# A parametrically excited nonlinear wave equation

Ferdinand Verhulst and Johan M. Tuwankotta

**Abstract** When considering nonlinear waves with periodic parametric forcing the geometry of the spatial domain plays a crucial part. If the spatial domain is a square we find an infinite number of $1:1$ resonances and in addition accidental resonances. Using Galerkin projection on 2 modes in $1:1$ resonance we find stable normal mode periodic solutions and unstable periodic solutions in general position; the location in phase-space is characterised as a triple resonance zone. In the limit case of vanishing dissipation we find neutral stability and strong recurrence of the orbits. Interaction of $1:1$ resonances shows a selection mechanism of the $1:1$ modes triggered off by the parametric forcing. In addition we analyse a number of prominent accidental resonances produced by the spectrum induced by our choice of a square in space.

## 1 Introduction

Consider the parametrically excited nonlinear wave equation formulated by Rand et al. [4] in the one-dimensional case; we will consider the equation on a square as two space dimensions often introduces new phenomena, in particular resonances.

$$u_{tt} - c^2(u_{xx} + u_{yy}) + \mu u_t + ({\omega_0}^2 + \beta\cos(\Omega t))u = \alpha u^3, \tag{1}$$

where $t \geq 0$ and $0 < x < \pi, 0 < y < \pi$. The boundary values are $\partial u/\partial n|_S = 0$.
The parameters $\mu, \beta$ are positive and small in a way to be specified.
The system of equations and conditions model the surface deflections $u(x, y, t)$ of a fluid in a square basin with parametric excitation and damping, $c$ is the wave speed.

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Resonant nonlinear waves in 2 spatial dimensions were also considered in [3] and [6]. We associate with the system the eigenfunctions:

$$v_{mn}(x, y) = \cos mx \cos ny, m, n = 0, 1, 2 \dots$$

with eigenvalues of the space-dependent operator:

$${\omega_{mn}}^2 = {\omega_0}^2 + (m^2 + n^2)c^2, \omega_{mn} = \omega_{nm} = \omega.$$

An early paper by W.T. van Horssen on the asymptotic approximation of solutions of nonlinear wave equations is [2]. The solutions of eq. (1) with boundary conditions can be approximated by projection of a finite sum of eigenfunctions (Galerkin projection) followed by averaging approximation. The process results in asymptotic approximations in the mathematical sense. The procedure is summarised with references in [6] section 1, we do not repeat this here.

The choice of eigenfunctions is determined by the initial values of eq. (1) while keeping an eye on the resonances of the eigenvalues. It turns out that for the geometry considered here, there are an infinite number of $1 : 1$ resonances. This will require our main attention. In addition we will briefly look at prominent accidental resonances.

## 2 The two-mode 1 : 1 resonance

We propose a two-mode expansion with:

$$u_p(x, y, t) = u_1(t) \cos mx \cos ny + u_2(t) \cos nx \cos my, \tag{2}$$

$m, n = 0, 1, 2 \dots, m \neq n$. Put $\omega_0 = 1$ and rescale $u = \sqrt{\varepsilon}\bar{u}$ (and its derivatives likewise) in eq.(1) with $\varepsilon$ a small positive parameter; we omit the bars. Substituting expansion (2) into eq. (1) and taking inner products with the eigenfunctions we find with $\omega_{mn} = \omega, m \neq n$:

$$\begin{cases} \ddot{u}_1 + \omega^2 u_1 &= -\mu\dot{u}_1 - \beta u_1 \cos(\Omega t) + \varepsilon\alpha(\frac{9}{16}{u_1}^3 + \frac{3}{4}u_1{u_2}^2), \\ \ddot{u}_2 + \omega^2 u_2 &= -\mu\dot{u}_2 - \beta u_2 \cos(\Omega t) + \varepsilon\alpha(\frac{9}{16}{u_2}^3 + \frac{3}{4}{u_1}^2 u_2). \end{cases} \tag{3}$$

We choose $\Omega = 2\omega$ to study prominent Floquet resonances; rescale $\mu = \varepsilon\bar{\mu}, \beta = \varepsilon\bar{\beta}$ after which we omit the bars. System (3) contains the $1 : 2$ Floquet resonance and in addition the $1 : 1$ resonance of the Hamiltonian interaction force .
Note that because of the symmetry of system (3) $u_1(t) = \pm u_2(t)$ satisfies the system. The coordinate planes $u_1, \dot{u}_1$ and $u_2, \dot{u}_2$ are invariant under the phase-flow, we start with the analysis of these normal mode planes.

