Pergamon

Int. J. Non-Linear Mechanics. Vol. 29, No. 2, pp. 225–232, 1994

0020-7462/94 $6.00 + 0.00

AUTOPARAMETRIC RESONANCE BY COUPLING OF LINEAR AND NON-LINEAR SYSTEMS

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(Received 6 January 1993)

Abstract—A pendulum is attached to one mass of a chain of n masses, connected by n linear springs. One of the masses is harmonically excited. The stability of the semi-trivial solutions, representing vibration of n masses without pendulum oscillation, is investigated in general. Using this approach, the occurrence of all autoparametric resonances can be determined. As an illustration, a two-mass subsystem with two degrees of freedom, where the pendulum is attached to the upper mass, is analysed.

1. INTRODUCTION

In this paper a rather general formulation is given of the problem of interaction between a chain of linear springs—with damping and forcing—and a non-linear system. This is a basic and important problem in mechanics (see [1–5]). It poses a formidable mathematical problem involving many phenomena, like bifurcations of various types and chaos; in this paper we shall be concerned only with the first steps, i.e. formulation of the problem, resonances and stability and an example.

Autoparametric systems represent an interesting group of excited systems with specific characteristic properties. Such a system is composed of two subsystems with a non-linear coupling and a periodic excitation which acts on one of the subsystems only; this is called the excited linear subsystem. In this paper we assume that the subsystem consists of n linear oscillators. The coupled subsystem is not affected by this, i.e. it does not oscillate, except for certain intervals of the excitation frequency. The solutions of the corresponding differential equations of motion with the non-excited subsystem at rest and the excited linear subsystem oscillating, will be called semi-trivial solutions.

Such a solution is stable, with the exception of the mentioned intervals of the excitation frequency. In these intervals the semi-trivial solution is destabilised by the oscillatory solution of the excited linear subsystem because the vibration of this subsystem produces parametric excitation of the non-excited subsystem. Autoparametric resonances occur in these intervals of the excitation frequency. This resonance is characterised by vibration of the whole system and by the differences in the character of vibrations of both subsystems: the dominant vibration component of the excited linear subsystem, for instance, has a frequency which is twice the frequency of the dominant component of the non-excited subsystem.

The aim of this contribution is to show that investigation of the semi-trivial solutions can disclose *all* intervals where autoparametric resonance arises and it forms the first step of the analysis of these systems. Leaving out this step of investigation some intervals of the autoparametric resonance occurrence could be missed. This can be demonstrated by a simple example of an autoparametric system which has been studied by several authors (see, e.g. [1–3]). This system consists of a mass, m_0, on a spring having stiffness k.

Contributed by P. B. Hagedorn.

A pendulum characterised by a mass m and length l is attached to this mass (see Fig. 1). Two alternatives are shown in Fig. 1: (a) the alternative with harmonic or periodic excitation of mass m_0, (b) the alternative with kinematic excitation. Most authors have dealt with alternative (a) with harmonic excitation and they have analysed directly the autoparametric resonance of the system when tuned into internal resonance, i.e. for the case when $\sqrt{k/(m+m_0)}/\sqrt{g/l} = 2$ and the excitation frequency ω is close to $\sqrt{k/(m+m_0)}$. In refs [1–3] the semi-trivial solution has not been investigated completely. In [4] the stability of the semi-trivial solution is investigated for the case of harmonic excitation of one mass m_0 with a constant amplitude as well as the case where the amplitude is proportional to the square of the excitation frequency; the case is studied when the spring, carrying mass m_0, is non-linear, the linear and cubic terms defining the spring characteristic. The stability investigation of the semi-trivial solution for the alternative (b) with one mass is investigated in [5]. In both these papers [4, 5], it is proved that the case of a system tuned into internal resonance is not the only possibility for autoparametric resonances to occur. In the present paper a more complicated system is analysed having an excited linear subsystem with arbitrarily many degrees of freedom.

