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HENRI POINCARÉ'S NEGLECTED IDEAS

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Abstract. The purpose of this article is to discuss two basic ideas of Henri Poincaré in the theory of dynamical systems. The first one, the recurrence theorem, got at first a lot of attention but most scientists lost interest when finding out that long timescales were involved. We will show that recurrence can be a tool to find complex dynamics in resonance zones of Hamiltonian systems; this is related to the phenomenon of quasi-trapping. To demonstrate the use of recurrence phenomena we will explore the $2 : 2 : 3$ Hamiltonian resonance near stable equilibrium. This will involve interaction of low and higher order resonance. A second useful idea is concerned with the characteristic exponents of periodic solutions of dynamical systems. If a periodic solution of a Hamiltonian system has more than two zero characteristic exponents, this points at the existence of an integral of motion besides the energy. We will apply this idea to examples of two and three degrees-of-freedom (dof), the Hénon-Heiles (or Braun's) family and the $1 : 2 : 2$ resonance.

1. **Introduction.** Henri Poincaré's books and papers are full of ideas, sometimes sketchily presented, sometimes worked out in great detail. After Poincaré's (1854-1912) time a large part of mathematics was developed in a different style of writing, think of Bourbaki. In this modern style the emphasis is on a concise definition-theorem-proof presentation. Poincaré's influence on mathematics and physics has been enormous but maybe this modern emphasis on the form of mathematics is the cause of the neglect of some of his ideas. The purpose of this paper is to trace two important ideas of Poincaré regarding dynamical systems and to illustrate this by examples. The intertwining of analysis and geometry is typical for the scientific work of Henri Poincaré. This will also become clear in the ideas and examples. In both cases the main source will be "Les Méthodes Nouvelles de la Mécanique Céleste" (1892-1899), [14]. On a wide range of topics, many other ideas can be found in his books and papers.

It may be a surprise that the first topic is the recurrence theorem for dynamical systems characterized by measure-preserving maps. This is of course a well-known result but there are historical reasons for its neglect. We will show that recurrence can be used as a tool to become aware of the fine-structure of resonance manifolds and zones. This topic involves passage through resonance problems for which there

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FIGURE 1. Henri Poincaré (1854-1912) at the time when he submitted his prize-essay for the birthday of the Swedish king Oscar II.

is a tremendous amount of literature. A survey is given by Kevorkian [12] where many applications are discussed; it also contains a reference to his original 1971 paper on this topic. An early asymptotic result by Sanders can be found in [17]. Seminal papers on the dynamics of passage including asymptotic estimates are by Neishtadt, see [13] with more references there. Haberman discussed passage through resonance both for the conservative and dissipative case; see for instance [10].

The second topic is the use of characteristic exponents to recognize the possibility of integrability. Here, the presence of non-generic behaviour may signal integrability.

General references for dynamical systems are [1], [4] and [5]; for Hamiltonian systems see also [23] and for an explicit analysis [8].

2. **The recurrence theorem.** To solve ODEs, in particular in problems of celestial mechanics, series expansions are often used to obtain local information. An important global result is the *recurrence theorem*, originally presented in his prize-essay for the birthday of the Swedish king, Oscar II, discussed again in [14], vol. 3, chapter 26.

Consider a dynamical system defined on a compact set in $\mathbb{R}^n$ with the property that the flow induced by the system is measure-preserving. Poincaré uses the term volume-preserving as the notion of measure did not exist at his time. Examples are found in billiard dynamics, the motion of an incompressible fluid in a non-deformable vessel or the phase-flow induced by a time-independent Hamiltonian system without singularities on a compact domain. Using the invariance of the domain volume under the flow, it is proved that most particles or fluid elements return an infinite number of times arbitrarily close to their initial position. The recurrence time is not specified but depends in general on the required closeness to the initial position and of course on the dynamical system at hand.

The interpretation of the recurrence theorem in the case of a chaotic system is interesting. In a two degrees-of-freedom Hamiltonian system near stable equilibrium, the KAM theorem guarantees in most cases the existence of an infinite number of two-dimensional invariant tori that separate the energy manifold into small chaotic

regions. In these systems the recurrence phenomena near stable equilibrium are quite strong. Moving further away from stable equilibrium the tori break up, the recurrence times will be more and more dependent on the initial positions.
In the case of more than two degrees-of-freedom, resonances will produce more active sets of chaotic orbits near stable equilibrium producing very different recurrence times.
It is easy to obtain an upper limit L for recurrence times, dependent on the Euclidean distance d to the initial condition. Consider a time-independent Hamiltonian of the $2n$ variables p, q expanded in the form

$$H(p,q) = H_2(p,q) + H_3(p,q) + H_4(p,q) + \dots, \tag{1}$$

where the index i indicates the degree of the homogeneous polynomials $H_i(p,q), i = 2, 3, \dots$ Assume that $H_2(p,q)$ is Morse at $(p,q) = (0,0)$ and that the quadratic part is definite, so the origin is a stable equilibrium of the equations of motion. In [22] it is argued that:

Proposition 1. *Each orbit near stable equilibrium of the system induced by Hamiltonian (1), except a number of orbits in a set of measure zero, reaches a size d neighborhood of its initial point with upper bound L of the recurrence time T_d:*

$$L = O\left(\frac{1}{d^{2n-1}}\right) \text{ as d} \to 0. \tag{2}$$

As the recurrence time T_d depends on so many characteristics of the dynamical system at hand, most researchers lost interest. It did not help that Ernst Zermelo used recurrence to throw doubt on the statistical mechanics of Ludwig Boltzmann; see for a review [18]. Zermelo, who was more a philosopher of mathematics than a mathematical physicist reasoned that a collection of molecules placed in a corner of an empty vessel would always return to this corner after some time. The implication, Zermelo inferred, is that Boltzmann's mechanics is wrong. However, a cm^3 of atmospheric air put into a closed vessel has more than 10^{20} dof which produces upper bounds L and probably recurrence times much longer than the lifetime of the universe. Non-integrability and in general complexity tends to increase recurrence times. On realistic timescales collisional gas dynamics has little use for the recurrence theorem.
In a sense, this discussion caused mathematical physicists to lose interest in the recurrence theorem. This is unfortunate as we shall see that the theorem can be used as an indicator for complex dynamics.

3. **Recurrence as a tool: the $2:2:3$ resonance.** To obtain a picture of the problem, think first of a two dof time-independent Hamiltonian system near stable equilbrium. The energy manifold for a fixed value of the energy is topologically the 3-dimensional sphere S^3. Suppose that the linear frequencies have a rational ratio, say $m:n$. The normal modes that in many resonance cases exist are embedded in the energy manifold and are lying on or near coordinate planes; they will have a linking dependent on m and n. Periodic orbits in $m:n$ resonance may exist in general position, a neighbourhood of these periodic solutions on the energy manifold will be called a resonance zone. Orbits starting outside a resonance zone may pass through this zone and will experience during passage the geometric structure of the zone. If $m = n = 2$ we have the $1:1$ resonance, but adding a coupled oscillator with frequency 3 will not change the dynamics much at first order approximation except that it adds dimensions to the resonance zone.