## 2.1 The invariant normal mode planes

The analysis for both coordinate planes runs exactly along the same lines with symmetric results so we consider only the $u_1, \dot{u}_1$ plane. We put $u_2 = \dot{u}_2 = 0$ and introduce amplitude-phase coordinates by:

$$u_1 = r_1 \cos(\omega t + \psi_1), \dot{u}_1 = -r_1 \omega \sin(\omega t + \psi_1).$$

Deriving the equations for $r_1, \psi_1$ and averaging over time we find the first order averaged system:

$$\dot{r}_1 = \frac{\varepsilon}{2} r_1 (-\mu + \frac{\beta}{2\omega} \sin 2\psi_1),\ \dot{\psi}_1 = \frac{\varepsilon}{4\omega}(\beta \cos 2\psi_1 - \alpha \frac{27}{32} {r_1}^2). \tag{4}$$

Here and in the sequel, the solutions of first order averaged equations with appropriate initial values approximate the solutions of the original system with error $O(\varepsilon)$ on a long interval of time of order $1/\varepsilon$. A critical point corresponding with an equilibrium of system (4) is given by:

$$\beta \sin 2\psi_1 = 2\mu\omega,\ \beta \cos 2\psi_1 = \alpha \frac{27}{32} {r_1}^2,\ 0 < \frac{2\mu\omega}{\beta} < 1.$$

A critical point of the averaged equations corresponds under certain conditions with a periodic solution of the original equations; see theorems 11.5–11.6 in [5] (this is sometimes called the 2nd Bogoliubov theorem). We can eliminate the phase angle to find:

$${r_1}^2 = {r_0}^2 = \frac{32}{27\alpha} \sqrt{\beta^2 - 4\mu^2\omega^2}.$$

Computing eigenvalues at the critical point shows that the periodic solution is stable within the invariant coordinate plane. For the eigenvalues we have:

$$\lambda_{1,2} = -\mu \pm \sqrt{5\mu^2 - \frac{\beta^2}{\omega^2}}. \tag{5}$$

If $\beta > \sqrt{5}\mu\omega$ the periodic solution is complex stable in the coordinate plane, if $2\mu\omega < \beta < \sqrt{5}\mu\omega$ the periodic solution is stable with real eigenvalues. If $\beta = 2\mu\omega$ the periodic solution vanishes.

An important question is whether the periodic solution is stable or unstable in the full 4-dimensional system. For $u_2, \dot{u}_2$ near zero we should not use polar coordinates. Instead we introduce in system (3) the variables $a, b$ by:

$$u_2 = a \cos \omega t + \frac{b}{\omega} \sin \omega t,\ \dot{u}_2 = -a\omega \sin \omega t + b \cos \omega t.$$

Introducing amplitude-phase variables for $u_1$ and $a, b$ variables for $u_2$ in system (3) we have to average the system. To determine the stability of the normal mode periodic

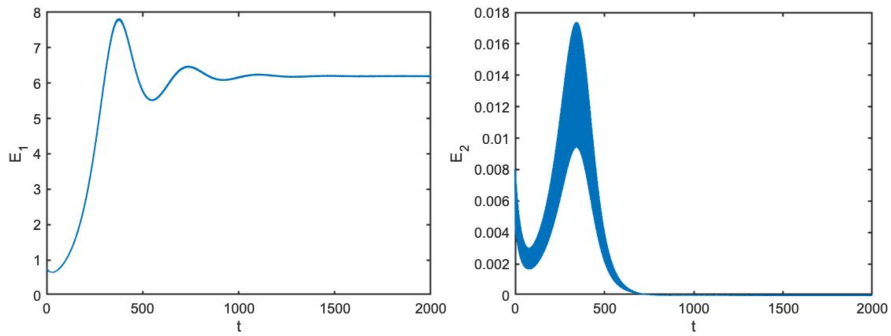


**Fig. 1** The behaviour of the solutions of system (3) near the invariant $u_1, \dot{u}_1$ coordinate plane is shown by plotting $E_1(t) = 0.5(\dot{u}_1^2(t)+6u_1^2(t))$ and $E_2(t) = 0.5(\dot{u}_2^2(t)+6u_2^2(t))$ for the parametrically excited oscillators. The initial conditions are $u_1(0) = 0.5, \dot{u}_1(0) = 0, u_2(0) = \dot{u}_2(0) = 0.05$; $\omega^2 = 6, \mu = 0.01, \beta = 0.1, \alpha = 0.05$.

solution we compute the Jacobian of the averaged system for $r_1, \psi_1, a, b$ and find the eigenvalues of the gradient of the Jacobian at the periodic $u_1(t)$ for $a = b = 0$. This means that we can leave out the quadratic and cubic expressions in $a, b$. For the averaged system in the variables $r_1, \psi_1, a, b$ we find:

$$\begin{cases} \dot{r}_1 &= \dfrac{\varepsilon}{2} r_1 \left(-\mu + \dfrac{\beta}{2\omega}\sin 2\psi_1\right) + \dots, \\ \dot{\psi}_1 &= \dfrac{\varepsilon}{2}\left(\dfrac{\beta}{2\omega}\cos 2\psi_1 - \dfrac{27\alpha}{64\omega} {r_1}^2\right) + \dots, \\ \dot{a} &= \dfrac{\varepsilon}{2}\left(-\mu a + \dfrac{\beta}{2\omega^2} b + \dfrac{3\alpha}{16\omega}{r_1}^2\left((\sin 2\psi_1)\, a + \left(\dfrac{2-\cos 2\psi_1}{\omega}\right) b\right)\right) + \dots, \\ \dot{b} &= \dfrac{\varepsilon}{2}\left(-\mu b - \dfrac{\beta}{2} a + \dfrac{3\alpha}{16}{r_1}^2\left((2+\cos 2\psi_1)\, a - \dfrac{\sin 2\psi_1}{\omega} b\right)\right) + \dots. \end{cases}$$

where the dots stand for the omitted higher order terms in $a, b$.The gradient of the Jacobian at the periodic solution in the coordinate plane becomes when omitting the factor $\varepsilon/2$:

$$\begin{pmatrix} 0 & \dfrac{r_0\beta\cos 2\psi_1}{\omega} & 0 & 0 \\ -\dfrac{27\alpha r_0}{32\omega} & -2\mu & 0 & 0 \\ 0 & 0 & \dfrac{-16\mu\omega + 3\alpha {r_0}^2\sin 2\psi_1}{16\omega} & \dfrac{8\beta + 3\alpha {r_0}^2(2-\cos 2\psi_1)}{16\omega^2} \\ 0 & 0 & \dfrac{-8\beta + 3\alpha {r_0}^2(2+\cos 2\psi_1)}{16} & \dfrac{-16\mu\omega - 3\alpha {r_0}^2\sin 2\psi_1}{16\omega} \end{pmatrix}$$

