The Dynamics of Slow Manifolds

Ferdinand Verhulst*and Taoufik Bakri
Mathematisch Instituut
University of Utrecht
PO Box 80.010, 3508 TA Utrecht
The Netherlands

Abstract

After reviewing a number of results from geometric singular perturbation theory, we discuss several approaches to obtain periodic solutions in a slow manifold. Regarding nonhyperbolic transitions we consider relaxation oscillations and canard-like solutions. The results are illustrated by prey-predator systems.

1 Introduction

In *singular perturbations* we have a small, positive parameter, ε, characterizing the size of perturbation terms, and the disturbing fact that putting $\varepsilon = 0$, the so-called unperturbed (or reduced) problem is not 'sufficient' to start a perturbation expansion. Let us illustrate this immediately with a nearly-trivial example. Consider the initial value problem for $t \geq 0$:

$$\begin{aligned} \dot{x} &= 1, \; x(0) = 1, \\ \varepsilon \dot{y} &= -y + \varepsilon f(x), \; y(0) = 1, \end{aligned}$$

with $f(x)$ a smooth scalar function. Putting $\varepsilon = 0$ we have $0 = -y, \dot{x} = 1$ with solution $x(t) = 1 + t, y(t) = 0$. The 'unperturbed solution' does not satisfy the initial condition for y, but in the theory of singular perturbations, techniques have been developed to handle such cases. In this example, the solution $y(t)$ changes quickly in a neighborhood of $t = 0$, a so-called *boundary layer* in time. For a recent survey of methods see [27].

In this paper we will review a number of the theorems available for singularly perturbed initial value problems of ordinary differential equations, while adding results on periodic solutions and examples for simple looking but surprisingly rich prey-predator systems. The numerics for autonomous two-dimensional systems is carried out by **pplane**, using MATLAB. The nonautonomous systems were integrated using Runge-Kutta 7(8).

In the actual constructions of asymptotic approximations, the Tikhonov theorem is basic for providing a boundary layer property of the solution. This leads naturally to a number of qualitative and quantitative results. Also certain attraction (or hyperbolicity) properties of the 'unperturbed solution' play an essential part in the construction of the asymptotic approximation, adding a geometric flavour to the analysis that is essential. In the case of our nearly-trivial example, as we shall see, the 'unperturbed solution' $x(t) = 0, y(t) = 1 + t$ is associated with the existence of a so-called slow manifold.

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2 Basic results

In this section we formulate the Tikhonov theorem, O'Malley-Vasil'eva expansion and we will summarize Fenichel's results.

2.1 The Tikhonov theorem

The following result was obtained in 1952 by Tikhonov [21]:

Theorem 2.1
Consider the initial value problem

$$\begin{aligned} \dot{x} &= f(x,y,t) + \varepsilon \cdots, \quad x(0) = x_0, \quad x \in D \subset \mathbb{R}^n, t \geq 0, \\ \varepsilon \dot{y} &= g(x,y,t) + \varepsilon \cdots, \quad y(0) = y_0, \quad y \in G \subset \mathbb{R}^m. \end{aligned}$$

For f and g, we take analytic vector functions in x, y, and t; the dots represent (analytic) higher-order terms in ε. We assume that:

a. A unique solution of the initial value problem exists and we suppose, this holds also for the reduced problem
$$\begin{aligned} \dot{x} &= f(x,y,t), \quad x(0) = x_0, \\ 0 &= g(x,y,t), \end{aligned}$$
with solutions $\bar{x}(t),\ \bar{y}(t)$.

b. The equation $0 = g(x,y,t)$ is solved by $\bar{y} = \phi(x,t)$, where $\phi(x,t)$ is a continuous function and an isolated root. Also suppose that $\bar{y} = \phi(x,t)$ is an asymptotically stable solution of the equation
$$\frac{dy}{d\tau} = g(x,y,t)$$
that is uniform in the parameters $x \in D$ and $t \in \mathbb{R}^+$.

c. $y(0)$ is contained in an interior subset of the domain of attraction of $\bar{y} = \phi(x,t)$ in the case of the parameter values $x = x(0),\ t = 0$.

Then we have
$$\begin{aligned} \lim_{\varepsilon \to 0} x_\varepsilon(t) &= \bar{x}(t), \quad 0 \leq t \leq L, \\ \lim_{\varepsilon \to 0} y_\varepsilon(t) &= \bar{y}(t), \quad 0 < d \leq t \leq L \end{aligned}$$
with d and L constants independent of ε.

