# Measures of Chaos in Hamiltonian Systems

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**Abstract**

Hamiltonian systems with two or more degrees of freedom are generally nonintegrable which usually involves chaotic dynamics. The size of the chaotic sets determines for a large part the nature and influence of chaos. Near stable equilibrium we can obtain normal forms that often produce 'formal integrability' of the Hamiltonian system and at the same time produce rigorous but not necessarily optimal upper limits for the size, the measure of chaotic sets. This is demonstrated for two and three degrees of freedom systems with attention to the role of symmetry.

## 1 Introduction

Most Hamiltonian systems are not integrable. However, as we shall see, this is a very deceptive statement although it is mathematically correct. To get this in the right perspective, we shall start by outlining suitable approximation methods. These are canonical normal form methods, sometimes called after Birkhoff-Gustavson, and averaging performed in a canonical way. The methods admit precise error estimates and enable us therefore to determine local measures of regularity and chaos. The methods also permit us to locate normal modes and other short-periodic solutions.
We recall that two degrees of freedom time-independent Hamiltonian systems near stable equilibrium can be normalized and that the normal form is always integrable to any order, see also section 2. The integrals are the Hamiltonian and its quadratic part. The integrable motion dominates phase-space and this result expresses that the amount of chaos near stable equilibrium is exponentially small. Explicitly: near stable equilibrium, the measure of chaos is $O(\varepsilon^a \exp(-1/\varepsilon^b)$ for suitable constants $a, b$ where the energy $E = O(\varepsilon^2)$. An example is studied in [6].

## 2 Approximations and normal forms

Consider the $n$ degrees of freedom time-independent Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} \omega_i \left(p_i^2 + q_i^2\right) + H_3 + H_4 + \cdots . \tag{1}$$

with $H_k$, $k \geq 3$ a homogeneous polynomial of degree $k$ and positive frequencies $\omega_i$. We introduce a small parameter $\varepsilon$ into the system by rescaling the variables by $q_i = \varepsilon \overline{q_i}, p_i = \varepsilon \overline{p_i}, i = 1, \cdots, n$ and dividing the Hamiltonian by $\varepsilon^2$. This implies that we localize near stable equilibrium with energy $O(\varepsilon^2)$.
We can define successive, nonlinear coordinate (or *near-identity*) transformations that will bring the Hamiltonian into the so-called Birkhoff normal form; see [3] and [14] for details and references. For a general dynamical systems reference see [1, 2], for symmetry in the context of Hamiltonian systems see [2, 8, 13]. A stimulating text on chaos and resonance is [5]. In action-angle variables $\tau, \phi$, a Hamiltonian $H$ is said to be in Birkhoff normal form of degree $2k$ if it can be written as

$$H = \sum_{i=1}^{n} \omega_i \tau_i + \varepsilon^2 P_2(\tau) + \varepsilon^4 P_3(\tau) + \cdots + \varepsilon^{2k-2} P_k(\tau),$$

where $\tau = (\tau_1, \cdots, \tau_n)$ and $P_i(\tau)$ is a homogeneous polynomial of degree $i$ in $\tau_i = \frac{1}{2}({p_i}^2 + {q_i}^2), i = 1, \cdots, n$. The variables $\tau_i$ are called actions; note that if Birkhoff normalization is possible, the angles have been eliminated. If a Hamiltonian can be transformed into Birkhoff normal form, the dynamics is fairly regular. The system is integrable with integral manifolds which are tori described by taking $\tau_i$ constant. The flow on the tori is quasi-periodic.

