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Resonance in a Rigid Rotor with Elastic Support

Für einen Rotor mit elastischer Halterung in axialer wie auch seitlicher Richtung wird ein Modell formuliert, das auf das Stabilitätsproblem für kleine Vertikalschwingungen in senkrechter Position führt. Dies ist ein autoparametrisches Erregungsproblem, das in Gestalt zweier gekoppelter Mathieu-ähnlicher Gleichungen formuliert werden kann. Die Analyse der Spezialfälle ohne und mit (linearer) Dämpfung sowie der Fall nichtlinearer Dämpfung werden ausgeführt, wobei Mittelung und numerische Bifurkationstechniken angewendet werden, die im nichtlinearen Fall auf Hysteresis und Phasensynchronisierung führen.

A model is formulated for a rotor with elastic support in axial and lateral directions which leads to the problem of stability of small vertical oscillations in the upright position. This is an autoparametric excitation problem which can be formulated as two coupled Mathieu-like equations. The analysis of the cases without and with (linear) damping and the case of nonlinear damping is carried out using averaging and numerical bifurcation techniques leading in the nonlinear case to hysteresis and phase-locking.

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Introduction

Some rotating machines, e.g. centrifuges, can be modelled by a rigid rotor which is elastically mounted in lateral and axial directions whose axis of rotation is vertical. It is assumed that the axial thrust bearing can be modelled as a joint. The elastic mounting in axial direction is due to the elasticity of the thrust bearing support. In some cases the elasticity of the floor on which the machine is situated can influence the elasticity in axial direction.

In this paper we study a basic model for this rigid rotor, which is assumed to be perfectly balanced. In particular, the stability of small vertical oscillations of the upright position will be considered. Taking the amplitude of this oscillation as the small parameter, introducing asymptotic expansions around the vertical oscillation leads, to first order, to a system with two degrees of freedom, consisting of two coupled Mathieu-like equations. Depending on the frequency of the oscillation and of the model's parameters (such as mass, moments of inertia, rotational speed) parametric resonance can occur. Using the method of averaging the frequency-range for which the motion becomes unstable is calculated in section 3.

In section 4 linear damping is added to the model, which leads to interesting changes in the stability domain.

The last two paragraphs are concerned with nonlinear damping. In section 5 the effects of various types of nonlinear damping on the one degree of freedom Mathieu equations are summarized. Finally, in section 6 nonlinear damping is added to the model of the rotor. Numerical bifurcation analysis of the averaged equation shows that the system then exhibits hysteresis and phase-locking.

An interesting aspect is that most of the asymptotic expansions for the stable periodic solutions, obtained by averaging in these problems, yield approximations which are valid for all time.

1. Models for a rigid rotor with elastic support in axial and lateral directions

The following formulation is based on [10]. Consider a rigid rotor, consisting of a heavy disk of mass M which is rotating around an axis (Fig. 1). The axis of rotation is elastically mounted on a foundation and has a joint in point A; the connections which are holding the rotor in an upright position are also elastic. To describe the position of the rotor we use the axial displacement u in the vertical (z-) direction and the angle of the axis of rotation with the z-axis, θ, and around the z-axis, φ. See Fig. 1.

The distance between the centre of gravity B of the rotating disk and the point A is R. The moments of inertia are I_1, I_2 and I_3 where, because of the symmetry of the rotor, $I_1 = I_2$. The equations of motion will be derived in a conservative frame-work, using Lagrange equations, after which we add various kinds of friction. The main purpose will be to study the stability of the upright position of the rotor, depending on the system's parameters; so the equations will be linearized around $\theta = 0$, the upright position.

2. The equations of motion

In order to derive the Lagrangean, the kinetic energy with respect to the origin O has to be established. This can be written as

$$T = T_A + \tfrac{1}{2} M\dot{u}^2 + \dot{u} \sum_i m_i v_i^{(3)} . \qquad (2.1)$$

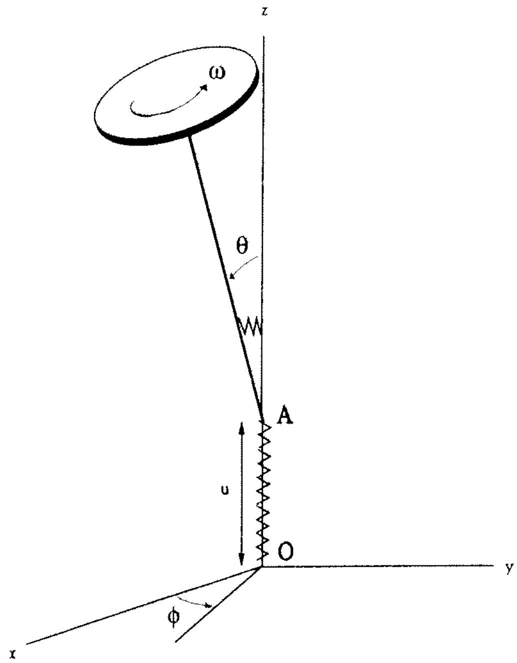

Fig. 1

where T_A is the kinetic energy with respect to A, and $v_i^{(3)}$ is the z-component of the velocity, with respect to A, of a particle; the sum being over all particles in the rotor (see Fig. 1). Because of the symmetry of the rotor, this is equal to the total mass of the rotor times the z-component of the velocity of the centre of mass, so that

$$\dot{u} \sum_i m_i v_i^{(3)} = -MR\dot{u}\dot{\theta} \sin \theta \,. \tag{2.2}$$

We wish to study the stability of the upright position of the rotor by considering small oscillations around $\theta = 0$. In order to have the centre of gravity actually passing through $\theta = 0$, we assume that the angular velocity with respect to the z-axis remains constant, say ω (see [5]). In that case we can write:

$$T_A = \tfrac{1}{2} I_1 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \tfrac{1}{2} I_3 (\omega + (\cos \theta - 1)\, \dot{\varphi})^2 \,. \tag{2.3}$$

The *kinetic energy* (2.1) with respect to O then becomes:

$$T = \tfrac{1}{2} I_1 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \tfrac{1}{2} I_3 (\omega + (\cos \theta - 1)\, \dot{\varphi})^2 + \tfrac{1}{2} M\dot{u}^2 - MR\dot{u}\dot{\theta} \sin \theta \,. \tag{2.4}$$

The *potential energy* is given by

$$V = Mg(R \cos \theta + u) + kR^2 \sin^2 \theta + k_0 u^2 \,, \tag{2.5}$$

where k and k_0 are the coefficients of the lateral stiffness of the mounting and of the vertical stiffness of the axis, respectively.