When using the theorem, the system

$$\dot{x} = f(x,y,t),\ 0 = g(x,y,t), \tag{1}$$

is usually called the *unperturbed*, *reduced* or *degenerate* system. The equation for x is called the slow equation, the equation for y the fast equation.
In assumption (b), t and x are parameters and not variables. The idea is that during the fast motion of the variable y in a boundary layer of time, the small variations of these parameters are negligible as long as the stability holds for values of the parameters $x \in D$ and $t \in \mathbb{R}^+$. As in our nearly-trivial example above, we have that the unperturbed solution only partially

satisfies the initial conditions and it is surprising that the unperturbed solution still represents an approximation of the x-component. For a more extensive discussion of the Tikhonov theorem see [27].

Example 2.1
It is natural to consider first our simple two-dimensional system for $t \geq 0$:

$$\begin{aligned} \dot{x} &= 1,\ x(0) = 1, \\ \varepsilon \dot{y} &= -y + \varepsilon f(x),\ y(0) = 1, \end{aligned}$$

with $f(x)$ an analytic scalar function (in fact, analyticity is a requirement that can be relaxed to C^1 in the Tykhonov theorem). Putting $\varepsilon = 0$, we have $\dot{x} = 1, 0 = -y$ with $y = 0$ an asymptotically stable solution of the equation $dy/d\tau = -y$. We conclude that we have for the solution $x_\varepsilon(t), y_\varepsilon(t)$:

$$\begin{aligned} \lim_{\varepsilon \to 0} x_\varepsilon(t) &= 1 + t, \quad 0 \leq t \leq L, \\ \lim_{\varepsilon \to 0} y_\varepsilon(t) &= 0, \quad 0 < d \leq t \leq L, \end{aligned}$$

with d and L constants independent of ε.
It is easy to extend this result to a system of the form $\dot{x} = g(x, y), \varepsilon \dot{y} = -y + \varepsilon f(x, y)$ with initial conditions.

More complicated examples will be considered in later sections.

2.2 The O'Malley-Vasil'eva expansion

We will use Tikhonov's theorem to obtain approximations of solutions of nonlinear initial value problems. The theorem does not state anything about the size of the boundary layer or the timescales involved to describe the initial behavior and the relative slow behaviour later on.
Asymptotic expansions are described as follows (for references see [27]):

Theorem 2.2
(O'Malley-Vasil'eva)
Consider the initial value problem in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+$

$$\begin{aligned} \dot{x} &= f(x, y, t, \varepsilon),\ x(0) = x_0, \quad x \in D \subset \mathbb{R}^n,\ t \geq 0, \\ \varepsilon \dot{y} &= g(x, y, t, \varepsilon),\ y(0) = y_0, \quad y \in G \subset \mathbb{R}^m, \end{aligned}$$

where f and g can be expanded in powers of ε to order $(m+1)$. Suppose that the requirements of Tikhonov's theorem have been satisfied and moreover that for the solution of the reduced system $0 = g(x, \bar{y}, t, 0),\ \bar{y} = \phi(x, t)$ we have, with μ a constant independent of ε,

$$\text{Re Sp} \quad g_y(x, \bar{y}, t) \leq -\mu < 0, x \in D, 0 \leq t \leq L,$$

(Sp is the spectrum of the matrix g_y). Then, for $t \in [0, L], x \in D, y \in G$, the formal approximation described above leads to asymptotic expansions of the form

$$\begin{aligned} x_\varepsilon(t) &= \sum_{n=0}^{m} \varepsilon^n a_n(t) + \sum_{n=1}^{m} \varepsilon^n \alpha_n \left(\frac{t}{\varepsilon}\right) + O(\varepsilon^{m+1}), \\ y_\varepsilon(t) &= \sum_{n=0}^{m} \varepsilon^n b_n(t) + \sum_{n=0}^{m} \varepsilon^n \beta_n \left(\frac{t}{\varepsilon}\right) + O(\varepsilon^{m+1}). \end{aligned}$$

The constant L that bounds the domain of validity in time is *in general* an $O(1)$ quantity determined by the vector fields f and g. There are cases where L extends to ∞.
An intermediate step in the analysis by O'Malley and Vasil'eva is an expansion of the form

$$y = \phi(x,t) + \varepsilon y_1(x,t) + \varepsilon^2 y_2(x,t) + \varepsilon^3 \cdots . \tag{2}$$

The expansion is derived from the fast equation and it is asymptotically valid on a timescale $O(1)$ outside the boundary layer in time where fast motion takes place.

