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Timescales and Error Estimates in Dynamical Systems

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Abstract. The method of multiple timescales is widely used in engineering and mathematical physics. In this note we draw attention to the literature on the techniques and comparison of various perturbation methods. The emphasis is on autonomous ODEs and ODEs with periodic coefficients. We indicate where we can obtain an advantage from the concept of timescales and we present examples from bifurcation theory where the anticipation of timescales is not straightforward and multiple timing is in danger of being deficient. The paper is tutorial but new results are presented in section 6.
Keywords: .

1 Introduction

Many problems in physics and engineering can be formulated as a perturbation problem, i.e. as a small perturbation of a problem that we know how to solve. Usually a small, positive parameter ε plays a part in the formulation; we will assume $0 \leq \varepsilon \ll 1$. We start with some examples to illustrate the concept of timescales.

Example 1 *Consider the harmonic equation with a slight perturbation (detuning) of the frequency* 1*:*

$$\ddot{x} + (1+\varepsilon)x = 0.$$

It is easy to solve the perturbed equation, we find the general solution

$$x(t) = A\cos(\sqrt{1+\varepsilon}t) + B\sin(\sqrt{1+\varepsilon}t)$$

with arbitrary constants A and B which are for instance determined by initial conditions. Expanding with respect to ε in a Taylor series, we find

$$\cos(\sqrt{1+\varepsilon}t) = \cos t - \frac{\varepsilon t}{2}\sin t + \frac{\varepsilon^2 t}{8}\sin t - \frac{\varepsilon^2 t^2}{8}\cos t + \varepsilon^3 \ldots$$

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and for $\sin t$ *a similar expression. The exact solution is periodic with respect to* t*, but the Taylor expansion with respect to* ε *is not. In fact, the expansion contains terms that are unbounded with time, so-called* secular *terms. These secular terms assume different forms and are called time-like variables or* timescales. *In this elementary problem, the timescales* $t, \varepsilon t$ *and* $\varepsilon^2 t$ *play a part, at higher order more timescales appear.*

Example 2 *We know the damped harmonic oscillator*

$$\ddot{x} + \mu\dot{x} + x = 0,\ \ \mu > 0,$$

and its solutions with usually μ *rather small to avoid quenching the oscillation too quickly. Suppose now that we are considering a mechanical process where, for some reason, the damping slowly increases from (say)* $\mu = \varepsilon$ *to* $\mu = 2\varepsilon$*. For this oscillator, we propose the equation*

$$\ddot{x} + \varepsilon(2 - e^{-\varepsilon t})\dot{x} + x = 0.$$

Note that already in the equation a timescale, εt*, is present, but maybe the dynamics of this oscillator will produce more timescales. If* $t = 0$*, we have the damped oscillator given above for* $\mu = \varepsilon$*; if we let* t *tend to infinity, we have this oscillator with* $\mu = 2\varepsilon$*. What happens for the time in between? If* $\varepsilon = 0$*, the independent variable is time* t*. It is natural to assume that as the damping varies with* εt*, an approximation of the problem can be achieved by assuming that two* timescales*, play a part:* t *and* εt*. We will show how we will handle such a problem.*

The picture of timescales as in the examples above is not always so simple. Consider for instance an example of the classical Euler equation:

Example 3

$$t^2\ddot{x} - t\dot{x} + (1 + \varepsilon)x = 0.$$

The so-called Euler-index λ *is obtained by substituting* $x(t) = t^\lambda$*. This produces the index-equation*

$$\lambda^2 - 2\lambda + 1 + \varepsilon = 0,$$

with

$$\lambda = 1 \pm i\sqrt{\varepsilon}.$$

So, independent solutions are $t\cos(\sqrt{\varepsilon}\ln t)$ *and* $t\sin(\sqrt{\varepsilon}\ln t)$ *with timescales* t *and* $\sqrt{\varepsilon}\ln t$*. However, ignoring the exact solution, and putting* $\varepsilon = 0$ *in the equation, gives the index-equation*

$$\lambda^2 - 2\lambda + 1 = 0,$$

with double roots 1*. The independent solutions if* $\varepsilon = 0$ *are* t *and* $t\ln t$ *which have not much in common with the perturbed solutions.*
This is a so-called bifurcation problem for the index-equation. This equation has two coincident solutions that bifurcate to two different solutions by adding a small parameter. Usually, problems in applications contain parameters that at specific values produce bifurcation phenomena. As these correspond with qualitative changes in the solutions and also in the physical applications, bifurcation phenomena merit special interest.

The examples until now are all concerned with linear equations; in some cases we have found so-called 'natural timescales', but sometimes we have already in a linear problem unexpected phenomena. Examples 1 and 2 will be typical for the theory to be developed in the sequel.

2 The general formulation of perturbation problems

A rather general problem formulation is to consider ordinary differential equations (ODEs) that contain a small positive parameter ε as in

$$\dot{x} = f(t, x, \varepsilon), \ x \in \mathbb{R}^n, \tag{1}$$

depending to some order smoothly on x and t for $t_0 \leq t < \infty$ and ε for $0 \leq \varepsilon \leq \varepsilon_0$; the dot represents differentiation with respect to t. We assume we can Taylor-expand:

$$\dot{x} = f_0(t, x) + \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \varepsilon^3 \ldots \tag{2}$$

For such a general problem, we usually cannot formulate explicit solutions of the equation in terms of elementary functions, but we assume that the equation can be solved, at least to some extent, if $\varepsilon = 0$. 'To some extent' may also mean that we are able to extract certain special solutions, equilibria or periodic, if $\varepsilon = 0$. By expanding in a neighbourhood of such a solution we can obtain so-called variational equations.

In our analysis we hope for the presence of certain typical timescales like $t, \varepsilon t, \varepsilon^2 t, \ldots$, which we called 'natural' in the Introduction, on which approximate solutions depend; in some problems we have similar choices for spatial variables. Contrasting with this approach of *anticipating timescales* is averaging, a normal form method, where no apriori assumption on the form of time-dependence is made. This contrasting approach also holds for the renormalization method. It will be clear that an apriori choice of timescales should be linked with apriori knowledge of the nature of the solutions.

The idea of anticipating timescales was introduced in Kiev by Krylov and Bogoliubov in 1935 [9]; the first application (as far as we are aware) was by Kuzmak in 1959 [10]. After 1960, the idea of multiple timescales was advocated and studied by Kevorkian [6], Cochran [4] and Nayfeh, see for instance [15]. The method, also called multiple timing, is intuitively clear and became very popular, especially in engineering.

The Kiev school of approximation theory for nonlinear ODEs was very influential so it is interesting to find out why they dropped the idea of multiple timing after the work of Kuzmak. When asked for a reason, Yu.A. Mitropolsky, a prominent member of the Kiev school, told me "because it is not a good method" [12]. This seems somewhat exaggerated as the validity of the method can be demonstrated in a great many cases. But it is true, as we shall see, that for a large number of important research problems, multiple timing can be misleading.