2. EQUATIONS OF MOTION

We consider an n-degrees-of-freedom linear system, coupled with a one-degree-of-freedom non-linear system. The linear system consists of n masses $m_1, \ldots, m_n$ connected in a chain by linear springs having stiffness k_i, $i = 1, \ldots, n$—the excited subsystem— and a pendulum of mass m and length l which is attached to mass m_j (in Fig. 2 m_j is m_1). One of the masses $m_1, \ldots, m_n$ is excited harmonically, the amplitude of excitation is either a constant or it is proportional to the square of frequency of excitation ω.

Let y_i denote the deflection of the mass m_i, ϕ the angular deflection of the pendulum and b and b_i the linear viscous damping coefficients of the damping forces acting on the pendulum on the pair of masses m_i, m_{i+1}, respectively.

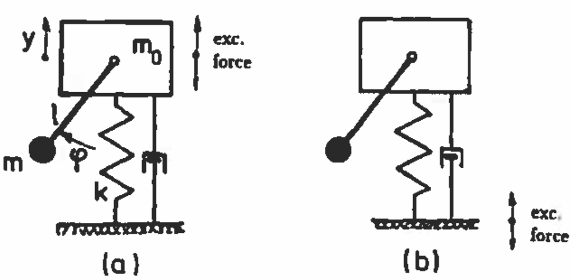

Fig. 1. System of one mass with a pendulum.

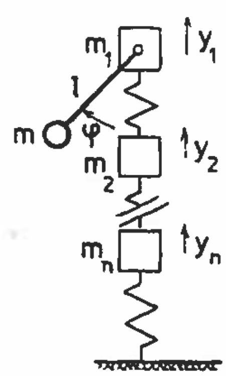

Fig. 2. System of n masses with a pendulum.

The kinetic energy T and potential energy V are defined by relations

$$T = \frac{m}{2}(\dot{y}_j + l\dot{\phi}\sin\phi)^2 + \frac{m}{2}(l\dot{\phi}\cos\phi)^2 + \sum_{i=1}^{n}\frac{1}{2}m_i\dot{y}_i^2,$$

$$V = mgl(1-\cos\phi) + \sum_{i=1}^{n}\frac{1}{2}k_i(y_i - y_{i+1})^2 + \frac{1}{2}k_n y_n^2.$$

Using the Lagrange equations of the second kind, with the Lagrange function $L = T - V$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0 \quad (i = 1, 2, \ldots, n)$$

and adding the damping and forcing terms, the following system of equations of motion is obtained:

$$\begin{aligned}
&ml^2\ddot{\phi} + b\dot{\phi} + mgl\sin\phi + ml\ddot{y}_j\sin\phi = 0,\\
&(m+m_j)\ddot{y}_j + b_j(\dot{y}_j - \dot{y}_{j+1}) - b_{j-1}(\dot{y}_{j-1} - \dot{y}_j) + k_j(y_j - y_{j+1}) - k_{j-1}(y_{j-1} - y_j)\\
&\qquad + ml(\ddot{\phi}\sin\phi + \dot{\phi}\cos\phi) = 0,\\
&m_1\ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_2) + k_1(y_1 - y_2) = 0,\\
&\quad\vdots\\
&m_k\ddot{y}_k - b_{k-1}(\dot{y}_{k-1} - \dot{y}_k) + b_k(\dot{y}_k - \dot{y}_{k+1}) - k_{k-1}(y_{k-1} - y_k)\\
&\qquad + k_k(y_k - y_{k+1}) = \begin{cases} a\cos\omega t\\ m_k e\omega^2\cos\omega t,\end{cases}\\
&m_n\ddot{y}_n - b_{n-1}(\dot{y}_{n-1} - \dot{y}_n) + b_n\dot{y}_n - k_{n-1}(y_{n-1} - y_n) + k_n y_n = 0.
\end{aligned} \tag{1}$$