2.3 The slow manifold: Fenichel's results

Tikhonov's theorem is concerned with the attraction, at least for some time, to the regular expansion that corresponds with a stable critical point (corresponding with an equilibrium) of the boundary layer equation. The theory is quite general and deals with nonautonomous equations.
In the case of autonomous equations, it is possible to associate with the regular expansions $\sum_{n=0}^{m} \varepsilon^n a_n(t)$ and $\sum_{n=0}^{m} \varepsilon^n b_n(t)$, a manifold in phase-space and to consider the attraction properties of the flow near this manifold. Such questions were addressed and answered in a number of papers by Fenichel [7]– [10], and other authors; the reader is referred to the survey papers [14] and [15]. See for an introduction also [27].
Consider the autonomous system

$$\begin{aligned} \dot{x} &= f(x,y) + \varepsilon \cdots , \quad x \in D \subset \mathbb{R}^n, \\ \varepsilon \dot{y} &= g(x,y) + \varepsilon \cdots , \quad y \in G \subset \mathbb{R}^m. \end{aligned}$$

In this context, one often transforms $t \to \tau = t/\varepsilon$ so that

$$\begin{aligned} x' &= \varepsilon f(x,y) + \varepsilon^2 \cdots , \quad x \in D \subset \mathbb{R}^n, \\ y' &= g(x,y) + \varepsilon \cdots , \quad y \in G \subset \mathbb{R}^m, \end{aligned}$$

where the prime denotes differentiation with respect to τ, G is a *compact* set.
As before, y is called the fast variable and x the slow variable. The zero set of $g(x,y)$ is given again by $y = \phi(x)$, which in this autonomous case represents a first-order approximation M_0 of the n-dimensional (slow) manifold M_ε. The flow on M_ε is to a first order approximation described by $\dot{x} = f(x, \phi(x))$.
Note that the assumption for the system to be autonomous is not essential for Fenichel's theory; it only facilitates the geometric interpretation.

Comparison of Tykhonov and Fenichel
In Tikhonov's theorem, we assumed asymptotic stability of the approximate slow manifold; in the asymptotic constructions we assume moreover that the eigenvalues of the linearized flow near M_0, derived from the equation for y, have negative real parts only.
In geometric singular perturbation theory, for which Fenichel's results are basic, we only assume that all real parts of the eigenvalues are nonzero. In this case of a slow-fast system, the slow manifold M_ε is called *normally hyperbolic.* A manifold is called hyperbolic if the local linearisation is structurally stable (real parts of eigenvalues all nonzero), and it is normally hyperbolic if in addition the expansion or contraction near the manifold in the transversal direction is larger than in the tangential direction (the slow drift along the slow manifold).

If M_0 is a compact manifold that is normally hyperbolic, it persists for $\varepsilon > 0$ (i.e., there exists for sufficiently small, positive ε a smooth manifold M_ε close to M_0). Corresponding with the signs of the real parts of the eigenvalues, there exist stable and unstable manifolds of M_ε which are smooth continuations of the corresponding manifolds of M_0, on which the flow is fast.

The compactness property
The compactness of the slow manifold is not just an artificial requirement. The condition guarantees the existence and uniqueness of the slow manifold M_ε as the following example shows.

Example 2.2
Consider the two-dimensional system

$$\begin{aligned} x' &= -x + \varepsilon y, \quad x(0) = x_0, \\ y' &= \varepsilon, \quad y(0) = y_0. \end{aligned}$$

In this system, x is the fast variable, y the slow one. Putting $\varepsilon = 0$, we have the solution $(x(t), y(t)) = (x_0 \exp(-t), y_0)$ with clearly $x = 0$ a candidate to be a first approximation M_0 of a slow manifold. However, this set is not compact. Usually, we are interested in a certain part of phase-space and we can remedy the non-compactness by bounding the flow 'far-away' from the region of interest. For instance by changing the system to

$$\begin{aligned} x' &= -x + \varepsilon y, \quad x(0) = x_0, \\ y' &= \varepsilon(1 - c^2 y^2), \quad y(0) = y_0, \end{aligned}$$

with c a positive constant, sufficiently small. We could for instance choose $c < 1$ fixed, with $\varepsilon \ll c$. We now have that $y(t)$ is bounded and a first approximation M_0 of the slow manifold is parametrized by y_0 in $(x, y) = (0, y_0)$. On the other hand, it is easy to see that

$$\lim_{t \to \infty} x_\varepsilon(t) = \frac{\varepsilon}{c},$$

which is in an ε-neighborhood of M_0, but depends on the choice of c. Using the system with $c \neq 0$ as an approximation of the system with $c = 0$, we have a unique slow manifold for c fixed. Because of the arbitrariness of the choice of c, the location of the slow manifold is then arbitrary in a neighborhood of M_0.

An interesting example of an application of Fenichel's results is given in [5] and [13]. We give a summary of the results.