The four eigenvalues are splitting up in two groups; the first group corresponds with the eigenvalues of eq. (5), the second group produces the eigenvalues $\lambda_{3,4}$ with $\lambda_3 + \lambda_4 = -2\mu$. We find:

$$\lambda_{3,4} = -\mu \pm \sqrt{\frac{13}{108}\frac{\beta^2}{\omega^2} - \frac{88}{81}\mu^2 - \frac{128}{81}\frac{\omega^2\mu^4}{\beta^2}}.$$

$\lambda_{3,4}$ depends on the parameters $\mu, \beta, \omega, \alpha$. We conclude that the 2 periodic normal mode solutions of the 1 : 1 resonances are asymptotically stable if

$$\sqrt[4]{\frac{39}{2}}\,\beta \leq \omega\mu.$$

See fig. 1.

## 2.2 First order averaging for the orbits in general position

Introducing amplitude-phase coordinates by:

$$u = r\cos(\omega t + \psi), \dot{u} = -r\omega\sin(\omega t + \psi),$$

we find by first order averaging:

$$\begin{cases} \dot{r}_1 &= \dfrac{\varepsilon}{2}\left(-\mu r_1 + \dfrac{\beta}{2\omega}r_1\sin 2\psi_1 - \dfrac{3\alpha}{16\omega}r_1{r_2}^2\sin 2(\psi_1 - \psi_2)\right), \\ \dot{\psi}_1 &= \dfrac{\varepsilon}{8\omega}\left(\beta\cos 2\psi_1 - \dfrac{27\alpha}{16}{r_1}^2 - \dfrac{3\alpha}{2}{r_2}^2 - \dfrac{3\alpha}{4}{r_2}^2\cos 2(\psi_1 - \psi_2)\right), \\ \dot{r}_2 &= \dfrac{\varepsilon}{2}\left(-\mu r_2 + \dfrac{\beta}{2\omega}r_2\sin 2\psi_2 + \dfrac{3\alpha}{16\omega}{r_1}^2 r_2\sin 2(\psi_1 - \psi_2)\right), \\ \dot{\psi}_2 &= \dfrac{\varepsilon}{8\omega}\left(\beta\cos 2\psi_2 - \dfrac{27\alpha}{16}{r_2}^2 - \dfrac{3\alpha}{2}{r_1}^2 - \dfrac{3\alpha}{4}{r_1}^2\cos 2(\psi_1 - \psi_2)\right). \end{cases} \tag{6}$$

The solutions of system (6) approximate the exact solutions with given initial values to $O(\varepsilon)$ on the timescale $1/\varepsilon$; with some abuse of notation we kept the notation $r, \psi$ for the approximating system.
It is important to note that the damping term (coefficient $\mu$) is not scaled by the frequency $\omega$, but on the other hand the parametric excitation (coefficient $\beta$) and the nonlinear interaction (coefficient $\alpha$) are reduced considerably for high frequency modes ($\omega$ large). If $\omega$ is $O(1/\varepsilon)$, system (6) is dominated by the damping terms.
Assuming that $\omega$ is $O(1)$ with respect to $\varepsilon$ we have for the resonant combination angle $\chi = \psi_1 - \psi_2$:

$$\dot{\chi} = \frac{\varepsilon}{8\omega}\left(\beta(\cos 2\psi_1 - \cos 2\psi_2) - \frac{3\alpha}{16}\left({r_1}^2 - {r_2}^2\right) - \frac{3\alpha}{4}\left({r_2}^2 - {r_1}^2\right)\cos 2\chi\right). \tag{7}$$

The resonance zones are corresponding with domains in phase-space where the three angles $\psi_1, \psi_2, \chi$ are not timelike, they are determined by the zeros of the equations for the 3 angles. From eq. (7) we find for angle $\chi$ two possible resonance zones $M_1, M_2$ given by:

$$r_1 = r_2,\ \psi_1 = \psi_2,\ \text{or}\ \psi_1 = \psi_2 + \pi, . \tag{8}$$

Dynamically most interesting is the case that we have intersection of resonance zones. For the angles $\psi_1, \psi_2$, using system (6), this leads in $M_1, M_2$ to the equations:

$$\beta \cos 2\psi_{1,2} = \frac{63\alpha}{16} {r_1}^2. \tag{9}$$

So for triple intersection of resonance zones we have the necessary condition: $r_1 = r_2$, $0 < \frac{63\alpha}{16\beta} {r_1}^2 < 1$.

## 3 Triple resonance for the 1 : 1 case

We will distinguish the dynamics for the dissipative and volume-preserving cases.