Apart from the literature cited above, introductions to the multiple timescale method can be found in [2], [15], [19] and [24]. A comparison of averaging and multiple timing by a number of important examples can be found in [7]. There have appeared many papers on the approximation of solutions of ODEs, we can cite only a few of them.

The relation between multiple timing and the renormalization method was discussed in [2], [3] and [13], however on a formal level only. In [16], Perko established the equivalence of the averaging method and multiple timing for standard equations like

$$\dot{x} = \varepsilon f(t, x)$$

on intervals of time of order $1/\varepsilon$. This was a major step forward. See also the extensive discussions in [14] and [19].

Asymptotic equivalence of methods would imply that, considering a solution $x(t)$ of a differential equation, expressions $\bar{x}_1(t)$ and $\bar{x}_2(t)$ obtained by different methods, would both represent an approximation of $x(t)$ with error $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$ on the same interval of time (for instance of size $1/\varepsilon$). Such results extend beyond the formal level.

Often, we will indicate that an approximation with error $\delta(\varepsilon)$ is valid on an interval of size $1/\varepsilon$. A more precise statement is that the error estimate is valid for $t_0 \leq \varepsilon t \leq t_0 + L$ with t_0, L constants independent of ε. It was shown in [16] that the approximations obtained by averaging and by multiple timing are equivalent to $O(\varepsilon)$ on intervals of time of order $1/\varepsilon$.

We will restrict ourselves to a discussion of ODEs. In [24] one can find a discussion and references of a number of PDE problems.

3 The basic idea for two timescales

As stated above, many small ε parameter problems are studied using timescales like $t, \varepsilon t, \varepsilon^2 t$ and in general $\varepsilon^n t$ with $n \in \mathbb{N}$. In the perturbation problem of eq. (2), the form of the solution for $\varepsilon = 0$ plays a part.

3.1 The variational equation

Ideally, we know the solution of the equation

$$\dot{x} = f_0(t, x)$$

explicitly, say $x(t)|_{\varepsilon=0} = \psi(t, c)$ with c a constant n-vector. We transform the solution of eq. (2) as follows. Put

$$x(t) = \psi(t, y),$$

and substitute into eq. (2) (this is Lagrange's method of variation of contants). We find:

$$\dot{x} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial y}\dot{y} = f_0(t, \psi(t, y)) + \varepsilon f_1(t, \psi(t, y)) + \varepsilon^2 \ldots$$

Assuming that we can invert the matrix $\partial\psi/\partial y$, we derive:

$$\dot{y} = \varepsilon \left(\frac{\partial \psi}{\partial y}\right)^{-1} f_1(t, \psi(t, y)) + \varepsilon^2 \ldots$$

This is the so-called variational equation in standard form.

In a number of problems we have less explicit knowledge of the solutions of the unperturbed problem. We may know an explicit solution which can be used to start a perturbation formulation. Another possibility is the presence of one or more integrals of motion of the unperturbed problem. These integrals can also be used as new variables for perturbation equations.

3.2 Two-timing

A simple but typical approach for two timescales runs as follows. Consider the variational equation in standard form

$$\dot{x} = \varepsilon f(t, x) \tag{3}$$

with $f(t, x)$ T-periodic in t, the initial value $x(0)$ is given. As we will see below, we can also start our multiple timing process directly for eq. (2) (direct two-timing). We will look for solutions of the form

$$x(t) = x_0(t, \tau) + \varepsilon x_1(t, \tau) + \varepsilon^2 \ldots \tag{4}$$

with $\tau = \varepsilon t$, the dots represent the higher order expansion terms. As the unknown functions $x_0, x_1, \ldots$ are supposed to depend on two variables, we have to transform the differential operator; we have to first order in ε:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}.$$

Using this differential operator and the expansion we find

$$\frac{\partial x_0}{\partial t} + \varepsilon \frac{\partial x_0}{\partial \tau} + \varepsilon \frac{\partial x_1}{\partial t} + \varepsilon^2 \ldots = \varepsilon f(t, x_0(t, \tau) + \varepsilon x_1(t, \tau) + \varepsilon^2 \ldots)$$

Suppose we can Taylor-expand the function f to a certain order, collecting then the terms of order 1 and ε, we find the simple partial differential equations

$$\frac{\partial x_0}{\partial t} = 0,$$
$$\frac{\partial x_1}{\partial t} = -\frac{\partial x_0}{\partial \tau} + f(t, x_0).$$

The first equation produces

$$x_0(t, \tau) = A(\tau), A(0) = x(0),$$

with $A(\tau)$ still an unknown function; A will be determined at the next step. For x_1 we find by integration

$$x_1(t, \tau) = \int_0^t \left(-\frac{\partial A(\tau)}{\partial \tau} + f(s, A(\tau)) \right) ds + B(\tau).$$

The function $B(\tau)$ is unknown and has to satisfy $B(0) = 0$. If we are looking for bounded solutions of eq. (3), or even for periodic solutions, the integral

$$\int_0^t \left(-\frac{\partial A(\tau)}{\partial \tau} + f(s, A(\tau)) \right) ds$$

has to be bounded. This is called the *secularity condition*. We can achieve this by determining $A(\tau)$ such that

$$\frac{dA}{d\tau} = \frac{1}{T} \int_0^T f(s, A(\tau)) ds. \tag{5}$$

Assuming that $f(t, x)$ has a Fourier expansion is a natural condition as it means that the 'constant' term of the expansion vanishes. The determination of $A(\tau)$ implies that satisfying the secularity condition corresponds with averaging the function $f(t, x)$ while keeping x constant. This idea can be traced to the end of the 18th century, for instance in the writings of Lagrange (see [19]).

The condition (5) is exactly the condition for averaging. Starting with the standard form (3), and initial condition $x(0) = x_0$, the initial value problem

$$\tilde{x} = \varepsilon \frac{1}{T} \int_0^T f(t, \tilde{x}) dt, \ \tilde{x}(0) = x_0,$$

produces an approximation $x(t) = \tilde{x}(t) + O(\varepsilon)$ on intervals of time $O(1/\varepsilon)$. This establishes the equivalence of two-timing and averaging to first order in ε.

Note that both two-timing and averaging assume boundedness of the solutions resulting in the secularity condition. If the solutions are unbounded it makes no sense to apply a secularity condition.