We will look at the projection of the centre of gravity on the x,y-plane, given by

$$x = R \sin \theta \cos \varphi \,, \qquad y = R \sin \theta \sin \varphi \,,$$

which leads to

$$\dot{\theta} = \frac{x\dot{x} + y\dot{y}}{\sqrt{(R^2 - x^2 - y^2)(x^2 + y^2)}} \,, \qquad \dot{\varphi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \,.$$

Inserting these expressions into the kinetic and potential energies (2.4–5) and retaining the linear and the quadratic terms around $x = y = 0$, gives:

$$T_{\text{lin}} = \frac{1}{2} \frac{I_1}{R^2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_3 \left(\omega^2 - \frac{\omega}{R^2} (x\dot{y} - y\dot{x}) \right) + \frac{1}{2} M\dot{u}^2 - M \frac{\dot{u}}{R} (x\dot{x} + y\dot{y}) \,. \tag{2.6}$$

$$V_{\text{lin}} = Mg(R + u) + \left(k - \frac{Mg}{2R} \right) (x^2 + y^2) + k_0 u^2 \,. \tag{2.7}$$

The Lagrange equations, $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}(T - V)\right) - \frac{\partial}{\partial x_i}(T - V) = 0$, $i = 1, 2, 3$, become:

$$\left.\begin{aligned} &I_1\ddot{x} + I_3\omega\dot{y} + (2kR^2 - MgR)\,x = MR\ddot{u}x\,, \qquad I_1\ddot{y} - I_3\omega\dot{x} + (2kR^2 - MgR)\,y = MR\ddot{u}y\,,\\ &\ddot{u} + \frac{2k_0}{M}u = \frac{1}{R}(\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}) - g\,. \end{aligned}\right\} \tag{2.8}$$

Dividing the first two equations by I_1 and using the scaling $\tau = \Omega t$ with $\Omega^2 = \frac{2kR^2 - MgR}{I_1}$ finally produces:

$$\left.\begin{aligned} &x'' + 2\alpha y' + x = \frac{MR}{I_1}u''x\,, \qquad y'' - 2\alpha x' + y = \frac{MR}{I_1}u''y\,,\\ &u'' + 4\eta^2 u = \frac{1}{R}(x'^2 + xx'' + y'^2 + yy'') - \frac{g}{\Omega^2} \end{aligned}\right\} \tag{2.9}$$

with $2\alpha = \frac{I_3}{I_1\Omega}\omega$, $4\eta^2 = \frac{2k_0}{\Omega^2 M}$.

A special solution of this equation, oscillation in the upright position, is given by $x = y = 0$ and

$$u_0(t) = a\cos 2\eta t - \frac{g}{4\eta^2\Omega^2} = a\cos 2\eta t - \frac{Mg}{2k_0}\,. \tag{2.10}$$

We will consider the situation that $a \ll 1$, and study the stability of this solution by postulating asymptotic expansions for x, y and u:

$$x = \varepsilon x_1 + \varepsilon^2 x_2 + \dots\,, \qquad y = \varepsilon y_1 + \varepsilon^2 y_2 + \dots\,, \qquad u = -\frac{Mg}{2k_0} + \frac{I_1}{MR}\varepsilon\cos 2\eta t + \varepsilon^2 u_2 + \dots\,, \tag{2.11}$$

with $\varepsilon = a\frac{MR}{I_1}$ a small positive parameter. Inserting these expressions into (2.9) yields up to first order in ε:

$$x_1'' + 2\alpha y_1' + x_1 = -4\varepsilon\eta^2\cos 2\eta t x_1\,, \qquad y_1'' - 2\alpha x_1' + y_1 = -4\varepsilon\eta^2\cos 2\eta t y_1\,. \tag{2.12}$$

In the equation for u all terms to order ε vanish.

3. The linear system (2.12) or (3.1)

We replace (x_1, y_1) by (x, y). Neglection of $O(\varepsilon^2)$ terms means neglection of all nonlinear terms in a neighbourhood of the trivial equilibrium solution $(x, \dot{x}, y, \dot{y}) = (0, 0, 0, 0)$ of system (2.9),

$$\ddot{x} + 2\alpha\dot{y} + (1 + 4\varepsilon\eta^2\cos 2\eta t)\,x = 0\,, \qquad \ddot{y} - 2\alpha\dot{x} + (1 + 4\varepsilon\eta^2\cos 2\eta t)\,y = 0\,. \tag{3.1}$$

System (3.1) constitutes a system of Mathieu-like equations; note that we have also neglected the effects of damping, see sections 4–6. The frequencies of the unperturbed, $\varepsilon = 0$, system (3.1) are $\omega_1 = \sqrt{\alpha^2 + 1} + \alpha$ and $\omega_2 = \sqrt{\alpha^2 + 1} - \alpha$. It is well-known that the frequency of the autoparametric excitation 2η being close to certain resonances with the eigenfrequencies of the unperturbed system ($\varepsilon = 0$) causes the trivial solution to be unstable. We shall determine the instability domains for ε small.