### 3.1 Periodic solutions in the dissipative case

Assume $\mu > 0$ and consider the resonance zones $M_1, M_2$ determined by eq. (8). An interesting type of periodic solution may arise if $\dot{r}_{1,2} = 0$ and simultaneously $\dot{\chi} = 0$. With these assumptions we find in $M_1, M_2$:

$$\begin{aligned} \dot{r}_{1,2} &= \frac{\varepsilon}{2} r_{1,2} \left( -\mu + \frac{\beta}{2\omega} \sin 2\psi_{1,2} \right) = 0, \\ \dot{\psi}_{1,2} &= \frac{\varepsilon}{8\omega} \left( \beta \cos 2\psi_{1,2} - \frac{63\alpha}{16} {r_{1,2}}^2 \right) = 0. \end{aligned} \tag{10}$$

Conditions (10) are satisfied if the periodic solutions are located in the triple resonance zone determined by eq. (9) and moreover:

$$\sin 2\psi_{1,2} = \frac{2\mu\omega}{\beta}, \left| \frac{2\mu\omega}{\beta} \right| \leq 1 \text{ or } \mu \leq \frac{\beta}{2\omega}, \tag{11}$$

which puts a bound on the size of the dissipation with respect to the other parameters. In a Galerkin projection of eq. (1) with large eigenvalues $\omega$, these periodic solutions vanish. From $\sin^2 \psi + \cos^2 \psi = 1$ we find for the amplitudes of the periodic solutions:

$${r_{1,2}}^2 = \frac{16}{63\alpha} \sqrt{\beta^2 - 4\mu^2\omega^2}. \tag{12}$$

### 3.2 Stability in the dissipative case, $\mu > 0$

To establish the stability of the periodic solutions in the triple resonance zone we use theorems 11.5–11.6 from [5] (the 2nd Bogoliubov theorem). We need the Jacobian of the vector field $F$ of the averaged system (6). Omitting the factor $\varepsilon/2$ we find for Jacobian $\nabla F$:

$$\begin{pmatrix} A_1 & -\dfrac{3\alpha r_1 r_2 \sin 2\chi}{8\omega} & B_1 & \dfrac{3\alpha r_1 {r_2}^2 \cos 2\chi}{8\omega} \\ \dfrac{3\alpha r_1 r_2 \sin 2\chi}{8\omega} & A_2 & \dfrac{3\alpha {r_1}^2 r_2 \cos 2\chi}{8\omega} & B_2 \\ -\dfrac{27\alpha r_1}{32\omega} & -\dfrac{3\alpha r_2 (2+\cos 2\chi)}{8\omega} & C_1 & -\dfrac{3\alpha {r_2}^2 \sin 2\chi}{8\omega} \\ -\dfrac{3\alpha r_1 (2+\cos 2\chi)}{8\omega} & -\dfrac{27\alpha r_2}{32\omega} & \dfrac{3\alpha {r_1}^2 \sin 2\chi}{8\omega} & C_2 \end{pmatrix}$$

where

$$\begin{aligned} A_1 &= -\mu + \frac{\beta}{2\omega}\sin 2\psi_1 - \frac{3\alpha}{16\omega}{r_2}^2 \sin 2\chi,\ A_2 = -\mu + \frac{\beta}{2\omega}\sin 2\psi_2 + \frac{3\alpha}{16\omega}{r_1}^2 \sin 2\chi, \\ B_1 &= \frac{\beta}{\omega} r_1 \cos 2\psi_1 - \frac{3\alpha}{8\omega} r_1 {r_2}^2 \cos 2\chi,\ B_2 = \frac{\beta}{\omega} r_2 \cos 2\psi_2 - \frac{3\alpha}{8\omega}{r_1}^2 r_2 \cos 2\chi, \\ C_1 &= -\frac{\beta}{2\omega}\sin 2\psi_1 + \frac{3\alpha}{8\omega}{r_2}^2 \sin 2\chi,\ \text{and } C_2 = -\frac{\beta}{2\omega}\sin 2\psi_2 - \frac{3\alpha}{8\omega}{r_1}^2 \sin 2\chi. \end{aligned}$$

Applying the Jacobian at the periodic solutions using eqs. (8), (9), (11) with notation $r_1 = r_2 = r$ we find the matrix:

$$J(r) = \begin{pmatrix} 0 & 0 & \dfrac{57\alpha}{16\omega} r^3 & \dfrac{3\alpha}{8\omega} r^3 \\ 0 & 0 & \dfrac{3\alpha}{8\omega} r^3 & \dfrac{57\alpha}{16\omega} r^3 \\ -\dfrac{27\alpha}{32\omega} r & -\dfrac{9\alpha}{8\omega} r & -\mu & 0 \\ -\dfrac{9\alpha}{8\omega} r & -\dfrac{27\alpha}{32\omega} r & 0 & -\mu \end{pmatrix}$$

It is easy to see that if $r > 0$ we have $|J(r)| > 0$. The implication is from [5] that periodic solutions obtained from nontrivial equilibria of the averaged system (6) do exist in an $\varepsilon$-neighbourhood of the equilibria. Note that this also holds if $\mu = 0$.
A Mathematica calculation produces the eigenvalues of matrix $J(r)$:

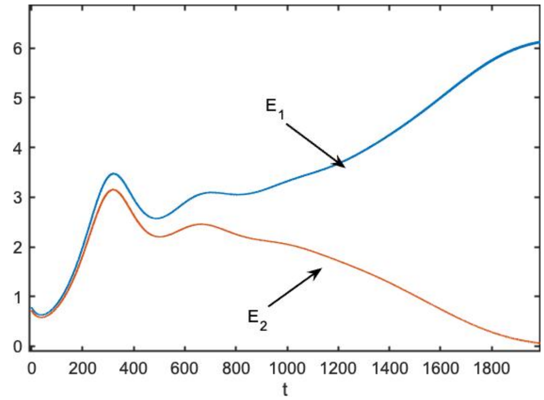


**Fig. 2** The behaviour of the solutions of system (3) near the general position solution $u_1(t) = u_2(t)$ is shown by plotting $E_1(t) = 0.5(\dot{u}_1^2(t)+6u_1^2(t))$ and $E_2(t) = 0.5(\dot{u}_2^2(t)+6u_2^2(t))$. The initial conditions are $u_1(0) = 0.51$, $\dot{u}_1(0) = 0.05$, $u_2(0) = 0.49$, $\dot{u}_2(0) = 0.05$; $\omega^2 = 6$, $\mu = 0.01$, $\beta = 0.1$, $\alpha = 0.05$.

$$\lambda_{1,2} = -\frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 + \frac{459\alpha^2}{128\omega^2}r^4}, \lambda_{3,4} = -\frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - \frac{3969\alpha^2}{128\omega^2}r^4}.$$

The eigenvalues $\lambda_{1,2}$ are real, the plus sign results in a positive eigenvalue so we have instability of the periodic solutions. The instability is caused by the parametric excitation, it is weakened for large $\omega$. In fig. 2 we show the instability by starting near the solution where $u_1(t) = u_2(t)$.

### 3.3 Stability in the volume-preserving case, $\mu = 0$

Without dissipation the flow in the time-extended phase-space is volume-preserving, the dynamics is more delicate. For the angle $\chi$ the resonance zones $M_1, M_2$ are unchanged. Looking for periodic solutions with constant, nontrivial amplitudes $r_1, r_2$ we find from system (6):

$$\beta \sin 2\psi_1 = \frac{3}{8}\alpha {r_2}^2 \sin 2\chi, \quad \beta \sin 2\psi_2 = -\frac{3}{8}\alpha {r_1}^2 \sin 2\chi.$$

These conditions lead in $M_1, M_2$ to the solutions:

$$\psi_1 = \psi_2 = 0 \text{ and } \psi_1 = 0, \psi_2 = \pi.$$

In system (6) we have $\dot{\psi}_1 = \dot{\psi}_2 = 0$ if:

$${r_1}^2 = {r_2}^2 = {r_0}^2 = \frac{16\beta}{63\alpha}.$$

Using this value of $r$ and the eigenvalues (1) we obtain for the eigenvalues in the volume-preserving case:

$$\lambda_{1,2} = \pm\frac{\beta}{21\omega}\sqrt{\frac{51}{2}},\ \lambda_{3,4} = \pm\frac{\beta}{2\omega}\sqrt{2}i. \tag{13}$$

We have again instability of the periodic solutions. The periodic solutions in the triple resonance zone are illustrated in fig. 3. The behaviour for the cases $\psi_2 = 0, \pi$ is identical. We use the expression $I_{1,2}(t) = \frac{1}{2}(\dot{u}_{1,2}^2 + \omega^2 u_{1,2}{}^2)$. The recurrence in the volume-preserving case $\mu = 0$ is illustrated by plotting the Euclidean distance $d(t)$ to the initial values, we have:

$$d^2(t) = \sum_{i=1}^{2}(u_i(t) - u_i(0))^2 + (\dot{u}_i(t) - \dot{u}_i(0))^2.$$

With $\varepsilon = 0.01$ the typical timescale of recurrence is 3500 timesteps.

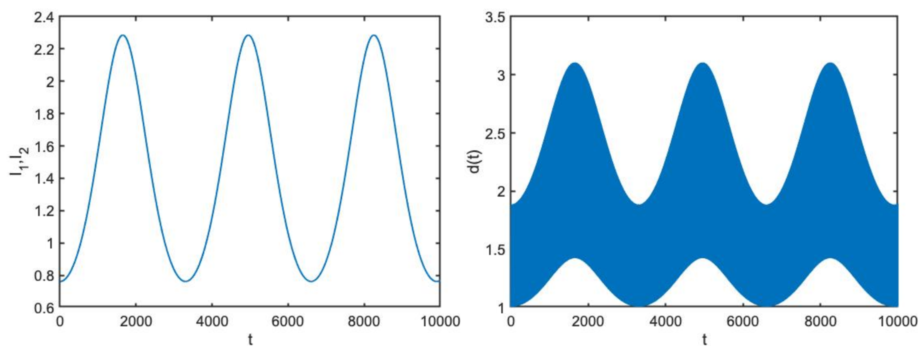


**Fig. 3** Left we illustrate the behaviour of the solutions of system (3) ($m = 1, n = 2$) by plotting $I_1(t) = I_2(t)$ in the case $\omega = \sqrt{6}, \mu = 0, \alpha = \beta = 1, \varepsilon = 0.01$ with initial conditons $u_1(0) = u_2(0) = r_0, \psi_1(0) = \psi_2(0) = 0$. Right we show the Euclidean distance $d(t)$ to the initial conditions.