The complexity of the resonance zones will generally increase with the dimension. We will find stable and unstable periodic solutions producing homo- and heteroclinic tangles, see [5] and [23] for the general theory and subsection 3.4 for analysis. Solutions that pass through a resonance zone will be delayed by the complexity of the dynamics, but they will always return to the zones, winding for some time around the tori associated with the stable periodic solutions and being perturbed by the tangles. This delay of orbits in resonance zones is called *quasi-trapping* in [23].
Apart from a measure zero number of orbits, passage *through* the resonance zone is the rule and both quasi-trapping and recurrence will take place. By studying the Euclidean deviation (or distance) d from the initial conditions in phasespace as a function of time one may get an indication of the dynamics of the resonance zone inviting further study. Relatively long recurrence times suggest complexity.

3.1. **Formulation of the $2:2:3$ resonance.** We illustrate the dynamics of the $2:2:3$ resonance for the Hamiltonian $H_2 + \varepsilon^2 H_4 + \varepsilon^3 H_5$.

$$\begin{cases} H = & \frac{1}{2}(\dot{q}_1^2 + 4q_1^2) + \frac{1}{2}(\dot{q}_2^2 + 4q_2^2) + \frac{1}{2}(\dot{q}_3^2 + 9q_3^2) - \frac{1}{4}\varepsilon^2(\alpha_1 q_1^4 + 2\alpha_2 q_1^2 q_2^2 + \alpha_3 q_2^4 + \alpha_4 q_3^4) \\ & -\varepsilon^3(b_1 q_1^3 q_3^2 + b_3 q_2^3 q_3^2 + b_3 q_1^2 q_2 q_3^2 + b_4 q_1 q_2^2 q_3^2). \end{cases} \tag{3}$$

The Hamiltonian was chosen to contain one combination angle at H_4 level: ($\chi_1 = \phi_1 - \phi_2$) and 4 combination angles at H_5 level. The equations of motion induced by (3) are:

$$\begin{cases} \ddot{q}_1 + 4q_1 = & \varepsilon^2(\alpha_1 q_1^3 + \alpha_2 q_1 q_2^2) + \varepsilon^3(3b_1 q_1^2 q_3^2 + 2b_3 q_1 q_2 q_3^2 + b_4 q_2^2 q_3^2), \\ \ddot{q}_2 + 4q_2 = & \varepsilon^2(\alpha_2 q_1^2 q_2 + \alpha_3 q_2^3) + \varepsilon^3(3b_2 q_2^2 q_3^2 + b_3 q_1^2 q_3^2 + 2b_4 q_1 q_2 q_3^2), \\ \ddot{q}_3 + 9q_3 = & \varepsilon^2 \alpha_4 q_3^3 + \varepsilon^3(2b_1 q_1^3 q_3 + 2b_2 q_2^3 q_3 + 2b_3 q_1^2 q_2 q_3 + 2b_4 q_1 q_2^2 q_3). \end{cases} \tag{4}$$

The figures we show in this section have been obtained using system (4). To make some explicit numerical calculations we choose in most cases:

$$\alpha_1 = 0.4, \alpha_2 = 1, \alpha_3 = 0.6, \alpha_4 = 4, b_1 = 1, b_2 = -1.5, b_3 = 1, b_4 = -1. \tag{5}$$

The choice of $\alpha_1, \ldots, \alpha_4$ excludes non-generic behaviour; for instance if $\alpha_2 = 0$, the Hamiltonian $H_2 + \varepsilon^2 H_4$ is integrable, there are other non-generic cases of the α coefficients. In the sequel we fix the α coefficients according to (5); this also holds for the b coefficients except in the discussion of section 3.4. Because of the resonance in H_2 we expect resonant interaction between the first two modes q_1, q_2 and no or very small interaction with the third mode.

3.2. **The primary resonance zones.** At H_5 four combination angles are possibly active:

$$\chi_2 = 3\phi_1 - 2\phi_3,\ \chi_3 = 3\phi_2 - 2\phi_3,\ \chi_4 = 2\phi_1 + \phi_2 - 2\phi_3,\ \chi_5 = \phi_1 + 2\phi_2 - 2\phi_3.$$

We will use polar (amplitude-phase) coordinates r, ϕ by transformation $q = r_1 \cos(\omega t + \phi)$, $\dot{q} = -\omega r \sin(\omega t + \phi)$. The transformation results in:

$$H_2 = 2r_1^2 + 2r_2^2 + \frac{9}{2} r_3^2.$$

We find after averaging to $O(\varepsilon^2)$:

$$\begin{cases} \dot{r}_1 = & -\varepsilon^2 \frac{1}{8} r_1 r_2^2 \sin 2\chi_1,\ \dot{\phi}_1 = -\varepsilon^2 \frac{1}{8}(\frac{6}{5} r_1^2 + 2r_2^2 + r_2^2 \cos 2\chi_1), \\ \dot{r}_2 = & +\varepsilon^2 \frac{1}{8} r_1^2 r_2 \sin 2\chi_1,\ \dot{\phi}_2 = -\varepsilon^2 \frac{1}{8}(2r_1^2 + r_1^2 \cos 2\chi_1 + \frac{9}{5} r_2^2), \\ \dot{r}_3 = & 0,\ \dot{\phi}_3 = -\varepsilon^2 \frac{3}{8} r_3^2. \end{cases} \tag{6}$$

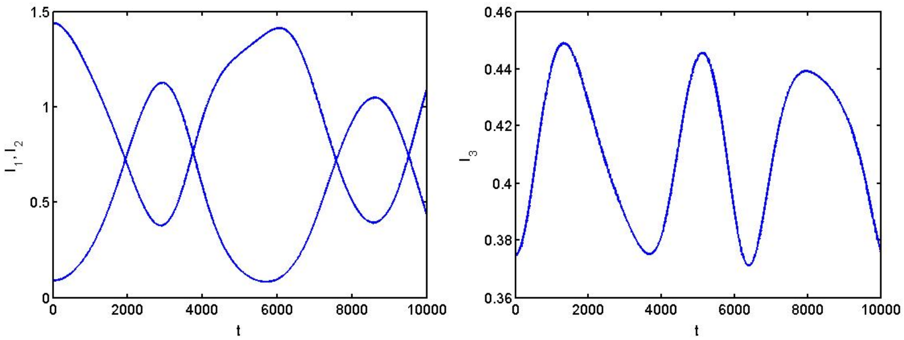

FIGURE 2. The $2:2:3$ resonance with the actions I_1, I_2, I_3 in 10 000 timesteps for Hamiltonian (3) starting outside the primary resonance zones; initial conditions $q_1(0) = 0.3, q_2(0) = 1.2, q_3(0) = 0.5$, velocities zero, and parameter values $\varepsilon = 0.1$ and (5). Left I_1, I_2 showing strong energy exchanges, the q_1 and q_2 normal modes are unstable. Right the action I_3 showing variations of order 0.04 (between 0.375 and 0.450).