Putting $z = x + iy$ system (3.1) can be written as

$$\ddot{z} - 2\alpha i\dot{z} + (1 + 4\varepsilon\eta^2\cos 2\eta t)\,z = 0\,. \tag{3.2}$$

Introducing

$$v = e^{-i\alpha t}z \tag{3.3}$$

and putting $\eta t = \tau$ we obtain

$$v'' + \left(\frac{1 + \alpha^2}{\eta^2} + 4\varepsilon\cos 2\tau\right)v = 0\,, \tag{3.4}$$

where ′ denotes differentiation with respect to τ. This is the Mathieu equation, see for instance [6, 11].

We conclude that the trivial solution is stable for ε small enough if $\sqrt{1+\alpha^2} \neq n\eta$, $n = 1, 2, 3, \ldots$. A primary resonance, $n = 1$, arises if

$$\sqrt{1+\alpha^2} = \eta\,. \tag{3.5}$$

If (3.5) is satisfied the trivial solution of equation (3.4) is unstable. So, because of (3.3), the trivial solution of system (3.2) and (3.1) is unstable. Note that instability arises if

$$\omega_1 + \omega_2 = 2\eta\,,$$

i.e. if the sum of the eigenfrequencies of the unperturbed system equals the autoparametric excitation frequency 2η. The domain of instability can be calculated as in [11], appendix 2; we find for the boundaries:

$$\eta = \sqrt{1+\alpha^2}\,(1 \pm \varepsilon) + O(\varepsilon^2)\,. \tag{3.6}$$

It must be kept in mind that α is proportional to the rotating frequency of the disk and to the ratio of the moments of inertia; see (2.9). A secondary resonance, $n = 2$, arises if

$$\sqrt{1+\alpha^2} = 2\eta\,, \tag{3.7}$$

i.e. $\omega_1 + \omega_2 = \eta$. As above we find the boundaries of the domains of instability:

$$2\eta = \sqrt{1+\alpha^2}\,(1 + \tfrac{1}{24}\varepsilon^2) + O(\varepsilon^4)\,, \qquad 2\eta = \sqrt{1+\alpha^2}\,(1 - \tfrac{5}{24}\varepsilon^2) + O(\varepsilon^4)\,. \tag{3.8}$$

Higher order resonances can be studied in the same way; the domains of instability in parameter space continue to narrow as n increases.

4. Effect of linear damping on the instability-interval

Adding small linear damping to system (3.1), with positive damping parameter $\mu = 2\varepsilon\varkappa$ leads to the equation

$$\ddot{z} - 2\alpha i\dot{z} + (1 + 4\varepsilon\eta^2\cos 2\eta t)\,z + 2\varepsilon\varkappa\dot{z} = 0\,. \tag{4.1}$$

Because of the damping term, we cannot reduce to a single second order real equation.

The solution of the unperturbed ($\varepsilon = 0$) equation can be written as

$$z(t) = z_1\,\mathrm{e}^{i\omega_1 t} + z_2\,\mathrm{e}^{-i\omega_2 t}\,; \qquad z_1, z_2 \in \mathbb{C}\,, \tag{4.2}$$

with $\omega_1 = \sqrt{\alpha^2+1} + \alpha$, $\omega_2 = \sqrt{\alpha^2+1} - \alpha$. Applying variation of constants, we find

$$\begin{aligned} &\dot{z}_1\,\mathrm{e}^{i\omega_1 t} + \dot{z}_2\,\mathrm{e}^{-i\omega_2 t} = 0\,,\\ &i\omega_1\dot{z}_1\,\mathrm{e}^{i\omega_1 t} - i\omega_2\dot{z}_2\,\mathrm{e}^{-i\omega_2 t} = -\varepsilon(2\varkappa(i\omega_1 z_1\,\mathrm{e}^{i\omega_1 t} - i\omega_2 z_2\,\mathrm{e}^{-i\omega_2 t}) + 4\varepsilon\eta^2\cos 2\eta t(z_1\,\mathrm{e}^{i\omega_1 t} + z_2\,\mathrm{e}^{-i\omega_2 t}))\,. \end{aligned} \tag{4.3}$$

From (4.3) we get the equations for z_1 and z_2:

$$\left.\begin{aligned} \dot{z}_1 &= \frac{i\varepsilon}{\omega_1+\omega_2}\,(2\varkappa(i\omega_1 z_1 - i\omega_2 z_2\,\mathrm{e}^{-i(\omega_1+\omega_2)t}) + 4\eta^2\cos 2\eta t(z_1 + z_2\,\mathrm{e}^{-i(\omega_1+\omega_2)t}))\,,\\ \dot{z}_2 &= \frac{-i\varepsilon}{\omega_1+\omega_2}\,(2\varkappa(i\omega_1 z_1\,\mathrm{e}^{i(\omega_1+\omega_2)t} - i\omega_2 z_2) + 4\eta^2\cos 2\eta t(z_1\,\mathrm{e}^{i(\omega_1+\omega_2)t} + z_2))\,. \end{aligned}\right\} \tag{4.4}$$

We want to calculate the instability-interval around $\eta = \eta_0 = \frac{1}{2}(\omega_1 + \omega_2) = \sqrt{\alpha^2+1}$. To this effect we put

$$\eta = \eta_0 + \varepsilon\sigma\,, \tag{4.5}$$

where σ is a parameter, independent of ε, which indicates the distance to exact resonance and calculate the values of σ for which the trivial solution of (4.4) becomes unstable.

