\infty)\times(-\pi/2, \pi/2)$ and

$$\det \bar{P}'(A_0, \theta_0) \neq 0, \tag{6}$$

then, for any $\varepsilon > 0$ sufficiently small, equation (1) has a unique π-periodic solution x_ε satisfying (3) with (x_0, v_0) given by (5). Assume additionally, that the matrix $\bar{P}'(A_0, \theta_0)$ is diagonalizable over $\mathbb{C}$, i.e. $\bar{P}'(A_0, \theta_0)$ admits two eigenvalues. Then the solution x_ε is attracting (asymptotically stable), repelling, or a saddle according to whether the real parts of the two eigenvalues of $\bar{P}'(A_0, \theta_0)$ are both negative, both positive, or have opposite signs.

Proof. Rewrite system (1) as follows

$$\ddot{x} + \frac{1}{(1-\varepsilon\omega_\varepsilon)^2}x = \varepsilon f(t, x, \dot{x}, \varepsilon) + 2\varepsilon\frac{\omega_\varepsilon}{(1-\varepsilon\omega_\varepsilon)^2}x - \varepsilon^2\frac{(\omega_\varepsilon)^2}{(1-\varepsilon\omega_\varepsilon)^2}x, \quad x > 0, \tag{7}$$
$$\ddot{x} + \frac{1}{\varepsilon^2(\omega_\varepsilon)^2}x = g(t, x, \dot{x}, \varepsilon), \quad x < 0, \tag{8}$$

and consider the reduced system

$$\ddot{x} + \frac{1}{(1-\varepsilon\omega_\varepsilon)^2}x = 0, \quad x > 0, \tag{9}$$
$$\ddot{x} + \frac{1}{\varepsilon^2(\omega_\varepsilon)^2}x = 0, \quad x < 0. \tag{10}$$

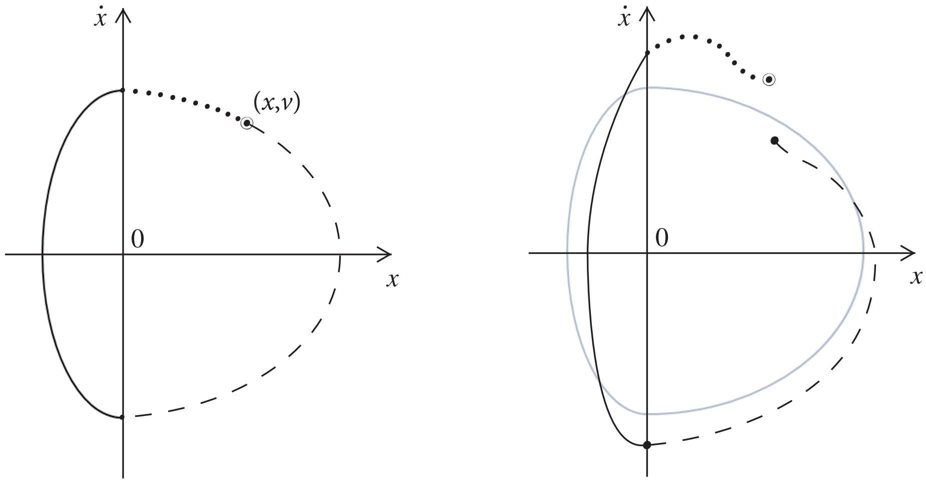

Figure 1: Left: The three pieces of the cycle of (9)–(10) with the initial condition at the dotted point: 1) from the initial condition to the time of switching from $x > 0$ to $x < 0$ (dashed), 2) from the latter switching to the switching from $x < 0$ to $x > 0$ (solid), 3) from the latter switching to π (dotted). Right: Illustration of the change of variables (15)–(17).

The idea behind separating the reduced system (9)–(10) is that any solution of (9)–(10) is π-periodic. To see this, observe that equations (9) and (10) are symmetric about $x = 0$. Therefore, the switching threshold $x = 0$ cuts exactly half of the cycle in equation (9) and exactly half of the cycle in equation (10). In other words, the period T of any cycle of (9)–(10) is the sum of (i) half of the period of cycles of (9) (denote this half by T^+) and (ii) half of the period of cycles of (10) (denote this half by T^-). Referring to Fig. 1 (left), the period T^+ represents the combined duration of dashed and dotted arcs, while the period T^- represents the duration of the solid arc. Now recall that the full period of cycles of a linear oscillator $\ddot{x} + kx = 0$ equals $2\pi/\sqrt{k}$, so that half of this period is $\pi/\sqrt{k}$. Therefore, $T^+ = \pi/\sqrt{1/(1-\varepsilon\omega_\varepsilon)^2} = (1-\varepsilon\omega_\varepsilon)\pi$ and $T^- = \varepsilon\omega_\varepsilon\pi$. Summing up, $T = T^+ + T^- = \pi$. Moreover, since equations (9)–(10) are linear we can construct periodic cycle with any initial condition $(\bar{x}(0), \dot{\bar{x}}(0)) = (x, v)$, $x > 0$, in closed-form to get

$$\begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix} = A \begin{pmatrix} \cos\left(\dfrac{1}{1-\varepsilon\omega_\varepsilon}(t+\theta)\right), \\ -\dfrac{1}{1-\varepsilon\omega_\varepsilon}\sin\left(\dfrac{1}{1-\varepsilon\omega_\varepsilon}(t+\theta)\right) \end{pmatrix} =: \Omega_{1,\varepsilon}(A, t+\theta), \quad t+\theta \in \left[\theta, \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon)\right], \tag{11}$$

$$\begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix} = \frac{A}{1-\varepsilon\omega_\varepsilon} \begin{pmatrix} \varepsilon\omega_\varepsilon \cos\left(\dfrac{1}{\varepsilon\omega_\varepsilon}(t+\theta-\dfrac{\pi}{2}(1-\varepsilon\omega_\varepsilon)) + \dfrac{\pi}{2}\right) \\ -\sin\left(\dfrac{1}{\varepsilon\omega_\varepsilon}(t+\theta-\dfrac{\pi}{2}(1-\varepsilon\omega_\varepsilon)) + \dfrac{\pi}{2}\right) \end{pmatrix} =: \Omega_{2,\varepsilon}(A, t+\theta),$$

$$t+\theta \in \left[\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon), \frac{\pi}{2}(1+\varepsilon\omega_\varepsilon)\right], \tag{12}$$

$$\begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix} = \Omega_{1,\varepsilon}(A, t+\theta+\pi+2\pi\varepsilon\omega_\varepsilon) =: \Omega_{3,\varepsilon}(A, t+\theta), \quad t+\theta \in \left[\frac{\pi}{2}(1+\varepsilon\omega_\varepsilon), \theta+\pi\right], \tag{13}$$

where (A, θ) computes from

$$\begin{pmatrix} x \\ v \end{pmatrix} = A \begin{pmatrix} \cos\left(\dfrac{1}{1-\varepsilon\omega_\varepsilon}\theta\right), \\ -\dfrac{1}{1-\varepsilon\omega_\varepsilon}\sin\left(\dfrac{1}{1-\varepsilon\omega_\varepsilon}\theta\right) \end{pmatrix}. \tag{14}$$

The three formulas (11), (12), and (13) describe the dashed, solid, and dotted pieces of the cycle as shown at Fig. 1(left).