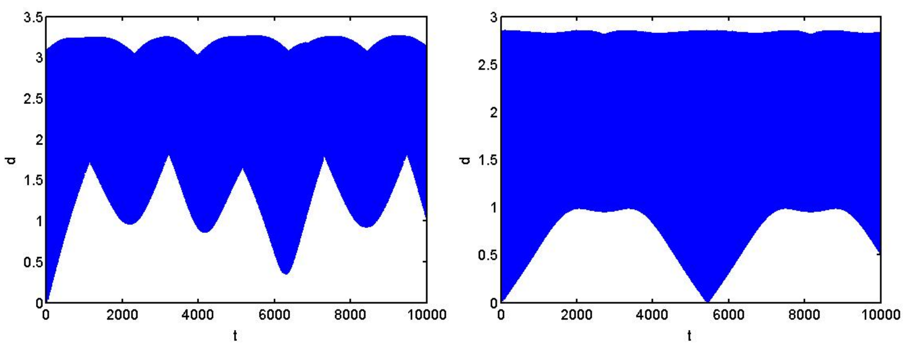

FIGURE 3. The $2:2:3$ resonance. Left the Euclidean distance d in 10 000 timesteps of the orbits to their initial conditions in phasespace outside the primary resonance zones and parameter values of fig. 2; recurrence takes many more timesteps. Right the Euclidean distance d for the same Hamiltonian (3) but starting at $q_1(0) = 0.3, q_2(0) = 1.2, q_3(0) = 0$ and velocities zero; we have now $q_3(t) = \dot{q}_3(t) = 0, t \geq 0$. The recurrence is different and faster in this case of pure q_1, q_2 interaction.

The combination angles $\chi_2, \ldots, \chi_5$ will arise at $O(\varepsilon^3)$. For the averaging-normal form (6) we have $2r_1^2 + 2r_2^2 = E_1 + O(\varepsilon)$ with constant $E_1 \geq 0$ and $r_3 - r_3(0) = O(\varepsilon)$ on the timescale $1/\varepsilon^2$. The normal form to $O(\varepsilon^2)$ is integrable with integrals $r_3 = r_3(0)$, $2r_1^2 + 2r_2^2 = E_1$ and the normalized Hamiltonian to quartic terms.
The equation for χ_1 depends on r_1, r_2 and χ_1. If the normal form (6) has a few solutions with $\sin 2\chi_1 = 0$, $\dot{\chi}_1 = 0$, these are approximate short-periodic solutions in general position (separate from the normal modes).

A *primary resonance zone* M is defined as a neighborhood of the solutions with $\sin 2\chi_1 = 0, \dot{\chi}_1 = 0$. In the $2:2:3$ problem such a resonance zone has for a chosen value E_0 with $H_2 = E_0, E_0 > 0$ the free parameter $r_3(0)$. We find with system (6) :

$$\dot{\chi}_1 = \frac{1}{8}\varepsilon^2 \left(\frac{4}{5} r_1^2 - \frac{1}{5} r_2^2 + (r_1^2 - r_2^2) \cos 2\chi_1 \right). \tag{7}$$

Putting the righthand side of eq. (7) zero we find two primary resonance zones M_1, M_2:

$$\begin{cases} M_1 : \ 2\chi_1 = & 0, 2\pi, r_1^2 = \frac{2}{3} r_2^2, \\ M_2 : \ 2\chi_1 = & \pi, 3\pi, r_1^2 = 4 r_2^2. \end{cases} \tag{8}$$

The periodic solutions of the two primary resonance zones are located on the energy manifold which is topologically the 5-dimensional sphere S^5 and its intersections with the elliptical cylinder $\frac{1}{2}(\dot{q}_1^2 + 4q_1^2) + \frac{1}{2}(\dot{q}_2^2 + 4q_2^2) = E_1$ and the hyperbolic cylinder $\dot{q}_1^2 + 4q_1^2 = 4r_1^2 = \mu(\dot{q}_2^2 + 4q_2^2)$ with respectively $\mu = \frac{2}{3}$ or 4; this defines two three-dimensional manifolds.
To study the complexity of the primary resonance zones we use the recurrence theorem. In fig. 2 we start outside the resonance zones to observe the resonant interaction of the first two modes; the variations of the third mode are small in accordance with the normal form (6). We will use the Euclidean distance (or deviation) d for three dof in fig. 3 defined by:

$$d^2 = \sum_{i=1}^{3} [(q_i(t) - q_i(0))^2 + (\dot{q}_i(t) - \dot{q}_i(0))^2)]. \tag{9}$$

Choosing $q_3(0) = \dot{q}_3(0) = 0$ the third mode remains zero, the recurrence is fairly strong and regular; see fig. 3 right. Starting with nonzero initial values of the third mode in fig. 3 left we find behaviour that suggests quasi-trapping in one or more resonance zones. This suggests that we have to study the resonance zones in more detail.

3.3. **Secondary resonance.** We will extend the theory of higher order resonance of [16], see also [17], to three dof. Higher order (or secondary) resonance may take place in a primary resonance zone. Consider as an example the primary zone M_1 with $2\chi_1 = 0, 2\pi, r_1^2 = \frac{2}{3} r_2^2$. To $O(\varepsilon^2)$ the higher order combination angles satisfy the equations:

$$\begin{cases} \dot{\chi}_2 = & -\frac{3}{4}\varepsilon^2 (\frac{3}{5} r_1^2 + \frac{3}{2} r_2^2 - r_3^2), \\ \dot{\chi}_3 = & -\frac{3}{4}\varepsilon^2 (\frac{3}{2} r_1^2 + \frac{9}{10} r_2^2 - r_3^2), \\ \dot{\chi}_4 = & -\frac{3}{4}\varepsilon^2 (\frac{9}{10} r_1^2 + \frac{13}{10} r_2^2 - r_3^2), \\ \dot{\chi}_5 = & -\frac{3}{4}\varepsilon^2 (\frac{6}{5} r_1^2 + \frac{11}{10} r_2^2 - r_3^2). \end{cases} \tag{10}$$

Outside the primary resonance zone we have terms in system (6) from the first two modes that dominate at $O(\varepsilon^2)$. In the zone M_1 this is not the case and other combination angles may play a part. If the derivative of a combination angle is sign-definite, the angle is timelike and we can average over the angle. A combination angle is not timelike if the righthand side of the equation contains zeros. In such a case a secondary resonance will be found in the primary resonance zone. In M_1 we find for the $1:1$ periodic solutions:

$$2r_1^2 + 2r_2^2 = E_1, \, r_1^2 = \frac{2}{3} r_2^2 \ \rightarrow \ r_1^2 = \frac{1}{5} E_1, r_2^2 = \frac{3}{10} E_1.$$

