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Editor

# The Foundations of Chaos Revisited: From Poincaré to Recent Advancements

Ch. 1: Ferdinand Verhulst
Henri Poincaré's inventions
in Dynamical systems and Topology
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# Chapter 1
# Henri Poincaré's Inventions in Dynamical Systems and Topology

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**Abstract** The purpose of this article is to trace the invention of images and concepts that became part of Poincaré's dynamical systems theory and the Analysis Situs. We will argue that these different topics are intertwined whereas for topology Riemann surfaces and automorphic functions play an additional part. The introduction explains the term *invention* in the context of Poincaré's philosophical ideas. Poincaré was educated in the school of Chasles and Darboux that emphasized the combination of analysis and geometry to perform mathematics fruitfully. This will be illustrated in the second section where we list his new concepts and inventions in dynamical systems, followed by the descriptions of theory available before Poincaré started his explorations and the theory he developed. The third section studies in the same way the development of Poincaré's topological thinking that took place in the same period of time as his research in dynamical systems theory.

## 1.1 Introduction

The purpose of this paper is to *trace the inventions of Poincaré regarding dynamical systems and topology* starting with the accepted knowledge of his time. As we will see, for topology we will have to discuss aspects of the theory of automorphic functions. The intertwining of analysis and geometry is typical for the scientific work of Henri Poincaré.

This paper will not be a systematic treatment of his achievements and their impact on later science. Such systematic descriptions and references can be found in the biographies [8] and [31].

The use of the word 'invention' in the title needs some explanation. One should note that the first meaning of *invention* in French, as Poincaré used it, is indeed the same as in English.

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A famous essay by Henri Poincaré in [16] has the title *"L'invention mathématique"*. In [31, p. 84] the essay is described as follows:

> In a wonderful piece of introspection, Poincaré describes in the essay how sudden insight came to him in solutions of mathematical problems. He conjectures that the unconscious mind, stimulated by intense but seemingly fruitless exploration of a problem by the conscious mind, considers many mathematical combinations and makes a choice on the basis of aesthetics and economy. An example that he gives of such an occurrence concerns the Fuchsian functions.

There has been a lot of confusion about the use of this term. The English translation of [16] uses "discovery" instead of "invention", see [18, p. 46]. Even recently in [8, p. 120] this produced the following mix-up:

> In 1908 Poincaré talked to the Société de Psychologie in Paris about the psychology of discovery of new results in mathematics. The published version in *L'enseignement mathématique*, "L'invention mathématique," became one of his more famous essays.

The mix-up of discovery and invention is repeated on p. 120 of [8].

People argue that, starting with a complete system of axioms, mathematics is not invented but *discovered*. Of course discovery applies to a result like:

Assuming Euclides' axioms in plane geometry, we have that the sum of the angles within a triangle is 180 degrees.

Such results were discovered by careful analysing and following up the given assumptions.

*When Poincaré uses the term* **invention** *he refers to the creation of new concepts or the identification of deep relations between different mathematical or physical concepts.*

Platonic reasoning would argue for instance that the integers or the prime numbers exist independently of the human mind. However, long after identifying three apples or nine trees, the human mind came up with the abstract notion of number, for instance 3 or 9, as an element of the set of integers. An integer (and the set of integers) has no relation to a physical phenomenon, it exists only as an abstraction in the human mind; it is an example of human invention. It became the inspiration for the concept of operations like multiplication, a subsequent invention. And this was followed by the concept of multiplication of elements of other sets, for instance complex numbers, quaternions, matrices, elements of vectorspaces, etc.

Closely related to the idea of invention is the importance Poincaré attributes to language in [15]. The scientist creates the language to describe phenomena; to find the most suitable verbal description is an essential element of understanding the phenomena, both in the natural sciences and in mathematics. The perception of the relation between concepts and phenomena needs expression in language, which is an ingredient of the process of scientific invention. An example given by Poincaré concerns the motion of the celestial bodies. Kepler's laws contain a description in terms of the motion of the planets in elliptical orbits; the geometric concept of an ellipse provided the language. The transition to Newton's laws produced a richer formulation resulting in deeper understanding; the analytic concept of differential equation provided the language.

A concept introduced in [17] gives another illustration of the language for a new concept. Poincaré reasons that the *classification* of scientific facts is a main part of the activity of scientists. One considers for instance in biology all living creatures on Earth and tries to classify them in various groups. Or one considers in mathematics the set of integers and tries to distinguish subsets as even or odd numbers. A classification makes sense if adding new elements to the set does not change the old classification. For instance in biology, the discovery of a new type of living creature in the deep oceans, does not change the 'definition' of birds or mammals. If a classification is not changed by adding new elements, it is called *predicative* by Poincaré. This is now an accepted concept in logic. A definition in mathematics is really a classification, a definition has to be predicative.

Poincaré gives a simple example. Consider the set of integers and as a subset $H$ the first hundred integers. Classify them in two subsets: $A$, the numbers one through ten and $B$, the numbers larger than ten. Embedding $H$ in a larger set, for instance the first 200 integers, does not change the classification in $A$ and $B$, so it is predicative.

When Poincaré (1854–1912) started his career, his educational background was as follows:

He was a student at the Lycée of Nancy (1871–1875) where classical geometry, analysis, algebra and the humanities were taught. After this he was a student at L'École Polytechnique (1873–1875) with courses in analysis, geometry, mechanics and physics, chemistry, celestial mechanics. Then he attended L'École des Mines (1875–1878) where technical and geophysical lectures were given.

His dissertation on singularities of solutions of first order nonlinear partial differential equations was accepted at the Sorbonne in 1879, he became 25 in that year.

Poincaré was an enthusiastic reader of novels but not of scientific papers. He read the classics on celestial mechanics and special functions of that time, papers by Betti, Hermite, Laguerre, Bonnet, Halphen, Darboux, and later the writings of Riemann and Weierstraß whom he admired.

In the sequel we will start each section with a list of Poincaré's inventions and ideas, followed by descriptions of what was known at that time and a sketch of his ideas.

## 1.2 Dynamical Systems

New concepts and inventions:

1. Algebroid functions.
2. Index theory for plane dynamical systems i.e. autonomous second-order ordinary differential equations (ODEs).
3. The Poincaré-Bendixson theorem for plane dynamical systems.
4. Convergence of series solutions of ODEs, the use of the implicit function theorem, bifurcation theory (the Hopf bifurcation).

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5. Asymptotic, divergent series.
6. Normalization, the Poincaré domain.
7. Fixed point theorems for dynamical systems.
8. The recurrence theorem for dynamical systems characterized by measure-preserving maps.
9. Homoclinic chaos.