Example 2.3
Consider the three-dimensional system

$$\begin{aligned} x' &= y, \\ y' &= x - x^2 + \varepsilon y(z^2 - a), \\ z' &= \varepsilon(1 + bx - cz^2). \end{aligned}$$

The parameters a and b strongly influence the nature of the flow, c is a positive constant, $c < 1$, and is used to guarantee compactness. Putting $\varepsilon = 0$, the flow is described by the integrable system $x' = y, y' = x - x^2, z = z(0)$. This system has two zero sets, $(0, 0, z(0))$ and $(1, 0, z(0))$. The first one will be called M_0 and it is normally hyperbolic (in the x, y phase-plane, it is a saddle); according to Fenichel it will persist for $\varepsilon > 0$. If $\varepsilon = 0$, a stable and an unstable manifold emanate from M_0, closing to form a (cylindrical) homoclinic manifold. These manifolds are still present as stable and unstable manifold of M_ε, but in general we expect them to intersect transversally. In [5] and [13] it is shown, using Melnikov methods, that for a certain value of a and $b \geq -1$, a homoclinic tangency arises, followed by a cascade of pulse orbits. If $b < -1$, the situation is dynamically more complex. A highly technical analysis shows that a Smale horseshoe is present in the flow, corresponding with chaotic dynamics in the system.

3 Periodic solutions

Early examples of periodic solution theorems can be found in [11] and [1]. If the reduced system (1), $\varepsilon = 0$, has a hyperbolic T-periodic solution, then under certain additional conditions, the full system has a unique T-periodic solution. Hyperbolicity of the periodic solution of the reduced system plays an essential part in both papers and this prohibits application to a number of interesting cases.
A major technical problem was the absence of a theorem on the existence of a manifold of solutions (the so-called slow manifold), corresponding with the solutions of the reduced system. This complicated the existence problem of the theorems of that time enormously.
The existence and smoothness of the slow manifold, in combination with the possibility of a regular expansion describing the slow manifold drift, enables us to take a fairly easy shortcut to obtain periodic solutions. If we restrict ourselves to periodic solutions located completely within a slow manifold, this excludes by definition the case of nonhyperbolic transitions as found in relaxation oscillations.

3.1 Restricting to the slow manifold

In [29] a theorem leading to periodic solutions for autonomous systems is formulated. Generalization to nonautonomous systems is straightforward.
Consider the system in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+$

$$\begin{aligned} \dot{x} &= f_0(x, y, t) + \varepsilon f_1(x, y, t) + \varepsilon^2 \cdots, \quad x \in D \subset \mathbb{R}^n, \ t \geq 0, \\ \varepsilon \dot{y} &= g_0(x, y, t) + \varepsilon g_1(x, y, t) + \varepsilon^2 \cdots, \quad y \in G \subset \mathbb{R}^m, \end{aligned}$$

where f_0, f_1, g_0, g_1 are analytic vector functions, the dots represent bounded and analytic higher order terms. All vector functions are T-periodic in t. Furthermore the assumptions of Tikhonov's and Fenichel's theorems apply for $0 \leq t \leq L$ with $L > T$.
For the solutions in the slow manifold we can apply the expansion (2) $y = \phi(x, t) + \varepsilon y_1(x, t) + \varepsilon^2 \cdots$ with $g_0(x, \phi(x, t), t) = 0$.
For $x(t)$ in the slow manifold this results in

$$\dot{x} = f_0(x, \phi(x, t), t) + \varepsilon \frac{\partial f_0}{\partial y}(x, \phi(x, t), t) y_1(x, t) + \varepsilon f_1(x, \phi(x, t), t) + \varepsilon^2 \cdots. \tag{3}$$

This is still a very general system and much depends on our knowledge of the reduced equation. Note, that if we would strictly apply the O'Malley-Vasil'eva expansion for the equations

governing the slow manifold flow, this may produce secular terms when approximating periodic solutions. Using eq. (3), secular terms can be avoided.
An application of this perturbation idea is found in [29], where one considers periodic solutions of an autonomous system of the form

$$\begin{aligned}\dot{x} &= A(y)x + \varepsilon f_1(x,y) + \varepsilon^2 \cdots, \quad x \in D \subset \mathbb{R}^n,\ t \geq 0,\\ \varepsilon\dot{y} &= g_0(x,y) + \varepsilon g_1(x,y) + \varepsilon^2 \cdots, \quad y \in G \subset \mathbb{R}^m.\end{aligned}$$

A second possibility is to use numerical bifurcation techniques. In a number of applications one can for instance identify the periodic solution as arising from a Hopf bifurcation.
We sketch briefly a third approach. Suppose that in the slow manifold no critical point is present. Sometimes we can associate with the flow on the slow manifold a continous map of a convex set into itself. According to the Brouwer fixed point theorem, this means that at least one fixed point of the map exists. As there is no critical point, this fixed point corresponds with a periodic solution. Of course a necessary condition for this is, that the solutions stay on the slow manifold for all time, excluding nonhyperbolic transitions.

For illustration, we will discuss now a nonautonomous problem from population dynamics.