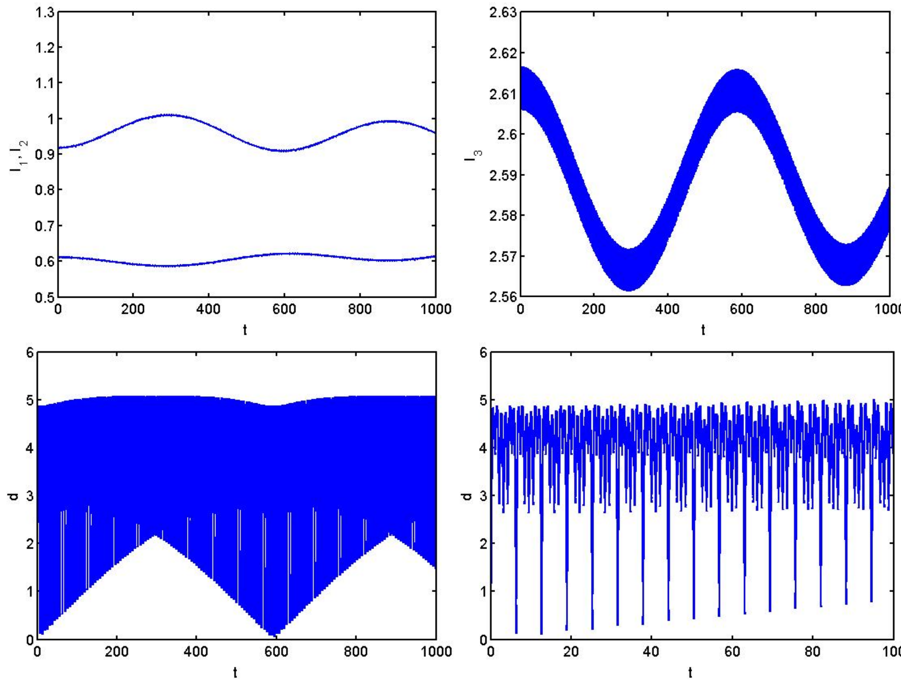

FIGURE 4. Recurrence when starting in the primary resonance zone M_1 in 1000 timesteps. $E_1 = 3.06$ as in fig. 3 with now $q_1(0) = 0.7823, q_2(0) = 0.9581$; $q_3(0) = 1.32068$, the value given by eq. (11) which puts the orbits in the secondary resonance zone. Initial velocities are zero. The top figure left shows the variations of the actions I_1, I_2, the top figure right shows I_3 with small variations as predicted. The recurrence d (below left) is strong and regular as the orbits are caught in the resonance zone M_1 with near fairly stable dynamics; the picture for 1000 timesteps shows some white segments but hides the fine-structure of recurrence transitions shown when enlarged for 100 timesteps (below right).

The combination angles in (10) admit a zero (critical value) if in M_1:

$$r_3^2 = \frac{57}{100}E_1. \tag{11}$$

We repeat the calculation for primary resonance zone M_2. We find for the 1 : 1 periodic solutions in M_2:

$$2r_1^2 + 2r_2^2 = E_1,\, r_1^2 = 4r_2^2 \;\to\; r_1^2 = \frac{2}{5}E_1, r_2^2 = \frac{1}{10}E_1.$$

The combination angles in (10) (we omit the details) admit a zero (critical value) if in M_2:

$$r_3^2 = \frac{29}{100}E_1. \tag{12}$$

To improve our insight in the dynamics we produce the actions corresponding with the two cases of fig. 5. In fig. 6 we present left I_1 and I_2 starting outside the primary resonance zones exchanging energy according to the 1 : 1 resonance; I_3

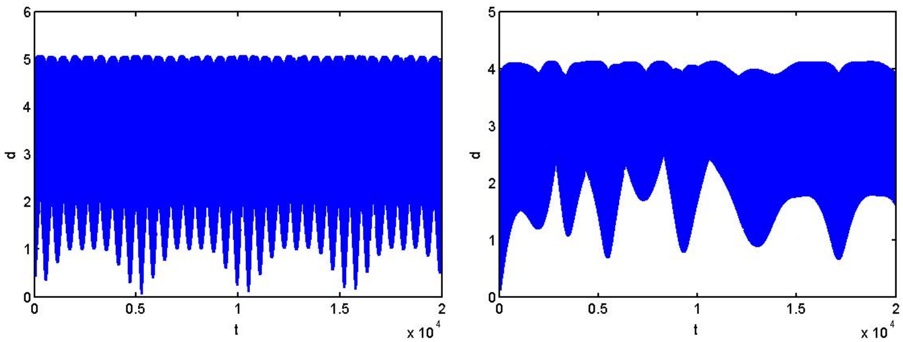

FIGURE 5. Recurrence when starting outside primary resonance zone M_1 and M_2 in 20 000 timesteps. $E_1 = 3.06$ as in fig. 3 with now left $q_1(0) = 0.3, q_2(0) = 1.2$; $q_3(0) = 1.32068$, the critical value in M_1 given by eq. (11). The orbits passing through M_1 encounter the critical case of $q_3(0)$, recurrence is slow and different from the case of fig. 3. Right the case $q_1(0) = 0.3, q_2(0) = 1.2$; $q_3(0) = 0.9420$, the critical value in M_2 given by eq. (12). The recurrence takes much longer which suggests that this location corresponds with unstable periodic solutions.

(right) varies within the error estimates. Next we give the actions in the unstable case of fig. 5 (right). It is interesting that $I_3(t)$ changes strongly (between 0.65 and 1.33) when outside the time-interval of validity, $1/\varepsilon^2$, of the normal form. Conservation of energy influences the exchanges of I_1 and I_2 at the same time.

3.4. **Normalization in the primary resonance zones.** To obtain a better understanding of the dynamics in the primary resonance zones we have to extend the averaging-normal forms to $O(\varepsilon^3)$. We find with $\sin 2\chi_1 = 0$ in M_1 and M_2:

$$\begin{cases} \dot{r}_1 = & -\frac{\varepsilon^3}{32}(3b_1 r_1^2 r_3^2 \sin\chi_2 + 2b_3 r_1 r_2 r_3^2 \sin\chi_4 + b_4 r_2^2 r_3^2 \sin\chi_5), \\ \dot{\phi}_1 = & -\frac{\varepsilon^2}{8}(\frac{6}{5}r_1^2 + 2r_2^2 + r_2^2\cos 2\chi_1) - \frac{\varepsilon^3}{32}(3b_1 r_1 r_3^2 \cos\chi_2 + 2b_3 r_2 r_3^2 \cos\chi_4 + b_2 \frac{r_2^2 r_3^2}{r_1}\cos\chi_5), \\ \dot{r}_2 = & -\frac{\varepsilon^3}{32}(3b_2 r_2^2 r_3^2 \sin\chi_3 + b_3 r_1^2 r_3^2 \sin\chi_4 + 2b_4 r_1 r_2 r_3^2 \sin\chi_5), \\ \dot{\phi}_2 = & -\frac{\varepsilon^2}{8}(2r_1^2 + r_1^2\cos 2\chi_1 + \frac{9}{5}r_2^2) - \frac{\varepsilon^3}{32}(3b_2 r_2 r_3^2 \cos\chi_3 + b_3 \frac{r_1^2 r_3^2}{r_2}\cos\chi_4 + 2b_4 r_1 r_3^2 \cos\chi_5), \\ \dot{r}_3 = & \frac{\varepsilon^3}{24}(b_1 r_1^3 r_3 \sin\chi_2 + b_2 r_2^3 r_3 \sin\chi_3 + b_3 r_1^2 r_2 r_3 \sin\chi_4 + b_4 r_1 r_2^2 r_3 \sin\chi_5, \\ \dot{\phi}_3 = & -\varepsilon^2 \frac{3}{8} r_3^2 - \frac{\varepsilon^3}{24}(b_1 r_1^3 \cos\chi_2 + b_2 r_2^3 \cos\chi_3 + b_3 r_1^2 r_2 \cos\chi_4 + b_4 r_1 r_2^2 \cos\chi_5). \end{cases} \tag{13}$$