### *1.2.1 Ordinary Differential Equations in the Nineteenth Century*

Scientific treatises discussing ordinary differential equations in the nineteenth century are of three different types: books or papers on mathematical physics, on special functions and separate treatises on differential equations as we know them nowadays. We will leave aside the books that are completely application oriented. These books are of great interest but they merit a special study.

Special functions like the elliptic ones pose many difficult analytic problems. A typical and important example is the monograph by Jacobi [9]. The book is devoted to the analysis of elliptic functions (generalization of solutions of the mathematical pendulum equation).

George Boole's [4] is a text that deals mainly with elementary methods; it can be compared with introductions as taught at present. It discusses exact first order equations, integrating factors, special solutions and equations (Riccati equation) and methods for linear equations (sometimes tricks), variation of constants, geometric methods (involutes, curvature, tangencies).

A similar elementary treatise was written by Duhamel [7]. Duhamel lectured at the École Polytechnique, where Poincaré studied. Henri acquainted himself already with this course while still at the Nancy Lycée (see [31]). Part 4 on the integration of ODEs contains material as in Boole [4] but with more geometric problems and elementary Taylor series expansions for solutions.

We will pay special attention to the extensive treatises by Jordan [10] and Laurent [13]. Although at the year of their publication, Poincaré had been publishing on differential equations since 1879, his results are still ignored here. The books [10] and [13] are typical for the knowledge of ordinary differential equations in the nineteenth century before Poincaré.

Camille Jordan (1838–1922), see Fig. 1.1, was professor at L'École Polytechnique where he taught analysis. His three volumes Cours d'Analyse are a rich and didactical account of the analysis of his time. In vol. 3, pp. 1–296, two chapters deal with ordinary differential equations. The first chapter introduces again exact equations and integrating factors with examples from classical equations (Bernoulli, Clairaut), but interestingly, Jordan extends this to the cases in dynamics where one knows a number of integrals but not enough to solve the system. The integrals can be used to reduce the dimension of the system.

Attention is given to series expansions of solutions near regular and near singular points. Cases like

$$\frac{dy}{dx} = \frac{1}{f(x,y)} \text{ or } x\frac{dy}{dx} = f(x,y)$$

with $f(0,0) = 0$ and series expansions near $(0,0)$ are discussed extensively, based on the theory of Briot and Bouquet [5]. It should be noted that in the subsequent chapter on partial differential equations, the topic of series expansions cannot be found (this would be the topic of Poincaré's doctoral thesis). The second chapter treats linear equations with variable and constant coefficients. The theory is illustrated by the discussion of a number of special functions.

Hermann Laurent (1841–1908) published his seven volumes Traité d'Analyse [13] in the period 1895–1891; he was "examinateur d'admission á l'École Polytechnique" and from 1889 on professor at the École Agronomique in Paris, see Fig. 1.1. Volume 5 of [13, pp. 1–320], contains an extensive didactical introduction to ordinary differential equations. It has also special value because of the many references and the exercises. The first three chapters follow the same path as present day introductions: special methods, first order equations, equations of Bernoulli, Clairaut, etc. The treatment of linear equations becomes more interesting as Laurent discusses for instance equations with periodic coefficients, Lamé's equation and Halphen's theory of invariants. Chapter 4 summarizes the theory of special functions but without the difficult questions raised by Riemann, see Sect. 1.3. Chapter 5 is on nonlinear equations with emphasis on special integrable cases. Interesting is the method attributed to Jacobi; consider the equation

$$\frac{d^2y}{dx^2} = F(x,y)$$

Fig. 1.1 Camille Jordan (1838–1922) and Hermann Laurent (1841–1908)

with first integral

$$\frac{dy}{dx} = \phi(x, y, c),$$

$c$ a constant of integration. Jacobi shows that in this case the differential equation can be solved by quadrature. It can be considered a generalization of the method of d'Alembert that solves a similar problem for linear equations. The last chapter considers systems of first order linear equations including Cauchy's introduction of characteristic equations.

The exercises give an idea of the level of teaching and the requirements for students. Many exercises are concerned with geometrical questions for instance involving the curvature of certain solutions.

### *1.2.2 Poincaré's Thesis*

Poincaré was educated in geometry and analysis, but he did not restrict himself to one particular mathematical discipline. His major contributions regarding dynamical systems, the *Mémoire* of 1881–82, the *prize essay* of 1889 and the *Méthodes Nouvelles de la Mécanique Célèste*, are clearly characterized by the interaction of analysis and geometry

The thesis [22] was presented in 1879 and is concerned with an extended study of the known concepts of critical points and singularities of nonlinear first-order partial differential equations of the form

$$F(z, x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}) = 0.$$

The method of characteristics reduces the problem to the integration of an $n$-dimensional system of nonlinear ODEs. If $n = 2$ we can write the phase-plane equation associated with the two characteristic equations as

$$x^m \frac{dy}{dx} = f(x, y)$$

with $f(x, y)$ a holomorphic function. If $m = 0$, $y(x)$ is holomorphic near $x = 0$ and can be described by a corresponding series expansion. If $m = 1$, we have a weakly singular case, if $m > 1$ and integer we have an irregular singularity. Poincaré introduces *algebroid function* as follows: The function $z$ of $n$ variables $x_1, \ldots, x_n$ is algebroid of degree $m$ near $(0, \ldots, 0)$ if $z$ satisfies an equation of the form

$$z^m + A_{m-1} z^{m-1} + \ldots + A_1 z + A_0 = 0,$$

where the functions $A_0, \ldots, A_{m-1}$ have a convergent power series in $x_1, \ldots, x_n$ near $(0, \ldots, 0)$. If we can prove that the solution of the partial differential equation is algebroid, we can formulate results on the existence of certain convergent series expansions near $(0, \ldots, 0)$.

This is a useful generalization of the results of Briot and Bouquet [5], but the thesis goes on with the treatment of more complicated cases. In this connection, Poincaré introduces series expansions that exclude resonances of the form

$$m_2\lambda_2 + m_3\lambda_3 + \ldots + m_n\lambda_n = \lambda_1,$$

where the $\lambda_i$ are determined by the differential equation, the $m_2, \ldots, m_n$ are positive integers. In addition, the idea of non-resonance in celestial mechanics is generalized to requiring that the convex hull of the $\lambda_i$ in the complex plane does not contain the origin. This precludes the theory of normal forms, see for instance Arnold [1], where for the location of the $\lambda_i$ we would nowadays say "the spectrum is in the Poincaré domain".