3.2 A prey-predator system

Consider two populations, consisting of a prey (population x) and a predator (population y), interacting with the prey by a Holling type II interaction term. The growth of the prey is also restricted by a logistic term. The equations are

$$\begin{aligned}\dot{x} &= x\left(r - \frac{rx}{K(t)} - \frac{p_1 y}{H_1 + x}\right),\\ \dot{y} &= y\left(\frac{c_1 x}{H_1 + x} - d_1\right).\end{aligned}$$

We have $x, y \geq 0$, r, p_1, H_1, c_1, d_1 are positive constants. The constant r indicates the growth rate of the prey, p_1 the predation rate, c_1 the growth rate of the predator. We assume that $r \gg c_1$ so that we can put $c_1/r = \varepsilon$; the assumption means that the population growth of the prey is relatively fast. H_1 is the saturation constant, $K(t)$ is the carrying capacity of the prey which is supposed to be a positive, continuous T-periodic function. These periodic variations arise from external, for instance seasonal, fluctuations. The case of constant carrying capacity is well-known, see [19] and further references there. Seasonal fluctuations in this model were considered in [23], where many bifurcations are found using numerical bifurcation techniques; this paper does not discuss the singular perturbation case.
We make the system nondimensional by putting $\delta = d_1/c_1, K(t) = K_0(k_0 + f(t)), \beta = H_1/K_0$ and rescaling $\bar{t} = c_1 t, x = K_0 \bar{x}, \bar{y} = p_1 y/(rK_0)$; $f(t)$ is a T-periodic function with average zero.
The system becomes, omitting the bars:

$$\begin{aligned}\varepsilon\dot{x} &= x\left(1 - \frac{x}{k_0 + f(t)} - \frac{y}{\beta + x}\right),\\ \dot{y} &= y\left(\frac{x}{\beta + x} - \delta\right).\end{aligned}$$

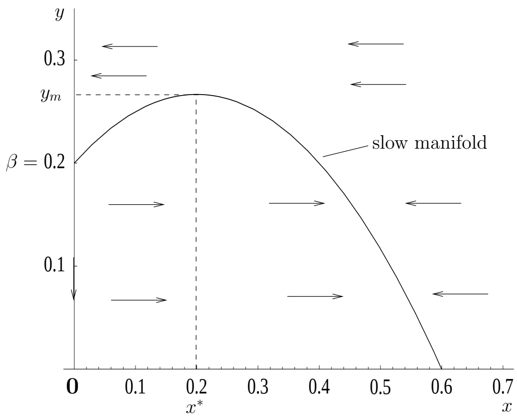

Figure 1: Slow manifold $x = 0$ and parabolic slow manifold M_p at $t = 0$ of the prey-predator system from section 3.2. The parameter values are in this case $k_0 = 0.6, \beta = 0.2$. To the right of the maximum at $x = x^*$, the parabolic slow manifold is stable.

Furthermore we assume $0 < \beta, \delta < 1$.
There are two slow manifolds, first the y-axis, given by $x = 0$ and secondly M_p given by

$$y = \beta + x - \frac{\beta + x}{k_0 + f(t)} x + O(\varepsilon)$$

with nonhyperbolic, transcritical intersection point $(x, y) = (0, \beta)$. The y-axis is stable for $y > \beta$ and unstable if $0 \leq y < \beta$. The second slow manifold, M_p, has for t fixed a parabolic cross-section (with accuracy $O(\varepsilon)$), it is a periodic surface in x, y, t-space. For arbitrary but fixed time t, the cross-section has a maximum $y = y_m$ if $x = x^* = (k_0 + f(t) - \beta)/2$. The slow manifold is stable if $x > x^*$, unstable if $0 < x < x^*$; see fig. 1.
A periodic solution arises in the following scenario. The unstable part of the periodic slow manifold M_p is chosen outside the first quadrant $x, y \geq 0$ by requiring that $x^* = (k_0 + f(t) - \beta)/2$ is negative for all time. The flow in the parabolic slow manifold induces a time T-map of the interval $(0, \beta + O(\varepsilon))$ in itself. According to the Brouwer fixed point theorem this map has a fixed point, corresponding with a T-periodic solution. Although the system is three-dimensional, the projection of the periodic solution on the x, y-plane is of course a closed curve, see fig. 2.

4 Nonhyperbolic transitions

Transitions arising from nonhyperbolicity arise often in applications and they have been studied in various contexts. For an interesting boundary value problem and references see [16], for a study motivated by atmospheric research see [24].