## 3.4 Interaction of 1 : 1 resonances

As we have an infinite number of 1 : 1 resonances it is natural to study a combination of $N$ eigenfunctions of the form:

$$u_p(x, y, t) = \sum_{i=1}^{N}(u_{1i}(t)\cos m_i x \cos n_i y + u_{2i}(t)\cos n_i x \cos m_i y), \tag{14}$$

where $m_j, n_j \in \{0, 1, 2, \ldots, N\}$. We choose $m_i \neq n_i$, $i = 0, 1, \ldots, N$ and avoid accidental resonances (to be discussed in the next section) in (14). Substitution in wave equation (1) and taking inner products with the individual eigenfunctions produces a system of $2N$ second order coupled ODEs. The $\varepsilon$-scaling as before enables us to apply averaging; the results depend on the choice of $\Omega$. Suppose that for one of the $(m_i, n_i)$ combinations we have the frequency $\omega_i$ with $\Omega = 2\omega_i$. The corresponding eigenfunction will show dynamics that is different from the other modes.
We will discuss the dynamics in a particular case of $N = 2$ as this shows the essential behaviour and avoids too much notation. The results can easily be generalised for $N > 2$. Choose $m_1 \neq n_1$ with corresponding $\omega_1$ and $\Omega = 2\omega_1$. Choose a different set $m_2 \neq n_2$ with corresponding $\omega_2$, $\Omega \neq 2\omega_2$. We associate with $m_1, n_1$ the time-dependent amplitudes $u_1, u_2$, with $m_2, n_2$ the amplitudes $u_3, u_4$. Substituting the expansion containing 4 modes:

$$\sum_{i=1}^{2} u_1(t) \cos m_i x \cos n_i y + u_{i+1}(t) \cos n_i x \cos m_i y$$

into the wave equation (1) and taking inner products with the eigenfunctions we obtain after the usual $\varepsilon$-scaling the system:

$$\begin{cases} \ddot{u}_1 + \omega_1^2 u_1 & = -\varepsilon\mu\dot{u}_1 - \varepsilon\beta u_1 \cos(2\omega_1\, t) + \varepsilon\alpha P(u_1, u_2, u_3, u_4), \\ \ddot{u}_2 + \omega_1^2 u_2 & = -\varepsilon\mu\dot{u}_2 - \varepsilon\beta u_2 \cos(2\omega_1\, t) + \varepsilon\alpha P(u_2, u_1, u_3, u_4), \\ \ddot{u}_3 + \omega_2^2 u_3 & = -\varepsilon\mu\dot{u}_3 - \varepsilon\beta u_3 \cos(2\omega_1\, t) + \varepsilon\alpha P(u_3, u_1, u_2, u_4), \\ \ddot{u}_4 + \omega_2^2 u_4 & = -\varepsilon\mu\dot{u}_4 - \varepsilon\beta u_4 \cos(2\omega_1\, t) + \varepsilon\alpha P(u_4, u_1, u_2, u_3), \end{cases}$$

where $P(u_1, u_2, u_3, u_4) = \frac{9}{16} {u_1}^3 + \frac{3}{4} u_1 {u_2}^2 + \frac{3}{4} u_1 {u_3}^2 + \frac{3}{4} u_1 {u_4}^2$. We assume that there are no accidental resonances as discussed in the next section. First order averaging produces in amplitude-phase coordinates for orbits in general position:

$$\begin{cases} \dot{r}_i & = \dfrac{\varepsilon}{2} \left( \left( -\mu + \dfrac{\beta}{2\omega_1} \sin 2\psi_i \right) r_i + (-1)^i \dfrac{3\alpha}{16\omega_1} {r_1}^i {r_2}^{3-i} \sin 2(\psi_1 - \psi_2) \right), \\ \dot{\psi}_i & = \dfrac{\varepsilon}{8} \left( \dfrac{\beta}{\omega_1} \cos 2\psi_i - \dfrac{27\alpha}{16\omega_1} {r_i}^2 - \dfrac{3\alpha}{2\omega_1} \left( {r_{3-i}}^2 + {r_3}^3 + {r_4}^2 \right) \right. \\ & \qquad \left. - \dfrac{3\alpha}{4\omega_1} {r_{3-i}}^2 \cos 2(\psi_1 - \psi_2) \right), \quad \text{for } i = 1, 2 \\ \dot{r}_j & = \dfrac{\varepsilon}{2} \left( -\mu r_j + (-1)^j \dfrac{3\alpha}{16\omega_2} {r_3}^{j-2} {r_4}^{5-j} \sin 2(\psi_3 - \psi_4) \right), \\ \dot{\psi}_j & = \dfrac{\varepsilon}{8} \left( -\dfrac{27\alpha}{16\omega_2} {r_j}^2 - \dfrac{3\alpha}{2\omega_2} \left( {r_1}^2 + {r_2}^2 + {r_{7-j}}^2 \right) \right. \\ & \qquad \left. - \dfrac{3\alpha}{4\omega_2} r_{7-j}^2 \cos 2(\psi_3 - \psi_4) \right), \quad \text{for } j = 3, 4 \end{cases}$$

From the equations for $r_3, r_4$ we find:

$$\frac{1}{2}\frac{d}{dt}(r_3{}^2 + r_4{}^2) = -\frac{\varepsilon}{2}\mu(r_3{}^2 + r_4{}^2),$$

so the amplitudes $r_3, r_4$ will vanish with time. For the wave equation (1) the behaviour of the eigenfunctions corresponding with $m_1, n_1$ will be prominent.

