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The Lyapunov exponents of the Van der Pol oscillator

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SUMMARY

Lyapunov exponents characterize the dynamics of a system near its attractor. For the Van der Pol oscillator these are quantities for which an approximation should be at hand. Similar to the asymptotic approximation of amplitude and period, expressions are derived for the non-zero Lyapunov exponent for both small and large parameter values. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: Lyapunov exponents; Van der Pol oscillator; weakly oscillation; relaxation oscillation

1. INTRODUCTION

The Van der Pol equation is probably the best analysed non-linear second order differential equation having an asymptotically stable periodic solution. Numerical and analytical methods have been used to approximate its amplitude and period. The equation reads

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \tag{1}$$

where μ is a parameter taking values on the semi-infinite interval $(0, \infty)$, see Reference [1]. The equation was introduced as a model for a self-sustained oscillation in an electric circuit. It turned out that this equation could serve as a prototype for a non-linear oscillator in many more applications such as in physiology (heartbeat) and mechanics. From this last field of application the need of computing of the Lyapunov exponents of the Van der Pol equation

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originated: quenching of undesirable oscillations requires quantitative information about these quantities.

From the asymptotically stable periodic solution of (1), period T and amplitude A can be computed numerically [2]. Using software packages that can carry out formal computations, power series expansions of both with respect to the parameter μ can be made:

$$A = 2 + \frac{1}{96}\mu^2 - \frac{1033}{552960}\mu^4 + \mathcal{O}(\mu^6) \quad \text{for } \mu \downarrow 0 \tag{2}$$

$$T = 2\pi\left[1 + \frac{1}{16}\mu^2 - \frac{5}{3072}\mu^4 + \mathcal{O}(\mu^6)\right] \quad \text{for } \mu \downarrow 0 \tag{3}$$

see References [3–5]. However, for μ large a large number of terms of the expansion is needed to obtain an approximation of this form with a reasonable accuracy.

The approach for μ large should reflect the particular behaviour of the periodic solution for μ large known as a relaxation oscillation [6]. In this parameter regime the phase, as it runs through the full period, can be divided in intervals where the solution has its typical behaviour which can be caught in a locally valid asymptotic solution that is based on the property that the parameter μ is large. Integration constants in such solutions are found by matching solutions valid for adjacent time intervals. From these solutions asymptotic expressions for the amplitude and period holding for $\mu \uparrow \infty$ can be derived:

$$A = A_0 + A_{2/3}\mu^{-4/3} + A_1(\mu)\mu^{-2} + \mathcal{O}(\mu^{-8/3}) \tag{4}$$

with

$$A_0 = 2, \quad A_{2/3} = \frac{1}{3}\alpha, \quad A_1(\mu) = -\frac{16}{27}\ln(\mu) + \frac{2}{9}\ln(2) - \frac{8}{9}\ln(3) + \frac{1}{3}b_1 - \frac{1}{9}$$

and

$$T = T_1\mu + T_{-1/3}\mu^{-1/3} + T_{-1}(\mu)\mu^{-1} + \mathcal{O}(\mu^{-4/3}\ln(\mu)) \tag{5}$$

with

$$T_1 = 3 - 2\ln(2), \quad T_{-1/3} = 3\alpha$$

$$T_{-1}(\mu) = -\frac{2}{3}\ln(\mu) + \ln(2) - \ln(3) + 3b_1 - 1 - \ln(\pi) - 2\ln(Ai'(-\alpha))$$

where α is the first zero of the Airy function and b_1 is a constant that can be approximated numerically:

$$\alpha = 2.33810741 \quad \text{and} \quad b_1 = 0.17235$$

In addition to the amplitude and the period the non-trivial Lyapunov exponent of the periodic solution can be approximated asymptotically for μ small as well as large. In this study we will construct the first few terms of such an asymptotic approximation. Moreover the result will be compared with numerical values obtained from a numerical approximation of the periodic solution.

As we mentioned above stable periodic solutions corresponding with a normal mode in an engineering system can be undesirable. Well-known examples are flow-induced vibrations like the galloping of overhead power lines or violent oscillations of movable dams immersed in a river or estuary. There are many other examples, for instance heave-roll motion of a ship, oscillations of masses on transporter belts caused by dry-friction or rotating machinery which is modelled by a parametrically excited rotor. A survey of such problems and a suitable analysis is given in Reference [7]. In flow-induced vibrations self-excitation plays a prominent part. Such vibrations are usually modelled by the Van der Pol or by the Rayleigh equation. The engineering treatment makes often use of energy absorbers which means mathematically that the equation with self-excitation is coupled to an oscillator which destabilizes the self-excited periodic solution. Two aspects are relevant here. First this coupling requires a specific tuning to the period of the self-excited oscillation. We note that for the Van der Pol equation the period, as it depends on the parameter, is well-known. A second aspect is the rate of attraction of the periodic solution that is to be balanced by the coupled oscillator. This rate of attraction is measured by its Lyapunov exponent which will be studied in this paper. We note that a first-order analysis which aims at destabilization of the Van der Pol relaxation oscillation is presented by Verhulst and Abadi [8].