### *1.2.3 The Mémoire of 1881–82*

The Mémoire [20] of 1881–82 is mainly concerned with two-dimensional problems and so is very different from his three volumes *Méthodes Nouvelles de la Mécanique Céleste* [14] where the first general theory of dynamical systems is found. The Mémoire is restricted to autonomous second order equations as many articles on ODEs are in the nineteenth century, but the research programme sketched by Poincaré breaks with the traditions of his time; it is very general and at present the programme still dominates research. In ODE research, it is the first study of global behaviour of solutions. Poincaré unfolds here the philosophy of studying nonlinear dynamics as it is still practiced today:

> Unfortunately it is evident that in general these equations [ODEs] can not be integrated using known functions, for instance using functions defined by quadrature. So, if we would restrict ourselves to the cases that we could study with definite or indefinite integrals, the extent of our research would be remarkably diminished and the vast majority of questions that present themselves in applications would remain unsolved.

And a few sentences on:

> The complete study of a function [solution of an ODE] consists of two parts:
>
> 1. Qualitative part (to call it like this), or geometric study of the curve defined by the function;
> 2. Quantitative part, or numerical calculation of the values of the function.

Consider the two-dimensional system

$$\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y)$$

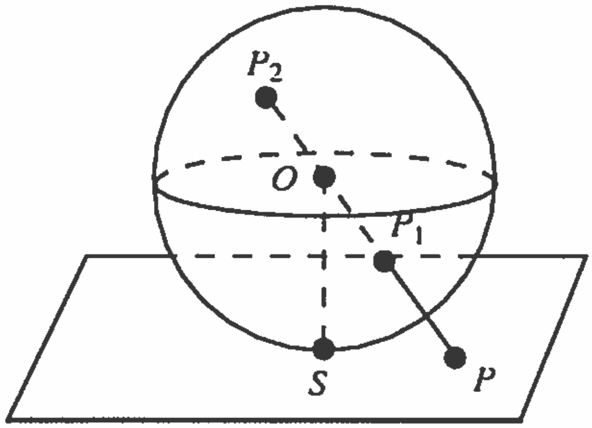


**Fig. 1.2** Gnomonic projection of a plane onto a sphere

with orbits in the Euclidean $(x, y)$-phaseplane. For the analysis of the system, Poincaré uses gnomonic projection; this is a cartographic projection of a plane onto a sphere (in cartography of course the other way around), see Fig. 1.2.

The plane is tangent to the sphere and each point of the plane is projected through the centre of the sphere, producing two points on the spherical surface, one on the Northern hemisphere, one on the Southern. The equatorial plane separates the two hemispheres. A point on the great circle in the equatorial plane corresponds with infinity.

Each straight line in the plane projects onto a great circle. So a tangent to an orbit in the plane projects onto a great circle that has at least one point in common with the projection of the orbit on the sphere. Such a point will be called a *contact*. The advantage of this projection is that the plane is projected on a compact set which makes global treatment easier. We have to consider with special attention the equatorial great circle which corresponds with the points at infinity of the plane. A bounded set in the plane is projected on two sets, symmetric with respect to the centre of the sphere and located in the two hemispheres.

If in a point $(x_0, y_0)$ we have not simultaneously $X = Y = 0$, $(x_0, y_0)$ is a regular point of the system and we can obtain a power series expansion of the solution near $(x_0, y_0)$.

If in a point $(x_0, y_0)$ we have simultaneously $X = Y = 0$, $(x_0, y_0)$ is a singular point. Under certain nondegeneracy conditions Poincaré finds four types for which he introduces the nowadays well-known names *saddle, node, focus and centre*. These are called singularities of first type. In the case of certain degeneracies we have singularities of the second type. Points on the equatorial great circle may correspond with singularities at infinity and can be investigated by simple transformations. The next section of the Mémoire is remarkable; it discusses the distribution and the number of singular points. Assuming that $X$ and $Y$ are polynomials and of the same degree and if $X_m$, $Y_m$ indicate the terms of the highest degree, while we have *not* $xY_m - yX_m = 0$, then the number of singular points is

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at least 2 (if the curves described by $X = 0$ and $Y = 0$ do not intersect on the two hemispheres after projection, there must be an intersection on the equatorial circle). In addition it is shown that a singular point on the equator has to be a node or a saddle, in the plane one cannot spiral to or from a singularity at infinity. An important new concept is *index*. Consider a closed curve, a cycle, located on one of the hemispheres. Taking one tour of the cycle in the positive sense, the expression $Y/X$ jumps $h$ times from $-\infty$ to $+\infty$, it jumps $k$ times from $+\infty$ to $-\infty$. We call $i$ with

$$i = \frac{h-k}{2}$$

the index of the cycle. It is then relatively easy to see that for cycles consisting of regular points one has:

- A cycle with no singular point in its interior has index 0.
- A cycle with exactly one singular point in its interior has index $+1$ if it is a saddle, index $-1$ if it is a node or a focus.
- If $N$ is the number of nodes within a cycle, $F$ the number of foci, $C$ the number of saddles, the index of the cycle is $C - N - F$.
- If the number of nodes on the equator is $2N'$, the number of saddles $2C'$, the index of the equator is $N' - C' - 1$.
- The total number of singular points on the sphere is $2 + 4n$, $n = 0, 1, \cdots$.

A solution of the ODE may touch a curve or cycle in a point, a contact. In such a point the orbit and the curve have a common tangent. An algebraic curve or cycle has only a finite number of contacts with an orbit. Counting the number of contacts and the number of intersections for a given curve contains information about the geometry of the orbits.

A useful tool is the 'théorie des conséquents', what is now called the theory of Poincaré maps. We start with an algebraic curve parametrized by $t$ so that $(x, y) = (\phi(t), \psi(t))$ with $\phi(t), \psi(t)$ algebraic functions; the endpoints $A$ and $B$ of the curve are given by $t = \alpha$ and $t = \beta$. Assume that the curve $AB$ has no contacts and so has only intersections with the orbits. Starting on point $M_1$ with a semi-orbit (the orbit traced for $t \geq t_0$), we may end up again on the curve $AB$ in point $M_1$ which is the 'conséquent' of $M_0$. Nowadays we would call $M_1$ the point generated by the Poincaré-map of $M_0$ under the phaseflow of the ODE. It plays an important part in understanding high-dimensional ODEs, anticipating the theory of fixed points of maps of differential topology.

If $M_0 = M_1$, the orbit is a cycle and Poincaré argues that returning maps correspond with either a cycle or a spiralling orbit. It is possible to discuss various possibilities with regards to the existence of cycles in which the presence or absence of singular points plays a part.