4.1 Relaxation oscillations

Classical phenomena are relaxation oscillations where jumps and fast transitions, take place after moving along a slow manifold that becomes unstable. For this topic see [12], [17], [20]

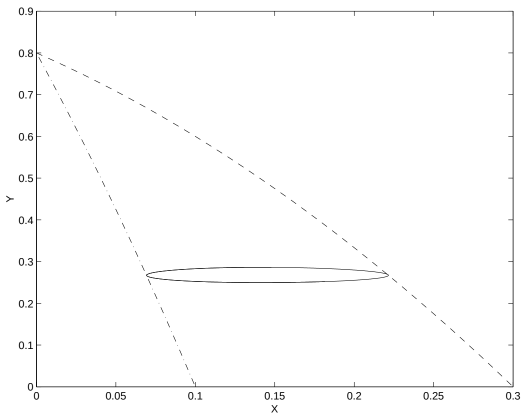

Figure 2: Periodic solution in the parabolic slow manifold of the prey-predator system from section 3.2. The parameter values are $\varepsilon = 0.05, k_0 = 0.2, \beta = 0.8, \delta = 0.15$. The carrying capacity is given by $k_0 + a \sin t$ with in this case $a = 0.1$. The dashed and dash-dotted parabolic curves represent the outer limits of the projection of the parabolic slow manifold on the x, y-plane.

and chapter 4 in [2]. Most rigorous analysis is carried out for two-dimensional autonomous and forced problems and it is not easy to extend this to more dimensions.

Example 4.1
The classical example is the Van der Pol-equation

$$\ddot{x} + x = \mu(1 - x^2)\dot{x},\ \mu \gg 0,$$

where we know apriori that a unique periodic solution exists for any positive μ.

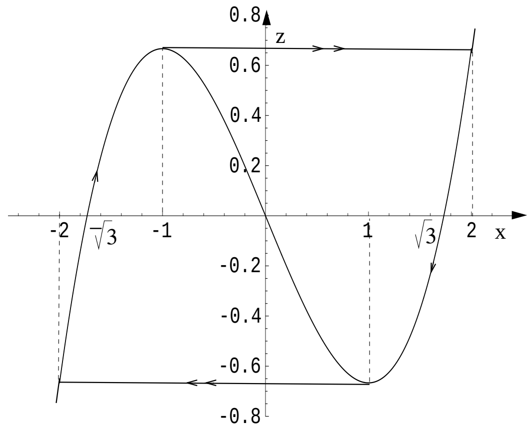

Figure 3: The relaxation oscillation of the Van der Pol-equation in the (x, z)-phaseplane; one arrow indicates motion along a stable slow manifold, double arrows a fast jump from one stable manifold to another stable one.

The phenomena are easiest understood in Liénard variables x, z (see [12]). In the (x, z)-phaseplane, fig. (3), the cubic curve corresponds with the slow manifold. For $-1 \leq x \leq 1$, the slow manifold is unstable. When the solution reaches the local extrema at $x \pm 1$ (fold points), the solution 'jumps' (moves very fast) to the stable part of the cubic curve. In the fold points, the slow manifold looses its hyperbolicity.

Example 4.2
A four-dimensional problem is discussed in [26] where the evidence is partly numerical. The paper is concerned with the system of coupled oscillators

$$\begin{aligned} \ddot{x} + x &= \mu(1 - x^2)\dot{x} + \mu c \dot{x} y^2, \ \mu \gg 0, \\ \ddot{y} + \kappa \dot{y} + q^2 y &= dxy. \end{aligned} \tag{4}$$

One finds slow manifolds in 4-space, periodic solutions and chaotic attractors.
We note that quenching of relaxation oscillators is discussed in [28].

4.2 Canards

Canard solutions play a special part. We shall use the following description.
Canard solutions are bounded solutions that, starting near an attracting normally hyperbolic slow manifold, cross a singularity (for instance a critical point or a stability change) of the system of differential equations and follow for an $O(1)$ time a normally hyperbolic repelling slow manifold.
Note that this is not a definition; depending on the dimension of the problem and the nature of the singularity, the description usually has to be more specific.
The first example of such behaviour was found by the Strassbourg group working in non-standard analysis for a perturbed Van der Pol-equation, see for instance [4]; see for details and more references [6]. In this first case, the singularity crossed is a fold point. In [19], second order slow-fast systems have been analyzed for homoclinic bifurcations; it contains a population dynamics application as discussed in section 3.2, with canard-like behaviour. Sticking of solutions to a repelling manifold is discussed in a general context in [22] where it is called 'delay in loss of stability'; this terminology follows Pontrjagin who was the first to observe the phenomenon. In [18] transitions through transcritical and pitchfork singularities are analyzed.
In general, canard analysis is highly technical. In [29] an example of transition through a transcritical singularity is discussed, the *logistic canard*, that can be calculated in an elementary way. As an example we will consider here the prey-predator system of section 3.2 with constant carrying capacity: $k(t) = k_0$.