2. THE LYAPUNOV EXPONENTS OF A DYNAMICAL SYSTEM

The Lyapunov exponents of a trajectory of a system of n coupled non-linear differential equations are defined as follows. Let $y(t)$ be the solution of the following vector differential equation with initial vector value

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y), \quad y(0) = y_0 \tag{6}$$

Then we formulate the corresponding tangent linear system by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(t)v, \quad F(t) = \left[\frac{\partial f_i}{\partial y_j}(y(t))\right] \tag{7}$$

For this system we follow the evolution of the set of initial points forming an n-dimensional unit sphere changing into an ellipsoid with principal axes $p_i(t)$, $i = 1, 2, \dots, n$. The Lyapunov exponents follow from the limit:

$$\lambda_i = \lim_{t \uparrow \infty} \frac{1}{t} \ln(p_i(t)), \quad i = 1, \dots, n \tag{8}$$

with an ordering such that $\lambda_i \geqslant \lambda_{i+1}$ [9]. This evolution of the ellipsoid also reflects the dynamics of the non-linear system near the above trajectory for a sufficiently small initial sphere. The Lyapunov exponents are a quantification of this local behaviour. It can be proved that at least one Lyapunov exponent has the value zero.

2.1. The Van der Pol oscillator

For the periodic solution of the Van der Pol equation it means that besides this vanishing exponent a real Lyapunov exponent with a negative value must exist. It is this exponent that will approximated asymptotically in this study. For a periodic solution of a two-dimensional system with period T the two Lyapunov exponents satisfy

$$\lambda_1 + \lambda_2 = \frac{1}{T}\int_0^T \operatorname{trace} F(t)\,\mathrm{d}t \tag{9}$$

[10]. Consequently, the second exponent of the periodic solution $x(t)$ of Van der Pol equation satisfies

$$\lambda_2 = \frac{1}{T}\int_0^T \operatorname{trace} F(t)\,\mathrm{d}t = -\frac{\mu}{T}\int_0^T \{x^2(t)-1\}\,\mathrm{d}t \tag{10}$$

This result is found by transforming the second order differential equation (1) to a system of two first order differential equations.

3. ALMOST LINEAR OSCILLATION

Assuming that the parameter in the differential equation is small, $\mu \ll 1$, we expand the periodic solution with respect to this small parameter:

$$x(t;\mu) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \cdots \tag{11}$$

Substitution in the differential equation (1) yields

$$\begin{aligned} &\frac{\mathrm{d}^2x_0}{\mathrm{d}t^2} + x_0 + \mu\left[\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + x_1 + (x_0^2-1)\frac{\mathrm{d}x_0}{\mathrm{d}t}\right] \\ &+\mu^2\left[\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + x_2 + (x_0^2-1)\frac{\mathrm{d}x_1}{\mathrm{d}t} + 2x_0x_1\frac{\mathrm{d}x_0}{\mathrm{d}t}\right] + \cdots = 0 \end{aligned}$$

This equation is satisfied if the coefficients of the powers over μ vanish leading to a recurrent system of linear differential equations for the coefficients of (11):

$$\begin{aligned} &\frac{\mathrm{d}^2x_0}{\mathrm{d}t^2} + x_0 = 0 \\ &\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + x_1 = -(x_0^2-1)\frac{\mathrm{d}x_0}{\mathrm{d}t} \\ &\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + x_2 = -(x_0^2-1)\frac{\mathrm{d}x_1}{\mathrm{d}t} - 2x_0x_1\frac{\mathrm{d}x_0}{\mathrm{d}t} \\ &\qquad \cdots\cdot \end{aligned}$$