Inserting (4.5) into (4.4) yields:

$$\left.\begin{aligned} \dot{z}_1 &= \frac{i\varepsilon}{\eta_0}\,(\varkappa(i\omega_1 z_1 - i\omega_2 z_2\,\mathrm{e}^{-2i\eta_0 t}) + \eta^2(z_1(\mathrm{e}^{2i\eta t} + \mathrm{e}^{-2i\eta t}) + z_2(\mathrm{e}^{2i\varepsilon\sigma t} + \mathrm{e}^{-2i(\eta+\eta_0)t})))\,,\\ \dot{z}_2 &= \frac{-i\varepsilon}{\eta_0}\,(\varkappa(i\omega_1 z_1\,\mathrm{e}^{2i\eta_0 t} - i\omega_2 z_2) + \eta^2(z_1(\mathrm{e}^{2i(\eta+\eta_0)t} + \mathrm{e}^{-2i\varepsilon\sigma t}) + z_2(\mathrm{e}^{2i\eta t} + \mathrm{e}^{-2i\eta t})))\,. \end{aligned}\right\} \tag{4.6}$$

After transforming

$$z_1 = v_1\, e^{i\varepsilon\sigma t}; \qquad z_2 = v_2\, e^{-i\varepsilon\sigma t} \tag{4.7}$$

we get equations that are in a suitable form for averaging over t (see [7]):

$$\left.\begin{aligned} \dot v_1 &= \frac{\varepsilon}{\eta_0}\left(-(\omega_1\varkappa + i\sigma\eta_0)\, v_1 + \varkappa\omega_2 v_2\, e^{-2i\eta t} + i\eta^2(v_1(e^{2i\eta t} + e^{-2i\eta t}) + v_2(1 + e^{-4i\eta t}))\right),\\ \dot v_2 &= \frac{\varepsilon}{\eta_0}\left(\varkappa\omega_1 v_1\, e^{2i\eta t} - (\omega_2\varkappa - i\sigma\eta_0)\, v_2 - i\eta^2(v_1(1 + e^{4i\eta t}) + v_2(e^{2i\eta t} + e^{-2i\eta t}))\right). \end{aligned}\right\} \tag{4.8}$$

The averaged equations for v_1 and v_2 become:

$$\dot v_1 = \frac{\varepsilon}{\eta_0}\left(-(\omega_1\varkappa + i\sigma\eta_0)\, v_1 + i\eta_0^2 v_2\right), \qquad \dot v_2 = \frac{\varepsilon}{\eta_0}\left(-i\eta_0^2 v_1 - (\omega_2\varkappa - i\sigma\eta_0)\, v_2\right). \tag{4.9}$$

The *stability of the trivial solution of* (4.9) and therefore of (4.6) is determined by the real parts of the eigenvalues of (4.9). Let λ' be an eigenvalue of (4.9). Define λ by $\lambda' = \frac{\varepsilon}{\eta_0}\lambda$. The eigenvalue equation for (4.9) becomes:

$$\lambda^2 + 2\eta_0\varkappa\lambda + \varkappa^2 - 2i\alpha\varkappa\sigma\eta_0 + \sigma^2\eta_0^2 - \eta_0^4 = 0 \tag{4.10}$$

which has the roots:

$$\lambda^{+,-} = -\eta_0\varkappa \pm \sqrt{(\alpha\varkappa + i\eta_0\sigma)^2 + \eta_0^4}\,. \tag{4.11}$$

For $\varkappa = 0$ (no damping), we find that $\lambda^+ = \sqrt{\eta_0^4 - \sigma^2\eta_0^2}$, and so the trivial solution is unstable if $|\sigma| < \eta_0$, as we already found in section 3. However, if $\varkappa > 0$ we find after some calculations that Re $(\lambda^+) > 0$ iff:

$$\sigma^2 < \eta_0^4 - \varkappa^2 \tag{4.12}$$

if $\eta_0^4 - \varkappa^2 > 0$, otherwise there is stability for all values of σ. There is a curious jump-phenomenon associated with these results which is connected with non-smooth dependence of parameters in eigenvalue equations. In Fig. 2 we have indicated the boundaries of stable-unstable behaviour in an ε,η-diagram. If $\varkappa = 0$ equation (3.6) of section 3 applies; if $\varkappa > 0$ we find with (4.5) and (4.12) to first order (we recall that $\mu = 2\varepsilon\varkappa$)

$$\eta_b = \sqrt{1 + \alpha^2}\left(1 \pm \varepsilon\sqrt{1 + \alpha^2 - \frac{\varkappa^2}{\eta_0^2}} + \dots\right) = \sqrt{1 + \alpha^2}\left(1 \pm \sqrt{(1 + \alpha^2)\,\varepsilon^2 - \left(\frac{\mu}{\eta_0}\right)^2} + \dots\right). \tag{4.13}$$

It follows from Fig. 2 and Fig. 3, that the domain of instability actually becomes larger when damping is introduced. We also note that for $\varkappa \to 0$, $\eta_b \to \sqrt{1 + \alpha^2}\,(1 \pm \varepsilon\sqrt{1 + \alpha^2})$, which differs from the result we found when $\varkappa = 0$: $\eta_b = \sqrt{1 + \alpha^2}\,(1 \pm \varepsilon)$.