The method we follow now is a standard tool of nonlinear dynamics used to analyze weakly nonlinear systems at resonance, see e.g. [5, Sections 14–15]. Since we expect that solutions of the full system (7)–(8) are close on the interval $[0,\pi]$ to the solutions of the reduced system (9)–(10), we will search for solutions of (7)–(8) in the form of (11)–(13), where the constant function $t\mapsto A$ and the linear function $t\mapsto t+\theta$ will be unknown functions $t\mapsto A(t)$ and $t\mapsto \theta(t)$ located ε-close to $t\mapsto A(0)$ and $t\mapsto t+\theta(0)$. To implement this idea, we now view A and θ as new variables (i.e. functions) and use formulas (11)–(13) to introduce the following change of the variables in the full system (7)–(8):

$$\begin{pmatrix} x \\ \dot x \end{pmatrix} = \Omega_{1,\varepsilon}(A,\theta),\ \ \theta\in\left[-\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon),\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon)\right], \tag{15}$$

$$\begin{pmatrix} x \\ \dot x \end{pmatrix} = \Omega_{2,\varepsilon}(A,\theta),\ \ \theta\in\left[\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon),\frac{\pi}{2}(1+\varepsilon\omega_\varepsilon)\right], \tag{16}$$

$$\begin{pmatrix} x \\ \dot x \end{pmatrix} = \Omega_{3,\varepsilon}(A,\theta),\ \ \theta\in\left[\frac{\pi}{2}(1+\varepsilon\omega_\varepsilon),\frac{\pi}{2}(3-\varepsilon\omega_\varepsilon)\right]. \tag{17}$$

Since $A(t)$ in (15)–(17) is no longer a constant (but close to a constant) and since $\theta(t)$ is no longer $t+\theta$ (but close to $t+\theta$), the three curves (15), (16), and (17) now slightly deviate from their unperturbed ($\varepsilon=0$) counterparts (11), (12), and (13), as Fig. 1(right) illustrates. The change of the variables (15)–(17), as expected, transforms equations (7)–(8) to the following system

$$\begin{pmatrix} \dot A \\ \dot\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon G_1(t,A,\theta,\varepsilon), \qquad \text{if}\quad \theta\in\left[-\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon),\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon)\right], \tag{18}$$

$$\begin{pmatrix} \dot A \\ \dot\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon G_2(t,A,\theta,\varepsilon), \qquad \text{if}\quad \theta\in\left[\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon),\frac{\pi}{2}(1+\varepsilon\omega_\varepsilon)\right], \tag{19}$$

$$\begin{pmatrix} \dot A \\ \dot\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon G_3(t,A,\theta,\varepsilon), \qquad \text{if}\quad \theta\in\left[\frac{\pi}{2}(1+\varepsilon\omega_\varepsilon),\frac{\pi}{2}(3-\varepsilon\omega_\varepsilon)\right], \tag{20}$$

where, for $j\in\{1,3\}$,

$$G_j(t,A,\theta,\varepsilon) = (1-\varepsilon\omega_\varepsilon)^2\begin{pmatrix} 0 & 1/A \\ -1/A^2 & 0 \end{pmatrix}\Omega_{j,\varepsilon}(A,\theta)\left(f(t,\Omega_{j,\varepsilon}(A,\theta),\varepsilon) - \frac{\omega_\varepsilon(-2+\varepsilon\omega_\varepsilon)}{(1-\varepsilon\omega_\varepsilon)^2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T\Omega_{j,\varepsilon}(A,\theta)\right),$$

$$\text{and}\ \ G_2(t,A,\theta,\varepsilon) = \frac{1}{\varepsilon}(1-\varepsilon\omega_\varepsilon)^2\begin{pmatrix} 0 & 1/A \\ -1/A^2 & 0 \end{pmatrix}\Omega_{2,\varepsilon}(A,\theta)g\left(t,\Omega_{2,\varepsilon}(A,\theta),\varepsilon\right).$$

Let (A_0,θ_0) be as given by the formulation of the theorem. In what follows, we restrict our analysis to some neighborhood $B_\Delta(A_0,\theta_0)$ of (A_0,θ_0), where $\Delta>0$ will be defined later. By $[y]_i$ we will denote the ith component of a vector y.

Step 1. *Construction of the Poincaré map P_ε for system (18)–(20).* First we show that solution $t\mapsto(\bar A(\cdot,A,\theta,\varepsilon),\bar\theta(\cdot,A,\theta,\varepsilon))$ of (18)–(20) on $[0,\pi]$ can be consequently sewed by solutions of systems (18), (19) and (20).

Denote by $t\mapsto(\bar A_i(\cdot,t_0,A,\theta,\varepsilon),\bar\theta_i(\cdot,t_0,A,\theta,\varepsilon))$, $i=1,2,3$, the solutions of (18), (19), (20) respectively with the initial condition (A,θ) at time t_0. Define $\Delta>0$ so that $[\Omega_{1,0}(A,\theta)]_1\ge 0$, for all $(A,\theta)\in B_{2\Delta}(A_0,\theta_0)$. Then, there exists $\varepsilon_0>0$ such that

$$[\Omega_{1,\varepsilon}(A,\theta)]_1\ge 0,\quad \text{for all}\ (A,\theta)\in B_\Delta(A_0,\theta_0),\ \varepsilon\in[0,\varepsilon_0].$$

Put

$$F_1(T, A, \theta, \varepsilon) = \frac{1}{1 - \varepsilon\omega_\varepsilon}\bar{\theta}_1(T, 0, A, \theta, \varepsilon) - \frac{\pi}{2}.$$

Since $F_1\left(\frac{\pi}{2} - \theta_0, A_0, \theta_0, 0\right) = \bar{\theta}_1\left(\frac{\pi}{2} - \theta_0, 0, A_0, \theta_0, 0\right) - \frac{\pi}{2} = 0$ and

$$(F_1)_T\left(\frac{\pi}{2} - \theta_0, A_0, \theta_0, 0\right) = \bar{\theta}_T\left(\frac{\pi}{2} - \theta_0, 0, A_0, \theta_0, 0\right) = 1$$

then by the implicit function theorem [17, Chapter X, Section 2, Theorems 1 and 2], $\Delta > 0$ and $\varepsilon_0 > 0$ can be reduced so that a continuously differentiable function $T_1(A, \theta, \varepsilon)$ is defined on $(A, \theta) \in B_\Delta(A_0, \theta_0)$ and $\varepsilon \in [0, \varepsilon_0]$, for which

$$T_1(A, \theta, \varepsilon) \to \frac{\pi}{2} - \theta_0, \quad \text{as} \quad (A, \theta, \varepsilon) \to (A_0, \theta_0, 0), \tag{21}$$

and

$$F_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon) = 0, \qquad (A, \theta) \in B_\Delta(A_0, \theta_0),\ \varepsilon \in [0, \varepsilon_0].$$

Or, equivalently,

$$\frac{1}{1 - \varepsilon\omega_\varepsilon}\bar{\theta}_1(T_1(A, \theta, \varepsilon), 0, A, \theta, \varepsilon) = \frac{\pi}{2}, \qquad (A, \theta) \in B_\Delta(A_0, \theta_0),\ \varepsilon \in [0, \varepsilon_0].$$

Therefore, the time when the solution of system (18) with the initial condition (A, θ) at $t = 0$ approaches the threshold between systems (18) and (19) equals $T_1(A, \theta, \varepsilon)$. Using that $\theta_0 \in (-\pi/2, \pi/2)$ (as given by the assumptions of the theorem) and (21), we can reduce $\Delta > 0$ and $\varepsilon_0 > 0$ further to have

$$T_1(A, \theta, \varepsilon) > 0, \quad \text{for all } (A, \theta) \in B_\Delta(A_0, \theta_0),\ \varepsilon \in [0, \varepsilon_0].$$

Now we show that, for any $(A, \theta) \in B_\Delta(A_0, \theta_0)$ and $\varepsilon \in [0, \varepsilon_0]$, the solution

$$\begin{pmatrix} \bar{A}_2 \\ \bar{\theta}_2 \end{pmatrix}\left(\cdot, T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon\right),$$

stays till some time $T_2(A, \theta, \varepsilon)$ in

$$[0, \infty) \times \left[\frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon)\right]$$

and that $T_2(A, \theta, \varepsilon)$ is given by

$$T_2(A, \theta, \varepsilon) = T_1(A, \theta, \varepsilon) + \varepsilon\widetilde{T}_2(A, \theta, \varepsilon),$$

where $\widetilde{T}_2(A, \theta, \varepsilon)$ is a continuously differentiable bounded function. To do this, consider

$$F_2(T, A, \theta, \varepsilon) = \begin{cases} \dfrac{1}{\varepsilon\omega_\varepsilon}\left(\bar{\theta}_2\left[T_1(A, \theta, \varepsilon) + \varepsilon T, T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \dfrac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon\right] - \dfrac{\pi}{2}(1 - \varepsilon\omega_\varepsilon)\right) - \pi, & \varepsilon > 0, \\ \dfrac{1}{\omega_0}T - \pi, \quad \varepsilon = 0. \end{cases}$$