In principle all solutions satisfying an arbitrary initial condition can be approximated asymptotically for a large (but finite) time interval. For approximating the periodic solution special conditions should be imposed to the differential equations for the coefficients. With the method of Poincare-Lindstedt, see Reference [10], the system of equations can be solved recursively. Carrying out the computations for the first three terms we find

$$x(\theta;\mu) = 2\cos(\theta) + \mu x_1(\theta) + \mu^2 x_2(\theta) + \mu^3 x_3(\theta) + \cdots, \quad \theta = \omega t \tag{12}$$

with

$$x_1(\theta) = \frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta)$$

$$x_2(\theta) = -\frac{1}{8}\cos(\theta) + \frac{3}{16}\cos(3\theta) - \frac{5}{96}\cos(5\theta)$$

$$x_3(\theta) = -\frac{7}{256}\sin(\theta) + \frac{21}{256}\sin(3\theta) - \frac{35}{576}\sin(5\theta) + \frac{7}{576}\sin(7\theta)$$

and

$$\omega = 1 - \frac{1}{16}\mu^2 + \frac{17}{3072}\mu^4 + \cdots \tag{13}$$

In the literature a number of studies deal with schemes to compute the coefficients of these expansions up to a very high order using software packages for symbolic calculations, see Reference [5]. Substitution of (12) in (10) yields the following asymptotic approximation for the non-trivial Lyapunov exponent of the periodic solution:

$$\lambda_2 = -\mu - \frac{1}{16}\mu^3 + \frac{263}{18432}\mu^5 + \mathcal{O}(\mu^7) \quad \text{for } \mu \downarrow 0 \tag{14}$$

4. RELAXATION OSCILLATION

In order to analyse the periodic solution of the differential equation (1) for large parameter values, $\mu \gg 1$, we make a transformation in the time scale and introduce a small parameter

$$\tau = t/\mu, \quad \varepsilon = 1/\mu^2 \tag{15}$$

Then (1) takes the form

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + (x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = 0, \quad \varepsilon \ll 1 \tag{16}$$

For the almost linear oscillation an asymptotic solution valid over the entire time interval could be made. For the relaxation oscillation that is not the case: three approximations are made valid for three time intervals of half the period (symmetry), see Figure 1. These local solutions should match meaning that in a small domain of overlap where two approximations are valid the expansions of both solutions should be identical. For a survey of the different methods in literature that handle this problem we refer to Reference [6]. The method of Carrier and Lewis [11] suits very well for our purpose of approximating the non-trivial Lyapunov exponent. Others, e.g. Dorodnicyn [12], put much more effort in approximating asymptotically the period and amplitude of the relaxation oscillation with (4), (5) as result.

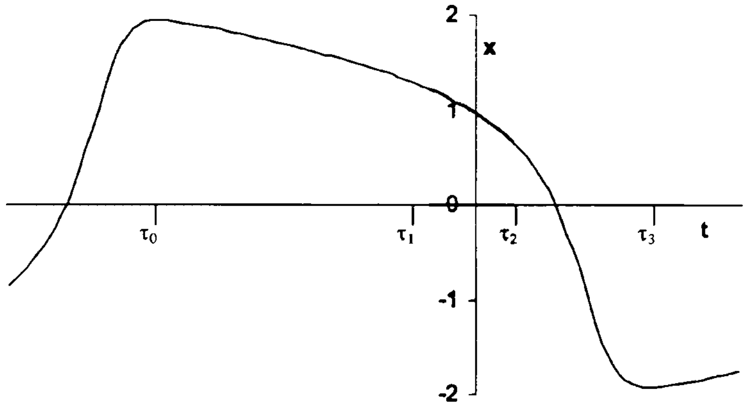

Figure 1. The periodic solution over half a period with the three intervals where a local asymptotic approximation is made.

The three time intervals mentioned above are separated by the points

$$\tau_0 = -\frac{T}{2\mu} + \alpha\varepsilon^{2/3} + r\varepsilon, \quad \tau_1 = -p\varepsilon^{2/3}, \quad \tau_2 = \alpha\varepsilon^{2/3} - q\varepsilon \quad \text{and} \quad \tau_3 = \alpha\varepsilon^{2/3} + r\varepsilon$$

In these expressions p, q and r denote parameters that are large but independent of ε. By varying these parameters the points τ_1, τ_2, and τ_3 move through the domain of overlap. The three local approximations are as follows [6]:
Interval 1 $[\tau_0, \tau_1]$:

$$x(\tau) = x_0(\tau) + x_1(\tau)\varepsilon + \cdots$$

with the coefficients satisfying

$$(x_0^2 - 1)\frac{\mathrm{d}x_0}{\mathrm{d}\tau} + x_0 = 0, \quad (x_0^2 - 1)\frac{\mathrm{d}x_1}{\mathrm{d}\tau} + 2x_0\frac{\mathrm{d}x_0}{\mathrm{d}\tau}x_1 + x_1 = -\frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2}$$