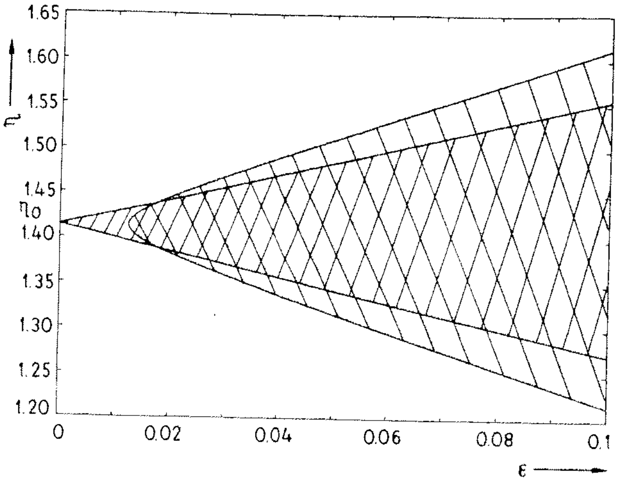

Fig. 2. Boundaries for stable-unstable behaviour
▨ domain of instability if $\varkappa = 0$
▧ domain of instability if $\varkappa > 0$, fixed

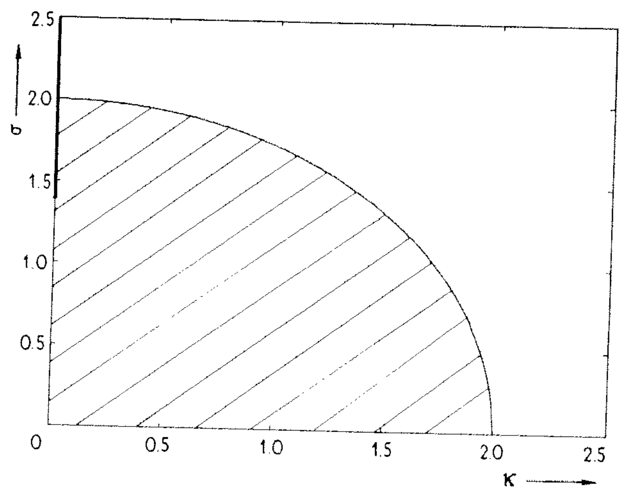

Fig. 3. Boundary for stable-unstable behaviour; domain of instability shaded. Thick line indicates stability interval for $\varkappa = 0$

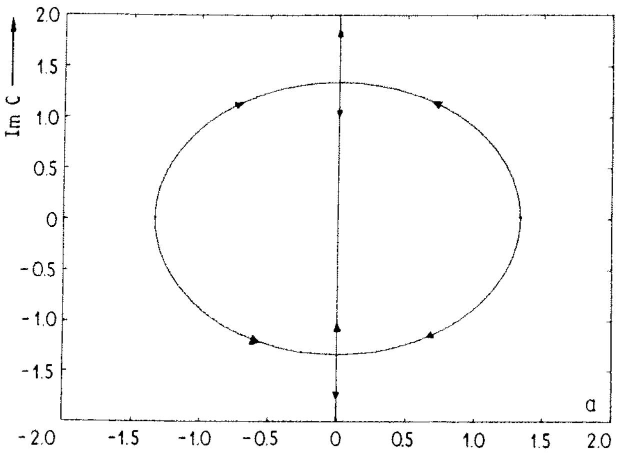

Fig. 4a. Eigenvalues of (4.9) parameterised by σ. $\varkappa = 0$, $\alpha = 1$

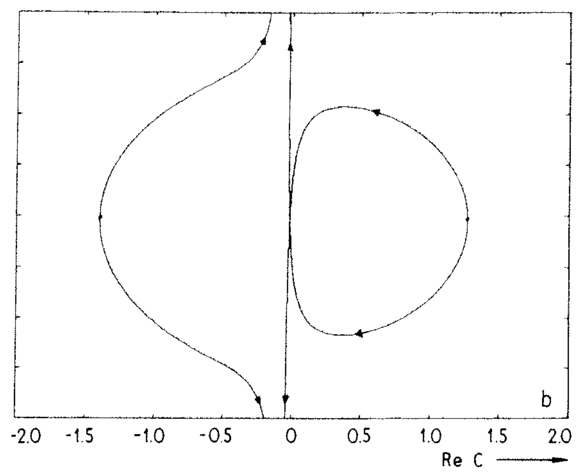

Fig. 4b. Eigenvalues of (4.9) parameterised by σ. $\varkappa = 0.3$, $\alpha = 1$

Another illustration of this phenomenon is given by Figs. 4a and 4b. These figures show the four eigenvalues λ_z of the averaged equations for z_1 and z_2 in the complex plane, as σ is varied. These eigenvalues can be calculated from (4.11) and transformation (4.7). This yields the four expressions:

$$\lambda_z = \varepsilon\left(-\varkappa \pm i\sigma \pm \sqrt{\left(\frac{\alpha\varkappa}{\eta_0} + i\sigma\right)^2 + \eta_0^2}\right).$$

In the figures the factor ε is scaled out and $\alpha = 1$. In Fig. 4a we see the situation for $\varkappa = 0$. Starting at $\sigma = 0$, there are two double eigenvalues ($+\eta_0$ and $-\eta_0$), which split up as σ starts to grow. The real parts of the eigenvalues (given by $\operatorname{Re}\lambda = \pm\sqrt{\eta_0^2 - \sigma^2}$) diminish in absolute value until they reach zero when $\sigma = \eta_0$. For $\sigma > \eta_0$ all eigenvalues are purely imaginary and the zero-solution of (4.9) is stable. In Fig. 4b we see how this situation is perturbed when $\varkappa > 0$. If $1 + \alpha^2 - \dfrac{\varkappa^2}{\eta_0^2} > 0$, the eigenvalues that split off of the positive eigenvalue reach the stability boundary (the imaginary axis) later than in the case $\varkappa = 0$. This corresponds to the broadening of the instability domain (Fig. 2). Also, when $\varkappa > 0$, the imaginary axis is crossed at the origin, whereas when $\varkappa = 0$, the imaginary axis is reached at $\pm i\eta_0$. This shows that the point in the complex domain where the real part of the eigenvalue becomes zero, does not depend continuously on $\varkappa$.

Mechanically this is caused by the coupling between the two degrees of freedom of the rotor in lateral directions which arises in the presence of damping. Such phenomena have been noted earlier in the literature, see [1, 2, 3].