3.3 Direct two-timing

The standard form eq. (3) was our starting point. In some cases, for instance for the perturbed harmonic equation:

$$\ddot{x} + x = \varepsilon f(t, x, \dot{x}),$$

it may be easier to transform the original equation using the differential quotients with respect to time. Assuming the presence of the timescales t and εt, we compute:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}, \text{ and } \frac{\mathrm{d}^2}{\mathrm{dt}^2} = \frac{\partial^2}{\partial \mathrm{t}^2} + 2\varepsilon \frac{\partial^2}{\partial \mathrm{t} \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}. \tag{6}$$

Substitution into the equation produces to first order in ε:

$$(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau})(x_0 + \varepsilon x_1) + x_0 + \varepsilon x_1 = \varepsilon f(t, x_0 + \varepsilon x_1, (\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau})(x_0 + \varepsilon x_1)) + \varepsilon^2 \ldots.$$

Collecting equal powers of ε we find to zero order

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0,$$

with general solution

$$x_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t.$$

To first order in ε we find:

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = 2(\frac{dA}{d\tau}\sin t - \frac{dB}{d\tau}\cos t) + f(t, x_0, \frac{\partial x_0}{\partial t}).$$

We have to apply the secularity condition to this first order equation to determine $A(\tau)$ and $B(\tau)$.

We demonstrate this for example 2.

Example 4 *Consider again the problem of example 2:*

$$\ddot{x} + \varepsilon(2 - e^{-\varepsilon t})\dot{x} + x = 0.$$

Introducing $\tau = \varepsilon t$, the differential operators (6) and the expansion (4) into the equation we find to zero order

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0,$$

with general solution

$$x_0(t,\tau) = A(\tau)\cos t + B(\tau)\sin t.$$

The unknown functions $A(\tau), B(\tau)$ will be determined at next order of ε:

$$\frac{\partial^2 x_1}{\partial t^2} + 2\frac{\partial^2 x_0}{\partial t \partial \tau} + (2 - e^{-\tau})\frac{\partial x_0}{\partial t} + x_1 = 0.$$

Using the expression for x_0 we can write this as:

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = 2(\frac{dA}{d\tau}\sin t - \frac{dB}{d\tau}\cos t) + (2 - e^{-\tau})(A\sin t - B\cos t).$$

The solutions of the inhomogeneous harmonic equation produces unbounded (secular) terms unless

$$2\frac{dA}{d\tau} + (2 - e^{-\tau})A = 0,\ 2\frac{dB}{d\tau} + (2 - e^{-\tau})B = 0.$$

Solving the equations for A and B we find to first order for $x(t)$:

$$e^{-\tau - \frac{1}{2}e^{-\tau} + \frac{1}{2}}(A(0)\cos t + B(0)\sin t).$$

$A(0)$ and $B(0)$ are determined by the initial conditions. As expected, the damping factor increases.

It is interesting to compare the two-timing result with the approximation obtained by averaging; see [19], introductions can be found in [23] and [24]. Averaging produces in the solution the same damping factor, as expected. The result of Perko [16] implies that the two methods both yield an $O(\varepsilon)$ approximation, valid on an interval of time of size $1/\varepsilon$. So any difference must be beyond this interval or at higher order approximations.

4 Algebraic timescales for bifurcations

Analytic and numerical approximation theory gives us useful details, but one of the basic questions of engineering and mathematical physics is to obtain a global picture of the behaviour of the dynamical system studied; this is tied in with the study of qualitative changes when the parameters of the system pass certain critical values. Such changes are called bifurcations, they may entail stability changes, branching or vanishing of solutions, transitions from periodic solutions to tori, emergence of chaos and other phenomena. As we shall see, it is important in these problems to avoid making apriori assumptions on timescales.

In the analysis of bifurcations, approximation theory is used, combined with linearization and matrix calculations. A typical computation for an equation of the form $\dot{x} = f(x, t, \varepsilon)$ will be to identify an equilibrium or special solution $\psi(t)$ and study the behaviour of this solution as the parameters are changing; this leads to the calculation of eigenvalues, Lyapunov exponents and characteristic multipliers.

A typical example of a corresponding bifurcation diagram is displayed in fig. 1 describing bifurcations in a three degrees of freedom mechanical system with damping parameter b and self-excitation magnitude β. The curves in the b, β-diagram correspond with bifurcations as for example Hopf (H), Chenciner (CH), Neimark-Sacker (NS) etc. The system is studied in [1].

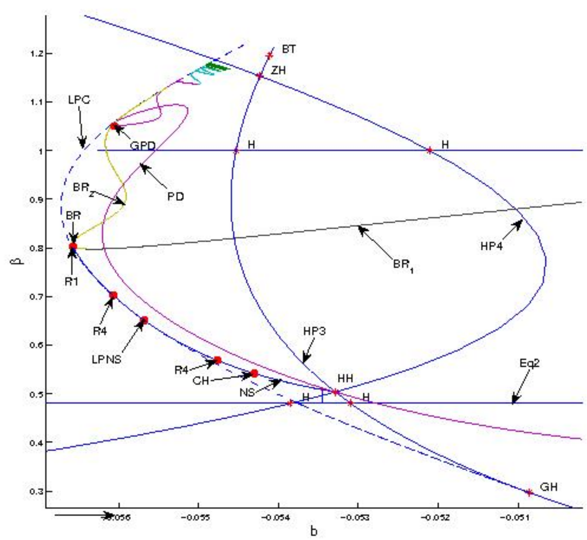

Fig. 1. Bifurcation diagram of a 6-dimensional system with damping b and self-excitation β. The curves correspond with bifurcations in parameter-space, see [1].

Bifurcation phenomena in ODEs lead by local linearization to studying systems of the form:

$$\dot{x} = A(\varepsilon)x, \tag{7}$$

where we can expand the $n \times n$-matrix A:

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 \dots$$

The $n \times n$-matrices A_n do not depend on ε; ε^n before a matrix should be interpreted as a diagonal $n \times n$-matrix with diagonal elements ε^n. If we have started with the standard form (3), we will have $A_0 = 0$. More in general, we have A_0 derived from the unperturbed problem, A_1 is produced by perturbation methods, by a special effort we may know A_2 and we will have some knowledge about higher order terms. An important question is then what the eigenvalues of A_0 and $A_0 + \varepsilon A_1$ tell us about the eigenvalues of $A(\varepsilon)$. This question is tied in with the structural stability of the matrices and whether eigenvalues are single or multiple. Failure of structural stability and the appearance of multiple eigenvalues is characteristic for bifurcation phenomena and so merit special attention. For instance in the bifurcation diagram of fig. 1, H corresponds with the presence of two purely imaginare eigenvalues, CH with one zero and two imaginary eigenvalues; for an extensive description see [11].

A $n \times n$ matrix is called *structurally stable* if it is nonsingular and all eigenvalues have nonzero real part. If we have a zero eigenvalue or purely imaginary eigenvalues, we can expect bifurcations. Apart from this, the presence of multiple eigenvalues affects the form of the expansions and the timescales.