This analysis has important consequences for the theory of limit cycles. Semi-orbits will be a cycle, a semi-spiral not ending at a singular point, or a semi-orbit going to a singular point. Interior and exterior to a limit cycle there has always to be

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at least one focus or one node. Of the various possibilities considered it is natural to select annular domains, not containing singular points and bounded by cycles without contact and so transversal to the phase-flow. Such annular domains are often used to prove the existence of one or more limit cycles (Poincaré-Bendixson theory).

In the Mémoire, the topology of two-dimensional domains, either $\mathbb{R}^2$ or for instance $S^2$, with the Jordan separation theorem as an ingredient, plays an essential role.

Poincaré gave a few examples that were reproduced in [31, pp. 116–117], however with disturbing misprints. We discuss the examples here.

*Example 1* Consider the system with Euclidean variables $x, y$:

$$\begin{cases} \dot{x} &= x(x^2 + y^2 - 2x - 3) - y, \\ \dot{y} &= y(x^2 + y^2 - 2x - 3) + x. \end{cases} \tag{1.1}$$

The origin $(0, 0)$ is a stable focus corresponding with two foci on the sphere. Using polar coordinates

$$x = r\cos\phi,\ y = r\sin\phi,$$

we find outside the origin:

$$\dot{r} = r(r^2 - 2r\cos\phi - 3),\ \dot{\phi} = 1.$$

Elimination of time produces the equation:

$$\frac{dr}{d\phi} = r(r^2 - 2r\cos\phi - 3).$$

As $dr/d\phi(r = 1) = 2(\cos\phi - 1) \leq 0$ and $r^2 - 2r\cos\phi < 3$ for $r < 1$, we have that within the circle $r = 1$ the flow is acyclic, the flow is contracting. As $dr/d\phi(r = 3) = 18(1 - \cos\phi) \geq 0$ and $r^2 - 2r\cos\phi > 3$ for $r > 3$, we have that the flow outside the circle $r = 3$ is also acyclic, the flow is expanding. Within the circle $r = 1$ and outside the circle $r = 3$ have opposite signs for $dr/d\phi$, so the annular region $1 < r < 3$ is cyclic. As $dr/d\phi$ changes sign only once in the annular region, the annular region is monocyclic and contains one (unstable) limit cycle.

The second example shows a different phenomenon.

*Example 2* Consider the system with Euclidean variables $x, y$:

$$\begin{cases} \dot{x} &= 2x(x^2 + y^2 - 4x + 3) - y, \\ \dot{y} &= 2y(x^2 + y^2 - 4x + 3) + x. \end{cases} \tag{1.2}$$

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The origin $(0, 0)$ is an unstable focus corresponding with two foci on the sphere. Using again polar coordinates we find outside the origin:

$$\dot{r} = 2r(r^2 - 4r\cos\phi + 3),\ \dot{\phi} = 1.$$

Elimination of time produces the equation:

$$\frac{dr}{d\phi} = 2r(r^2 - 4r\cos\phi + 3).$$

Rewrite the equation as

$$\frac{dr}{d\phi} = 2r[(r-1)(r-3) + 4r(1-\cos\phi)].$$

As $dr/d\phi(r = 1) = 8(1 - \cos\phi) \geq 0$ and $r^2 - 4r\cos\phi > -3$ within the circle $r = 1$, we have that within the circle $r = 1$ the flow is acyclic, the flow is expanding.

At $r = 3$ we have again $dr/d\phi \geq 0$; if $r > 3$, we have $dr/d\phi > 0$. The flow outside the circle $r = 3$ is acyclic, the flow is also there expanding. The annular region $1 < r < 3$ has to be considered more closely. By analyzing the expression $(r^2 - 4r\cos\phi + 3)$, we see that $dr/d\phi$ cannot change sign in the annular region, so the annular region is also acyclic. There exist no limit cycles in a finite domain of the system.

We add a note on the behaviour near infinity of the solutions of the two examples. The systems (1.1) and (1.2) can be written as:

$$\begin{cases} \dot{x} &= xA(x, y) - y, \\ \dot{y} &= yA(x, y) + x. \end{cases} \tag{1.3}$$

We add the initial conditions $x(0) = x_0 \neq 0,\ y(0) = y_0$. Putting $\eta = y/x$ we find:

$$\frac{d\eta}{dt} = 1 + \eta^2,\ \eta(0) = \frac{y_0}{x_0},$$

with solution

$$\eta(t) = \frac{y(t)}{x(t)} = \frac{y_0}{x_0} + tan(t).$$

Inside the limit cycle of Example 1, the rotation of the orbits toward the origin causes the orbits to cross the positive $y$-axis with period $2\pi$ (alternating with crossing the negative $y$-axis).

In both examples, the solutions starting at a point $r(0) > 3$ tend to infinity. The equation $\dot{\phi} = 1$ suggests rotation, but this is not the case as the solutions tend to

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infinity in a finite time. Assuming $r(0) > 3$, this can be seen from the following estimates:

Example 1, $\dot{r} \geq r(r^2 - 2r - 3) \geq (r - 3)^3$.

Example 2, $\dot{r} \geq 2r(r^2 - 4r + 3) \geq 2(r - 3)^3$.

Integration of the differential inequalities (with $r(0) > 3$) gives the desired result.

At the equator of the Poincaré sphere, we find no limit cycles. Transforming $x = 1/u$ and $y = 1/v$, we find from the transformed system that the singularities at the equator are not regular.

### *1.2.4 The Prize Essay for Oscar II, 1888–89*

The famous prize awarded by King Oscar II of Sweden and Norway on the occasion of his 60th birthday in 1889 has become a well-known story, mainly because Henri Poincaré, who won the prize (see [23]), had to admit and to correct an error after the event. For detailed accounts see [2] and [31]. Not so well-known is that apart from the error to be corrected, the first version of the prize essay contained already fundamental theorems. Important results from the prize essay involve series expansions, periodic solutions and bifurcations. Series expansions with respect to a small parameter were the main tool in celestial mechanics of that time, but these expansions were formal. Comparison with results of various authors was not easy as many different transformations of the equations of motion were in use. Poincaré gave explicit criteria for the convergence and divergence of such series based on holomorphic expansion theorems of differential equations and the implicit function theorem. At the same time, his insight in the causes of the break-up of validity of expansion procedures, inspired him to the first set-up of a very important field: bifurcation theory. All these topics would be treated more extensively in [14].

Series expansion produce always local information. An important global result is the *recurrence theorem*:

Consider a dynamical system defined on a compact set in $\mathbb{R}^n$ with the property that the flow induced by the system is measure-preserving. Poincaré uses the term volume-preserving as the notion of measure does not exist at this time. Examples are the motion of an incompressible fluid in a nondeformable vessel or the phase-flow induced by a time-independent Hamiltonian system without singularities on a compact domain. Using the invariance of the domain volume, it is proved that most particles or fluid elements return an infinite number of times arbitrarily close to their initial position. The recurrence time is not specified but depends in general on the required closeness to the initial position and of course on the dynamical system at hand.