We remark finally that it cannot be excluded that the boundary curve in the $\varkappa,\sigma$-diagram (Fig. 3), when nearing the σ-axis if $\varkappa$ tends to zero, suddenly bends away from the values $\sigma = \pm\eta_0^2$ to approach $\sigma = \pm\eta_0$. Because of the asymptotic validity of our calculations, this bending should take place in a boundary layer of thickness ε along the σ-axis. By performing calculations to second order — we omit the technical details — we have shown that the jump phenomenon is still there to $O(\varepsilon^2)$.

5. One-dimensional autoparametric excitation with nonlinear damping

The one-dimensional case can be seen as the limit case as α tends to zero which decouples the two degrees of freedom of system (3.1). It can also be seen as a one-dimensional oscillator with variable support.

The one-dimensional case with nonlinear damping is also a prototype for the full system. The results are well-known, see [9], but as there is no adequate survey available, we summarize the results in three significant cases. Each of them contains saddle-node bifurcations leading to stable periodic solutions.

a. Quadratic damping $|x|\,\dot{x}$.

The equation becomes

$$\ddot{x} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)\, x + \varepsilon\varkappa\dot{x} + \varepsilon\delta f(x, \dot{x})\,\dot{x} = 0 \tag{5.1}$$

with $\varkappa > 0$, $\delta \geqq 0$; first we take $f(x, \dot{x}) = |x|$, so we have progressive nonlinear damping. Putting again $\tau = \eta t$ we have

$$x'' = \left(\frac{1}{\eta^2} + 4\varepsilon \cos 2\tau\right) x + \varepsilon\frac{\varkappa}{\eta}\, x' + \varepsilon\frac{\delta}{\eta}\, |x|\, x' = 0 \tag{5.2}$$

with primary resonance if η is near to one:

$$\eta = 1 + \varepsilon\sigma\,. \tag{5.3}$$

To perform averaging, see [11], we use an amplitude-phase representation $x = r\cos(\tau + \psi)$, $x' = -r\sin(\tau + \psi)$ and to study the trivial equilibrium solution Cartesian coordinates $u = r\cos\psi$, $v = r\sin\psi$; note that we are not permitted to use polar coordinates near (0, 0).

Introducing the Lagrange standard form, averaging and omitting the terms of order ε^2 we find

$$r' = \varepsilon(r\sin 2\psi - \tfrac{1}{2}\varkappa r - \tfrac{1}{2}\delta r^2), \qquad \psi' = \varepsilon(-\sigma + \cos 2\psi). \tag{5.4}$$

In Cartesian coordinates system (5.4) becomes

$$\left.\begin{aligned} u' &= \varepsilon(-\tfrac{1}{2}\varkappa u + (1+\sigma)v - \tfrac{1}{2}\delta(u^2+v^2)^{1/2}u), \\ v' &= \varepsilon(-\tfrac{1}{2}\varkappa v + (1-\sigma)u - \tfrac{1}{2}\delta(u^2+v^2)^{1/2}v). \end{aligned}\right\} \tag{5.5}$$

Apart from the trivial solution $(u, v) = (0, 0)$ two non-trivial equilibrium solutions exist if

$$\sigma^2 < 1 - \tfrac{1}{4}\varkappa^2, \qquad \delta > 0. \tag{5.6}$$

Solution (0, 0) is unstable (2 real eigenvalues) if (5.6) is satisfied; at $\sigma^2 = 1 - \frac{1}{4}\varkappa^2$ the non-trivial solutions vanish, (0, 0) becomes asymptotically stable (2 negative eigenvalues) for increasing σ.

At the two points $\sigma = \pm(1 - \frac{1}{4}\varkappa^2)^{1/2}$ we have a saddle-node bifurcation producing a stable non-trivial equilibrium solution. Returning to the original coordinates x, x', equilibrium solutions presented here correspond with 2π-periodic solutions in τ of the original rotor system.

What happens mechanically is this. Near the primary resonance (5.3) the basic rotor motion becomes unstable but, because of progressive nonlinear damping, the motion remains bounded and tends towards a stable periodic solution. An $O(\varepsilon)$ approximation valid for all time (see [7], chapter 4) is

$$x_p(t) = \frac{2\sqrt{1-\sigma^2} - \varkappa}{\delta}\cos(2\eta t + \psi_p), \qquad \cos 2\psi_p = \sigma, \qquad \sin 2\psi_p > 0 \tag{5.7}$$

with two solutions for ψ_p.

b. Cubic damping $\dot{x}^3$.

In (5.1) we have $f(x, \dot{x}) = \dot{x}^2$. Performing averaging to first order near the primary resonance (5.3), only the terms with coefficients δ are changing. In (5.4) the last term in the equation for r becomes $-\frac{3}{8}\delta r^3$. In (5.5) the last terms become $-\frac{3}{8}\delta(u^2+v^2)u$ and $-\frac{3}{8}\delta(u^2+v^2)v$, respectively.

Again we have that apart from the trivial solution $(u, v) = (0, 0)$ two non-trivial equilibrium solutions exist if

$$\sigma^2 < 1 - \tfrac{1}{4}\varkappa^2, \qquad \delta > 0. \tag{5.6}$$

The stability behaviour of the equilibrium solutions is exactly as in the preceding case. If the non-trivial equilibrium solutions exist, an $O(\varepsilon)$ approximation of the corresponding periodic solution, valid for all time, is

$$x_p(t) = 2\left(\frac{2\sqrt{1-\sigma^2} - \varkappa}{3\delta}\right)^{1/2}\cos(2\eta t + \psi_p), \qquad \cos 2\psi_p = \sigma, \qquad \sin 2\psi_p > 0 \tag{5.8}$$

with two solutions for ψ_p.

c. Cubic damping $x^2\dot{x}$.