Let us verify that the function F_2 satisfies the assumptions of the implicit function theorem at the point $(T, A, \theta, \varepsilon) = (\omega_0\pi, A_0, \theta_0, \varepsilon)$. Since

$$\frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) = \bar{\theta}_2\left(T_1(A, \theta, \varepsilon), T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon\right),$$

by the mean-value theorem,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} F_2(T, A, \theta, \varepsilon) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon\omega_\varepsilon} (\bar{\theta}_2)_T \left(T_1(A,\theta,\varepsilon) + \lambda(A,\theta,\varepsilon)\varepsilon T, T_1(A,\theta,\varepsilon), \bar{A}_1(T_1(A,\theta,\varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon), \varepsilon \right) \varepsilon T - \pi \\
&= \frac{1}{\omega_0} T - \pi,
\end{aligned}
$$

that is F_2 is continuous at $\varepsilon = 0$ (here $\lambda(A, \theta, \varepsilon) \in [0, 1]$ is the constant given by the mean-value theorem). Furthermore, we have

$$
(F_2)_T \left(\frac{\pi}{2} - \theta_0 + \pi, A_0, \theta_0, 0 \right) = \frac{1}{\omega_0} \neq 0.
$$

Therefore, the implicit function theorem allows as to again reduce $\Delta > 0$ and $\varepsilon_0 > 0$ to have a continuously differentiable function $\widetilde{T}_2(A, \theta, \varepsilon)$ defined on $(A, \theta) \in B_\Delta(A_0, \theta_0)$ and $\varepsilon \in [0, \varepsilon_0]$, such that

$$
\widetilde{T}_2(A, \theta, \varepsilon) \to \omega_0 \pi \quad \text{as } \varepsilon \to 0
$$

and

$$
\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}_2(T_1(A,\theta,\varepsilon) + \varepsilon\widetilde{T}_2(A,\theta,\varepsilon), T_1(A,\theta,\varepsilon), A_1(T_1(A,\theta,\varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon), \varepsilon) - \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon) \right) = \pi,
$$

for all $(A, \theta) \in B_\Delta(A_0, \theta_0)$, $\varepsilon \in [0, \varepsilon_0]$. Since

$$
\begin{aligned}
&\bar{\theta}_3 \left(\pi, T_1(A,\theta,\varepsilon) + \varepsilon\widetilde{T}_2(A,\theta,\varepsilon), \mathcal{A}, \frac{\pi}{2}(1+\varepsilon\omega_\varepsilon), \varepsilon \right) \to \bar{\theta}_3 \left(\pi, \frac{\pi}{2} - \theta, A, \frac{\pi}{2}, 0 \right) = \pi + \theta, \quad \text{as} \quad \varepsilon \to 0, \\
&\quad \text{where } \mathcal{A} = \bar{A}_2 \left(T_1(A,\theta,\varepsilon) + \varepsilon\widetilde{T}_2(A,\theta,\varepsilon), T_1(A,\theta,\varepsilon), \bar{A}_1(T_1(A,\theta,\varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon), \varepsilon \right),
\end{aligned}
$$

the constants $\Delta > 0$ and $\varepsilon_0 > 0$ can be reduced further so that the solution $(A(t), \theta(t))$ of (20) with the initial condition $\left(\mathcal{A}, \frac{\pi}{2}(1+\varepsilon\omega_\varepsilon)\right)$ satisfies $[\Omega_{3,\varepsilon}(A(t), \theta(t))]_1 \geq 0$ on the time-interval $(T_1(A,\theta,\varepsilon) + \varepsilon\widetilde{T}_2(A,\theta,\varepsilon), \pi]$.

Summarizing, we can define the solution $t \mapsto \left(\bar{A}(t, A, \theta, \varepsilon), \bar{\theta}(t, A, \theta, \varepsilon)\right)$ of system (18)–(20) as follows

$$
\begin{aligned}
\begin{pmatrix} \bar{A} \\ \bar{\theta} \end{pmatrix} (t, A, \theta, \varepsilon) &= \begin{pmatrix} \bar{A}_1 \\ \bar{\theta}_1 \end{pmatrix} (t, 0, A, \theta, \varepsilon), \quad \text{if } \ t \in [0, T_1(A, \theta, \varepsilon)], \\
\begin{pmatrix} \bar{A} \\ \bar{\theta} \end{pmatrix} (t, A, \theta, \varepsilon) &= \begin{pmatrix} \bar{A}_2 \\ \bar{\theta}_2 \end{pmatrix} \left(t, T_1(A,\theta,\varepsilon), \bar{A}(T_1(A,\theta,\varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon), \varepsilon \right), \quad \text{if } \ t \in (T_1(A,\theta,\varepsilon), \tau], \\
\begin{pmatrix} \bar{A} \\ \bar{\theta} \end{pmatrix} (t, A, \theta, \varepsilon) &= \begin{pmatrix} \bar{A}_3 \\ \bar{\theta}_3 \end{pmatrix} \left(t, \tau, \bar{A}(\tau, A, \theta, \varepsilon), \frac{\pi}{2}(1+\varepsilon\omega_\varepsilon), \varepsilon \right), \quad \text{if } \ t \in (\tau, \pi], \quad \tau = T_1(A,\theta,\varepsilon) + \varepsilon\widetilde{T}_2(A,\theta,\varepsilon).
\end{aligned}
$$

Define the Poincaré map of system (18)–(20) as

$$
P_\varepsilon(A, \theta) = \begin{pmatrix} \bar{A} \\ \bar{\theta} \end{pmatrix} (\pi, A, \theta, \varepsilon).
$$

There should be no confusion between the usage of subindex to denote a partial derivative and a parameter throughout the paper. The subindex of a function stays for the parameter, when the given subindex is not listed as a variable of the function. In other words ε denotes a parameter in the notation $P_\varepsilon(A, \theta)$ and ε would denote a partial derivative with respect to ε in the notation $P_\varepsilon(A, \theta, \varepsilon)$.

The solution $x(t)$ of system (7)–(8) with the initial condition (x, v) can be computed from the solution $\begin{pmatrix} \bar{A} \\ \bar{\theta} \end{pmatrix} (t, \Omega_{1,\varepsilon}^{-1}(x, v))$ by applying the change of the variables (15)–(17).

Step 2. *Computing the leading order term of the expansion of P_ε in ε.* Let us decompose P_ε as

$$P_\varepsilon(A,\theta) = \begin{pmatrix} A \\ \theta+\pi \end{pmatrix} + \varepsilon(\bar{P}_{\varepsilon,1}(A,\theta) + \bar{P}_{\varepsilon,2}(A,\theta) + \bar{P}_{\varepsilon,3}(A,\theta)),$$

where

$$\bar{P}_{\varepsilon,1}(A,\theta) = \int\limits_0^{T_1(A,\theta,\varepsilon)} G_1(\tau, \bar{A}(\tau,A,\theta,\varepsilon), \bar{\theta}(\tau,A,\theta,\varepsilon),\varepsilon)\mathrm{d}\tau,$$

$$\bar{P}_{\varepsilon,2}(A,\theta) = \int\limits_{T_1(A,\theta,\varepsilon)}^{T_1(A,\theta,\varepsilon)+\varepsilon\widetilde{T}_2(A,\theta,\varepsilon)} G_2(\tau, \bar{A}(\tau,A,\theta,\varepsilon), \bar{\theta}(\tau,A,\theta,\varepsilon),\varepsilon)\mathrm{d}\tau,$$

$$\bar{P}_{\varepsilon,3}(A,\theta) = \int\limits_{T_1(A,\theta,\varepsilon)+\varepsilon\widetilde{T}_2(A,\theta,\varepsilon)}^{\pi} G_3(\tau, \bar{A}(\tau,A,\theta,\varepsilon), \bar{\theta}(\tau,A,\theta,\varepsilon),\varepsilon)\mathrm{d}\tau.$$

Since sin, cos and g are bounded on any bounded set then from system (18)–(20), we have that

$$\begin{pmatrix} \bar{A}(t,A,\theta,\varepsilon) \\ \bar{\theta}(t,A,\theta,\varepsilon) \end{pmatrix} \to \begin{pmatrix} A \\ t+\theta \end{pmatrix} \quad \text{as} \quad \varepsilon \to 0$$

uniformly with respect to $t \in [0,\pi]$ and $(A,\theta) \in B_\Delta(A_0,\theta_0)$. This gives

$$\begin{aligned} \bar{P}_{\varepsilon,1}(A,\theta) &\to \int\limits_0^{T_1(A,\theta,0)} G_1(\tau, A, \tau+\theta, 0)\mathrm{d}\tau, \\ \bar{P}_{\varepsilon,3}(A,\theta) &\to \int\limits_{T_1(A,\theta,0)}^{\pi} G_3(\tau, A, \tau+\theta, 0)\mathrm{d}\tau, \quad \text{as} \quad \varepsilon \to 0, \end{aligned} \tag{22}$$

uniformly with respect to $(A,\theta) \in B_\Delta(A_0,\theta_0)$. Since we proved that T_1 is continuously differentiable, then (22) implies that

$$\begin{aligned} (\bar{P}_{\varepsilon,1})'(A,\theta) &\to (P_{0,1})'(A,\theta), \\ (\bar{P}_{\varepsilon,3})'(A,\theta) &\to (P_{0,3})'(A,\theta), \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

uniformly with respect to $(A,\theta) \in B_\Delta(A_0,\theta_0)$.