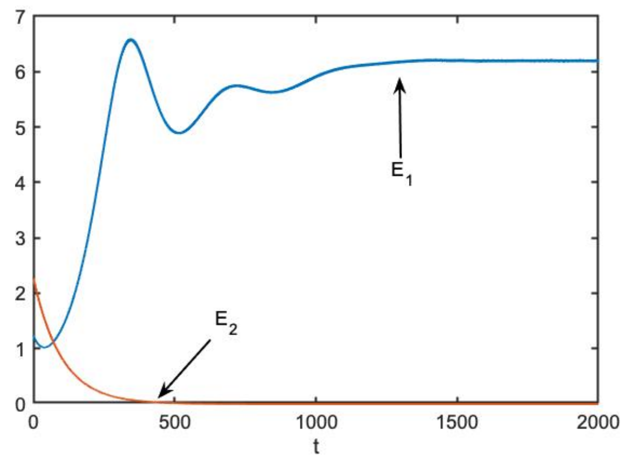


**Fig. 4** The behaviour of the solutions of system (3.4) by plotting $E_1(t)$ for the parametrically excited oscillators $u_1, u_2$ and $E_2(t)$ for the oscillators $u_3, u_4$. The initial conditions are $u_1(0) = u_2(0) = 0.5, u_3(0) = u(_4(0) = 0.4$ with initial velocities zero; $\omega_1^2 = 6, \omega_2^2 = 11, \mu = 0.01, \beta = 0.1, \alpha = 0.05$.

We illustrate the results for an explicit case. Consider the combination $(m, n) \in \{(1, 2), (2, 1)\}$ (coefficients $u_1(t), u_2(t)$) and $\{(1, 3), (3, 1)\}$ (coefficients $u_3(t), u_4(t)$). We have $\omega_1 = \sqrt{6}, \omega_2 = \sqrt{11}$, the parametric excitation frequency $\Omega = 2\sqrt{6}$. We introduce as measures for the energy of the oscillators $u_1, u_2$ the quantity

$$E_1 = \frac{1}{2}(\dot{u_1}^2 + 6u_1{}^2 + \dot{u_2}^2 + 6u_2{}^2)$$

and similarly for $u_3, u_4$ the quantity

$$E_2 = \frac{1}{2}(\dot{u_3}^2 + 11u_3{}^2 + \dot{u_4}^2 + 11u_4{}^2).$$

The initial values of the two groups of oscillators are equal, the first group is excited, the second group is damped out; see fig. 4.

## 4 Remarks on accidental resonances

The instability of periodic solutions in general position in the case of two modes with symmetric eigenfunctions (1) suggests the question whether energy can be transferred to other modes by accidental resonance. We consider a few prominent cases, the topic can be extended considerably. Choose for the eigenvalues (1) $\omega_0 = c = 1$.

### 4.1 The 1 : 1 : 3 resonance

Consider the 3 eigenfunctions with $(m, n) \in \{(1, 3), (3, 1), (7, 7)\}$. In this case the frequencies of the linear oscillations are given by 11, 11, and 99, producing the 1 : 1 : 3 resonance. The eigenfunction expansion of the corresponding 3 modes is:

$$u_p(x, y, t) = u_1(t) \cos x \cos 3y + u_2(t) \cos 3x \cos y + u_3(t) \cos 7x \cos 7y. \quad (15)$$

Substitution of expansion (15) into eq. (1) and taking inner products with the eigenfunctions we find with $\omega = \sqrt{11}, \omega_1 = 2\omega$:

$$\begin{cases} \ddot{u}_1 + \omega^2 u_1 & = -\varepsilon\mu\dot{u}_1 - \varepsilon\beta u_1 \cos(2\omega t) + \varepsilon\alpha P(u_1, u_2, u_3), \\ \ddot{u}_2 + \omega^2 u_2 & = -\varepsilon\mu\dot{u}_2 - \varepsilon\beta u_2 \cos(2\omega t) + \varepsilon\alpha P(u_2, u_1, u_3), \\ \ddot{u}_3 + 9\omega^2 u_3 & = -\varepsilon\mu\dot{u}_3 - \varepsilon\beta u_3 \cos(2\omega t) + \varepsilon\alpha P(u_3, u_2, u_1), \end{cases} \quad (16)$$

where $P(u_1, u_2, u_3) = \frac{9}{16}{u_1}^3 + \frac{3}{4}u_1{u_2}^2 + \frac{3}{4}u_1{u_3}^2$. Although we have a primary resonance it turns out that because of the symmetries of system (16) the 1 : 1 : 3 resonance is not effective. First order averaging produces for the amplitude $r_3$ of $u_3(t)$ the equation

$$\dot{r}_3 = -\varepsilon\frac{\mu}{2}r_3,$$

so there is no interaction with the modes $u_1(t), u_2(t)$ and no quenching or transfer of energy of the first two modes. Higher order approximation will not change this picture qualitatively.