In (5.1) we have now $f(x, \dot{x}) = x^2$. Performing averaging near the primary resonance (5.3) again only terms with coefficients δ are changing. In (5.4) the last term in the equation for r becomes $-\frac{1}{8}\delta r^3$. The bifurcation behaviour, (5.6), and the stability behaviour is the same as before. If the non-trivial equilibrium solutions exist, an $O(\varepsilon)$ approximation of the corresponding periodic solution, valid for all time, is

$$x_p(t) = 2\left(\frac{2\sqrt{1-\sigma^2} - \varkappa}{\delta}\right)^{1/2}\cdot\cos(2\eta t + \psi_p), \qquad \cos 2\psi_p = \sigma, \qquad \sin 2\psi_p > 0 \tag{5.9}$$

with two solutions for ψ_p.

6. Hysteresis and phase-locking in two degrees of freedom

We will take as damping function: $f(z, \dot{z}) = \varkappa\dot{z} + \delta|z|^2\dot{z}$. After scaling of z by a factor $(k/\delta)^{1/2}$, the equation becomes

$$\ddot{z} - 2i\alpha\dot{z} + (1 + 4\varepsilon\eta^2\cos 2\eta t)z + \varepsilon\varkappa\dot{z}(1 + |z|^2) = 0 \tag{6.1}$$

and after transforming $w = e^{-i\alpha t}z$ and scaling $\eta\tau = \tau$, $\eta = \sqrt{1+\alpha^2}(1 + \varepsilon\sigma)$:

$$w'' + w + \varepsilon\left[4\cos 2\tau\cdot w - 2\sigma w + \varkappa\left(\frac{w'}{\sqrt{1+\alpha^2}} + \frac{i\alpha w}{1+\alpha^2}\right)(1 + |w|^2)\right] = 0. \tag{6.2}$$

For $\varepsilon = 0$, the solution is $w = A\,\mathrm{e}^{i\tau} + B\,\mathrm{e}^{-i\tau}$, $A, B \in \mathbb{C}$. Applying variation of constants to A and B leads to the equations

$$A' = \frac{i\varepsilon}{2}\, g(A, B, \tau)\,\mathrm{e}^{-i\tau}, \qquad B' = \frac{-i\varepsilon}{2}\, g(A, B, \tau)\,\mathrm{e}^{i\tau} \tag{6.3}$$

with $g(A, B, \tau) = 4\cos 2\tau \cdot w - 2\sigma w + \varkappa\left(\frac{w'}{\sqrt{1+\alpha^2}} + \frac{i\alpha w}{1+\alpha^2}\right)(1 + |w|^2)$, where for w and w' we have to substitute

$$w = A\,\mathrm{e}^{i\tau} + B\,\mathrm{e}^{-i\tau} \quad \text{and} \quad w' = i(A\,\mathrm{e}^{i\tau} - B\,\mathrm{e}^{-i\tau})\,.$$

The right-hand-side of (6.3) is 2π-periodic in τ and can therefore be averaged over τ. This leads to the equations

$$\left.\begin{aligned} A' &= \frac{i\varepsilon}{2}\left[2B - 2\sigma A + \frac{i\varkappa}{\eta_0^2}(\eta_0 + \alpha)\,A + \frac{i\varkappa}{\eta_0^2}\left((\eta_0 + \alpha)\,A|A|^2 + 2\alpha A|B|^2\right)\right], \\ B' &= \frac{-i\varepsilon}{2}\left[2A - 2\sigma B - \frac{i\varkappa}{\eta_0^2}(\eta_0 - \alpha)\,B - \frac{i\varkappa}{\eta_0^2}\left((\eta_0 - \alpha)\,B|B|^2 - 2\alpha B|A|^2\right)\right], \end{aligned}\right\} \tag{6.4}$$

where $\eta_0 = \sqrt{1 + \alpha^2}$.

Introducing polar coordinates, $A = r_1\,\mathrm{e}^{i\varphi_1}$, $B = r_2\,\mathrm{e}^{i\varphi_2}$, yields

$$\left.\begin{aligned} r_1' &= \varepsilon\left[r_2 \sin(\varphi_1 - \varphi_2) - \frac{\varkappa\omega_1}{2\eta_0^2}\, r_1(1 + r_1^2) - \frac{\varkappa\alpha}{\eta_0^2}\, r_1 r_2^2\right], & \varphi_1' &= \varepsilon\left[\frac{r_2}{r_1}\cos(\varphi_1 - \varphi_2) - \sigma\right]; \\ r_2' &= \varepsilon\left[r_1 \sin(\varphi_1 - \varphi_2) - \frac{\varkappa\omega_2}{2\eta_0^2}\, r_2(1 + r_2^2) + \frac{\varkappa\alpha}{\eta_0^2}\, r_2 r_1^2\right], & \varphi_2' &= \varepsilon\left[\frac{r_1}{r_2}\cos(\varphi_1 - \varphi_2) - \sigma\right]. \end{aligned}\right\} \tag{6.5}$$

It is easily seen that (6.5) only has a fixed point if (in this point) $r_1 = r_2$; this can only be the case if $\alpha = 0$. From the right-hand-side of (6.5), it is seen that only the phase difference $\psi = \varphi_1 - \varphi_2$ is relevant, so we can study the reduced system

$$\left.\begin{aligned} r_1' &= \varepsilon\left[r_2 \sin\psi - \frac{\varkappa\omega_1}{2\eta_0^2}\, r_1(1 + r_1^2) - \frac{\varkappa\alpha}{\eta_0^2}\, r_1 r_2^2\right], \\ r_2' &= \varepsilon\left[r_1 \sin\psi - \frac{\varkappa\omega_2}{2\eta_0^2}\, r_2(1 + r_2^2) + \frac{\varkappa\alpha}{\eta_0^2}\, r_2 r_1^2\right], \\ \psi' &= \varepsilon\left[\frac{(r_1^2 + r_2^2)}{r_1 r_2}\cos\psi - 2\sigma\right]. \end{aligned}\right\} \tag{6.6}$$