The normal form has the integral corresponding with H_2:

$$2r_1^2 + 2r_2^2 + \frac{9}{2}r_3^2 = E_0,\ E_0 \geq 0. \tag{14}$$

Using eq. (10) we can analyse the equations of the combination angles $\chi_2, \ldots, \chi_5$. Differentiating the equations and substituting $\dot{r}_i, i = 1, 2, , 3$ from (10) we find in the primary resonance zone M_1 the system:

$$\begin{cases} \ddot{\chi}_i - \varepsilon^5[c_{i1} b_1 r_1^3(0) r_3^2(0)\sin\chi_2 + c_{i2} b_2 r_1^3(0) r_3^2(0)\sin\chi_3 + \\ \qquad c_{i3} b_3 r_1^2 r_2(0)(0) r_3^2(0)\sin\chi_4 + c_{i4} b_4 r_1 r_2^2(0)(0) r_3^2(0)\sin\chi_5] = 0 \end{cases} \tag{15}$$

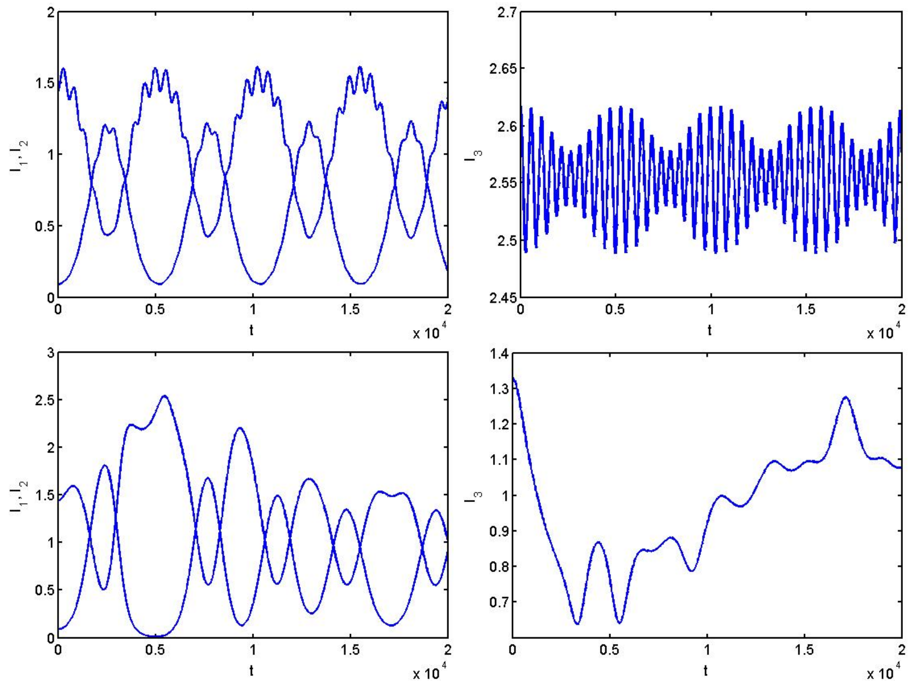

FIGURE 6. The actions when starting outside primary resonance zone M_1 and M_2 in 20 000 timesteps. Each orbit starts outside the primary resonance zones with $q_1(0) = 0.3$, $q_2(0) = 1.2$ and all velocities zero. In the first two figures (top) we took $q_3(0) = 1.32068$, the critical value in M_1 given by eq. (11). $I_3(t)$ varies with magnitude 0.06 in according with the error estimate. The instability of the normal modes forces considerable exchange of energy of the first two modes. The next two figures show the instability of M_2. When passing this primary resonance zone the critical value of $q_3(0) = 0.9420$ taken from eq. (12) plays a part.

with $i = 2, \ldots, 5$ and coefficients $c_{ij} \geq 0, i = 2, \ldots, 5, j = 1, \ldots, 4$. If for instance $b_2 = b_3 = b_4 = 0$ we have:

$$\ddot{\chi}_2 - \varepsilon^5 \frac{47}{320} b_1 r_1^3(0) r_3^2(0) \sin \chi_2 = 0. \tag{16}$$

The solutions of system (15) are valid only in the primary resonance zone M_1 and on a timescale $1/\varepsilon^{5/2}$. The equilibrium solutions correspond with higher order periodic solutions. In the case of only one angle, for instance χ_2 in eq. (16), we have equilibria $0, \pi$ with $\chi_2 = 0$ unstable, $\chi_2 = \pi$ stable. Around the stable periodic solutions we expect to find tori in the resonance zone. The unstable periodic solutions, for instance associated with the saddle point of (16), will produce tangles that complicate the dynamics.

Fig. 4 shows the behaviour of the actions and the Euclidean distance d to the initial values in zone M_1; the figure below right suggests quasi-trapping around tori alternating with excursions to a neighbourhood of the initial conditions in M_1.

Figs. 5 and 6 illustrate these computations. In fig. 8 we show the instability in