Let us now study the behavior of $\bar{P}_{\varepsilon,2}$ and $(\bar{P}_{\varepsilon,2})'$ as $\varepsilon \to 0$. We have

$$\bar{P}_{\varepsilon,2}(A,\theta) = -(1-\varepsilon\omega_\varepsilon) \int\limits_{T_1(A,\theta,\varepsilon)}^{T_1(A,\theta,\varepsilon)+\varepsilon\widetilde{T}_2(a,\theta,\varepsilon)} \begin{pmatrix} \dfrac{1}{\varepsilon}\sin\xi(\tau) \\ \dfrac{1}{\bar{A}(\tau,A,\theta,\varepsilon)}(1-\varepsilon\omega_\varepsilon)\omega_\varepsilon\cos\xi(\tau) \end{pmatrix}$$

$$\times g\left(\tau, \varepsilon\bar{A}(\tau,A,\theta,\varepsilon)\frac{\omega_\varepsilon\cos\xi(\tau)}{1-\varepsilon\omega_\varepsilon}, -\bar{A}(\tau,A,\theta,\varepsilon)\frac{\sin\xi(\tau)}{1-\varepsilon\omega_\varepsilon},\varepsilon\right)\mathrm{d}\tau, \quad \xi(\tau) = \frac{1}{\varepsilon\omega_\varepsilon}\left(\bar{\theta}(\tau,A,\theta,\varepsilon) - \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon)\right) + \frac{\pi}{2}.$$

Scaling the time in the integral as $\tau = T_1(A,\theta,\varepsilon) + \varepsilon\omega_\varepsilon s$, we get

$$\bar{P}_{\varepsilon,2}(A,\theta) = -\omega_\varepsilon(1-\varepsilon\omega_\varepsilon) \int\limits_0^{\widetilde{T}_2(A,\theta,\varepsilon)/\omega_\varepsilon} \begin{pmatrix} \sin\xi(\tau) \\ \varepsilon\dfrac{1}{\bar{A}(\zeta(s),A,\omega,\varepsilon)}(1-\varepsilon\omega_\varepsilon)\omega_\varepsilon\cos\xi(s) \end{pmatrix}$$

$$\times g\left(\zeta(s), \varepsilon\bar{A}(\zeta(s),A,\theta,\varepsilon)\frac{\omega_\varepsilon}{1-\varepsilon\omega_\varepsilon}\cos\xi(s), -\bar{A}(\zeta(s),A,\theta,\varepsilon)\frac{1}{1-\varepsilon\omega_\varepsilon}\sin\xi(s),\varepsilon\right)\mathrm{d}s,$$

$$\xi(s) = \left(\frac{1}{\varepsilon\omega_\varepsilon}\left(\bar{\theta}(T_1(A,\theta,\varepsilon)+\varepsilon s\omega_\varepsilon, A,\theta,\varepsilon) - \frac{\pi}{2}(1-\varepsilon\omega_\varepsilon)\right) + \frac{\pi}{2}\right), \quad \zeta(s) = T_1(A,\theta,\varepsilon) + \varepsilon\omega_\varepsilon s.$$

Put

$$K(A,\theta,\varepsilon)=\frac{1}{\varepsilon}\left(\bar\theta(T_1(A,\theta,\varepsilon)+\varepsilon\omega_\varepsilon s,A,\theta,\varepsilon)-\bar\theta(T_1(A,\theta,\varepsilon),A,\theta,\varepsilon)\right).$$

Since

$$\frac{1}{\varepsilon}\left(\bar\theta(T_1(A,\theta,\varepsilon)+\varepsilon\omega_\varepsilon s,A,\theta,\varepsilon)-\frac{\pi}{2}(1-\varepsilon\omega_\varepsilon)\right)=K(A,\theta,\varepsilon)\to\bar\theta_T(T_1(A,\theta,0),A,\theta,0)\omega_0 s=\omega_0 s,\quad \text{as}\quad \varepsilon\to 0,$$

then

$$\bar P_{\varepsilon,2}(A,\theta)\to-\omega_0\int\limits_0^\pi\begin{pmatrix}\sin\left(s+\frac{\pi}{2}\right)\\ 0\end{pmatrix}g\left(\frac{\pi}{2}-\theta,0,A\sin\left(s+\frac{\pi}{2}\right),0\right)\mathrm{d}s,\qquad \text{as}\quad \varepsilon\to 0,$$

uniformly with respect to $(A,\theta)\in B_\Delta(A_0,\theta_0)$. Since $K(A,\theta,\varepsilon)$ converges as $\varepsilon\to 0$ uniformly in $(A,\theta)\in B_\Delta(A_0,\theta_0)$, then $K_{(A,\theta)}(A,\theta,\varepsilon)\to K_{(A,\theta)}(A,\theta,0)$ as $\varepsilon\to 0$. Therefore,

$$(\bar P_{\varepsilon,2})'(A,\theta)\to(\bar P_{0,2})'(A,\theta)\quad \text{as}\quad \varepsilon\to 0$$

uniformly with respect to $(A,\theta)\in B_\Delta(A_0,\theta_0)$.

To conclude, we proved that the Poincaré map $P_\varepsilon(A,\theta)$ of system (18)–(20) can be decomposed as

$$P_\varepsilon(A,\theta)=\begin{pmatrix}A\\ \theta+\pi\end{pmatrix}+\varepsilon\bar P_\varepsilon(A,\theta),\ \text{ where } \bar P_\varepsilon(A,\theta)\to\bar P(A,\theta)\ \text{ and }\ (\bar P_\varepsilon)'(A,\theta)\to\bar P'(A,\theta)\ \text{ as } \varepsilon\to 0,\quad (23)$$

uniformly with respect to $(A,\theta)\in B_\Delta(A_0,\theta_0)$.

Step 3. *Linking π-periodic solutions of initial system (7)–(8) to the properties of the Poincaré map P_ε.* For given $\varepsilon\in[0,\varepsilon_0]$ and $(\bar x,\bar v)\in\Omega_{1,\varepsilon}(B_\Delta(A_0,\theta_0))$, let $x(t)$ be the solution of system (7)–(8) with the initial condition $(x(0),\dot x(0))=(\bar x,\bar v)$. Define the Poincaré map $\mathcal{P}_\varepsilon$ of system (7)–(8) as $\mathcal{P}_\varepsilon(\bar x,\bar v)=(x(\pi),\dot x(\pi))$. Fixed points of the map $\mathcal{P}_\varepsilon$ are the initial conditions of π-periodic solutions of system (7)–(8). Furthermore, stability of fixed fixed points of $\mathcal{P}_\varepsilon$ coincides with stability of corresponding π-periodic solutions of system (7)–(8). To investigate fixed points of $\mathcal{P}_\varepsilon$ we recall that according to the change of the variables (15)–(17), $\mathcal{P}_\varepsilon$ and P_ε are linked by the formula

$$\mathcal{P}_\varepsilon(x,v)=\Omega_{3,\varepsilon}(P_\varepsilon(\Omega_{1,\varepsilon}^{-1}(x,v))).\qquad (24)$$