In system (1), we envisage two natural types of harmonic excitation: the cases with constant amplitude and with the amplitude proportional to the square of the excitation frequency. With respect to further analysis, it is useful to transform these equations into dimensionless form. Putting

$$\begin{aligned}
&\omega_0 = \sqrt{g/l}, \qquad \omega/\omega_0 = \eta, \qquad \kappa = b/\omega_0 ml^2, \qquad \kappa_j = b_j/\omega_0(m+m_j),\\
&\kappa_i = b_i/\omega_0 m_i, \qquad q_i^2 = k_i/\omega_0^2 m_i, \qquad q_j^2 = k_j/\omega_0^2(m+m_j),\\
&\mu_i = m_{i-1}/m_i, \qquad \mu_j = m_{j-1}/(m+m_j), \qquad \mu_{j+1} = (m+m_j)/m_{j+1},\\
&\varepsilon = \begin{cases}\omega/m_k g\\ e/l\end{cases}, \qquad \text{resp.}\ \begin{cases}\omega/(m+m_j)g\\ m_j e/(m+m_j)l\end{cases}, \qquad w_i = y_i/l,
\end{aligned} \tag{2}$$

system (1) is transformed (using $\omega_0 t = \tau$) to the form:

$$\begin{aligned}
&\phi'' + \kappa\phi' + \sin\phi + w_j''\sin\phi = 0,\\
&w_j'' + \kappa_j(w_j' - w_{j+1}') - \mu_j\kappa_{j-1}(w_{j-1}' - w_j') + q_j^2(w_j - w_{j+1}) - \mu_j q_{j-1}^2(w_{j-1} - w_j)\\
&\qquad + \frac{m}{m+m_j}(\phi''\sin\phi + \phi'^2\cos\phi) = 0,\\
&w_1'' + \kappa_1(w_1' - w_2') + q_1^2(w_1 - w_2) = 0,\\
&w_k'' - \mu_k\kappa_{k-1}(w_{k-1}' - w_k') + \kappa_k(w_k' - w_{k+1}') - \mu_k q_{k-1}^2(w_{k-1} - w_k)\\
&\qquad + q_k^2(w_k - w_{k+1}) = \begin{cases}\varepsilon\cos\eta\tau\\ \varepsilon\eta^2\cos\eta\tau\end{cases},\\
&w_n'' - \mu_n\kappa_{n-1}(w_{n-1}' - w_n') + \kappa_n w_n' - \mu_n q_{n-1}^2(w_{n-1} - w_n) + q_n^2 w_n = 0.
\end{aligned} \tag{3}$$

3. SEMI-TRIVIAL SOLUTION OF THE SYSTEM AND ITS STABILITY

A semi-trivial solution $(\phi, \omega_1, \ldots, \omega_n) = (0, u_1, \ldots, u_n)$ of system (3) is defined by the condition

$$\phi(t) = 0, \quad t \geq 0. \tag{4}$$

Note that for a semi-trivial solution, system (3) decouples into a linear system of the (matrix) form

$$\mathbf{u}'' + \boldsymbol{\kappa}\mathbf{u}' + \mathbf{q}\mathbf{u} = \begin{cases} \boldsymbol{\varepsilon} \cos \eta\tau \\ \boldsymbol{\varepsilon}\eta^2 \cos \eta\tau \end{cases}, \tag{5}$$

with vectors

$$\mathbf{u} = (u_1, \dots, u_n),\ \boldsymbol{\varepsilon}^T = (0, \dots, 0, 1, 0, \dots, 0) \cdot \begin{cases} \varepsilon \\ \varepsilon\eta^2 \end{cases}$$

and matrices

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_1 & -\kappa_1 & 0 & \cdots & 0 \\ -\mu_2\kappa_1 & \mu_2\kappa_1 + \kappa_2 & -\kappa_2 & & \\ 0 & -\mu_3\kappa_2 & & & \\ \vdots & & & & -\kappa_{n-1} \\ 0 & & & -\mu_n\kappa_{n-1} & \mu_n\kappa_{n-1} + \kappa_n \end{bmatrix},$$

$$\mathbf{q} = \begin{bmatrix} q_1^2 & -q_1^2 & \cdots & 0 \\ -\mu_2 q_1^2 & \mu_2 q_1^2 + q_2^2 & & \\ \vdots & & & \\ 0 & & -\mu_n q_{n-1}^2 & \mu_n q_{n-1}^2 + q_n^2 \end{bmatrix}. \tag{6}$$