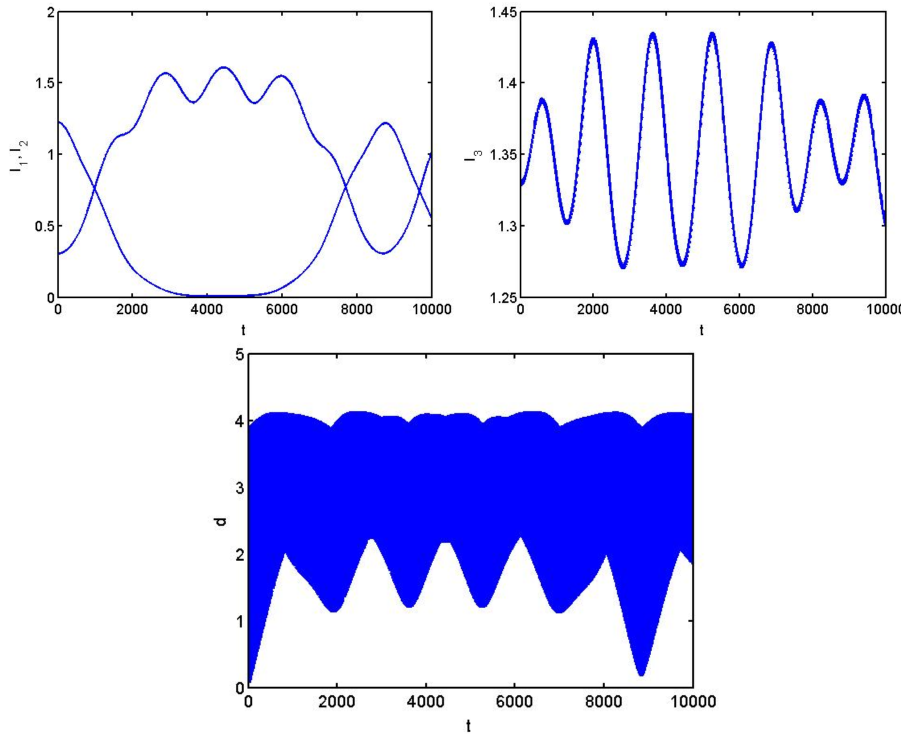

FIGURE 7. Dynamics when starting in the primary resonance zone M_2 in (because of the instability) 10 000 timesteps. $E_1 = 3.06$ as in fig. 3 with now $q_1(0) = 1.1063, q_2(0) = 0.5532$; $q_3(0) = 0.94202$, values putting the orbits in M_2. Initial velocities are zero. The top figure left shows strong variations of the actions I_1, I_2, the top figure right shows I_3; the variations are strong as we start near an unstable secondary resonance. The orbits are leaving the unstable resonance zone M_2. Below the corresponding recurrence d.

resonance zone M_1 associated only with combination angle χ_2 described by eq. (16); we chose $b_1 = 10$ to illustrate the phenomena more clearly.

Summary of results for the $2:2:3$ resonance. For the $2:2:3$ resonance the phase-flow is dominated by H_4. However, in the resonance zones M_1 and M_2 the actions I_1, I_2 are by definition varying very little so that H_5 terms become important. We have normalized the equations of motion in the resonance zones to H_5. In figs 2 and 3 we started to explore the dynamics of the $2:2:3$ resonance by numerically integration; the results suggest passage through resonance involving the third mode, especially as the results are changing qualitatively when excluding the third mode in fig. 4, right.

After identifying the resonance zones M_1 and M_2 we find in fig. 4 fairly regular behaviour in M_1 and more irregularity in M_2, see fig. 7. This illustrates the presence of stable and unstable secondary resonances as obtained by studying the normal forms in the primary resonance zones for 4 combination angles at H_5. Figs 5 and 6 show transition through resonance.

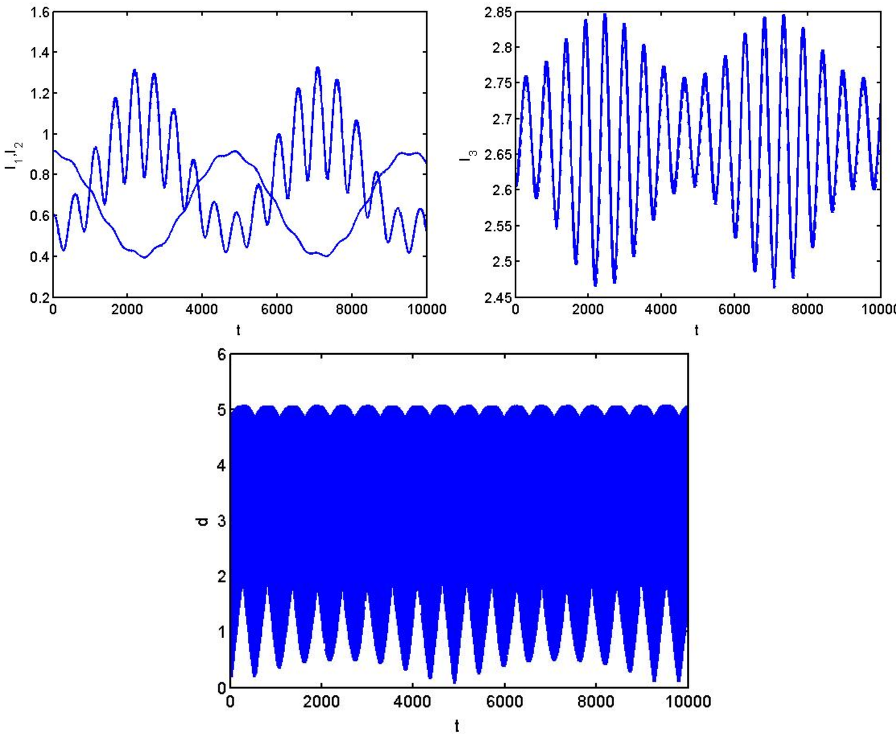

FIGURE 8. Recurrence starting in the primary resonance zone M_1 in 10 000 timesteps. $E_1 = 3.06$ (as in fig. 3) with $q_1(0) = 0.7823, q_2(0) = 0.9581$; $q_3(0) = 1.32068$, the value given by eq. (11) which puts the orbits in the secondary resonance zone. Initial velocities are zero. We have chosen $b_1 = 10, b_2, = b_3 = b_4 = 0$ to demonstrate the instability caused by the secondary resonance corresponding with χ_2. The top figure left shows the variations 0.5 and 0.3 of the actions I_1, I_2, the top figure right shows I_3 with variation 0.2. The corresponding recurrence d is shown in the third figure.

In fig. 8 instability in M_1 is highlighted by restricting H_5 to $b_1 \neq 0$, so only the combination angle χ_2 is involved. The actions show considerable fluctuations, as predicted.

4. **Second useful idea: genericity of periodic solutions.** The presence of an isolated periodic solution in a dynamical system or alternatively the presence of families of periodic solutions is related to the existence of integral manifolds of the system. Poincaré discussed such aspects extensively in [14]. The presence of a familiy of periodic solutions on the energy manifold may even signal integrability. In the subsequent section we will consider examples from two and three dof Hamiltonian systems.

4.1. **The Poincaré-Birkhoff theorem.** It bothered Poincaré, see [21], that so many results in dynamical systems are local, based on series expansions, normal forms, bifurcations, and he formulated a more global geometric theorem in [15]. Its publication was postponed as he found his reasoning not satisfactory; the actual proof was given by Birkhoff [2]. The theorem shows existence of periodic solutions but also that isolated periodic solutions on the energy manifold represents the generic case for two dof Hamiltonian systems.

The idea is to characterize certain dynamical systems by an area-preserving, continuous twist map of an annular region into itself. Such a map has at least two fixed points corresponding with periodic solutions of the dynamical system. The applications Poincaré had in mind were the global characterization of periodic solutions of time-independent Hamiltonian systems with two degrees of freedom. The dynamics of such a system restricted to a compact energy manifold is three-dimensional. The Poincaré map of the orbits can provide the two-dimensional twist map of the theorem. Poincaré's reasoning and the actual proof are constructive and show that generically the number of fixed points is even and that the map corresponds with a large number of periodic orbits emerging in pairs. This illustrates the complexity of the annular twist map. The results that we describe below using characteristic exponents are concerned with a much more general geometric-analytic setting but with the restriction that the results are local.

After 1912, fixed point theorems play an important part in general and differential topology and in dynamical systems.