Therefore, (x,v) is a fixed point of $\mathcal{P}_\varepsilon$ if and only if

$$P_\varepsilon(\Omega_{1,\varepsilon}^{-1}(x,v))=\Omega_{3,\varepsilon}^{-1}(x,v).\qquad (25)$$

In order to make use of (23) we now want to rewrite (25) in terms of

$$\begin{pmatrix}A\\ \theta\end{pmatrix}=\Omega_{1,\varepsilon}^{-1}(x,v).\qquad (26)$$

Therefore, given (26) we need to find $\Omega_{3,\varepsilon}^{-1}(x,v)$. This means that given $(A,\theta)\in B_\Delta(A_0,\theta_0)$ we have to find (A_*,θ_*) with $\theta_*\in\left[\frac{\pi}{2}(1+\varepsilon\omega_\varepsilon),\pi\right]$ such that

$$\Omega_{1,\varepsilon}(A,\theta)=\Omega_{3,\varepsilon}(A_*,\theta_*).$$

Using (15) and (16), the above equality leads to $A=A_*$ and $\frac{1}{1-\varepsilon\omega_\varepsilon}\theta+2\pi=\frac{1}{1-\varepsilon\omega_\varepsilon}(\theta_*-2\pi\varepsilon\omega_\varepsilon+\pi)$, yielding $\theta=\theta_*-\pi$. Therefore, (25) can be rewritten as

$$P_\varepsilon(A,\theta)=\begin{pmatrix}A\\ \theta+\pi\end{pmatrix}.\qquad (27)$$

And we can finally conclude that $(x,v) \in \Omega_{1,\varepsilon}(B_\Delta(A_0,\theta_0))$ is a fixed point of $\mathcal{P}_\varepsilon$ if and only if $(A,\theta) \in B_\Delta(A_0,\theta_0)$ given by (26) satisfies (27).

Step 4. *Proof of the necessity part.* Let x_ε be a π-periodic solution of (1) satisfying (3). Therefore x_ε is a solution of (7)–(8). According to (26), define $(A_\varepsilon,\theta_\varepsilon)$ as

$$\left(\begin{array}{c} A_\varepsilon \\ \theta_\varepsilon \end{array}\right) = \Omega_{1,\varepsilon}^{-1}(x_\varepsilon(0),\dot{x}_\varepsilon(0)) \tag{28}$$

By continuity of $\Omega_{1,\varepsilon}^{-1}$, we have that $(A_\varepsilon,\theta_\varepsilon)$ converges as $\varepsilon \to 0$. Put $(A_0,\theta_0) = \lim_{\varepsilon\to 0}(A_\varepsilon,\theta_\varepsilon)$. Plugging (23) into (27), we get

$$\left(\begin{array}{c} A_\varepsilon \\ \theta_\varepsilon+\pi \end{array}\right) = \left(\begin{array}{c} A_\varepsilon \\ \theta_\varepsilon+\pi \end{array}\right) + \varepsilon \bar{P}_\varepsilon(A_\varepsilon,\theta_\varepsilon),$$

from where $\bar{P}_\varepsilon(A_\varepsilon,\theta_\varepsilon) = 0$ for all $\varepsilon > 0$ sufficiently small. Passing to the limit as $\varepsilon \to 0$ in the latter relation, we conclude $\bar{P}_0(A_0,\theta_0) = 0$. The relation (5) follows by passing to the limit as $\varepsilon \to 0$ in (28).

Step 5. *Proof of the sufficiency part.* As in the proof of the necessity part, we plug (23) into (27) to get the following equation for (A,θ)

$$\bar{P}_\varepsilon(A,\theta) = 0. \tag{29}$$

Thanks to assumptions (4) and (6) we can now apply the implicit function theorem to conclude that equation (29) has a unique solution

$$(A_\varepsilon,\theta_\varepsilon) \to (A_0,\theta_0), \quad \text{as } \varepsilon \to 0.$$

To investigate stability of the fixed point

$$\left(\begin{array}{c} x_\varepsilon \\ v_\varepsilon \end{array}\right) = \Omega_{1,\varepsilon}(A_\varepsilon,\theta_\varepsilon),$$

denote by λ_0 an eigenvalue (real or complex) of $(\bar{P})'(A_0,\theta_0)$. Then the matrix $(\bar{P}_\varepsilon)'(A_\varepsilon,\theta_\varepsilon)$ admits an eigenvalue λ_ε such that

$$\lambda_\varepsilon \to \lambda_0 \quad \text{as} \quad \varepsilon \to 0.$$

Recalling (23), we have that $1+\varepsilon\lambda_\varepsilon$ is an eigenvalue of $(P_\varepsilon)'(A_\varepsilon,\theta_\varepsilon)$. From (24),

$$(\mathcal{P}_\varepsilon)'(x,v) = (\Omega_{3,\varepsilon})'\left(P_\varepsilon(\Omega_{1,\varepsilon}^{-1}(x,v))\right)(P_\varepsilon)'(\Omega_{1,\varepsilon}^{-1}(x,v))(\Omega_{1,\varepsilon}^{-1})'(x,v).$$

By the formula for the derivative of the inverse function, $(\Omega_{1,\varepsilon}^{-1})'(x,v) = \left[(\Omega_{1,\varepsilon})'(\Omega_{1,\varepsilon}^{-1}(x,v))\right]^{-1}$. Therefore,

$$\begin{aligned}(\mathcal{P}_\varepsilon)'(x_\varepsilon,v_\varepsilon) &= (\Omega_{3,\varepsilon})'(P_\varepsilon(A_\varepsilon,\theta_\varepsilon))(P_\varepsilon)'(A_\varepsilon,\theta_\varepsilon)\left[(\Omega_{1,\varepsilon})'(A_\varepsilon,\theta_\varepsilon)\right]^{-1} \\ &= (\Omega_{3,\varepsilon})'(A_\varepsilon,\theta_\varepsilon+\pi)(P_\varepsilon)'(A_\varepsilon,\theta_\varepsilon)\left[(\Omega_{1,\varepsilon})'(A_\varepsilon,\theta_\varepsilon)\right]^{-1}.\end{aligned}$$

Now we compute $(\Omega_{3,\varepsilon})'(A_\varepsilon,\theta_\varepsilon+\pi)$ and see that

$$(\Omega_{3,\varepsilon})'(A_\varepsilon,\theta_\varepsilon+\pi) = (\Omega_{1,\varepsilon})'(A_\varepsilon,\theta_\varepsilon),$$

which implies that the eigenvalues of $(\mathcal{P}_\varepsilon)'(x_\varepsilon,v_\varepsilon)$ coincide with the eigenvalues of $(P_\varepsilon)'(A_\varepsilon,\theta_\varepsilon)$. Therefore, $1+\varepsilon\lambda_\varepsilon$ is an eigenvalue of $(\mathcal{P}_\varepsilon)'(x_\varepsilon,v_\varepsilon)$. In other words, $1+\varepsilon\lambda_\varepsilon$ is a Floquet multiplier (or characteristic multiplier) of the π-periodic solution of (7)–(8) with the initial condition $(x(0),\dot{x}(0)) = (x_\varepsilon,v_\varepsilon)$. It remains to notice that when $\mathrm{Re}(\lambda_0) \neq 0$ (which is an assumption of the theorem), the number $1+\varepsilon\lambda_\varepsilon$ is inside of the unit circle for all $\varepsilon > 0$ sufficiently small or outside of the unit circle for all $\varepsilon > 0$ sufficiently small according to whether $\lambda_0 < 0$ or $\lambda_0 > 0$. This concludes the proof of the stability statement.

The proof of the theorem is complete. □

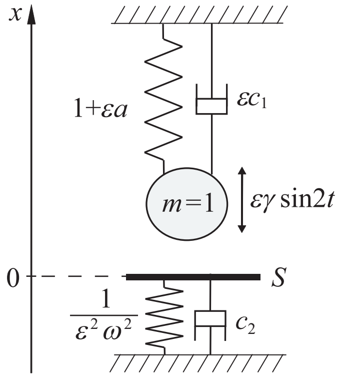

Figure 2: Model of a preloaded ball (body of mass $m = 1$) bouncing against a nearly elastic surface S which is represented by a spring of stiffness $\frac{1}{\varepsilon^2\omega^2}$ with $\varepsilon > 0$ small. The rest coordinate of the mass is assumed to coincide with the origin of the x-axis on the one hand and with the rest coordinate of surface S on the other hand. We also suppose that the viscous friction coefficient equals εc_1 outside the contact with S and that it takes the value $c_2 + \varepsilon c_1$ during the contact.