Suppose a Hamiltonian is in Birkhoff normal form to degree $m$, but the frequencies are satisfying a resonance relation of order $m+1$. This means that $H_{m+1}, H_{m+2}$ etc. may contain resonant terms which can not be transformed away. The procedure is now to split $H_{m+1}, H_{m+2}$ etc. in resonant terms and terms to which the Birkhoff normalization process can be applied. The resulting normal form will generally contain resonant terms and is called Birkhoff-Gustavson normal form. It contains terms dependent on the action $\tau$ and on resonant combination angles of the form $\chi_i = k_1\phi_1 + \cdots + k_n\phi_n$. In practice we have to consider a truncation of the Birkhoff-Gustavson normal form $\bar{H}$ at some degree $p \geq m$:

$$\bar{H} = H_2 + \varepsilon\bar{H}_3 + \varepsilon^2\bar{H}_4 + \cdots + \varepsilon^{p-2}\bar{H}_p. \tag{2}$$

Because of the construction we have the following results:

- $\bar{H}$ is conserved for the original Hamiltonian system (1) with error $O(\varepsilon^{p-1})$ for all time.
- $H_2$ is conserved for the original Hamiltonian system (1) with error $O(\varepsilon)$ for all time. So the normal form has at least two integrals. Symmetry can enhance the regularity, see [10].
- If we find other integrals of the Birkhoff-Gustavson normal form, we have slightly weaker error estimates. Explicitly, suppose that $F(p,q)$ is an independent integral of the truncated Hamiltonian system (2), we have for the solutions of the original Hamiltonian system (1) the estimate

$$F(p,q) - F(p(0), q(0)) = O(\varepsilon^{p-1}t).$$

An important consequence is the following statement: if the phaseflow induced by the truncated Hamiltonian (2) is completely integrable, the flow of the original Hamiltonian (1) is approximately integrable in the sense described above. In this case the original system is called *formally integrable*. This implies that the irregular, chaotic component in the flow of the original Hamiltonian is limited by the given error estimates and must be a small-scale phenomenon on a long timescale. For details see [14].

# 3 Normal modes and short-periodic solutions

Liapunov proved that if the frequencies $\omega_i$ satisfy no resonance relation, the normal modes, obtained by linearization, can be continued for the full, nonlinear Hamiltonian system (1), resulting in at least $n$ short-periodic solutions with periods $\varepsilon$-close to $2\pi/\omega_i$.

Weinstein [15] proved an important generalization: even in the case of resonance, there exist at least $n$ short-periodic solutions of Hamiltonian system (1). Note, that these periodic solutions are not necessarily continuations of the linear modes, the term 'normal modes' in this context can be confusing. Another important point is that $n$ short-periodic solutions is really the minimum number. For instance in the case of two degrees of freedom, 2 short-periodic solutions are guaranteed to exist by the Weinstein theorem. But in the $1:2$ resonance case one finds generically 3 short-periodic solutions for each (small) value of the energy. One of these is a continuation of a linear normal mode, the other two are not. For higher-order resonances like $3:7$ or $2:11$, there exist for an open set of parameters 4 short-periodic solutions of which two are continuations of the normal modes.

Of course symmetry and special Hamiltonian examples may change this picture drastically. For instance in the case of the famous Hénon-Heiles Hamiltonian

$$H(p,q) = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) + \frac{1}{3}q_1^3 - q_1q_2^2,$$

because of symmetry, there are 8 short-periodic solutions. For the relation between symmetry and periodic solutions and references see [4].

# 4 Three degrees of freedom

The question of asymptotic integrability is different for more than two degrees of freedom. First we consider the genuine first order resonances of three degrees of freedom systems.

Table 1: Integrability of the normal forms of the four genuine first order resonances.

| Resonance | Assumptions | $H_3$ | $H_4$ | Remarks |
|---|---|---|---|---|
| 1:2:1 | general | 2 | 2 | no analytic third integral |
| | discr.symm. $q_1$ | 2 | 2 | no analytic third integral |
| | discr.symm. $q_2$ | 3 | 3 | $\overline{H}_3 = 0$; 2 subsystems at $\overline{H}_4$ |
| | discr.symm. $q_3$ | 2 | 2 | no analytic third integral |
| 1:2:2 | general | 3 | 2 | no cubic third integral at $\overline{H}_4$ |
| | discr.symm. $q_2$ and $q_3$ | 3 | 3 | $\overline{H}_3 = 0$; 2 subsystems at $\overline{H}_4$ |
| 1:2:3 | general | 2 | 2 | no analytic third integral |
| | discr.symm. $q_1$ | 3 | 3 | 2 subsystems at $\overline{H}_3$ and $\overline{H}_4$ |
| | discr.symm. $q_2$ | 3 | 3 | $\overline{H}_3 = 0$ |
| | discr.symm. $q_3$ | 3 | 3 | 2 subsystems at $\overline{H}_3$ and $\overline{H}_4$ |
| 1:2:4 | general | 2 | 2 | no cubic third integral |
| | discr.symm. $q_1$ | 2 | 2 | no cubic third integral |
| | discr.symm. $q_2$ or $q_3$ | 3 | 3 | 2 subsystems at $\overline{H}_3$ and $\overline{H}_4$ |