4.2. **Characteristic exponents.** In chapter 4 of volume 1 of [14] pp. 162-232, Poincaré developed his theory of characteristic exponents of solutions of n-dimensional autonomous systems of the form $\dot{x} = X(x)$. Starting with a particular T-periodic solution of the system, $x = \phi(t)$, we can obtain an associated variational solution by considering neighbouring solutions of the form:

$$x = \phi(t) + \xi.$$

Substituting this into the equation, expanding and linearizing, we find:

$$\dot{\xi} = \frac{\partial X}{\partial x}|_{x=\phi(t)}\xi. \tag{17}$$

One of the independent n solutions of eq. (17) is $\dot{\phi}(t)$. In section 59 of [14] he shows that one can write the solutions of eq. (17) in the form:

$$\xi = e^{\alpha t} S(t), \tag{18}$$

with α a complex number and $S(t)$ a T-periodic $n \times n$ matrix. The constants α are called characteristic exponents, it is clear that at least one of them is zero. Poincaré discusses in great detail the case when the original equation has an integral and continues then to study the consequences in the case of Hamiltonian systems (section 69).

In this case with n-dimensional momentum and position variables, we have $2n$ characteristic exponents that appear in pairs $\alpha, -\alpha$; so in the time-independent Hamiltonian case at least two characteristic exponents are zero. Each extra independent integral adds two exponents zero if a transversality condition using a functional determinant is satisfied. In the case that the Hamiltonian system has no other independent integral besides the energy, a periodic solution on an energy manifold has in the generic case two characteristic exponents zero only. In section 69 of [14], Poincaré discusses exceptional cases.

Keeping an eye on applications in celestial mechanics, Poincaré also considers the case when the dynamical system contains a small parameter. He proves that if the solutions of the system can be expanded with respect to the small parameter following his expansion theorem, the characteristic exponents can also be expanded with respect to the small parameter. Historically the first example of non-integrability was discovered by Poincaré in his famous prize essay for King Oscar II (1890); it is discussed in more detail in [14] and [21].

We will discuss a few examples to illustrate the use of these results.

5. **Periodic solutions and integrability.** To compute characteristic exponents used to be difficult. Nowadays, normal form methods enable us rather easily to find approximations while recent developments in numerical bifurcation methods produce by standard path-following techniques explicit series of numbers. We will present a few results using averaging-normalization to two- and three dof systems.

5.1. **Braun's or Hénon-Heiles family of Hamiltonians.** The material in this subsection is based on part of [20] with a new discussion added. For historical reasons we use x, y instead of q_1, q_2. Consider the two dof Hamiltonian with parameters a_1, a_2:

$$H = \frac{1}{2}(\dot{x}^2 + x^2) + \frac{1}{2}(\dot{y}^2 + y^2) - \frac{a_1}{3}x^3 - a_2 xy^2. \tag{19}$$

The corresponding equations of motion are:

$$\ddot{x} + x = a_1 x^2 + a_2 y^2, \ \ddot{y} + y = 2a_2 xy. \tag{20}$$

If $a_2 = 0$ the system decouples and is trivially integrable; we exclude this case. If $a_1 = -1, a_2 = 1$ we have the famous Hénon-Heiles problem which gave in 1964 the first modern example of a non-integrable Hamiltonian system; see [11] and for more references [7] and [20].

Exact periodic solutions, see [20], in the form of elliptic functions on the energy manifold are the x-normal mode $y = \dot{y} = 0$ and the solutions:

$$(2a_2 - a_1)x^2 = a_2 y^2, \ a_2(2a_2 - a_1) > 0. \tag{21}$$

Introducing amplitude-phase variables r_1, ϕ_1, r_2, ϕ_2, combination angle $\chi = \phi_1 - \phi_2$ and second order averaging-normalization we find:

$$\begin{cases} \dot{r}_1 = & +\frac{1}{2}a_2(\frac{1}{6}a_1 - a_2)r_1 r_2^2 \sin 2\chi, \\ \dot{r}_2 = & -\frac{1}{2}a_2(\frac{1}{6}a_1 - a_2)r_1^2 r_2 \sin 2\chi, \\ 2\dot{\chi} = & (-\frac{5}{6}a_1^2 + a_1 a_2 + \frac{2}{3}a_2^2)r_1^2 - a_2(a_1 - \frac{1}{6}a_2)r_2^2 + a_2(\frac{1}{6}a_1 - a_2)(r_2^2 - r_1^2)\cos 2\chi. \end{cases} \tag{22}$$

Here, we consider the possibility of more than two zero characteristic exponents and the integrability of the original system (20). In [20] it was shown that the normal form system (22) is integrable and contains in some parameter cases families of periodic solutions on the energy manifold. Such cases yield more than two characteristic exponents zero; they merit special attention as this may possibly signal integrability. Short-periodic solutions are obtained by considering solutions of $\sin 2\chi = 0, \dot{\chi} = 0$. We find the following cases:

- $\chi = 0, \pi, \ a_1 = -a_2$.
  Substituting these values in the 3rd equation of the normal form system (22) produces $\dot{\chi} = 0$. Any combination of r_1 and r_2 will produce for fixed energy this type of periodic solution of system (22). However, one has to check by higher order normalization whether this phenomenon persists. This was carried out by Gustavson [9] who showed that this family of periodic solutions

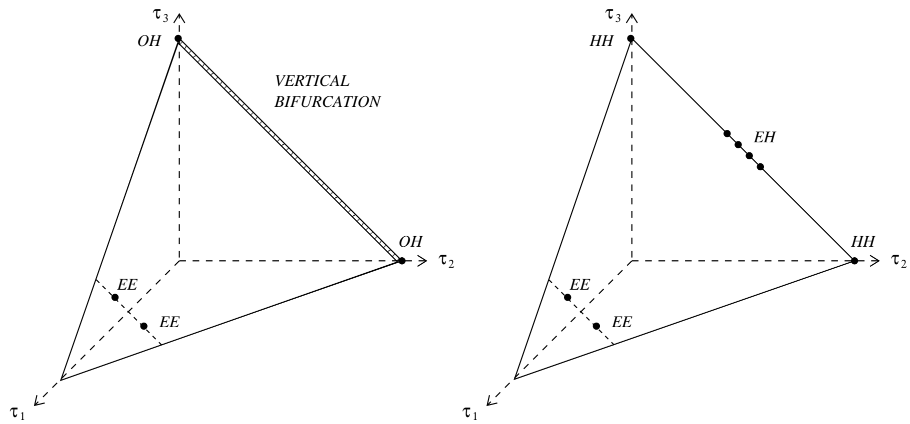

FIGURE 9. The actions of the $1:2:2$ resonance can be displayed on a simplex where the front plane contains the solutions on an energy manifold. Dots represent periodic solutions. The figure left is based on first order normalization and shows a continuous family of periodic solutions at action value $\tau_1 = 0$. Right shows the simplex obtained by second order normalization; the continuous family breaks up into six unstable periodic solutions. The figures are from [17] and [21].

breaks up into a finite number of periodic solutions at higher order. This is not surprising as $a_1 = -a_2$ represents the Hénon-Heiles Hamiltonian.