The interpretation of the recurrence theorem in the case of a chaotic system is interesting. In a two degrees-of-freedom Hamiltonian system near stable equilibrium, the KAM theorem guarantees in most cases the existence of an infinite number of two-dimensional invariant tori that separate the energy manifold into small chaotic regions. In these systems the recurrence phenomena near stable equilibrium are quite strong. Moving further away from stable equilibrium, the recurrence times will be more and more dependent on the initial positions.

In the case of more than two degrees-of-freedom, resonances will produce more active sets of chaotic orbits near stable equilibrium producing very different recurrence times.

Another basic result is the *non-integrability of conservative systems*.

In the corrected version of the prize essay [23], Poincaré overturned the general philosophy that Lagrangian or Hamiltonian systems are always integrable. The traditional idea was that if one could not find the integrals of for instance the gravitational three-body problem, this was caused only by lack of analytic skill. In fact, in his first submission of the prize essay, Poincaré set out to prove integrability of the circular, plane, restricted three-body problem. This can be written as a two degrees of freedom Hamiltonian system which takes the form of four first-order equations with periodic coefficients. He identified an unstable periodic solution and approximated its stable and unstable manifolds by series expansions. Poincaré calls these invariant manifolds "surfaces asymptotiques". He concluded (incorrectly in the first version) that the continuations of stable and unstable manifolds could be glued together to form integral surfaces corresponding with a second first integral of the system.

After a query of the editor of the Acta Mathematica asking for more details, Poincaré found out that this gluing was not possible in this particular example. He found an infinite number of intersections instead of merging of the manifolds. These results preclude the existence of homoclinic manifolds that would indicate the presence of a second integral. In the prize essay, the description of the geometry of the dynamics of the two degrees-of-freedom circular, plane, restricted three-body problem is tied in with the non-integrability results. In [14], the analysis will grow to its full generality for $n$ degrees of freedom Hamiltonian systems.

### *1.2.5 Les Méthodes Nouvelles de la Mécanique Céleste 1892–1899*

The three volumes of the Méthodes Nouvelles appeared in the same period (1892–1899) as the Analysis Situs and its supplements (1892–1905). The reference to celestial mechanics in the title of the three volumes is misleading, they contain the first general theory of dynamical systems describing both conservative and dissipative systems by analytical and geometric methods. Celestial mechanics is often used in [14] as an illustration of the theory.

To solve ODEs, in particular in problems of celestial mechanics, the use of series expansions is ubiquitous. Poincaré formulated and proved a basic series expansion theorem in vol. 1, Chap. 2 of [14]. At the same time he demonstrates how the convergence of such series can break down. This involves conditions of the implicit function theorem with consequences for the bifurcation of solutions.

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The use of the implicit function theorem was known at that time for sets of polynomial equations, but to apply these ideas to ODEs was new. Poincaré introduces the notion of *bifurcation set* with modifications for the dissipative and the conservative case (for more details see [31]). Particularly interesting is that in Chap. 3 a very general discussion is presented of what is now called the Hopf bifurcation.

The flexibility of Poincaré's mind shows again when he introduces divergent or asymptotic series in Chap. 8 as a legitimate tool. This went against the general mathematical philosophy of that time that required series to be convergent, but it agreed with the practice of many scientists working in applications. Divergent series can be used to obtain approximations of solutions but the difficult question of concluding existence of solutions and other qualitative questions from asymptotic approximations were not touched upon by Poincaré, this came after his time.

In [14], the fundamental non-integrability theorem is formulated and proved in the general case of the time-independent $2n$ dimensional Hamiltonian equations of motion

$$\dot{x} = \frac{\partial F}{\partial y}, \ \dot{y} = -\frac{\partial F}{\partial x}$$

with small parameter $\mu$ and the convergent expansion $F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$; $F_0$ depends on $x$ only and its Jacobian is non-singular, $|\partial F_0/\partial x| \neq 0$. Suppose $F = F(x, y)$ is analytic and periodic in $y$ in a domain $D$; the first integral $\Phi(x, y)$ of the system is analytic in $x, y$ in $D$, analytic in $\mu$ and periodic in $y$:

$$\Phi(x, y) = \Phi_0(x, y) + \mu\Phi_1(x, y) + \mu^2\Phi_2(x, y) + \cdots$$

*The statement is then that with these assumptions, $\Phi(x, y)$ can not be an independent first integral of the Hamiltonian equations of motion unless we impose further conditions.*

In the *Méthodes Nouvelles* [14, Chap. 5 of vol. 1], chapter 5 of volume 1, the technique is first analytic: a second integral should Poisson-commute with and be independent of the Hamiltonian; expanding the second integral with respect to a suitable small parameter and applying these conditions leads to a contradiction unless additional assumptions are made (see also [31]). It is understandable that the geometric aspects of non-integrability could not be understood at that time for more than two degrees of freedom. Very few contemporaries of Poincaré understood these aspects, even for two degrees of freedom (phase-space dimension 4). It is not clear whether Elie Cartan [6] understood non-integrability or, if he did, knew what to make of it. In his book [6] he recalls Poincaré's definition of integral invariant but he ignores existence questions.

There are more geometric details given in vol. 3, Chap. 32 of [14]. As in the prize essay, the analysis is inspired by the actual Hamiltonian dynamics of stable and unstable manifolds. Here we find the famous description of chaotic dynamical behaviour when considering the Poincaré-section of an unstable periodic solution in a two degrees of freedom Hamiltonian system:

> If on tries to represent the figure formed by these two curves with an infinite number of intersections whereas each one corresponds with a double asymptotic solution, these intersections are forming a kind of lattice-work, a tissue, a network of infinite closely packed meshes. Each of the two curves must not cut itself but it must fold onto itself in a very complex way to be able to cut an infinite number of times through each mesh of the network.
>
> One will be struck by the complexity of this picture that I do not even dare to sketch. Nothing is more appropriate to give us an idea of the intricateness of the three-body problem and in general all problems of dynamics where one has not a uniform integral and where the Bohlin series are divergent.

In this case of two degrees of freedom, the energy manifold is 3-dimensional in 4-dimensional phase-space.The flow on the energy manifold is visualized by the corresponding Poincaré-maps ("théorie des conséquents"). The double asymptotic solutions are the remaining homoclinic solutions that are produced by the intersections. The Bohlin series mentioned in the citation are *formal* series obtained by Bohlin for periodic solutions in celestial mechanics.