3. An application

In this section we apply the result of section 2 to an impact oscillator of Fig. 2. In this model, a body of mass $m = 1$ is bouncing against a nearly elastic surface S (of stiffness $1/\varepsilon^2\omega^2$). Assuming in addition that the body is subjected to viscous friction (linear damping), Rayleigh excitation (nonlinear damping, see [32]), and periodic forcing, the equation of motions can be written in the form (1) as follows

$$\begin{aligned} &\ddot{x} + x = -\varepsilon a x - \varepsilon c_1 \dot{x} + \varepsilon\mu_1 \dot{x}(1-\dot{x}^2) + \varepsilon\gamma \sin 2t, && \text{if} \quad x \geqslant 0, \\ &\ddot{x} + \frac{1}{\varepsilon^2\omega^2}x = -(c_2 + \varepsilon c_1)\dot{x} + (\mu_2 + \varepsilon\mu_1)\dot{x}(1-\dot{x}^2) + \varepsilon\gamma \sin 2t, && \text{if} \quad x < 0. \end{aligned} \tag{30}$$

There is no obvious reason why the Rayleigh excitation should be $O(1)$ during impact, but as this does not complicate the analysis, we admit this possibility. Following the strategy of the averaging method (see [5, 29]), we view system (30) as a damped periodic excitation of the following autonomous system

$$\begin{aligned} &\ddot{x} + x = -\varepsilon a x, && \text{if} \quad x \geqslant 0, \\ &\ddot{x} + \frac{1}{\varepsilon^2\omega^2}x = 0, && \text{if} \quad x < 0, \end{aligned}$$

which oscillates at the period $T_\varepsilon = \pi/\sqrt{1+\varepsilon a} + \varepsilon\pi\omega$. In other words, the period T_ε corresponds to the natural frequency of system (30) and the period π corresponds to the frequency of the excitation. The period T_ε is close to π but is different from π. We are going to prove the existence of asymptotically stable π-periodic oscillations in (30) meaning that π-periodic excitation in (30) overrides the natural period of oscillations and makes the full system oscillating at the period of the excitation (called period locking).

The averaging function $\bar{P}$ computes as

$$\begin{aligned}\bar{P}(A,\theta) &= -\int_0^{\pi/2-\theta}\begin{pmatrix}\sin(\tau+\theta)\\(1/A)\cos(\tau+\theta)\end{pmatrix}(-aA\cos(\tau+\theta)+c_1A\sin(\tau+\theta)\\
&\quad -\mu_1A\sin(\tau+\theta)(1-A^2\sin^2(\tau+\theta))+\gamma\sin 2\tau\,+\,2\omega A\cos(\tau+\theta))\mathrm{d}\tau\\
&\quad -\int_{\pi/2-\theta}^{\pi}\begin{pmatrix}\sin(\tau+\theta+\pi)\\(1/A)\cos(\tau+\theta+\pi)\end{pmatrix}(-aA\cos(\tau+\theta+\pi)+c_1A\sin(\tau+\theta+\pi)\\
&\quad -\mu_1A\sin(\tau+\theta+\pi)(1-A^2\sin^2(\tau+\theta+\pi))+\gamma\sin 2\tau\,+\,2\omega A\cos(\tau+\theta+\pi))\mathrm{d}\tau\\
&\quad -\omega\int_0^{\pi}\begin{pmatrix}\sin\left(\tau+\frac{\pi}{2}\right)\\0\end{pmatrix}\left(c_2A\sin\left(\tau+\frac{\pi}{2}\right)-\mu_2A\sin\left(\tau+\frac{\pi}{2}\right)\left(1-A^2\sin^2\left(\tau+\frac{\pi}{2}\right)\right)\right)\mathrm{d}\tau\\
&= \begin{pmatrix}-\frac{\pi}{2}A(c_1+c_2\omega)+\frac{\pi}{2}(\mu_1+\mu_2\omega)A\left(1-\frac{3}{4}A^2\right)-\frac{4}{3}\gamma\cos 2\theta\\ \frac{\pi}{2}(a\,-\,2\omega)\,+\,\frac{2}{3A}\gamma\sin 2\theta\end{pmatrix}.\end{aligned}$$

To compute zeros of $\bar{P}(A,\theta)$ we assume

$$\left|\frac{3\pi A}{4\gamma}(a-2\omega)\right|\le 1, \tag{31}$$

and solve the second equation of $\bar{P}(A,\theta)=0$ for θ obtaining two solutions

$$2\widetilde{\theta}(A)=-\arcsin\left(\frac{3\pi A}{4\gamma}(a-2\omega)\right)\quad\text{or}\quad 2\widehat{\theta}(A)=\pi+\arcsin\left(\frac{3\pi A}{4\gamma}(a-2\omega)\right), \tag{32}$$

which correspond to $\cos(2\widetilde{\theta}(A))\ge 0$ and $\cos(2\widehat{\theta}(A))\le 0$, respectively. We will stick to the second solution because, as we will see, it leads us to a positive A solving $\bar{P}(A,\theta)=0$. The reader can examine the first solution by analogy. Finding zeros of $\bar{P}(A,\theta)$ now reduces to finding zeros of a scalar function

$$M(A)=\left[\bar{P}(A,\widehat{\theta}(A))\right]_1=-\frac{\pi}{2}A(c_1+c_2\omega)+\frac{\pi}{2}(\mu_1+\mu_2\omega)A\left(1-\frac{3}{4}A^2\right)+\frac{1}{3}\sqrt{(4\gamma)^2-(3\pi A)^2(a-2\omega)^2},$$

where we used that

$$\cos(2\widehat{\theta}(A))\le 0 \tag{33}$$

by (32), to determine the sign in front of the square root. Observe that saying that $M(A)$ is defined implies that (31) is satisfied. Therefore, for any $A_0>0$ such that $M(A_0)=0$, the pair $(A_0,\widehat{\theta}(A_0))$ provides a zero of $\bar{P}$. We will now use the following lemma to compute $\det\bar{P}'(A,\widehat{\theta}(A))$.

Lemma 1. *Consider $F,G\in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$. Assume that $G(A,\widehat{\theta}(A))=0$ for some open interval of values of A where $\widehat{\theta}$ is a differentiable function. Then*

$$\frac{d}{dA}F(A,\widehat{\theta}(A))=\frac{1}{G_\theta(A,\widehat{\theta}(A))}\det\begin{pmatrix}F_A(A,\widehat{\theta}(A)) & F_\theta(A,\widehat{\theta}(A))\\ G_A(A,\widehat{\theta}(A)) & G_\theta(A,\widehat{\theta}(A))\end{pmatrix}.$$

The proof of the lemma comes by expressing $\widehat{\theta}'(A)$ from $G(A,\widehat{\theta}(A))=0$ and plugging the result to

$$\frac{d}{dA}F(A,\widehat{\theta}(A))=F_A(A,\widehat{\theta}(A))+F_\theta(A,\widehat{\theta}(A))\widehat{\theta}'(A).$$

Using Lemma 1, property (33), and the computation

$$\left[\bar{P}_\theta(A,\theta)\right]_2=\frac{4\gamma}{3A}\cos 2\theta,$$

we conclude that, when $\gamma > 0$ and $A > 0$,

$$\operatorname{sgn}(\det \bar{P}'(A, \widehat{\theta}(A))) = -\operatorname{sgn}(M'(A)).$$

The next lemma will be used to determine the signs of the eigenvalues of $\bar{P}'(A, \widehat{\theta}(A))$.

Lemma 2. *Consider a 2×2 real matrix D. Assume that matrix D is diagonalizable over $\mathbb{C}$, i.e. D admits two eigenvalues λ_1 and λ_2. Then the following statements hold up to the change of the indices of lambdas.*

If $\det D > 0$ *and* $\operatorname{tr} D < 0$ *then* $\operatorname{Re}\lambda_1 < 0$ *and* $\operatorname{Re}\lambda_2 < 0$*;*

If $\det D > 0$ *and* $\operatorname{tr} D > 0$ *then* $\operatorname{Re}\lambda_1 > 0$ *and* $\operatorname{Re}\lambda_2 > 0$*;*

If $\det D < 0$ *then* $\lambda_1\lambda_2 < 0$*.*

The statement of the lemma follows by direct computation of the eigenvalues of D according to the standard formula for the roots of a quadratic equation.