Because of the forcing ($\varepsilon \neq 0$), the deflections of the linear springs $\mathbf{u}$ will never be collectively zero for $t > 0$. The solution of equation (5) consists of a homogeneous part which vanishes with time and an inhomogeneous part of the form

$$\mathbf{A} \cos \eta\tau + \mathbf{B} \sin \eta\tau.$$

Substitution of this expression into equation (5) yields for the vectors $\mathbf{A}$ and $\mathbf{B}$ the $2n$-dimensional algebraic system

$$\begin{bmatrix} \mathbf{q} - \eta^2\mathbf{I} & \eta\boldsymbol{\kappa} \\ -\eta\boldsymbol{\kappa} & \mathbf{q} - \eta^2\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \mathbf{0} \end{bmatrix}. \tag{7}$$

To investigate the asymptotic stability of the semi-trivial solutions we use the well-known results of Poincaré and Lyapunov (see [6, Chap. 7]). We introduce

$$\phi = 0 + \psi, \qquad \mathbf{w} = \mathbf{u} + \mathbf{v},$$

$$A_j = \varepsilon A, \qquad B_j = \varepsilon B. \tag{8}$$

From system (3), we obtain in linear approximation

$$\psi'' + \kappa\psi' + [1 + \varepsilon(A \cos \eta\tau + B \sin \eta\tau)]\psi = 0. \tag{9}$$

The equation for $\mathbf{v}$ is of the form

$$\mathbf{v}'' + \mathbf{k}\mathbf{v}' + \mathbf{q}\mathbf{v} = \mathbf{f}(\cdots),$$

where $\mathbf{f}(\cdots)$, after elimination of ψ'', is non-linear, starting with quadratic terms. It follows from the Poincaré–Lyapunov theory that the asymptotic stability of the $2(n + 1)$-dimensional semi-trivial solution is completely determined by the asymptotic stability of $\psi = 0$ in equation (9). This is a solved problem (see for instance [1] or [6]).

The solution on the stability boundary can be approximated by (see [1, 2, 6])

$$\psi = C \cos \tfrac{1}{2}\eta\tau + S \sin \tfrac{1}{2}\eta\tau. \tag{10}$$

For the non-trivial solution of C, S, the following condition must be met:

$$(1 - \tfrac{1}{4}\eta^2)^2 + \tfrac{1}{4}\kappa^2\eta^2 - \tfrac{1}{4}\varepsilon^2\eta^4(A^2 + B^2) = 0. \tag{11}$$

The values of ε on the stability boundary are therefore determined by the relation

$$\varepsilon = \frac{2}{\eta^2}\sqrt{\frac{(1-\frac{1}{4}\eta^2)^2 + \frac{1}{4}\kappa^2\eta^2}{A^2+B^2}} = \varepsilon(\eta). \tag{12}$$

It follows from the analysis of relation (12) that minimum values of ε can be expected at $\eta = 2$ and for those values of η, where $A^2 + B^2$ reaches a maximum, i.e. close to the values of η corresponding to the relative natural frequencies of the excited subsystem and the pendulum.

Note that a similar approach can be used when several masses of the excited subsystem are harmonically excited. Only vector ε will be different from that in expression (6).