The picture Poincaré sketches destroys the possibility of a complete foliation into tori of the energy manifold, topologically $S^3$, induced by a second independent integral of motion.

### *1.2.6 The Poincaré-Birkhoff Theorem*

This theorem appeared in 1912, a long time after the Analysis Situs and its supplements. However, it is typical for Poincaré's interest in the global character of dynamical systems. It bothered him that so many results in this field are local, series expansions, normal forms, bifurcations, and he formulated a more global geometric theorem [24]. The reason to postpone its publication was that he found his reasoning not satisfactory; the actual proof was given by Birkhoff [3].

The idea is to characterize certain dynamical systems by an area-preserving, continuous twist-map of an annular region into itself. Such a map has at least two fixed points corresponding with periodic solutions of the dynamical system. The applications Poincaré had in mind were the global characterization of periodic solutions of time-independent Hamiltonian systems with two degrees of freedom. The dynamics of such a system restricted to a compact energy manifold is three-dimensional. The Poincaré maps of the orbits can provide the twist map described by the theorem. After 1912, fixed point theorems would play an important part in general and differential topology and in dynamical systems.

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## 1.3 Topology

A number of topological concepts were known before Poincaré's time, but, as in the case of the theory of dynamical systems, he invented its questions and the modern form of this field single-handedly. Poincaré used the term *Analysis Situs* ('analysis of place') for topology in a paper that appeared in 1892. It was followed up in typical Poincaré 'second-thoughts' style by five supplements, the last one in 1905. A translation into English and an introduction can be found in [26]. As stated before, the three volumes on dynamical systems [14] and the Analysis Situs were written in the same period of time. Before this period, Poincaré started his work on automorphic (Fuchsian) functions. We will argue that automorphic functions and dynamical systems, in particular the step from local to global considerations, were both instrumental in the creation of the Analysis Situs.

New concepts and inventions:

1. Triangularization of manifolds, the Euler-Poincaré invariant
2. Homology
3. The fundamental group
4. Algebraic topology

### *1.3.1 Topology Before Poincaré*

We will briefly describe topology before Poincaré and we will discuss in subsequent subsections various topics in Poincaré's work of the period 1878–1892 that might have inspired his ideas. We conclude with discussing some of his inventions of the Analysis Situs, see also [30]. A few aspects of our reasoning can be found in [32].

**Leibniz**

The term 'Analysis Situs' is attributed to Gottfried Wilhelm Leibniz (1648–1716) whose optimistic view considered our world the optimal one among possible worlds. The symbolism that he successfully applied in calculus was probably an inspiration for him to wish for symbolic 'calculus' in philosophy, sociology and geometry. For geometry this would imply an extension to forms and spaces characterized by algebraic symbols; this extension was called analysis situs, but the idea, although interesting, got no substance in Leibniz' subsequent work.

**Euler**

One of the mathematicians who thought about structures and forms in geometry was Leonhard Euler (1707–1783), see Fig. 1.3. He considered a convex two-dimensional polyhedron in Euclidean 3-space with $V$ the number of vertices, $E$ the number of edges and $F$ the number of faces. The Euler characteristic for polyhedrons $\chi$ is an invariant of the form:

$$\chi = V - E + F = 2.$$


**Fig. 1.3** Leonhard Euler (1707–1783), drawing by Giovanni Batista Bosio (1764–1827), engraved by Francesco Rebagli (courtesy private collection)

Interestingly, the Euler characteristic was generalized by Poincaré to more general closed, non-convex surfaces like tori or spheres with handles.

**Abel, Möbius and Jordan**

A handle, a 'look-through hole', in a surface is not so easy to characterize mathematically. Niels Henrik Abel (1802–1829) called the number of handles $g$, the genus of a surface in 3-space; for a sphere $g = 0$, for a torus $g = 1$ etc. August Ferdinand Möbius (1790–1868) developed ideas about non-orientable surfaces in Euclidean 3-space. Both Möbius and Camille Jordan (1838–1922) thought and formulated ideas about topological maps of surfaces. In their view, correspondence ("Elementarverwandschaft" in Möbius view) between two surfaces was not primarily characterized by point mappings but by considering the surfaces dissected in infinitesimal elements where neighboring elements of one surface correspond with neighboring elements of the other surface. For more details and references see [29].

**Betti**
Enrico Betti (1823–1892) gave a more precise description of tori and handles by defining his so-called Betti numbers. Betti uses the idea of connectivity and the number of closed curves separating a closed surface to characterize handles and more complicated structures.

**The Influence of Riemann**
The successes of analysis in dynamics, in particular in celestial mechanics, had its counterpart in applied mathematics in Germany, but meanwhile geometric thinking went there its autonomous course. This becomes clear in the mathematics of Bernhard Riemann (1826–1866), see Fig. 1.4. Poincaré notes in *La valeur de la science* [15]:

> Among the German mathematicians of this century, two names are particularly famous; these are the two scientists who have founded the general theory of functions, Weierstrass and Riemann. Weierstrass reduces everything to the consideration of series and their analytical transformations. To express it better, he reduces analysis to a kind of continuation of arithmetic; one can go though all his books without finding a picture. In contrast with this, Riemann calls immediately for the support of geometry, and each of his concepts presents an image that nobody can forget once he has understood its meaning. ([15], essay 'L'intuition et la logique en mathématiques')

It is interesting to consider Riemann's papers in the light of Poincaré's remarks.

At the occasion of his 'Habilitation' in Göttingen (1854), Riemann lectured on the foundations of geometry [28], see also [27] and for the historical context [29]. Riemann starts with experience and notes that the Euclidean foundations are not necessary, but that they have an acceptable certainty. He formulates a research plan for $n$-dimensional manifolds and spaces without precise descriptions. Weyl [28] links these considerations with later results in geometry, for instance by Klein, and with general relativity.

The collected works of Riemann [27] start with a treatise on the foundations of complex function theory, without figures but, as noted by Poincaré, "each of

**Fig. 1.4** Bernhard Riemann (1826–1866) and Henri Poincaré (1854–1912)

its concepts presenting an image". The interpretation of a complex function in the neighbourhood of a singularity plays a prominent part. In Riemann's articles, analysis and geometry go hand in hand, producing new insights in both fields.