Example 5 *We start with an example derived from an equation in standard form (3) where we have the expansion of $A(\varepsilon)$ until A_2:*

$$\dot{x} = \varepsilon^2 y, \ x(0) = 0,$$
$$\dot{y} = -\varepsilon x, \ y(0) = 1.$$

A_0 has vanished, A_1 has zero eigenvalues, $\varepsilon A_1 + \varepsilon^2 A_2$ has eigenvalues $\pm \varepsilon^{\frac{3}{2}} i$. The solution of the initial value problem is

$$x(t) = \varepsilon^{\frac{1}{2}} \sin(\varepsilon^{\frac{3}{2}} t), \ y(t) = \cos(\varepsilon^{\frac{3}{2}} t).$$

As can be seen from the eigenvalues, the timescale $\varepsilon^{\frac{3}{2}} t$ plays a part. Expanding the trigonometric functions on an interval of time of size $1/\varepsilon$, we find that the timescales t and εt can be used to obtain asymptotic estimates. On a longer interval of time, for instance $1/\varepsilon^2$, we need the timescale $\varepsilon^{\frac{3}{2}} t$ to obtain asymptotic estimates.

For bifurcations, local linearization leads to eigenvalue problems associated with eq. (7), so algebraic timescales are natural phenomena. Can we predict the form $\varepsilon^q t$ with q rational of such algebraic timescales? The following questions and results are classical.

Consider the matrix expansion obtained by a perturbation method:

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 \dots$$

- Can the eigenvalues be expanded in a convergent series of the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2 \cdots ,$$

where λ_0 is an eigenvalue of the matrix A_0 ? If this is the case, we expect timescales of the form $t, \varepsilon t, \varepsilon^2 t \dots$
- If we are in the critical case of bifurcations where λ_0 is zero or purely imaginary, how do the perturbations affect the eigenvalues and thus the qualitative behaviour of the solutions of the differential equations?

If A_0 vanishes, we extract ε and treat A_1 as perturbed matrix. We refer to [24] for references and summarize some basic results:

- If λ_0 is single, we have

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2 \dots$$

If $\lambda_0 = 0$, this means we have an $O(\varepsilon)$ size eigenvalue.
- According to Newton and Puisieux:
If λ_0 is multiple, the expansion is in fractional powers of ε.

Example 6 *Consider the equation $\dot{x} = A(\varepsilon)x$ with for the matrix $A(\varepsilon)$:*

$$A(\varepsilon) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The characteristic equation to $O(\varepsilon)$ is:

$$\lambda^3 - 3\varepsilon\lambda = 0$$

with eigenvalues $\lambda_1 = 0, \lambda_{2,3} = \pm 3\sqrt{\varepsilon}$. The matrix $A_0 + \varepsilon A_1$ is not structurally stable so we add the $O(\varepsilon^2)$ term. This leads to the characteristic equation:

$$\lambda^3 - 3\varepsilon\lambda + \varepsilon^3 = 0.$$

with Newton-Puisieux expansion for the eigenvalues

$$\lambda_1 = \frac{1}{3}\varepsilon^2 + \frac{1}{81}\varepsilon^5 + \cdots , \lambda_{2,3} = \pm 3\sqrt{\varepsilon} - \frac{1}{6}\varepsilon^2 + \cdots .$$

Including the $O(\varepsilon^2)$-terms we have structural stability. Solving the equation we have time-like variables (timescales) of the form

$$\sqrt{\varepsilon}t, \varepsilon^2 t, \varepsilon^5 t$$

and from the expansion also higher order timescales.

The discussion has some relevance for the analysis of the nonlinear problem

$$\dot{x} = A(\varepsilon)x + \varepsilon f(x),$$

where $f(x)$ is a nonlinear vector field with an expansion starting with quadratric terms near $x = 0$. The zero eigenvalue to $O(\varepsilon)$ suggests the presence of a center manifold associated with the corresponding eigenvector. The calculation of eigenvalues to $O(\varepsilon^2)$ destroys this picture.
Similar problems may arise for other codimension one and two bifurcations triggered by the matrix $A(\varepsilon)$.

In the sequel we will consider a number of bifurcation problems arising in applications.

5 Application: the Mathieu-equation

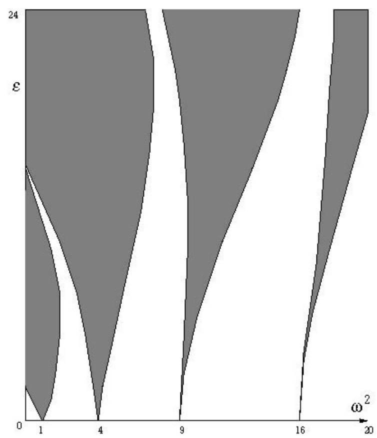

Fig. 2. The gray Floquet tongues denote for which parameter values ω and ε the trivial solution of the Mathieu equation is unstable. In our approximations we have described the lower part of the tongue emerging from $\omega = 1$ as in eq. (8).

We consider the Mathieu equation which plays a part in many engineering problems:

$$\ddot{x} + (\omega^2(\varepsilon) + \varepsilon \cos \nu t)x = 0,$$

in its fundamental 1 : 2-resonance with a slight detuning:

$$\ddot{x} + (1 + \varepsilon a + \varepsilon^2 b + \varepsilon \cos 2t)x = 0, \tag{8}$$

a and b are free parameters independent of ε, $\omega^2 = 1 + \varepsilon a + \varepsilon^2 b$. We apply Lagrange variation of constants

$$x = y_1 \cos t + y_2 \sin t, \ \dot{x} = -y_1 \sin t + y_2 \cos t.$$

The slowly-varying equations for (y_1, y_2) are, after averaging, of the form $\dot{y} = A(\varepsilon)y$; this (averaging) normal form approach produces to first order in ε:

$$A(\varepsilon) = +\varepsilon \begin{pmatrix} 0 & \frac{1}{2}(a - \frac{1}{2}) \\ -\frac{1}{2}(a + \frac{1}{2}) & 0 \end{pmatrix} + O(\varepsilon^2).$$

The eigenvalues are

$$\lambda_{1,2} = \pm \frac{1}{2}\sqrt{\frac{1}{4} - a^2},$$

the two approximate independent solutions for (y_1, y_2) can be written as

$$e^{\pm\frac{1}{2}\sqrt{\frac{1}{4}-a^2}\,\varepsilon t}.$$

This leads to the well-known result that for $a^2 > \frac{1}{4}$ the solutions of the Mathieu equation are stable (the approximate solutions are trigonometric) and for $a^2 < \frac{1}{4}$ they are unstable. The approximations with appropriate initial values have error estimate $O(\varepsilon)$ on a long time-interval $O(1/\varepsilon)$. In this approximation, the timescales for $x(t)$ are t and εt. The boundary of the instability domains, the Floquet tongues, are the bifurcation curves where the transition from unstable to stable solutions takes place in (ω^2, ε)-parameter space; see fig. 2.