Equations (6.6) have been investigated with the help of the software package AUTO [4]. This computer program is, among other things, able to track the fixed points of a system as a parameter (in our case: σ) is changed, calculate its stability and, most importantly, detect bifurcations. For system (6.6) the results are as follows: the zero solution $r_1 = r_2 = 0$ is unstable iff $|\sigma| \leqq \sigma_1 = \sqrt{1 + \alpha^2 - \frac{1}{4}\varkappa^2}$, as we already know from linear analysis; for $0 \leqq |\sigma| \leqq \sigma_1$ there is also an asymptotically stable fixed point. For $|\sigma| > \sigma_2$ (see Figs. 5a and 5b), only the zero-solution is stable, and there are no other fixed points. For $\sigma_1 < |\sigma| < \sigma_2$ we have two non-zero solutions, one asymptotically stable and one unstable.

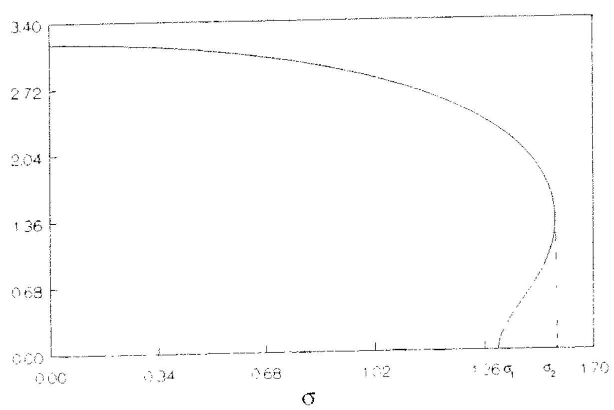

Fig. 5a. Amplitude r_1 of the steady-state periodic solution of (6.1) as function of the detuning σ

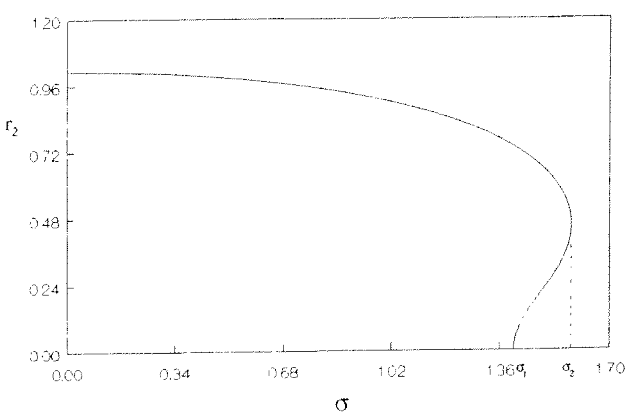

Fig. 5b. Amplitude r_2 of the steady-state periodic solution of (6.1) as function of the detuning σ

In Figs. 5a and 5b, the r_1 and r_2 components of the fixed points are plotted as a function of σ. The lower half of the 'fold', $\sigma_1 < |\sigma| < \sigma_2$, corresponds to the unstable fixed point. This bifurcation diagram shows that there is hysteresis in system (6.6) and therefore also in systems (6.5) and (6.1). When $|\sigma| < \sigma_1$, system (6.6) will tend to the non-zero fixed point. As $|\sigma|$ is increased, this fixed point will remain an attractor until σ_2 is reached, after which this fixed point disappears and the solution will suddenly 'jump' to the zero solution. System (6.5) also exhibits phase-locking. When the reduced system (6.6) tends to a non-trivial fixed point, this implies that the phase difference $\psi = \varphi_1 - \varphi_2$ will converge to a fixed (and in general non-zero) value ψ_0. It then follows from (6.5) that for the asymptotically fixed point we can write:

$$\varphi_1(t) = \varepsilon v t\,, \qquad \varphi_2(t) = \varepsilon v t - \psi_0 \quad \text{with} \quad v = \frac{r_1}{r_2}\cos\psi_0 - \sigma\,.$$

We can now reconstruct the solution of the original equation (6.1) by inverting the various transformations, and we find that for $|\sigma| \leqq \sigma_2$ there exists a stable solution of (6.1) given by

$$z(t) = r_1\,\mathrm{e}^{i(\omega_1 t + \varepsilon v_1 t)} + r_2\,\mathrm{e}^{i(-\omega_2 t - \psi_0 + \varepsilon v t)} + \mathcal{O}(\varepsilon) \tag{6.7}$$

on time-scale $1/\varepsilon$, with $\omega_1 = \sqrt{\alpha^2 + 1} + \alpha$, $\omega_2 = \sqrt{\alpha^2 + 1} - \alpha$ and r_1, r_2 and ψ_0 (the fixed points of (6.6)) depending only on σ. Solution (6.7) consists of two dominant vibration components, one with forward precession frequency ω_1, the second with backward precession frequency $-\omega_2$. For $|\sigma| \geqq \sigma_2$, this solution suddenly disappears and only $z(t) = 0$ remains as an asymptotically stable solution.

7. Conclusions

The motion in axial direction can initiate the whirling motion of the rotor around the axis of rotation. This whirling motion has two components: one consisting of forward, the other of backward precession. The frequencies of these components are different due to the gyroscopic effect of the rotor.

The models analysed in this paper are of course a simplification of real rotor systems. One of the first extensions we have in mind, is to include a slight unbalance of the rotor. Furthermore it would be interesting to consider the effects of autoparametric excitation in other dynamical states of the rotor system, in particular of solutions with precession.

The use of averaging, in combination, at some stage, with a numerical bifurcation analysis to follow the branching off and stability of periodic solutions, turns out to be very effective in these problems.

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