- $\chi = \pi/2, 3\pi/2,\ a_1 = a_2$.
  Substituting these values produces equally $\dot{\chi} = 0$. Inspection of the original equations (20) shows that the equations are separable when introducing $u = x + y, v = x - y$. The original system is integrable.
- A degeneration of the normal form takes place if $a_1 = 6a_2$.
  Substituting this value in the normal form system (22) produces: $\dot{r}_1 = \dot{r}_2 = 0$ whereas χ vanishes from the righthand side of the equation for χ in system (22). We conclude that any permitted choice of r_1, r_2 on the energy manifold keeps the amplitudes constant while $\dot{\chi} =$ constant. We can find families of periodic solutions. There is no need to compute higher order normal forms as this is a well-known case for integrability of system (20); see [3].

In these cases the presence of families of periodic solutions on the energy manifold corresponding with more than two zero characteristic exponents are indications of possible integrability of the original system. A more detailed analysis of these cases has settled the question of integrability. It is relatively easy to extend this method to other two dof Hamiltonian systems.

5.2. **The $1:2:2$ Hamiltonian resonance.** The following results are from [19]. The general time-independent Hamiltonian for three dof contains 56 cubic terms. Using the notations from [17] and [21] for the actions and angles (τ, ϕ) the first order normal form is:

$$H(\tau, \phi) = \tau_1 + 2\tau_2 + 2\tau_3 + 2a_1\tau_1\sqrt{2\tau_2}\cos(2\phi_1 - \phi_2 - a_2) + 2a_3\tau_1\sqrt{2\tau_3}\cos(2\phi_1 - \phi_3 - a_4), \tag{23}$$

with $a_1, \dots, a_4$ real constants. Remarkably enough it was shown in [19] that this normal form is integrable, it is the only case of a general three dof Hamiltonian normal form in first order resonance where this happens; see [6]. It is tied in with a symmetry in the normal form which produces the continuous familty of periodic solutions in the submanifold $\tau_1 = 0$. The symmetry is broken at the second order normal form, the continuous family breaks up into six periodic solutions including two normal modes, see fig. 9. The implication of the result is that the original (general) Hamiltonian in $1:2:2$ resonance behaves with error $O(\varepsilon)$ as an integrable system on the timescale $1/\varepsilon$. In addition, apart from the Hamiltonian itself, one other integral (H_2) is valid with error $O(\varepsilon)$ for all time.
The integrability at first order was signalled by the presence of the continuous family. It should be noted that a three dof Hamiltonian system in normal form has already two independent integrals, the normalized Hamiltonian and H_2. It is expected that the presence of certain integrals persists if one applies directly certain symmetries to the original cubic $1:2:2$ Hamiltonian. We give two examples, each with

$$H_2 = \frac{1}{2}(\dot{x}^2 + x^2) + \frac{1}{2}(\dot{y}^2 + 4y^2) + \frac{1}{2}(\dot{z}^2 + 4z^2).$$

Example 1. *Consider the Hamiltonian*

$$H = H_2 - (\frac{a_1}{3}x^3 + a_2 x(y^2 + z^2) + a_3 x^2(y^2 + z^2)), \tag{24}$$

with coefficients a_1, a_2, a_3. The equations of motion are:

$$\begin{cases} \ddot{x} + x = & a_1 x^2 + a_2(y^2 + z^2) + 2a_3 x(y^2 + z^2), \\ \ddot{y} + 4y = & 2a_2 xy + 2a_3 x^2 y, \\ \ddot{z} + 4z = & 2a_2 xz + 2a_3 x^2 z. \end{cases} \tag{25}$$

We know that the first order normal form is integrable with a family of periodic solutions in the submanifold $x(t) = \dot{x}(t) = 0$ (it is a matter of taste to rescale first $x \to \varepsilon x$ etc.). The normal form in this case is very degenerate. For the system (25) induced by Hamiltonian (24) we have the angular momentun integral: $y\dot{z} - \dot{y}z =$ constant as a second independent integral.

Example 2. *Consider the Hamiltonian*

$$H = H_2 - (a_1 x^2 y + a_2 x^2 z), \tag{26}$$

with coefficients a_1, a_2. The equations of motion are:

$$\begin{cases} \ddot{x} + x = & 2a_1 xy + 2a_2 xz, \\ \ddot{y} + 4y = & a_1 x^2, \\ \ddot{z} + 4z = & a_2 x^2. \end{cases} \tag{27}$$

Again, we know that the first order normal form is integrable with a family of periodic solutions in the submanifold $x(t) = \dot{x}(t) = 0$. In this case we obtain this family exactly by putting $x(t) = \dot{x}(t) = 0$ in system (27) to obtain a family of harmonic solutions. A second independent integral is easily obtained by eliminating x:

$$\frac{1}{2}(a_2\dot{y} - a_1\dot{z})^2 + \frac{1}{2}(a_2 y - a_1 z)^2 = \text{constant}.$$

Summary of integrability results. In subsection 5.1 we compare well-known results for the Hénon-Heiles (Braun's) family of Hamiltonians with the idea of considering characteristic exponents. Finding more than two zero exponents by normalization is an indication for the existence of an extra independent integral. However, higher order normalization may produce small nonzero exponents. This happens for the Hénon-Heiles system which is not integrable, but we find two (known) cases of integrability.

In subsection 5.2 we consider the $1:2:2$ Hamiltonian resonance. In [19] it was shown that the first order normal form is integrable and that a family of periodic solutions on the energy manifold exists; these solutions have 4 zero exponents. In [19] it was also shown that certain second-order perturbations will break up the family of periodic solutions destroying the integrability at the same time. In subsection 5.2 we present two examples where this break-up does not happen. This does not mean that the corresponding Hamiltonian is integrable but that a second independent integral exists besides the energy.

6. Conclusions.

- Recurrence can be a fairly regular phenomenon in dynamical systems or, alternatively, it can show patterns that are difficult to predict. If the system is Hamiltonian, this is often related to quasi-trapping of orbits in resonance zones of systems with three or more dof; for more discussion see [23]. In such a case, irregular and long-time recurrence can be a tool to find resonance zones with complex dynamics.
- The analysis of the Hamitonian $2:2:3$ resonance demonstrates the use of recurrence as a tool. However, this specific example merits a more detailed discussion than can be given here as in this example the higher order phenomena are arising already from H_5. In particular, the geometric structure of the resonance zones and the bifurcations arising from choices of the coefficients $b_1, \ldots, b_4$ present interesting open problems.
- The analysis of the $2:2:3$ resonance produces surprisingly strong interaction between low and higher order resonance. This is an extension of the theory of higher order resonance as put forward in [16] and [17].
- To compute accurate approximations of characteristic exponents of periodic solutions of dynamical systems with either normal form methods or by numerical bifurcation techniques is nowadays relatively easy. Generically we find for autonomous differential equations one zero exponent, for autonomous Hamiltonian systems two zero exponents. The presence of more zero characteristic exponents can be used as an indication for the existence of independent integrals of motion.

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