5.1 What happens at the tongue boundary?

What happens at the transition values, for instance at $\omega^2 = 1 + \varepsilon a$ where $a = \frac{1}{2}$? In this case, we have for the normal form to first order:

$$A_1 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix},$$

a typical degenerate matrix from bifurcation theory. Following [19] or [24] we perform a second-order averaging normalization to find:

$$A_2 = \begin{pmatrix} 0 & \frac{1}{64} + \frac{1}{2}b \\ \frac{7}{64} - \frac{1}{2}b & 0 \end{pmatrix}.$$

We find for the eigenvalues of $A(\varepsilon)$ to this order of approximation

$$\lambda^2 = -\frac{1}{4}\left(b + \frac{1}{32}\right)\varepsilon^3 + \frac{1}{4}\left(b + \frac{1}{32}\right)\left(\frac{7}{32} - b\right)\varepsilon^4.$$

The $O(\varepsilon^3)$-term dominates, $b = -\frac{1}{32}$ produces a more precise location of the Floquet tongue.

If $b > -\frac{1}{32}$ we have stability, if $b < -\frac{3}{32}$ we have instability.
The second order approximations of the solutions for (y_1, y_2) are a linear combination of $\exp.(+\lambda t)$ and $\exp.(-\lambda t)$. With appropriate initial values they yield approximations of the solutions of the Mathieu equation (8) with error estimate $O(\varepsilon^2)$ on a long time-interval $O(1/\varepsilon)$.

It is remarkable that the timescale $\varepsilon^{\frac{3}{2}}t$ plays a part in this problem because near the boundary of the Floquet tongue we have that $\lambda^2 = O(\varepsilon^3)$. The timescales characterizing the flow near the Floquet tongue are until second order

$$t,\ \varepsilon t,\ \varepsilon^{\frac{3}{2}}t,\ \varepsilon^2 t.$$

The presence of the timescale $\varepsilon^{\frac{3}{2}}t$ was noted for the Mathieu equation in [2], using the renormalization method. It is also noted in [2] that, using multiple timing with timescales $t, \varepsilon t, \varepsilon^2 t$, this extra timescale is not discovered. It arises from a bifurcation problem with multiplicity two eigenvalues

The tongue boundaries in fig. 2 correspond with parameter values where the Mathieu equation has periodic solutions. They indicate the transition from unstable to stable trivial solution, the eigenvalues on the boundaries show the nature of the bifurcations.

6 Application: resonance manifolds

Many dynamical systems, both dissipative and conservative, can be put in the form:

$$\begin{cases} \dot{x} = \varepsilon X(\phi, x) + \varepsilon^2 \dots, \\ \dot{\phi} = \Omega(x) + \varepsilon \dots \end{cases} \tag{9}$$

x is an n-vector (amplitudes) and ϕ an angle-vector (think of gyroscopic systems or in the case of Hamiltonian systems of actions and angles).
ϕ is time-like in domains where $\Omega(x) \neq 0$.

In a neighbourhood of $\Omega(x) = 0$, ϕ is not time-like and we have a resonance manifold.

6.1 Simple examples

Consider as an example a one degree of freedom system:

Example 7 *The equation to be studied is*

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}),$$

with (positive) constant frequency ω*. Putting* $\dot{x} = \omega y$ *and introducing amplitude-angle variables* $x, y \to r, \phi$ *by*

$$x = r \sin \phi, \;\; y = r \cos \phi,$$

we find the equations

$$\begin{aligned} \dot{r} &= \varepsilon \frac{\cos \phi}{\omega} f(r \sin \phi, \omega r \cos \phi), \\ \dot{\phi} &= \omega - \frac{\sin \phi}{\omega r} f(r \sin \phi, \omega r \cos \phi). \end{aligned}$$

One observes that the righthand sides are 2π*-periodic in* ϕ *and a perturbation scheme can be started, for instance by averaging over* ϕ*.*
Apply this for instance to the damped, Duffing equation where $f(x, \dot{x}) = -a\dot{x} - bx^3$*.*

For the theory we refer to [19] and [24]. New phenomena may emerge in the case of more degrees of freedom. We borrow some examples from [24].

Example 8 *After suitable transformations in a problem, we have obtained the system:*

$$\begin{aligned} \dot{x} &= \varepsilon x(\cos \phi_1 + \cos \phi_2 + \cos(2\phi_1 - \phi_2)), \\ \dot{\phi}_1 &= x, \\ \dot{\phi}_2 &= 2. \end{aligned}$$

We have one amplitude, x, two angles ϕ_1 and ϕ_2; in addition the combination angle $\psi = 2\phi_1 - \phi_2$. We could consider the angles $\phi_1, \phi_2, 2\phi_1 - \phi_2$ as time-like variables and average over them; this is also called 'averaging over a torus'. This would result in an average zero for the righthand side of $\dot{x}$. Is this a correct strategy? The answer is affirmative in the cases that the three angles are indeed time-like but not in the cases when

$$\dot{\phi}_1 = 0,\ \dot{\phi}_2 = 0,\ 2\dot{\phi}_1 - \dot{\phi}_2 = 0.$$

As $2\dot{\phi}_1 - \dot{\phi}_2 = 2(x-1)$ we have to consider separately the cases $x = 0$ and $x = 1$. The domains near $x = 0$ and $x = 1$ are called the resonance zones *in x-space. Outside the resonance zones, the average of $\dot{x}$ over the two angles and the combination angle vanishes, so $x(t)$ is nearly constant there. What happens in a resonance zone? In this example $x = 0$ is an exact solution, consider instead a neighbourhood of $x = 1$ by rescaling:*

$$x - 1 = \delta(\varepsilon)\xi.$$

Here, ξ is the new, local variable; $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$, but we still have to find out what the size of $\delta(\varepsilon)$ and the resonance zone is. Introducing ξ and ψ in the equations produces:

$$\begin{aligned}
\delta(\varepsilon)\dot{\xi} &= \varepsilon(\cos\phi_1 + \cos\phi_2 + \cos\psi) + O(\varepsilon\delta(\varepsilon)),\\
\dot{\phi}_1 &= 1 + O(\delta(\varepsilon)),\\
\dot{\phi}_2 &= 2\\
\dot{\psi} &= 2\delta(\varepsilon)\xi.
\end{aligned}$$

To first order, ϕ_1 and ϕ_2 are time-like in this resonance zone, ψ is not. The equations for ξ and ψ show the same size of terms on choosing $\delta(\varepsilon) = \sqrt{\varepsilon}$. The equations become with this choice:

$$\begin{aligned}
\dot{\xi} &= \sqrt{\varepsilon}(\cos\phi_1 + \cos\phi_2 + \cos\psi) + O(\varepsilon),\\
\dot{\phi}_1 &= 1 + O(\sqrt{\varepsilon}),\\
\dot{\phi}_2 &= 2\\
\dot{\psi} &= 2\sqrt{\varepsilon}\xi.
\end{aligned}$$

We average now over the time-like variables ϕ_1 and ϕ_2 to find the leading equations and terms in this resonance zone:

$$\dot{\xi} = \sqrt{\varepsilon}\cos\psi,\ \dot{\psi} = 2\sqrt{\varepsilon}\xi.$$

Differentiating ψ we get the pendulum equation for the combination angle:

$$\ddot{\psi} - 2\varepsilon\cos\psi = 0.$$

The pendulum equation has a centre point and a saddle. It can be shown that the stationary solutions of this resonance zone equation correspond with a stable and an unstable periodic solution of the original system. Note that in this example we had to localize in space to size $O(\sqrt{\varepsilon})$, the natural timescale in the resonance zone is $\sqrt{\varepsilon}t$, outside the resonance zones it is εt.