To compute $\operatorname{tr} \bar{P}'(A, \widehat{\theta}(A))$ we recall that

$$\operatorname{tr} \bar{P}'(A, \widehat{\theta}(A)) = -\frac{\pi}{2}(c_1 + c_2\omega) + \frac{\pi}{2}(\mu_1 + \mu_2\omega)\left(1 - \frac{9}{4}A^2\right) + \frac{4\gamma}{3A}\cos 2\theta. \tag{34}$$

Assuming that $A_0 > 0$ is found such that $\left[\bar{P}(A_0, \widehat{\theta}(A_0))\right]_1 = 0$, we can express $\cos 2\widehat{\theta}(A_0)$ from the latter equality and plug to (34) obtaining

$$\operatorname{tr} \bar{P}(A_0, \widehat{\theta}(A_0)) = -\pi(c_1 + c_2\omega) + \pi(\mu_1 + \mu_2\omega)\left(1 - \frac{3}{2}A_0^2\right). \tag{35}$$

The above findings can be summarized as the following proposition.

Proposition 1. *Assume that all the parameters in equation (30) are non-negative. If there exists $A_0 > 0$ such that $M(A_0) = 0$ and $M'(A_0) \neq 0$, then, for all $\varepsilon > 0$ sufficiently small, equation (30) has exactly one π-periodic solution*

$$(x_\varepsilon(0), \dot{x}_\varepsilon(0)) \to (A_0 \cos \widehat{\theta}(A_0), -A_0 \sin \widehat{\theta}(A_0)) \ \text{as } \varepsilon \to 0.$$

The solution x_ε is attracting, repelling, or a saddle according to whether

(a) $M'(A_0) < 0$ *and (35) is negative,*

(b) $M'(A_0) < 0$ *and (35) is positive,*

(c) $M'(A_0) > 0$*.*

Remark 1. *Assume $a \neq 2\omega$. Let A_1 be the largest value for which the square root in the definition of function $A \mapsto M(A)$ is defined, i.e. let $A_1 = \frac{2\gamma}{3\pi|a-2\omega|}$. Since we always have $M(0) > 0$, the property $M(A_1) < 0$ is a sufficient condition for the function M to admit a zero $A_0 \in [0, A_1]$ with $M'(A_0) < 0$.*

Simulation results in Figures 3 and 4 show that the amplitude of the attracting limit cycle of equation (30) does indeed approaches the zero of function $A \mapsto M(A)$ as $\varepsilon \to 0$. Furthermore, the two figures shows that the shape of the limit cycle can approach limiting shape differently for different sets of parameters.

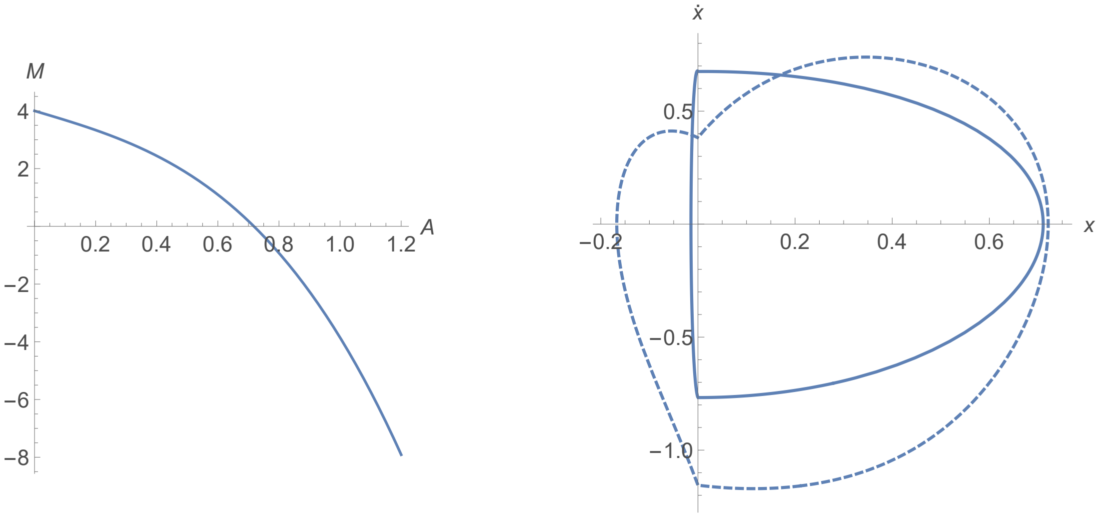

Figure 3: Simulation results for the parameters $c_1 = 1$, $c_2 = 5$, $\mu_1 = 2.5$, $\mu_2 = 1.5$, $\gamma = 3$, $a = 2$, $\omega = 1$. Left: The graph of function $M(A)$, which shows that $M'(A_0) < 0$. Computation in Mathematica returns $A_0 = 0.718$ and estimates (35) as -16.0007. Right: Attracting cycles of system (30) for $\varepsilon = 0.3$ (dashed) and for $\varepsilon = 0.02$ (solid).

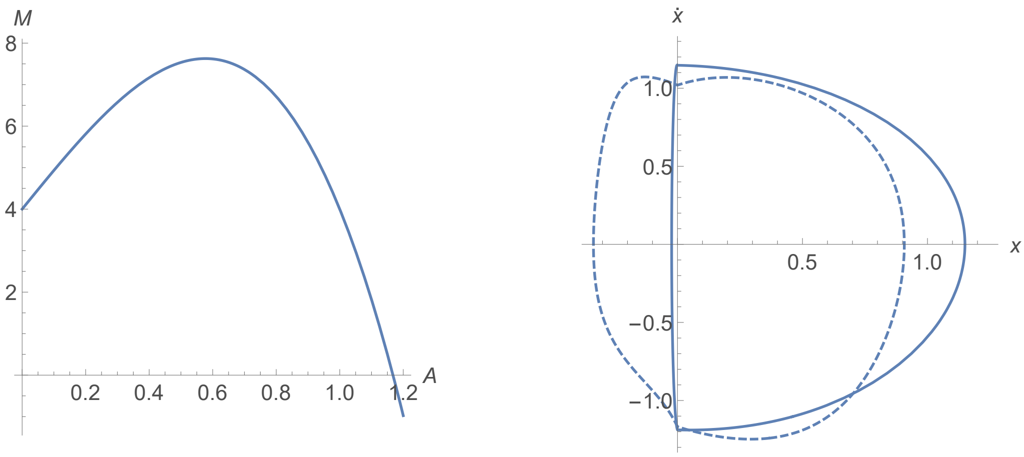

Figure 4: Simulation results for the parameters $c_1 = 1$, $c_2 = 1$, $\mu_1 = 3$, $\mu_2 = 5$, $\gamma = 3$, $a = 2$, $\omega = 1$. Left: The graph of function $M(A)$, which shows that $M'(A_0) < 0$. Computation in Mathematica returns $A_0 = 1.1677$ and estimates (35) as -32.552. Right: Attracting cycles of system (30) for $\varepsilon = 0.3$ (dashed) and for $\varepsilon = 0.02$ (solid).

4. Conclusion

In this paper we proved the existence of stable fixed points for a Poincaré map (also called period-map or stroboscopic map) with small parameter $\varepsilon > 0$ in the case where the flow of the system corresponding to $\varepsilon > 0$ is smooth while the flow for $\varepsilon = 0$ contains an impact. To carry out the proof, we rescale the time of the flow on the fast time-interval, which allows to expand the Poincaré map as in the standard averaging principle. The first term of the expansion provides a bifurcation function (also called averaging function or subharmonic Melnikov function, see Theorem 1), which explains how contributions of the slow and fast right-hand-sides of the bilinear oscillator need to be counted.