A second remark is that in system (3) the explicit form of the non-linear oscillator is not important for the determination of the asymptotic stability of the semi-trivial solution. For instance, $\sin\phi$ in system (3) may be replaced by $f(\phi)$ as long as it can be expanded, starting with a linear term: $f(\phi) = \alpha\phi + \cdots$ ($\alpha \neq 0$). The explicit form of the non-linearity is important if one wants to study the behaviour of the solutions in the case of an unstable semi-trivial solution.

4. EXAMPLES

In [5] the case is studied where $n = 1$, i.e. a system in which one linear spring is coupled to a pendulum. This case models the phenomenon of a ship rolling in longitudinal or oblique sea waves. The instability threshold in (ε, η)-parameter space turns out to have two minima in ε; for values of η close to 2 or q, the possibility of unstable parametric excitation of the semi-trivial solution is maximal.

Here, we consider the case $n = 2$, i.e. we have a subsystem of two linear springs and masses coupled to a pendulum (see Fig. 3). Two alternatives are considered: in case (a) the upper mass m_1 is excited; in case (b) the exciting force is applied to the lower mass m_2. The exciting force is also considered in two alternatives: with a constant amplitude of the exciting harmonic force or with an amplitude proportional to the square of the exciting frequency. This system is governed by the following differential equations, transformed into the dimensionless form:

Alternative (a)

$$\phi'' + \kappa\phi' + \sin\phi + w_1''\sin\phi = 0,$$

$$w_1'' + \kappa_1(w_1' - w_2') + q_1^2(w_1 - w_2) + \frac{m}{m+m_1}(\phi''\sin\phi + \phi'^2\cos\phi) = \begin{cases}\varepsilon\cos\eta\tau,\\ \varepsilon\eta^2\cos\eta\tau,\end{cases}$$

$$w_2'' - \frac{m+m_1}{m_2}[\kappa_1(w_1' - w_2') + q_1^2(w_1 - w_2)] + \kappa_2 w_2' + q_2^2 w_2 = 0. \tag{13}$$

Alternative (b)

$$\phi'' + \kappa\phi' + \sin\phi + w_1''\sin\phi = 0,$$

$$w_1'' + \kappa_1(w_1' - w_2') + q_1^2(w_1 - w_2) + \frac{m}{m+m_1}(\phi''\sin\phi + \phi'^2\cos\phi) = 0,$$

$$w_2'' - \frac{m+m_1}{m_2}[\kappa_1(w_1' - w_2') + q_1^2(w_1 - w_2)] + \kappa_2 w_2' + q_2^2 w_2 = \begin{cases}\varepsilon\cos\eta\tau,\\ \varepsilon\eta^2\cos\eta\tau,\end{cases} \tag{14}$$

where

$$q_1^2 = \frac{k_1}{(m+m_1)\omega_0^2}, \quad q_2^2 = \frac{k_2}{m_2\omega_0^2}, \quad \kappa = \frac{b}{m\omega_0 l}, \quad \omega_0 = \sqrt{\frac{g}{l}},$$

$$\kappa_1 = \frac{b_1}{(m+m_1)\omega_0}, \quad \kappa_2 = \frac{b_2}{m_2\omega_0}, \quad \text{(a)}\ \varepsilon = \begin{cases}\dfrac{a}{(m+m_1)g},\\ \dfrac{e}{l}\dfrac{m_1}{m+m_1}\end{cases} \quad \text{(b)}\ \varepsilon = \begin{cases}\dfrac{a}{m_2 g}\\ \dfrac{e}{l}\end{cases}.$$

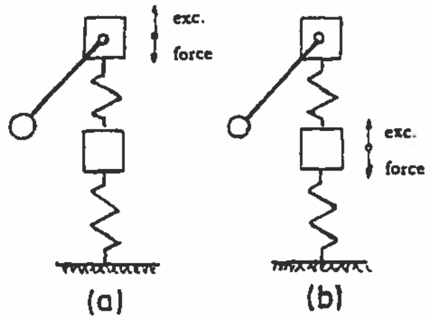

Fig. 3. Systems, consisting of a two degrees of freedom linear subsystem attached to a pendulum, analysed in the examples of Section 4.