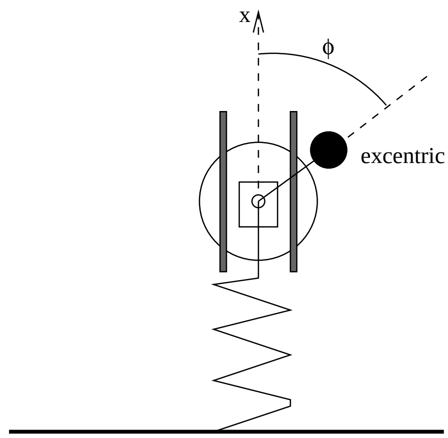

Fig. 3. Excentric flywheel, rotating on elastic foundation

6.2 Rotation of an excentric flywheel

An application in [24], example 12.11 (with more references there), describes a slightly excentric flywheel, see fig. 3; the analysis is based on the thesis of Van den Broek [22], see also [21]. The vertical displacement x of a small mass on the flywheel and its rotation angle ϕ are given by

$$\begin{cases} \ddot{x} + x = \varepsilon(-x^3 - \dot{x} + \dot{\phi}^2 \cos\phi) + O(\varepsilon^2), \\ \ddot{\phi} = \varepsilon(\frac{1}{4}(2 - \dot{\phi}) + (1 - x)\sin\phi) + O(\varepsilon^2). \end{cases} \tag{10}$$

To analyse the system and to put it in standard perturbation form, we introduce:

$$x = r\sin\phi_2, \dot{x} = r\cos\phi_2, \phi = \phi_1, \dot{\phi}_1 = \Omega,$$

with $r, \Omega > 0$. We find to $O(\varepsilon)$ a system with two angles, ϕ_1, ϕ_2, and slowly varying variables r and Ω:

$$\begin{aligned} \dot{r} &= \varepsilon\cos\phi_2(-r^3\sin^3\phi_2 - r\cos\phi_2 + \Omega^2\cos\phi_1), \\ \dot{\Omega} &= \varepsilon(\frac{1}{4}(2 - \Omega) + \sin\phi_1 - r\sin\phi_1\sin_2), \\ \dot{\phi}_1 &= \Omega, \\ \dot{\phi}_2 &= 1 + \varepsilon(r^2\sin^4\phi_2 + \frac{1}{2}\sin 2\phi_2 - \frac{\Omega^2}{r}\cos\phi_1\sin\phi_2). \end{aligned}$$

Evaluating the trigonometric terms in the slowly varying equations for r and Ω we find the angles $\phi_1, \phi_2, 2\phi_2, 4\phi_2$ and the combination angles $\phi_1 - \phi_2, \phi_1 + \phi_2$. The righthand sides of the equations for the angles are positive, so the only resonance zone that can be active is when $\dot{\phi}_1 - \dot{\phi}_2 \approx 0$. As

$$\frac{d}{dt}(\phi_1 - \phi_2) = \Omega - 1 + O(\varepsilon),$$

this happens if Ω is near 1. Note that the analysis included $O(\varepsilon)$ terms only, if we add higher order terms, more (but smaller) resonance zones may be found. Outside the resonance zone we average over the angles to find an approximation from

$$\begin{cases} \dot{r} = -\frac{\varepsilon}{2}r, \\ \dot{\Omega} = \frac{\varepsilon}{4}(2-\Omega). \end{cases} \tag{11}$$

Although simple looking, this result is already of interest. The deflection x of the flywheel will go exponentially fast to zero outside the resonance zone; outside resonance, $\Omega(t)$, the rotation speed, will tend to 2, but if $\Omega(0) < 1$, the flywheel will pass through the resonance zone, the averaged equations 11 do not apply in this zone. What happens there? As in example 8 above, we rescale locally in a neighbourhood of $\Omega = 1$ and introduce the combination angle ψ:

$$\Omega = 1 + \delta(\varepsilon)\omega, \ \psi = \phi_1 - \phi_2.$$

We find

$$\begin{aligned} \dot{r} &= O(\varepsilon), \\ \delta(\varepsilon)\dot{\omega} &= \varepsilon(\frac{1}{4} + \sin\phi_1 - \frac{1}{2}r\cos\psi + \frac{1}{2}r\cos(2\phi_1 - \psi)) + \ldots, \\ \dot{\phi}_1 &= 1 + \ldots, \\ \dot{\psi} &= \delta(\varepsilon)\omega + \ldots \end{aligned}$$

The dots represent higher order terms. The equations for ω and ψ show the same size of terms if

$$\delta(\varepsilon) = \sqrt{\varepsilon},$$

which determines the size of the resonance zone. Averaging over the remaining angle ϕ_1 and noting that $r(t)$ varies $O(\varepsilon)$ in the resonance zone, we find to $O(\sqrt{\varepsilon})$ (neglecting terms of $O(\varepsilon\sqrt{\varepsilon})$):

$$\begin{aligned} \dot{\omega} &= \sqrt{\varepsilon}(\frac{1}{4} - \frac{1}{2}r\cos\psi), \\ \dot{\psi} &= \sqrt{\varepsilon}\omega. \end{aligned}$$

Differentiating the equation for ψ we find again a pendulum equation describing the dynamics in the resonance zone:

$$\ddot{\psi} + \frac{1}{2}\varepsilon r(0)\cos\psi = \frac{1}{4}\varepsilon.$$

So it turns out that the resonance zone near $\Omega = 1$ is of size $O(\sqrt{\varepsilon})$, the timescale of the dynamics is $\sqrt{\varepsilon}t$. The centre equilibrium of the pendulum equation corresponds with a stable periodic solution, the saddle with an unstable one. A periodic solution that is Lyapunov stable only does not attract. By including second order aproximations, we find that if we start at $0 < \Omega(0) < 1$, there exist initial values $\Omega(0)$ so that the solution is trapped in the resonance zone, resulting in periodic deflections of the flywheel. To describe this behaviour analytically, we have to obtain a second order approximation with respect to ε (described in [21]). This second order approximation adds negative real values $O(\varepsilon^2)$ to the purely imaginary eigenvalues. For a numerical illustration see fig. 4.