The first equation is not solved but used for the purpose of changing the integration variable in (10) from t to x_0 for the leading term of (10) coming from this interval. Next the second equation is solved with x_0 as the integration variable by eliminating τ:

$$x_1(x_0) = \frac{x_0}{x_0^2 - 1}\left\{\frac{1}{2}\ln(x_0^2 - 1) - \frac{1}{2}\ln(x_0^2) + \frac{1}{x_0^2 - 1}\right\}$$

Interval 2 $[\tau_1, \tau_2]$:

$$x(\tau) = 1 + \varepsilon^{1/3}v(\xi) + \cdots, \quad \tau = \varepsilon^{2/3}\xi$$

with

$$v(\xi) = -\frac{Ai'(-\xi)}{Ai(-\xi)}; \quad v(\xi) \approx \sqrt{-\xi} - \frac{1}{4\xi} + \mathcal{O}(\xi^{-5/2}) \quad \text{for } \xi \downarrow -\infty$$

Interval 3 $[\tau_2, \tau_3]$:

$$x(\tau) = w(\eta) + \cdots, \quad \tau = \varepsilon^{2/3}\alpha + \varepsilon\eta$$

satisfying

$$\frac{\mathrm{d}w}{\mathrm{d}\eta} + \frac{1}{3}w^3 - w + \frac{1}{3}A^3 - A = 0$$

where A is the amplitude given by (4).

The Lyapunov exponent λ_2 is composed of three components that correspond with the periodic solution as it holds in the three intervals:

$$\lambda_2 = -\frac{\mu}{T}\int_0^T \{x^2(t) - 1\}\,\mathrm{d}t = -\frac{2\mu^2}{T}(I_1 + I_2 + I_3) \tag{17}$$

with

$$I_1 = \int_{\tau_0}^{\tau_1} \{x^2(\tau) - 1\}\,\mathrm{d}\tau, \quad I_2 = \int_{\tau_1}^{\tau_2} \{x^2(\tau) - 1\}\,\mathrm{d}\tau, \quad I_3 = \int_{\tau_2}^{\tau_3} \{x^2(\tau) - 1\}\,\mathrm{d}\tau$$

These integrals can be approximated using the locally valid approximations of the solution:

$$I_1 \approx \int_{A_r}^{1+\varepsilon^{1/3}\sqrt{p}} (x_0^2 - 1 + 2\varepsilon x_0 x_1)\frac{\mathrm{d}\tau}{\mathrm{d}x_0}\,\mathrm{d}x_0, \quad A_r = 2 + \frac{1}{3}\alpha\varepsilon^{2/3} + \left(\frac{1}{3}T_{-1} - \frac{2}{3}r\right)\varepsilon$$

$$I_2 \approx -2\varepsilon\int_{-p}^{\alpha - q\varepsilon^{2/3}} \frac{Ai'(-\xi)}{Ai(-\xi)}\,\mathrm{d}\xi$$

$$\approx \varepsilon\left[2\ln(Ai'(-\alpha) + 2\ln(2q) + \frac{2}{3}\ln(\varepsilon) + \ln(\pi) + \frac{1}{2}\ln(p) + \frac{4}{3}p^{3/2}\right]$$

$$I_3 \approx \varepsilon\int_{1-1/q}^{-A+s(r)} \frac{x^2 - 1}{x - \frac{1}{3}x^3 + A - \frac{1}{3}A^3}\,\mathrm{d}x, \quad s(r) \approx 3\varepsilon^{-3r-1}$$

Working out the expression for the local contribution for interval I we obtain the following result:

$$I_1 = I_{10} + I_{11}$$

with

$$I_{10} = \int_{A_r}^{1+\varepsilon^{1/3}\sqrt{p}} (x_0^2 - 1)\frac{\mathrm{d}\tau}{\mathrm{d}x_0}\,\mathrm{d}x_0 \approx \ln(2) + \frac{3}{4} + \frac{3}{2}\alpha\varepsilon^{2/3} + \left(\frac{3}{2}T_{-1} - \frac{4}{3}p^{3/2} - 3r\right)\varepsilon$$

and

$$I_{11} = \int_{A_r}^{1+\varepsilon^{1/3}\sqrt{p}} 2\varepsilon x_0 x_1 \frac{\mathrm{d}\tau}{\mathrm{d}x_0}\,\mathrm{d}x_0 \approx -5\ln(2) - \frac{5}{2}\ln(3) - \frac{1}{3}\ln(\varepsilon) - \frac{1}{2}\ln(p)$$