## 4.1 First-order resonances

It turns out that the normal form of the $1:2:2$-resonance is integrable; this is caused by a hidden symmetry which reveals itself by normalization. The $1:2:1$-resonance and the $1:2:3$-resonance on the other hand are not integrable for an open set of parameters of the Hamiltonian. The results are illustrated in the table.

If three independent integrals of the normalized system can be found, the normalized system is integrable. The integrability depends in principle on how far the normalization is carried out ($\overline{H}_k$ represents the normal form of $H_k$, the homogeneous part of the Hamiltonian of degree $k$). The formal integrals have a precise asymptotic meaning as discussed in section 2. We use the following abbreviations: *no cubic integral* for no quadratic or cubic third integral; *discr. symm.* $q_i$ for discrete (or mirror) symmetry in the $p_i, q_i$-degree of freedom; *2 subsystems at* $\overline{H}_k$ for the case that the normalised system decouples into a one and a two degrees of freedom subsystem upon normalising to $H_k$. In the second and third column one finds the number of known integrals when normalizing to $\overline{H}_3$ respectively $\overline{H}_4$.

The remarks which have been added to the table reflect some of the results known on the non-existence of third integrals. Note that the results presented here are for the general Hamiltonian and that additional assumptions, in particular involving symmetry, may change the results. In this respect it is interesting that in a number of applications, chaotic dynamics appears to be of relatively small size. An example is the dynamics of elliptical galaxies that display three-axial symmetry. Astrophysical observations suggest highly nonlinear but integrable motion. The statements above with indication 'Assumptions': 'general', are for Hamiltonian systems in general form near stable equilibrium; in this case $H_3$ has 56 terms, $H_4$ counts 126 terms. To illustrate the analysis we will look now at specific potential problems of the form

$$\begin{aligned}
\ddot{q}_1 + \omega_1^2 q_1 &= \varepsilon R_1(q_1, q_2, q_3, \varepsilon),\\
\ddot{q}_2 + \omega_2^2 q_2 &= \varepsilon R_2(q_1, q_2, q_3, \varepsilon),\\
\ddot{q}_3 + \omega_1^2 q_3 &= \varepsilon R_3(q_1, q_2, q_3, \varepsilon),
\end{aligned}$$

where $R_1, R_2, R_3$ represent the quadratic and higher-order potential terms.

*Example: the* $1:2:2$*-resonance*
Considering a general potential starting with cubic terms, only the terms $a_1 q_1^2 q_2$ and $a_2 q_1^2 q_3$ show up in the normalized $H_3$. The resulting mirror symmetry makes the normal form $\overline{H} = H_2 + \varepsilon \overline{H}_3$ integrable. According to section 2, the asymptotic validity of the third integral (which is quadratic) is $O(\varepsilon^2 t)$. The implication is, that chaotic dynamics in the system is restricted to size $O(\varepsilon)$ on the long timescale $1/\varepsilon$.