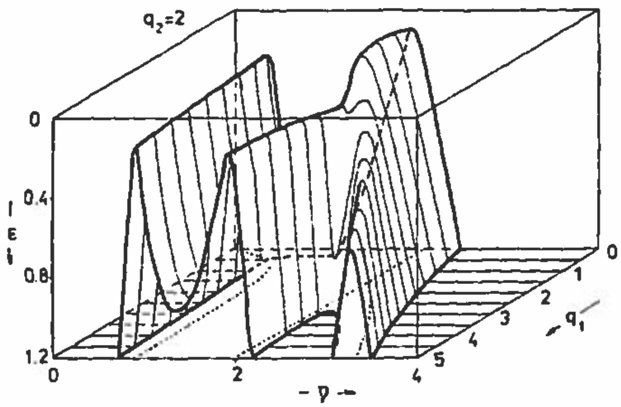

Fig. 4. Stability threshold surface $\varepsilon(\eta, q_1)$ of the semi-trivial solution; alternative (a) excitation with constant amplitude, $q_2 = 2$.

Note that, comparing with equations (1) and (3), the excitation $m_1 e\omega^2 \cos \omega t$—alternative (a)—or $m_2 e\omega^2 \cos \omega t$—alternative (b)—is considered for the case where the amplitude is proportional to the square of the excitation frequency.

The results of the stability investigation of the semi-trivial solutions are presented in the form of stability threshold surfaces where ε is expressed as a function of the relative exciting frequency η and of the tuning coefficient q_1 for several values of the tuning coefficient q_2. In these diagrams, for convenience of presentation, the positive direction of the ε-axis is changed by orienting it downwards. In this way, the stability region is located under the surface and the minima appear as maxima. In the stability threshold surfaces (Figs 4–7) the two relative natural frequencies (relative to the pendulum natural frequency) of the excited subsystem without damping, are indicated by dotted lines. The periphery of the instability threshold surfaces is marked by broken lines. In all examples, the following parameter values were used:

$$\kappa = 0.05, \qquad \kappa_1 = \kappa_2 = 0.1, \qquad m + m_1 = m_2.$$

The results for alternatives (a) and (b) with the two types of excitation are presented in Figs 4–7 ($q_2 = 2$).

We can see that, for a given value of q_1, the curve representing a section of the threshold surface can have three local minima of ε which appear as maxima when viewing the diagrams: at $\eta \approx 2$ and at the values of η corresponding to the relative natural frequencies of the excited subsystem. The absolute minimum of ε lies in most cases at the values of η, q_1 when the relative natural frequency equals 2 (see, e.g. Fig. 4 at $q_1 \approx 1.5$, Fig. 5 at $q_1 \approx 2$). Generally, alternative (a) and the excitation having the amplitude proportional to the square of the excitation frequency manifest themselves to be less resistant to the instability of the semi-trivial solution. These results are in full agreement with general experience. For

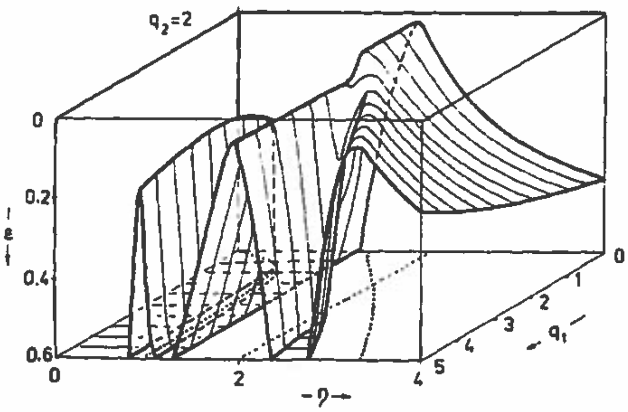

Fig. 5. Stability threshold surface $\varepsilon(\eta, q_1)$ of the semi-trivial solution; alternative (a), excitation with the amplitude proportional to the square of the excitation frequency, $q_2 = 2$.