in which the parameters p and q (as well as r) cancel out in the final summation,

$$I_1 + I_2 + I_3 = L_0 + L_{2/3}\varepsilon^{2/3} + L_1(\varepsilon)\varepsilon + o(\varepsilon)$$

where

$$L_0 = \ln(2) + \frac{3}{4}, \quad L_{2/3} = \frac{3}{2}\,\alpha$$

$$L_1(\varepsilon) = \frac{1}{3}\,\ln(\varepsilon) + \frac{3}{2}\,T_{-1}(\varepsilon) - 3\ln(2) + 2\ln(Ai'(-\alpha)) + \ln(\pi) - \frac{5}{2}\,\ln(3)$$

Consequently, using (5), (15) and (17) we obtain for $\mu \uparrow \infty$

$$\lambda_2 = -2\mu\left[\frac{L_0}{T_1} + \frac{1}{T_1^2}(T_1 L_{2/3} - T_{-1/3}L_0)\mu^{-4/3} + \frac{1}{T_1^2}(T_1 L_1(\mu) - T_{-1}(\mu)L_0)\mu^{-2} + \cdots\right] \tag{18}$$

5. COMPARISON WITH THE NUMERICAL APPROXIMATION

The periodic solution can be approximated numerically over its full period T by taking as starting value

$$x(0) = A, \quad x'(0) = 0 \tag{19}$$

The numerical value of T, as it depends upon μ, is taken over from Reference [2]. In Figure 2 this numerical approximation of $-\lambda_2/\mu$ is compared with the asymptotic

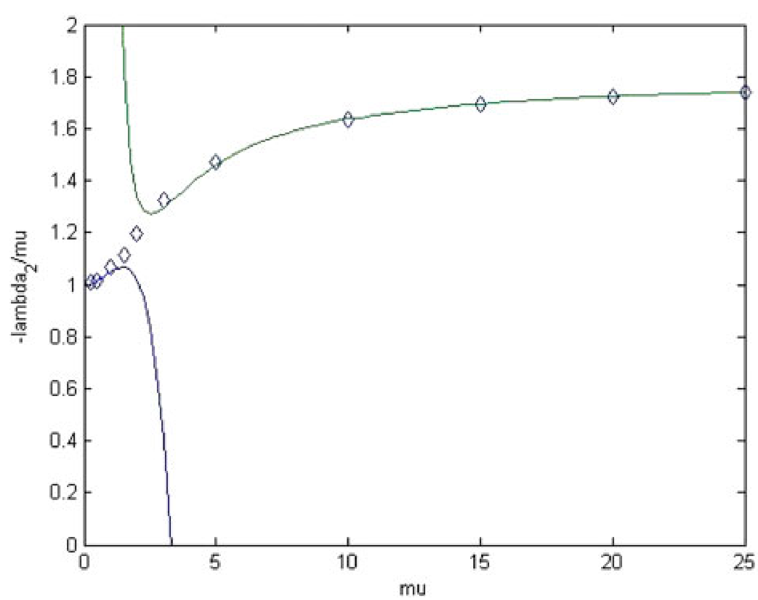

Figure 2. The dependence of $-\lambda_2/\mu$ upon μ: the asymptotic approximations are compared with the numerical approximation.

Table I. The Lyapunov exponent λ_2 for large μ: by numerical integration and by asymptotic approximation (18). The difference is $R(\mu) = \mu^{-1}\{\lambda_2^{\mathrm{as}}(\mu) - \lambda_2^{\mathrm{num}}(\mu)\}$.

μ	$-\lambda_2^{\mathrm{num}}/\mu$	$-\lambda_2^{\mathrm{as}}/\mu$	$R(\mu)$
1	1.0648	2.6957	1.6309
5	1.4724	1.4757	0.0033
10	1.6358	1.6423	0.0065
25	1.7398	1.7418	0.0020
50	1.7691	1.7697	0.0006

expressions (14) and (18). It is noted that both asymptotic approximations breakdown suddenly and that there is no overlap. Such an overlap comes within reach if in the regular expansion (12) a larger number of terms is included [5]. The decrease of the remainder term $R(\mu)$ for increasing μ is as expected. It is remarked that it does not exactly keep pace with the asymptotic order of the first neglected term. From one side this comes from the fact that the coefficients of the expansion tend to have larger values for higher order terms. At the other side the accuracy of the numerical solution tends to decrease slightly because of the increasing stiffness of the differential equation for increasing μ. The difference of approximation (18) with the sufficiently accurate numerical approximation of the Lyapunov exponent is given in Table I.

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