A long article on Abelian functions in [27] is written in the same style, it contains four figures. The integration of differential equations leads more often than not to solutions that are defined implicitly. We are then faced with an inversion problem to find the explicit solution. Consider for instance a simple implicit relation in complex variables: $w = z^2$ with inversion, $z = \sqrt{w}$; this leads to the well-known problem that, starting, say on the real axis, and moving on a circle around the origin (the singularity), will produce a different value when arriving again at the real axis. An ingenious solution for the problem of many-valuedness to obtain unique continuation of such a function was proposed by Riemann. Using several sheets (complex planes) when moving around the singularity and joining them, one obtains the so-called Riemann surface. In the example of the quadratic equation above, one needs two sheets to be joined. For more general algebraic implicit equations, one needs for such an inversion a finite number of sheets and so a more complicated Riemann surface. A clear and systematic treatment of Riemann surfaces with historical remarks can be found in [12].

A prominent mathematician after Riemann was Felix Klein (1849–1925). His papers, books and lectures have a strong intuitive and geometric flavor. His work on automorphic functions, although considerable, was overshadowed by the results of Poincaré at the same time; see also [8] and [31]. Both mathematicians elaborated on the geometric aspects of Riemann surfaces.

### *1.3.2 Local Versus Global in Poincaré's Fuchsian Functions*

Many results on the local behaviour of functions were known in the 18th and 19th centuries. A few mathematicians aimed at a more global understanding ; Poincaré shared this ambition with Felix Klein (1849–1925). In his lecture notes on linear differential equations [11] Klein notes that we can make series expansions near the singularities of the coefficients, but this does not help global understanding. A basic tool for these problems is the geometric theory of automorphic functions developed both by Klein and Poincaré. Klein, while referring to an earlier lecture, states in the beginning of [11] (lecture of April 24, 1894):

> ... für hypergeometrische Functionen trat in meiner Vorlesung das Bestreben hervor den Gesamtverlauf der durch die Differentialgleichung definirten Funktionen zu erfassen.
>
> (... for hypergeometric functions, I wished to get a grip on the overall behaviour of the functions defined by the differential equation.)

The theory of Fuchsian (automorphic) functions is a successful synthesis of function theory and geometry, at the same time the concepts that were developed stimulated the emergence of topological concepts. Poincaré started to publish about Fuchsian functions in 1881, see vol. 2 of [19] and [25]. He was inspired by the

German mathematician Fuchs (1833–1902) who considered a second order, linear, ordinary differential equation of the form

$$y'' + A(z)y' + B(z)y = 0$$

with $A(z)$ and $B(z)$ holomorphic functions of the complex variable $z$ in a region $S \subset \mathbb{C}$. There are two independent solutions $y_1(z)$ and $y_2(z)$ and Fuchs started to consider the ratio $\eta = y_1/y_2$. He was interested in the behaviour of the solutions near singular points of $A(z)$ and $B(z)$ and performed analytic continuation of $y_1(z)$ and $y_2(z)$ along a closed curve around such a singularity and inversion of the function $\eta(z)$. This led him to consider a linear transformation of $\eta$ and, more in general, to look for functions that are invariant under a substitution of the form

$$z \to \frac{az+b}{cz+d}, \tag{1.4}$$

with coefficients $a, b, c, d$. So we have

$$f\left(\frac{az+b}{cz+d}\right) = f(z).$$

The substitution (1.4) (or transformation as we call it nowadays) is very rich; it consists of translations and rotations in the complex plane or, in the language of dynamical systems, expansions and contractions. The ratio $\eta(z)$ of the solutions should be invariant under these linear substitutions which is a more general property than periodicity that corresponds with the special case $a = c = d = 0$, $b \neq 0$ and real.

### *1.3.3 Fuchsian Groups*

Poincaré put the results at a higher level of abstraction. He called the functions which are invariant under transformation (1.4) Fuchsian, they are now called automorphic. The group of transformations acts usually on the upper complex half-plane $Im(z) > 0$ or on the disk $|z| < 1$. It is still removed from our present abstract concept of a group as a set of elements with certain operations defined on it. For his analysis, Poincaré had to distinguish between continuous and discontinuous transformation groups. He understood by a flash of intuition that the continuation of these complex functions, the use of Riemann surfaces and transformations in the complex plane correspond with geometric structures that can be understood only in terms of non-Euclidean geometry. In fact, until Poincaré looked at these problems, non-Euclidean geometry was considered as an artificial playground without much relevance to mathematics in general.

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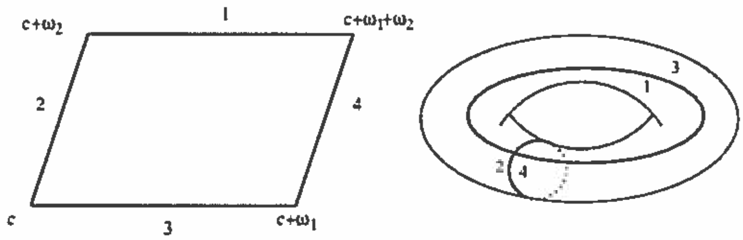


**Fig. 1.5** Fundamental parallelogram corresponding with a torus

In the analysis of functions with singularities, fundamental polygons and coverings of Riemann surfaces by polygons play an important part. For some functions a covering by triangles is suitable. In the case of elliptic functions we have the inversion of an elliptic integral that produces double periodicity:

$$\eta(x + m_1\omega_1 + m_2\omega_2) = \eta(x)$$

which keeps $\eta(x)$ invariant ($m_1, m_2 \in \mathbb{Z}$). In this case one uses a covering of parallelograms. Identifying several parts of the boundary of a fundamental polygon leads for the triangle to genus zero (one can deform to a sphere), for the parallelogram to $g = 1$ (identifying the opposite sides two by two leads to a torus, see Fig. 1.5). Poincaré shows that fundamental polygons bounded by more sides lead to arbitrary large genus. For an introduction to polygon coverings see [12, Chap. 12]. In [11] Klein discusses the relations between a fundamental parallelogram and a torus (see Fig. 1.5) and between a fundamental octagonal and a surface with genus two.

Closely related to this is Poincaré's theory of uniformization problems. Differential equations lead to the integration and inversion of algebraic functions; their analytic continuations produce multi-valued analytic functions. Uniformization of such functions corresponds to obtaining a parametrization by single-valued meromorphic functions. The development has led to the relation between complex function theory and hyperbolic geometry, and also to many results in the study of quadratic forms and arithmetic surfaces. The theory of uniformization contains still many fundamental open questions.

### *1.3.4 Covering an Analytic Curve in 1883*

In [21] Poincaré considers a complex vector function $y_1(x), y_2(x), \ldots, y_n(x)$; he lets the complex variable $x$ describe a closed contour $C$ on a Riemann surface $S$. When $x$ traces the contour $C$, the function is restricted to an analytic curve on $S$. The idea is to show that there exists a transformation $x \to z$ such that after applying the transformation, the vector field $y$ can be parametrized by single-valued meromorphic functions of $z$. There are two types of contours:

1. When $x$ traces $C$ once, at least one of the components of $y$ does not return to its starting value.
2. All components return to their starting value when tracing the contour $C$ once. There are two subcases:
    1. By slight deformation of $C$ this property persists;
    2. Applying slight deformation of $C$ the property does not persist.