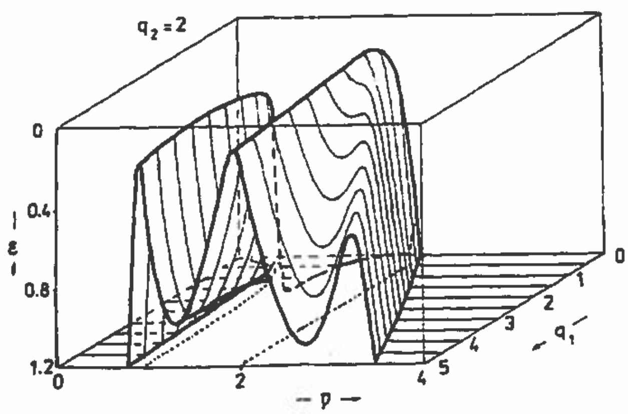

Fig. 6. Stability threshold surface $\varepsilon(\eta, q_1)$ of the semi-trivial solution; alternative (b), excitation with constant amplitude, $q_2 = 2$.

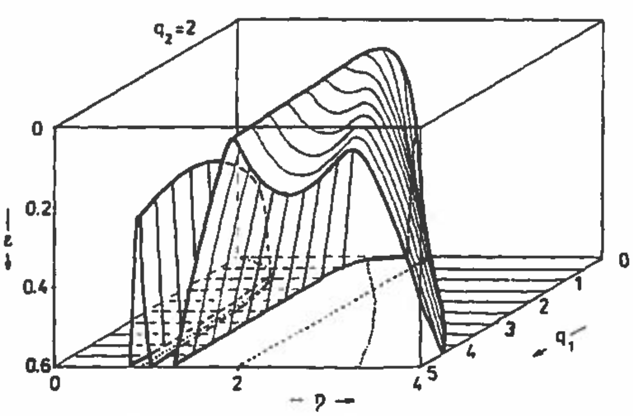

Fig. 7. Stability threshold surface $\varepsilon(\eta, q_1)$ of the semi-trivial solution; alternative (b), excitation with the amplitude proportional to the square of the excitation frequency, $q_2 = 2$.

alternative (a), the limit case $q_1 = 0$ represents a pendulum whose hinge point is harmonically excited in the vertical direction. Of course, only one instability interval of η can exist in this limit case.

When the semi-trivial solution is unstable an autoparametric resonance occurs which is characterised by the oscillation of the whole system. For certain values of the tuning coefficients q_1, q_2 even three autoparametric resonances can occur.

It is clear that for resonance instability to arise, the system need not be restricted to tuning into internal resonance frequencies as was conjectured by various investigators.

5. CONCLUSIONS

Studying autoparametric systems consisting of linear and non-linear subsystems, it is convenient, as a first step, to investigate the stability of the semi-trivial solutions in which the non-linear system is at rest. In this case all possible autoparametric resonances, which initiate instability, can be determined. The analysis here has demonstrated that, apart from tuning into internal resonance, there are more possibilities to generate autoparametric excitation. In particular, when attaching a pendulum subsystem to a certain mass of the excited linear subsystem consisting of n masses with n degrees of freedom altogether the initiation of autoparametric resonances can be expected at the following values of the excitation frequency: at double frequency of the pendulum subsystem's natural frequency and at frequencies close to any natural frequency of the excited linear subsystem. This means that for certain open parameter sets of the system, when changing the excitation frequency, there exists $n + 1$ intervals where autoparametric resonance can be initiated.

A possible application of these results might be the quenching of oscillators in linear systems by coupling them to a non-linear subsystem. To realise this we have to investigate the behaviour of the excited non-linear system. This will be the subject of future investigations.

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