It is easy to obtain the short-periodic solutions from the normal form. There are two stable general position families of periodic solutions ($\tau_1\tau_2\tau_3 \neq 0$ and with multi-frequencies $1, 2, 2$). Moreover there exists a continuous set of periodic solutions in the 2-d.o.f. submanifold $\tau_1 = 0$. This is a nongeneric phenomenon that is expected to break up when normalizing to higher-order. To observe this break-up, we include one of the terms which vanishes in the normalization to $H_3$. In particular consider the Hamiltonian system

$$H = H_2 + \varepsilon(a_1 q_1^2 q_2 + a_2 q_1^2 q_3 + a_3 q_1 q_2 q_3), \tag{3}$$

producing the equations of motion

$$\begin{aligned} \ddot{q}_1 + q_1 &= -\varepsilon(2a_1 q_1 q_2 + 2a_2 q_1 q_3 + a_3 q_2 q_3), \\ \ddot{q}_2 + 4q_2 &= -\varepsilon(a_1 q_1^2 + a_3 q_1 q_3), \\ \ddot{q}_3 + 4q_3 &= -\varepsilon(a_2 q_1^2 + a_3 q_1 q_2). \end{aligned}$$

After normalization to $H_4$ we find that the symmetry which created the continuous set of periodic solutions is broken. The submanifold $\tau_1 = 0$ of the normal form still exists, but now the continous set has broken up into 6 unstable periodic solutions that include the two normal modes $\tau_1 = \tau_2 = 0$ and $\tau_1 = \tau_3 = 0$. At this point the results are robust, i.e. there will be no qualitative change to this picture of short-periodic solutions by adding higher-order perturbations.
Interestingly, the integrability of the normal form to $H_4$ is still an open problem. Details of the calculations can be found in [11].
*Mirror symmetry in* $q_1$.
If, on replacing $q_1$ by $-q_1$, the Hamiltonian remains invariant, the picture changes drastically as terms like $a_3 q_1 q_2 q_3$ are absent. The problem has not been analyzed in its full generality. Consider for instance the system induced by (3) with $a_3 = 0$. The normal form in action-angle coordinates to $H_4$ reads

$$\begin{aligned} \bar{H} &= \tau_1 + 2\tau_2 + 2\tau_3 + \frac{1}{2}\varepsilon\tau_1\Big(a_1\sqrt{\tau_2}\sin(2\phi_1 - \phi_2) + a_2\sqrt{\tau_3}\sin(2\phi_1 - \phi_3)\Big) - \\ & \quad \frac{1}{4}\varepsilon^2\Big(\frac{9}{16}(a_1^2 + a_2^2)\tau_1^2 + \frac{1}{4}a_1^2\tau_1\tau_2 + \frac{1}{4}a_2^2\tau_1\tau_3 + \frac{1}{2}a_1 a_2 \tau_1\sqrt{\tau_2\tau_3}\cos(\phi_2 - \phi_3)\Big). \end{aligned}$$

The analysis of the normal form to $H_4$ shows structurally unstable phenomena like eigenvalues zero. The implication is that higher-order normal forms have to be computed to analyze a system with this symmetry.
*Mirror symmetry in* $q_1, q_2$.
In this case, the low-order normal form calculation simplifies drastically: $\bar{H}_3 = 0$. The normal form to $H_4$ is integrable, so chaotic dynamics is restricted in this case to size $O(\varepsilon^2)$ on the long timescale $1/\varepsilon$ or $O(\varepsilon)$ on the long timescale $1/\varepsilon^2$. The structurally stability analysis of the periodic solutions requires higher-order normal form calculations.

*Example: the* $1:2:3$*-resonance*
This resonance was analyzed in [7] and [12]. We will summarize some results and formulate some open problems. When normalizing to $H_4$ one finds 7 short-periodic (families of) solutions. One of them is for an open set of parameters complex unstable (for the complementary set it is unstable of saddle type). This complex instability is a source of chaotic behaviour. Using Šilnikov-Devaney theory, it is shown in [7] that a horseshoe map exists in the normal form to $H_4$ which makes the normal form chaotic.
Numerics indicate that the normal form $\bar{H} = H_2 + \bar{H}_3$ is already chaotic, but a proof is missing. Also the dynamics of the case where the periodic solution is unstable, but of saddle type, has still to be characterized.
Discrete symmetry in either the first or the last degree of freedom makes the normal form to $H_4$ integrable.