Example 4.3
Consider system

$$\begin{aligned} \varepsilon \dot{x} &= x\left(1 - \frac{x}{k_0} - \frac{y}{\beta + x}\right), \\ \dot{y} &= y\left(\frac{x}{\beta + x} - \delta\right). \end{aligned}$$

This autonomous system has in the first quadrant two or three critical points: $(0, 0), (k_0, 0)$

and $x_0 = \beta\delta/(1-\delta)$ (requirement $0 \leq x_0 \leq k_0$) with (x_0, y_0) located on the slow manifold

$$y(x) = (\beta + x)\left(1 - \frac{x}{k_0}\right).$$

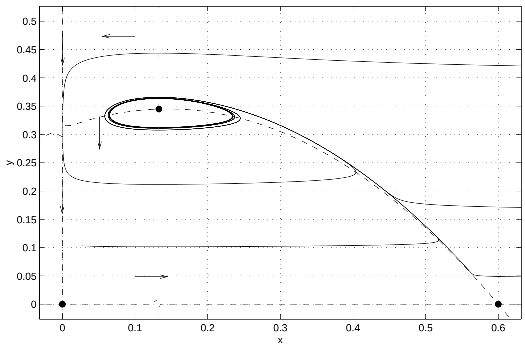

Figure 4: The prey-predator system with constant carrying capacity of example 4.3 just after the Hopf bifurcation. The three black dots correspond with three critical points. A transient starts to the right and approaches a (relatively) small periodic solution around the critical point on the slow manifold. We have $\varepsilon = 0.05, k_0 = 0.6, \beta = 0.31, \delta = 0.3$.

Suppose that the slow manifold (parabola) has a maximum in the first quadrant at $x^* = (k_0 - \beta)/2$. Linearization of the vector field at (x_0, y_0) shows that the critical point is an attractor for $x^* < x_0 \leq k_0$. At the value $x_0 = x^*$, the eigenvalues are purely imaginary and we have a Hopf bifurcation. At this point a stable periodic solution is created and if (x_0, y_0) moves slightly to the left of the maximum on the slow manifold by a small change of parameters, a small periodic solution is present. This is illustrated in fig. 4, the scenario for this problem is described in detail in [19] and in a general context in [3]. If the critical point moves further to the left, we meet an interesting combination of relaxation and canard behaviour in the following way. The periodic solution first moves along the stable part of the parabolic slow manifold. At the maximum of the parabola (fold point), the solution will jump to a neighbourhood of the y-axis (first slow manifold). Here $y(t)$ will start to decrease until, after passing the value $y = \beta$, the y-axis becomes unstable. Note again that $(0, \beta)$ is a transcritical singularity; see for the theory [18] and for a simple example [29].
As the solution has become exponentially close to the y-axis, it takes time to leave the now unstable axis (canard behaviour), but at some time the solution will jump again to the parabolic slow manifold. Then the process starts again, see fig. 5. The lift off position from the unstable slow manifold can be calculated with a certain precision; see [4] and [19].

In the nonautonomous case of the prey-predator system of section 3.2 we have no critical point on the parabolic slow manifold, but we can still identify periodic solutions, relaxation and canard behaviour. We illustrate this by an example, leaving the theoretical details for subsequent investigations.

Consider the system of section 3.2 and a choice of parameters such that the maxima of the parabolic slow manifold are in the first quadrant. The projection of the periodic solution on

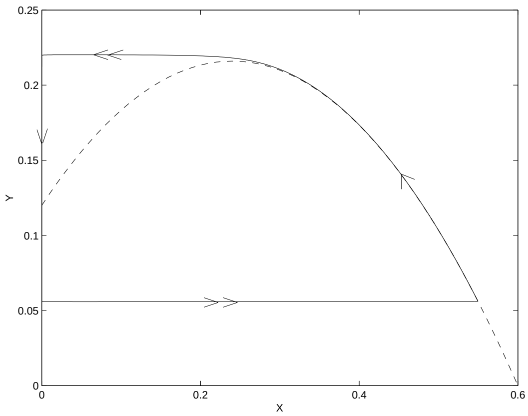

Figure 5: The prey-predator system with constant carrying capacity of example 4.3 in which the periodic solution produced by the Hopf bifurcation shows both relaxation (fast motion, indicated by double arrows) and canard behaviour. Single arrows indicate motion along a slow manifold. Near the top of the parabolic slow manifold, the solution jumps to a neighbourhood of the y-axis (also slow manifold) and moves down. At the value $y = \beta\,(= 0.12$ in this case), the y-axis becomes unstable, but delay of instability causes the solution to move for some time along the unstable y-axis (canard); then the solution jumps to the stable part of the parabolic slow manifold. We have $\varepsilon = 0.001, k_0 = 0.6, \beta = 0.12, \delta = 0.3$.

the x, y-plane will show a double loop as the periodic solution follows the time-varying slow manifold, see fig. 6. This shows again relaxation and canard behaviour.
To illustrate this, we compute the difference between $y(t)$ of the periodic solution and the corresponding slow manifold values. The nearly horizontal parts near $y = 0$ represent the motion along the time-varying slow manifold. The deviations from zero correspond with relaxation jumps and canard behaviour respectively; see fig. 7.