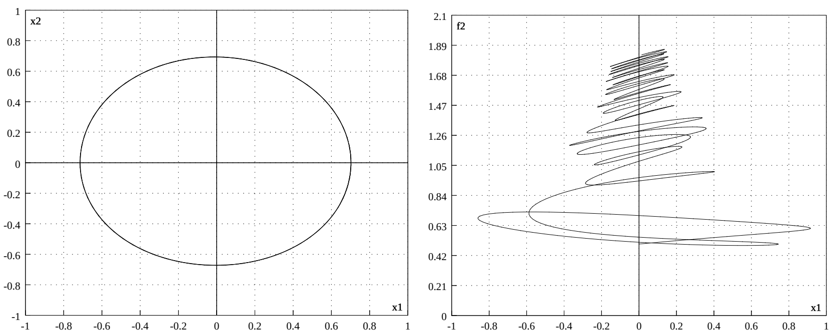

Fig. 4. Orbits for the excentric flywheel. Left: capture into resonance ($x1 = x, x2 = \dot{x}$); $\phi_1(0) - \phi_2(0) = 1.13, \phi_2(0) = 0, \varepsilon = 0.1$. Right: transition through the resonance zone, vertical $f2 = \phi_2; \phi_1(0) - \phi_2(0) = 0.5, \phi_2(0) = 0.5, \varepsilon = 0.1$.

It turns out that at this level of approximation, there are three open sets of initial values of the combination angle ψ that lead the corresponding solutions to trapping into the resonance zone. If $\varepsilon = 0.01$, the sets are for $\phi_1(0) - \phi_2(0) = \psi(0)$: $[1.049, 1.232]$, $[2.840, 3.047]$ and $[4.763, 4.863]$. In [21] this result is established analytically and confirmed numerically.

The results are dependent on the value of ε. It is an open question how the number of 'channels' leading to trapping in the resonance zone depends on the level of approximation; narrower channels may exist at higher order.

Problems where averaging over angles (a torus) has to be used, arise in many fields of application, for instance in gyroscopic systems, also in Hamiltonian mechanics. Algebraic timescales of the form $\varepsilon^q t$ with q a rational number, are natural in this context; see also [24] for the general theory and more examples.

6.3 Application: resonance manifolds in Hamiltonian systems

Higher order resonance turns out to be a natural application of the asymptotics of resonance manifolds. For an application in two degrees-of-freedom Hamiltonian systems, in particular the elastic pendulum, see [20].

Consider the two degrees-of-freedom Hamiltonian in local coordinates with Taylor-expansion:

$$H = H_2 + \varepsilon H_3 + \varepsilon^2 H_4 + \dots,$$

with H_k homogeneous of degree k in position and momentum (p, q). H_2 takes the standard form

$$H_2 = \frac{m}{2}(q_1^2 + p_1^2) + \frac{n}{2}(q_2^2 + p_2^2),$$

with the integers m, n positive and relative prime. The phase-flow in a neighbourhood of the origin takes place on compact manifolds parametrised by the Hamiltonian (energy) integral.

Most of the attention in the literature went to the primary resonance 1 : 2 and to the secondary resonances 1 : 1 and 1 : 3. In these resonance cases, the dominant part of the phase-flow is characterised by the timescales $t, \varepsilon t, \varepsilon^2 t$ and the time intervals of validity of approximation $1/\varepsilon$ and $1/\varepsilon^2$, see [19].

6.4 The higher order normal form

The cases where $m + n \geq 5$ are called higher order resonances. Studying these resonances requires the computation of higher order normal forms and involves intervals of time longer than $1/\varepsilon^2$. In the Hamiltonian normal form, the first resonant term, involving both actions and angles, arrives from H_{m+n} at $O(\varepsilon^{m+n-2})$.

The first basic approach to higher order resonance was given in [17] with applications in [18]. In [20] an improvement of the estimates has been given, together with a number of applications, among which the elastic pendulum (a pendulum where the suspending, inflexible string is replaced by a linear spring). Introducing action-angle variables $p_i, q_i \to \tau_i, \phi_i$, $i = 1, 2$, the normal form will be of the form:

$$H = m\tau_1 + n\tau_2 + \varepsilon_2 P_2(\tau_1, \tau_2) + \ldots + \varepsilon^{m+n-2} D(\tau_1^n \tau_2^m)^{\frac{1}{2}} \cos\chi,$$

with resonance combination angle $\chi = n\phi_1 - m\phi_2 + \alpha$. The dots represent terms depending on τ_1, τ_2 only, the terms in so-called Birkhoff normal form. A consequence from the corresponding equations of motion is that the actions are constant until terms of order $O(\varepsilon^{m+n-2})$ are taken into account, for the combination angle we have

$$\dot{\chi} = \varepsilon^2 \left(n\frac{\partial P_2}{\partial \tau_1} - m\frac{\partial P_2}{\partial \tau_2} \right) + \varepsilon^3 \ldots$$

6.5 The phase-flow of higher order resonance

Consider the *higher order* resonances defined by $m + n \geq 5$. It turns out there are two domains in phase-space where the dynamics is very different and is characterised by different timescales:

- The *resonance domain* D_I, which is a neighborhood of the resonance manifold M. The resonance manifold, if it exists, arises from the condition that $P_2(\tau_1, \tau_2)$ and maybe higher order Birkhoff normal forms vanish. In D_I the variations of the actions and the combination angle may interact significantly. In terms of singular perturbations, this is the *inner* boundary layer of the Hamiltonian system. In [20] it has been shown that the size of the resonance domain is $O(\varepsilon^{\frac{m+n-4}{2}})$, the interaction of the actions takes place on a time interval of order $O(\varepsilon^{-\frac{m+n}{2}})$

- The remaining part of phase-space, outside the resonance domain, is D_0, the *outer* domain. In the domain D_0, there is, to a certain approximation, little variation of the actions, and so hardly any exchange of energy between the two degrees of freedom.

It is shown in [20] that for Hamiltonians derived from a potential, we have $\alpha = 0$, and that for the elastic pendulum, after the first order 2 : 1-resonance, the higher order 4 : 1-resonance is the most prominent one with resonance manifold of size $O(\varepsilon^{\frac{1}{2}})$ and time interval of interaction $O(\varepsilon^{-\frac{5}{2}})$; for a Poincaré map of the 1 : 6-resonance of the elastic pendulum see fig. 5.