### 4.2 The 1 : 1 : 1 resonance

Consider the 3 eigenfunctions with $(m, n) \in \{(1, 7), (7, 1), (5, 5)\}$. In this case the frequencies of the linear oscillations are the same, i.e. $\omega = \sqrt{51}$, producing the 1 : 1 : 1 resonance. The eigenfunction expansion of the corresponding 3 modes is:

$$u_p(x, y, t) = u_1(t) \cos x \cos 7y + u_2(t) \cos 7x \cos y + u_3(t) \cos 5x \cos 5y. \quad (17)$$

We substitute expansion (17) into eq. (1) and we take inner products with the eigenfunctions. Put $\omega_1 = 2\sqrt{51}$ and rescale $\sqrt{51}\,t \longmapsto t$, $\varepsilon/51 \longmapsto \varepsilon$; we find the system:

$$\begin{cases} \ddot{u}_1 + u_1 & = -\varepsilon\mu\dot{u}_1 - \varepsilon\beta u_1\cos(2t) + \varepsilon\alpha P(u_1, u_2, u_3), \\ \ddot{u}_2 + u_2 & = -\varepsilon\mu\dot{u}_2 - \varepsilon\beta u_2\cos(2t) + \varepsilon\alpha P(u_2, u_1, u_3) \\ \ddot{u}_3 + u_3 & = -\varepsilon\mu\dot{u}_3 - \varepsilon\beta u_3\cos(2t) + \varepsilon\alpha P(u_3, u_1, u_1), \end{cases}$$

where $P(u_1, u_2, u_3) = \frac{9}{16}{u_1}^3 + \frac{3}{4}u_1{u_2}^2 + \frac{3}{4}u_1{u_3}^2$. Because of the symmetry of the system we can recover the solutions of the preceding 1 : 1 resonances in 2 degrees-of-freedom invariant manifolds. This means that we find periodic solutions in the 3 normal mode planes and unstable periodic solutions in 3 invariant 4-dimensional manifolds when putting successively the initial conditions of one mode equal to zero.

However, we are interested in general position orbits. We can extend the averaging by adding to system (6) 2 equations, the angle $\psi_3$ and the combination angles $\psi_1 - \psi_3$ and $\psi_2 - \psi_3$. Apart from the normal mode solutions and because of the symmetry of system (18) we can enumerate a number of exact solutions in general position, for instance:

$$u_1(t) = u_2(t) = u_3(t). \tag{18}$$

We can also put $u_2(t) = -u_1(t), u_3(t) = -u_1(t)$ or $u_3(t) = -u_1(t), u_2(t) = -u_3(t)$. In the case of eq. (18) we have the special solution from:

$$u_1(t) = u_2(t) = u_3(t) = u(t), \, , \ddot{u} + u = -\varepsilon\mu\dot{u} - \varepsilon\beta u\cos(2t) + \varepsilon\alpha\frac{33}{16}u^3.$$

With amplitude-phase coordinates as before the solutions are approximated by averaging:

$$\dot{r} = \frac{\varepsilon}{2}r(-\mu + \frac{\beta}{2}\sin 2\psi), \; \dot{\psi} = \frac{\varepsilon}{4}(\beta\cos 2\psi - \alpha\frac{99}{32}r^2). \tag{19}$$

In system (19) $r = r_1 = r_2 = r_3$ and $\psi = \psi_1 = \psi_2 = \psi_3$. A critical point of the averaged vector field is determined by:

$$\mu = \frac{\beta}{2}\sin 2\psi, \; \beta\cos 2\psi = \alpha\frac{99}{32}r^2.$$

Elimination of the phase-angle yields:

$$r^2 = \frac{32}{99\alpha}\sqrt{\beta^2 - 4\mu^2}. \tag{20}$$

### 4.3 The 1 : 1 : 1 : 1 resonance

Consider the 4 eigenfunctions with $(m, n) \in \{(3, 4), (4, 3), (0, 5), (5, 0)\}$ . In this case the frequencies are $\omega = \sqrt{26}$, producing the $1 : 1 : 1 : 1$ resonance. The eigenfunction expansion of the corresponding 4 modes is:

$$u_p(x, y, t) = u_1(t)\cos 3x \cos 4y + u_2(t)\cos 4x \cos 3y + u_3(t)\cos 5y + u_4(t)\cos 5x.$$

The analysis by averaging and of exact solutions runs as before. This is left to the reader.

## 5 Conclusions

1. The analysis of a nonlinear wave equation with 2 spatial dimensions introduces many new problems involving resonances. The $1 : 1$ resonance dominates the dynamics in the case of a square domain.
2. In contrast to the case of systems without forcing, see [6], the excitation forces a strong selection of modes. This has become clear in the analysis of interaction of $1 : 1$ resonances.
3. An important aspect of the analysis is the choice of the parametric excitation frequency $\Omega = 2\omega$ in eq. (1). In the cases of modes with $\Omega \neq 2\omega$ and $\mu > 0$ we expect reduction of these modes by damping. This becomes already clear for the $1 : 1 : 3$ resonance in section 4.1.
4. We have omitted the analysis of detuning. Inspection of the frequencies generated by the space-dependent operator suggests a number of interesting cases. The formulation of the initial-value problem for eq. (1) raises many more questions that will hopefully be discussed in later papers.

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