We have formulated an unusual type of singular perturbation problem. Putting $\varepsilon = 0$, we have a non-smooth impact, for $\varepsilon > 0$ we have fast motion in a neighborhood of the subset $x = 0$. For $x > 0$ slow motion takes place but this is not described by standard slow manifold theory, see [32]. Still, the dynamics for $x > 0$ can be considered as taking place in an explicitly formulated slow manifold. On the other hand, the solutions for $x < 0$ have as slow manifold the boundary $x = 0$. This does not satisfy the necessary hyperbolicity condition, but the solutions for $x > 0$ are forced to the manifold $x = 0$ and, after passing by a fast transition through the domain $x < 0$ they are forced again to leave $x = 0$. We note also that sliding along the slow manifold, as happens for instance in dry friction problems, is not possible. This simplifies the analysis.

Regarding the averaging result obtained in this paper, the interested reader might consult the papers [22, 24] and further references there. In [24], a framework of differential inclusions is used, in [22] explicit estimates of the vector field and the solutions are given in the case of impulsive forces. Our approach doesn't estimate the general solution behavior as we aim at just periodic solutions. However, the averaging function of Theorem 1 can be used for the construction of a reduced (averaged) system that will describe the global dynamics (as opposed to just local asymptotic stability) near the periodic solution obtained.

The authors' future plans include studying the occurrence of asymptotically stable periodic solutions in system (1) when (2) admits a family of cycles of varying periods. Another topic of our future research is the case where the unperturbed system (2) admits a limit cycle whose period is ε-close to the period of perturbation in (1) (complementing the existence result of Battelli-Feckan [8]). The non-resonant case where the unperturbed system (2) comes with an asymptotically stable limit cycle whose period is separated from the period of the perturbation is a particularly interesting research direction with regard to system (2). We expect that similar to the classical result by Levinson [19], it must be possible to prove that periodic perturbation of (1) transforms the limit cycle of (2) into an asymptotically stable manifold invariant under the action of the Poincaré map (i.e. stroboscopic map) of (1), where ergodic solutions are possible.

Acknowledgments.

The authors are grateful to anonymous Reviewers and the Associate Editor, whose comments helped to improve the paper. The main part of this paper has been written while the first author visited the Institute of Mathematics at the University of Utrecht. He expresses his deep gratitude for the excellent working conditions provided by the Institute. The work of the first author is supported by NSF grant CMMI-1916876.

References

[1] U. Andreaus, L. Placidi, G. Rega, Numerical simulation of the soft contact dynamics of an impacting bilinear oscillator. Commun. Nonlinear Sci. Numer. Simul. 15 (2010), no. 9, 2603–2616.

[2] J. Awrejcewicz, C.-H. Lamarque, Bifurcation and chaos in nonsmooth mechanical systems. World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 45. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. xviii+543 pp.

[3] V.I. Babitsky, V.L. Krupenin, "Vibration of strongly nonlinear discontinous systems," Springer, Berlin etc., (2001).

[4] B. Blazejczyk-Okolewska, K. Czolczynski, T. Kapitaniak, Hard versus soft impacts in oscillatory systems modeling. Commun. Nonlinear Sci. Numer. Simul. 15 (2010), no. 5, 1358–1367.

[5] N. N. Bogoliubov, Y. A. Mitropolsky, Asymptotic methods in the theory of non-linear oscillations, 1961, Translated from the second revised Russian edition. International Monographs on Advanced Mathematics and Physics Hindustan Publishing Corp., Delhi, Gordon and Breach Science Publishers, New York.
[6] A. Buica, J. Llibre, O. Makarenkov, Asymptotic stability of periodic solutions for nonsmooth differential equations with application to the nonsmooth van der Pol oscillator, SIAM J. Math. Anal. 40 (2009), no. 6, 2478–2495.
[7] V. Burd, Method of averaging for differential equations on an infinite interval. Theory and applications. Lecture Notes in Pure and Applied Mathematics, 255. Chapman & Hall/CRC, Boca Raton, FL, 2007. xii+343 pp.
[8] F. Battelli, M. Feckan, Fast-slow dynamical approximation of forced impact systems near periodic solutions. Bound. Value Probl. 2013, 2013:71, 33 pp.
[9] F. Battelli, M. Feckan,On the chaotic behaviour of discontinuous systems. J. Dynam. Differential Equations 23 (2011), no. 3, 495–540.
[10] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, Piecewise-smooth dynamical systems. Theory and applications. Applied Mathematical Sciences, 163. Springer-Verlag London, Ltd., London, 2008. xxii+481 pp.
[11] A. Fidlin, Nonlinear Oscillations in Mechanical Engineering, Springer, Berlin, Heidelberg, 2005.
[12] M. Franca, M. Pospisil, New global bifurcation diagrams for piecewise smooth systems: transversality of homoclinic points does not imply chaos. J. Differential Equations 266 (2019), no. 2-3, 1429–1461.
[13] J. Glover, A.C. Lazer, P.J. McKenna, Existence and stability of large scale nonlinear oscillations in suspension bridges. Z. Angew. Math. Phys. 40 (1989), no. 2, 172–200.
[14] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. Revised and corrected reprint of the 1983 original. Applied Mathematical Sciences, 42. Springer-Verlag, New York, 1990. xvi+459 pp.
[15] S. J. Hogan, K. Uldall Kristiansen, On the regularization of impact without collision: the Painleve paradox and compliance. Proc. A. 473 (2017), no. 2202, 20160773, 18 pp.
[16] A. Ivanov, Bifurcations in impact systems, Chaos, Solitons & Fractals Vol. 7, No. 10. pp. 1615-1634, 1996.
[17] A. N. Kolmogorov, S. V. Fomin, "Elements of the theory of functions and functional analysis," Fourth edition, revised. Izdat. "Nauka", Moscow, 1976 (in Russian); transl. 1st ed. Dover Publ., New York (1996).
[18] M. A. Krasnosel'skii, "The operator of translation along the trajectories of differential equations," Translations of Mathematical Monographs, 19. Translated from the Russian by Scripta Technica, American Mathematical Society, Providence, R.I. (1968).
[19] N. Levinson, Small periodic perturbations of an autonomous system with a stable orbit. Ann. of Math. (2) 52 (1950), 727–738.
[20] O. Makarenkov, J. S. W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey. Phys. D 241 (2012), no. 22, 1826–1844.
[21] J. Mawhin, Resonance and nonlinearity: a survey. Ukrainian Math. J. 59 (2007), no. 2, 197–214.
[22] Yu.A. Mitropolsky, A.M. Samoilenko, "Forced oscillations of systems with impulsive force", Int. J. Non-Linear Mechanics 20, pp. 419-426 (1985)
[23] J. Newman, Regularization of a Disk in a Frictionable Wedge, IFAC Proceedings Volumes 45 (2012), issue 2, 830–835.
[24] V.A. Plotnikov, "The averaging method for differential inclusions and its application to optimal-control problems", Differential Equations 15, pp. 1427-1433 (1979)
[25] S. R. Pring, C. J. Budd, The dynamics of regularized discontinuous maps with applications to impacting systems. SIAM J. Appl. Dyn. Syst. 9 (2010), no. 1, 188–219.
[26] O. Payton, A.R. Champneys, M.E. Homer, L. Picco, M.J. Miles, Feedback-induced instability in tapping mode atomic force microscopy: theory and experiment, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467 (2130) (2011) 1801–1822.
[27] M. Sayli, Y. M. Lai, R. Thul, S. Coombes, Synchrony in networks of Franklin bells. IMA J. Appl. Math. 84 (2019), no. 5, 1001–1021.
[28] J. Sotomayor, M. A. Teixeira, Regularization of discontinuous vector fields. International Conference on Differential Equations (Lisboa, 1995), 207–223, World Sci. Publ., River Edge, NJ, 1998.
[29] J. J. Stoker, Nonlinear Vibrations in Mechanical and Electrical Systems. Wiley, New York, 1992.
[30] J.J. Thomsen, A. Fidlin, Near-elastic vibro-impact analysis by discontinuous transformations and averaging, J. Sound Vib. 311 (12) (2008) 386–407.
[31] P. Thota, H. Dankowicz, Continuous and discontinuous grazing bifurcations in impacting oscillators, Physica D 214 (2006) 187–197.
[32] F. Verhulst, Methods and applications of singular perturbations, boundary layers and multiple timescale dynamics, Springer, New York etc., (2005).
[33] K. Yagasaki, Nonlinear dynamics of vibrating microcantilevers in tapping-mode atomic force microscopy, Physical Review B 70, 245419 (2004).