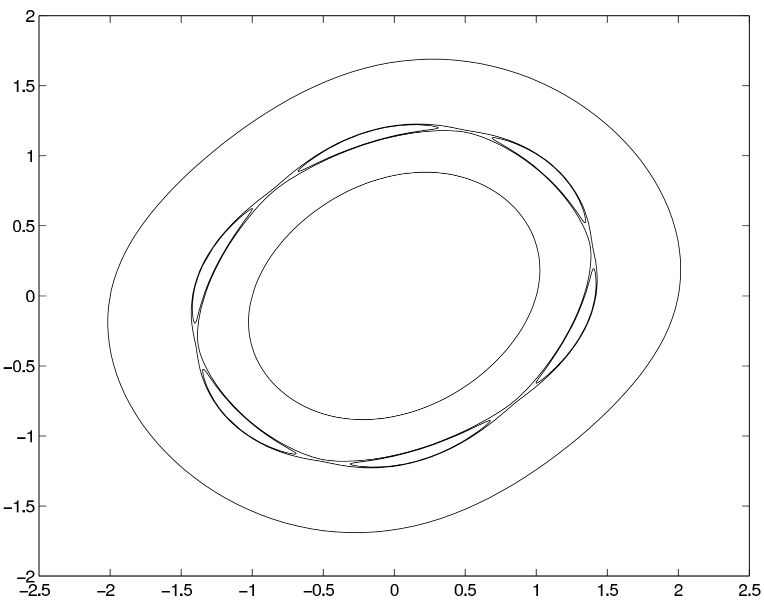

Fig. 5. The Poincaré map for the 1 : 6-resonance of the elastic pendulum ($\varepsilon = 0.75$, large for illustration purposes). In the resonance domain, the saddles are connected by heteroclinic cycles and inside the cycles are 6 center fixed points, see [20].

6.6 The Hénon-Heiles family

A well-known model for orbits in axi-symmetric galaxies is the family of Hénon-Heiles potential problems

$$H = \frac{m}{2}(q_1^2 + p_1^2) + \frac{n}{2}(q_2^2 + p_2^2) - \varepsilon \left(\frac{a_1}{3} q_1^3 + a_2 q_1 q_2^2 \right). \qquad (12)$$

In the literature, most of the attention is on the $m : n = 1 : 1$- and $2 : 1$-resonances. In [20] it is noted that the $m : n = 1 : 2$-resonance degenerates because of the discrete symmetry in the second degree of freedom, it is treated as a 2 : 4 higher order resonance. In this case the resonance manifold, for the parameter values where it exists, has size $O(\varepsilon)$, the timescale of interaction is $\varepsilon^3 t$.

Is the degenerate 1 : 2-resonance the most prominent higher order resonance? Other candidates are the 2 : 3- and the 4 : 1-resonances, the 3 : 2- and 1 : 4-resonances are degenerate because of the discrete symmetry of the potential.

If $a_2 = 0$, the equations decouple, so we assume $a_2 \neq 0$. Assuming $m+n \geq 5$ we find the normal form from [18] or [20]. As explained above, we find for the actions $\tau_1 = \frac{1}{2}(q_1^2 + p_1^2), \tau_2 = \frac{1}{2}(q_2^2 + p_2^2)$:

$$\dot{\tau}_1 = O(\varepsilon^3),\ \dot{\tau}_2 = O(\varepsilon^3).$$

For the combination angle $\chi = n\phi_1 - m\phi_2$ we have:

$$\dot{\chi} = \varepsilon^2 \left(-\frac{5n}{12}a_1^2 + \frac{m}{4}a_1 a_2 + \frac{m}{30}a_2^2 \right) 2\tau_1 + \varepsilon^2 \left(-\frac{n}{2}a_1 a_2 - \frac{n}{15}a_2^2 + \frac{29m}{120}a_2^2 \right) 2\tau_2. \tag{13}$$

For the Hénon-Heiles family, one usually puts $\lambda = a_1/3a_2$, producing:

$$\dot{\chi} = 6a_2^2\varepsilon^2 \left(-\frac{5n}{4}\lambda^2 + \frac{m}{4}\lambda + \frac{m}{90} \right) \tau_1 + 6a_2^2\varepsilon^2 \left(-\frac{n}{2}\lambda - \frac{n}{45} + \frac{29m}{360} \right) \tau_2. \tag{14}$$

The resonance manifold, if it exists, is determined by the equation

$$\left(-\frac{5n}{4}\lambda^2 + \frac{m}{4}\lambda + \frac{m}{90} \right) \tau_1 + \left(-\frac{n}{2}\lambda - \frac{n}{45} + \frac{29m}{360} \right) \tau_2 = 0. \tag{15}$$

The approximate energy integral is given by

$$m\tau_1 + n\tau_2 = E_0,\ 0 \leq \tau_1 \leq \frac{E_0}{m},\ 0 \leq \tau_2 \leq \frac{E_0}{n}.$$

We will consider the prominent higher order resonances for the original Hénon-Heiles problem [5] and the potential often used by Contopoulos, see [18]. As mentioned, the candidates for this are the 2 : 3- and the 4 : 1-resonances. If they exist, the size of the resonance manifolds are in these cases $O(\varepsilon^{\frac{1}{2}})$, the interaction of the degrees of freedom in the resonance manifold takes place on an interval of order $O(\varepsilon^{-\frac{5}{2}})$.

6.7 The Hénon-Heiles case

In this model we have $a_1 = 1$, $a_2 = -1$, $\lambda = -1/3$ (in the original problem we have $m = n = 1$, see [5]). From eq. (15) and the approximate energy integral we find if $m + n \geq 5$ the conditions:

$$-\left(\frac{5n}{18} + \frac{13m}{90} \right) \tau_1 + \left(\frac{13n}{45} + \frac{29m}{180} \right) \tau_2 = 0$$

and

$$m\tau_1 + n\tau_2 = E_0,\ 0 \leq \tau_1 \leq \frac{E_0}{m},\ 0 \leq \tau_2 \leq \frac{E_0}{n}.$$

The 2 : 3-resonance
Putting $m = 2$, $n = 3$ we find that the resonance manifold exists near:

$$\tau_1 = \frac{107}{517}E_0, \ \tau_2 = \frac{101}{517}E_0.$$

The 4 : 1-resonance
Putting $m = 4$, $n = 1$ we find that this resonance manifold also exists; it is found near:

$$\tau_1 = \frac{84}{413}E_0, \ \tau_2 = \frac{77}{413}E_0.$$

In both resonance cases we find islands with stable and unstable periodic solutions. Generically, the stable and unstable manifolds of the unstable solution will cross, producing homoclinic chaos.

6.8 The Contopoulos case

In this model we have $a_1 = 0$, $\lambda = 0$. From eq. (15) we find the condition:

$$\frac{m}{2}\tau_1 + \left(-n + \frac{29m}{8}\right)\tau_2 = 0.$$

So we have for existence the requirement $n > 29m/8$; the 2 : 3- and the 4 : 1-resonances will not be present at this potential; also not the degenerate 1 : 2-resonance which can be seen as a 2 : 4-resonance. The higher order resonances that exist have to satisfy the requirement and will have smaller resonance manifolds than in the Hénon-Heiles potential. An example is the 2 : 9-resonance with resonance manifold size $O(\varepsilon^{\frac{7}{2}})$, time interval of interaction $O(\varepsilon^{-\frac{11}{2}})$. The homoclinic chaos in the resonance zones will be smaller in size than in the Hénon-Heiles model.

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