## 4.2 Higher-order resonances

Higher order resonances abound in applications. The results discussed thus far are mostly general, but, with regards to applications, it is very important to look again at the part played by symmetries. This will be illustrated for the $1:3:7$-resonance and will be discussed in some detail. This also serves as an example that resonances with odd resonance numbers are particularly sensitive to symmetries.
*Example: the* $1:3:7$*-resonance*
We start with the general Hamiltonian with this resonance in $H_2$:

$$H_2 = \tau_1 + 3\tau_2 + 7\tau_3.$$

At $H_3$ level there is no resonance and we find after normalization, $\bar{H}_3 = 0$.
There are two combination angles active at $H_4$ level: $\chi_1 = 3\phi_1 - \phi_2$ and $\chi_2 = \phi_1 + 2\phi_2 - \phi_3$. At $H_5$ level

no combination angles are added, $H_5$ can be brought in Birkhoff normal form. We list the consequences of mirror symmetry:

- In the first d.o.f: $\chi_1$ and $\chi_2$ not active; formal integrability until $\bar{H}_5$, chaotic dynamics has measure $O(\varepsilon^4 t)$.
- In the second d.o.f: $\chi_1$ not active; formal integrability until $\bar{H}_7$, chaotic dynamics has measure $O(\varepsilon^6 t)$.
- In the third d.o.f: $\chi_2$ not active; formal integrability until $\bar{H}_7$, chaotic dynamics has measure $O(\varepsilon^6 t)$.
- The case of mirror symmetry in all three d.o.f. is discussed below.

One can continue the analysis to higher order normal forms to obtain more precise estimates of the remaining chaotic dynamics. We discuss an example.
*Three-axial elliptical galaxies*
In this case we have discrete (mirror) symmetry in three degrees of freedom. Until $H_7$ the system can be brought into Birkhoff normal form, chaotic dynamics has measure $O(\varepsilon^6 t)$ which predicts regular behaviour on a long timescale. The Birkhoff-Gustavson normal form $\bar{H}_8$ contains the combination angles $6\phi_1 - 2\phi_2$ and $2\phi_1 + 4\phi_2 - 2\phi_3$.
The situation needs a very high degree of normalization as becomes clear when considering the analysis of periodic solutions. Because of the discrete symmetry $\tau_i = 0, i = 1, 2, 3$ each corresponds with a two d.o.f. submanifold of the original (symmetric) Hamiltonian. The normal modes are exact periodic solutions of the normal form and the original Hamiltonian. The normal forms in these 4-dimensional submanifolds are all integrable (section 2) and chaotic behaviour takes place in exponentially small sets. Consider the question of how far we have at least to normalize the flow in these submanifolds.
Case $\tau_1 = 0$. This is the worst case, as it involves the 3 : 7-resonance. In the symmetric case this system has to be normalized to $H_{20}$ to characterize the periodic solutions.
Case $\tau_2 = 0$ involving the 1 : 7-resonance. The system has to be normalized to $H_{16}$ to characterize the periodic solutions.
Case $\tau_3 = 0$ involving the 1 : 3-resonance. The relatively well-known system has to be normalized to $H_8$ to characterize the periodic solutions. In [10] it is described how to deal with such higher-order cases.

# 5 A remark on chains of oscillators

Our knowledge of chains of oscillators is still restricted. Remarkably enough the normal form of the $1 : 2 : \cdots : 2$-resonance with $n$ degrees of freedom is integrable. Consider the Hamiltonian

$$H(p, q) = \frac{1}{2}(p_1^2 + q_1^2) + \sum_{i=2}^{n}(p_i^2 + q_i^2) + H_3 + \cdots,$$

where $H_3 + \cdots$ represents the general cubic and higher order terms. The Hamiltonian is formally integrable and the proof runs along the lines of the analysis of the 1 : 2 : 2-resonance, displaying again hidden symmetry.
A spectacular result arises for the classical Fermi-Pasta-Ulam problem which is a chain of identical oscillators coupled by nearest neighbour interaction. At low energy levels the chain shows recurrence and no chaos. Recently it was shown in [9] by normal form methods and symmetry considerations, that a nearby integrable system exists which make the KAM-theorem applicable. This solves the recurrence phenomenon at low energy.

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