5 A remark on resonance manifolds

In mechanics, an important part is played by slow-fast sytems, usually called amplitude-angle or action-angle systems, of the form

$$\begin{aligned} \dot{x} &= \varepsilon X(\phi, x) + \varepsilon^2 \cdots , \\ \dot{\phi} &= \Omega(x) + \varepsilon \cdots , \end{aligned}$$

with $x \in \mathbb{R}^n, \phi \in \mathbb{T}^m$. The so-called spatial variable x is derived from a system of oscillators, ϕ indicates the corresponding angles, defined on the m-torus. Averaging over the torus (the angles) is possible outside the resonance manifolds. The latter correspond with the zeros of the righthand side of the equation for ϕ written out in all the possible combination angles.
However, it is already clear that, because of the form of the righthand side of the angle equation, such a resonance manifold will not be hyperbolic. One might expect, that localizing around such a resonance manifold might resolve this, but it is shown quite generally in [25], section 11.7, that up to first order, the equations determining the dynamics in the resonance manifold are not structurally stable, *even* if the original oscillator system is dissipative. A

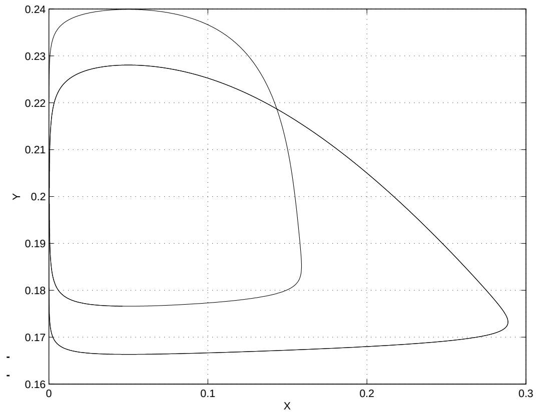

Figure 6: The prey-predator system with time-varying carrying capacity of section 3.2 in which the periodic solution shows both relaxation and canard behaviour. We have $\varepsilon = 0.01, k_0 = 0.4, a = 0.1\beta = 0.2, \delta = 0.2$.

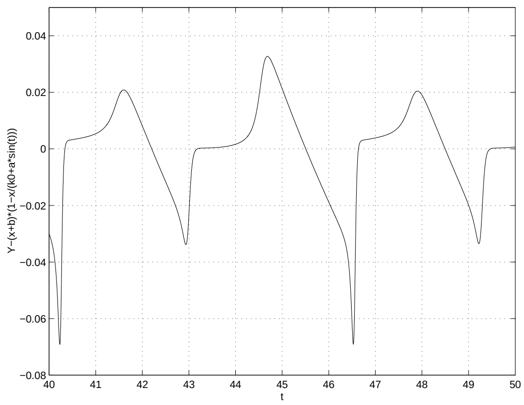

Figure 7: The difference $y(t) - (x(t) + \beta)(1 - x/k(t))$ for the periodic solution $x(t), y(t)$ of the prey-predator system with time-varying carrying capacity $k(t)$ of section 3.2 . We have $\varepsilon = 0.01, k_0 = 0.4, a = 0.1\beta = 0.2, \delta = 0.2$.

second order approximation produces hyperbolicity in a number of cases.
We conclude that slow-fast systems of this type are of a different nature and have only a superficial similarity to systems with slow manifolds of Fenichel type.

6 Discussion

Slow-fast systems, exhibiting various qualitatively different timescales, arise often in applications. In section 3 we discussed the existence of periodic solutions within slow manifolds. The idea used here, is related to the much older analysis of the dynamics in center manifolds. One localizes to a slow manifold after which one of the known approaches to obtain periodic solutions may be applied. It is rather straightforward to extend these results to existence results for tori within slow manifolds of dimension 3 or higher.
The modified logistic equation (see [29]) with alternating negative and positive growth rates

is a simple metaphor for more complex models. It is interesting that the solutions of this equation show sudden 'population explosions' related to canard behaviour.
Both slow manifolds as discussed in this paper and resonance manifolds in dynamical systems represent slow-fast dynamics. However the similarity is superficial as in the equations for resonance manifolds nonhyperbolic features are so typical that a different approach is needed.
Finally we note that some of the examples we have shown, also exhibit more complex phenomena like tori, torusbifurcations and chaos. For all these problems, a combination of asymptotic analysis and geometric theory is very effective.

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