The proof that such a transformation exists rests on two ideas. First, one knows that if $C$ is a closed contour, one can find a holomorphic function $u(\xi, \eta)$ inside $C$ which takes prescribed values on $C$. This is based on solving the Dirichlet problem of the Laplace equation in two dimensions.

The second point concerns us here. Poincaré states in the proof that the analytic curve on the Riemann surface $S$ is covered by an infinite number of *feuillets*, the infinitesimal elements of Möbius and Jordan. This construction of the covering is later used and extended by Poincaré as a general covering procedure for manifolds.

### *1.3.5 The Analysis Situs and Its Supplements*

On reading the Analysis Situs of 1895 and its later supplements [26], one notes that the conciseness and abstraction of modern mathematics is missing; reading the text is relatively easy. This is deceptive as the ideas and new concepts go very deep. Its readability is misleading.

Introductions to Poincaré's topological papers are found in [29] and [26]. We will discuss a number of basic concepts from the papers referring sometimes to his earlier work. Poincaré was not an avid reader but usually gave carefully credit to ideas and results of colleagues if he knew about them. There are not many references in the Analysis Situs as the material was so new.

1. Introduction of the concept of manifold in arbitrary dimension (by construction).

    The idea of a manifold has a long history with contributions from many mathematicians. Poincaré introduced the covering of an analytic curve in [21]. It is generalized to two and higher-dimensional manifolds.

    In the first section of the Analysis Situs, manifolds are described by sets of algebraic equations in $\mathbb{R}^n$. A new approach is given in the third section where manifolds are defined by continuous parametrizations; they can be replaced by analytic parametrizations as we can approximate continuous functions by analytic ones. In this way, manifolds of the same dimension that have a common part can be considered an analytic continuation of each other.

    Thus far, the analysis of Poincaré of the treatment of manifolds was a natural extension of ideas of older mathematicians and the theory of complex functions on Riemannian surfaces, see [29].

2. The use of local parametrizations that become global by overlap like in analytic continuation was a new idea. Another new element arises in Sect. 10 of the Analysis Situs [26]: geometric representation by gluing together polyhedra identifying faces and manifolds. Consider a manifold $M$ and replace the manifold by approximating simplexes with adjacent boundaries, forming a simplicial complex. In this way, using polygons like triangles, we obtain a triangulation of a manifold that makes it easier to apply homology (the next item).
3. Homology.

   Suppose a manifold $M$ contains $r$-dimensional submanifolds, Poincaré calls them cycles. If $M$ has a $(r+1)$-dimensional submanifold with as a boundary one given $r$-dimensional cycle, the cycle is homologous to zero in $M$. Consider as an illustration an annular region in the plane where $r = 1$, see Fig. 1.6.
4. Homology theory and the fundamental group.

   In Sect. 11 of [26], Poincaré considers domains in 4-space with 3-dimensional surfaces as boundaries that can be subdivided and homeomorphically transformed into polygons. Regarding such transformations, the inspiration from Fuchsian groups becomes explicit in the Sects. 10–14. In Sect. 11, Poincaré writes

   > The analogy with the theory of Fuchsian groups is too evident to need stressing. (transl. J. Stillwell [26])

   One of the results is the emergence of algebraic structures between Betti numbers and a generalized topological Euler invariant (usually called now Euler-Poincaré invariant). Consider a group $\Gamma$ of translations $\gamma$ of the complex plane $\mathbb{C}$ (or a suitable other domain) which is fixed-point free. A typical case is when $\Gamma$ is generated by two Euclidean translations in different directions.

   Associated with $\Gamma$ is a fundamental domain D which is a polygon. In the case of $\mathbb{C}$ we can take for the fundamental domain a parallelogram. The translations of this polygon in two directions fill $\mathbb{C}$, see Fig. 1.5.

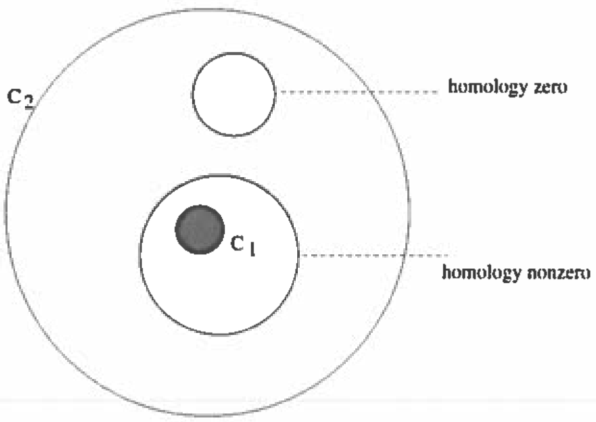


**Fig. 1.6** Consider the annular region bounded by $C_1$ and $C_2$. A *closed curve* with interior in the annular region has homology zero, a *closed curve* encircling $C_1$ has nonzero homology

Another aspect brings us to algebraic topology: we can identify opposite sides of the fundamental parallelogram to obtain a torus which is in this special case $\mathbb{C}/\Gamma$. Considering other domains and polygons we may find manifolds with genus higher than one.

5. Associated with homology is also Poincaré duality. It was stated in terms of Betti numbers: The $k$th and $(n-k)$th Betti numbers of a closed, orientable n-manifold are equal. Criticism of his work by Poul Heegaard led him to discuss (so-called) torsion in the second supplement.

### *1.3.6 Conclusions*

The Analysis Situs was created as a completely new mathematical theory. Its inventions are geometrical representation, triangulation of manifolds, homology and algebraic topology. In particular:

1. To study the connectedness of a manifold Poincaré developed a calculus of submanifolds. The relations involved were called homologies, they could be handled as ordinary equations. This started algebraic topology and what Leibniz would have called "an algebra of surfaces".
2. Technically, Fuchsian transformations and the fundamental group played an inspiring and important part in the set-up of the Analysis Situs.
3. Geometrically, the picture is more complex. Riemann surfaces, global considerations from ODEs and Hamiltonian dynamics were another inspiration. In the dynamical systems theory of Poincaré [20] and [14], an important part of the considerations are local like series expansions, bifurcation theory etc. The development of global insight in dynamical systems like the reasoning needed to describe homoclinic chaos and the use of fixed point results to find periodic solutions (Sect. 1.2.6) was new, it needed consideration of the dynamics on 3-dimensional compact manifolds embedded in 4-space.

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