Ferdinand Verhulst
Mathematical Institute
Utrecht University
P.O. Box 80.010
3508 TA Utrecht
f.verhulst@uu.nl

The interaction of geometry and analysis in Henri Poincaré's conceptions

Henri Poincaré had many interests, both inside and outside science. His special attention in this was devoted to the interaction between different fields of knowledge. In this article Ferdinand Verhulst goes into the interaction between mathematical disciplines, where he concentrates on geometry and analysis.

To appreciate the enormous amount of scientific results obtained by Henri Poincaré (1854–1912), it helps to realise how diverse and important the various influences were on the activities of his mind. His talents were both in the humanities and in science. When he was young he wrote hundreds of letters to his family and friends, mixing them with his own poetry; he got honourable mentions for his essays at school, and he also wrote a novel. In psychology, his observations of how the human mind works to create and invent became textbook accounts. Later in life he became known to the general public because of his philosophical essays. In science, his interests were in geology, physics and astronomy, and of course mathematics. All these fields influenced his ideas and work. It is not surprising that also in mathematics his creative attention was directed at a great many different topics. In this article we will be concerned with two mathematical disciplines, geometry and analysis, the historical development of their interaction and their role in Poincaré's mathematics.

Geometry and the rise of modern science

The rise of modern science in the seventeenth century cannot be dissociated from mathematical thinking and the mathematisation of the description of natural phenomena, called mathematical modeling. The mathematical thinking and writing in that period had a strong geometrical flavour. The *Principia* [11] of Isaac Newton (1643–1727) is not only full of geometric arguments and pictures, but contains actually many geometric theorems. A similar characterization applies to the fundamental treatise by Christiaan Huygens (1629–1695) *Traité de la lumière* [6]. Many examples of geometric reasoning can be found in the work of other prominent scientists, for instance Johann Bernoulli (1667–1748).

Although the dominance of traditional geometric formulations posed a certain restriction, modern science started to develop. Poincaré notes that with Newton, a change of language took place, which enabled 'natural philosophers' to expand science dramatically. Generalization became then possible which had not been guessed before:

"When Kepler's laws were replaced by Newton's, one knew only elliptic motion. Regarding this motion, the two laws differ in form only, going from one to the other is simply by differentiation. From Newton's laws, however, one could by immediate generalization deduce all perturbation effects in celestial mechanics as a whole. If, on the other hand, one would have kept the formulation of Kepler, one would not have considered the orbits of perturbed planets, these complicated curves of which nobody has written down the equations." [17, essay 'L'analyse et la physique']

In Kepler's laws one saw the geometrical and dynamical movements of the planets in elliptical orbits. After the formulation of

Newton's laws one started to use differential equations which enabled us to describe much more complicated gravitational systems than the two-body problem. It is the change of languages that opens up new generalizations which had not been seen before.

Two mathematical cultures

The beginning of the nineteenth century saw the continuation of two cultures, one based on the impressive success of analytic computations and one in which geometric thinking reached a new level. As a striking representative of the analytic culture, Lagrange was very influential. We will discuss Riemann as a representative of geometric thinking.

Lagrange

The use of the analytic ideas of Newton was not always straightforward in practical problems. A simple example is the differentiation of the product of two functions, say $f(x)g(x)$. One can do this by using the definition of differential quotient, but using Leibniz's rule of differentiation $d/dx(fg) = f'g + fg'$ leads quickly to a correct result. The enormous amount of calculations in celestial mechanics and other parts of mechanics in the eighteenth century required such algorithmic rules. To calculate the ephemerides of comets and planets it was essential to have algorithms and more or less 'automatic computation' at one's disposal. A rather extreme example is found in the work of Joseph-Louis Lagrange (1736–1813). Lagrange writes for instance in the introduction to his *Mécanique Analytique* [8]:

"One will not find figures in this work. The methods that I explain therein require neither geometric nor mechanical constructions or arguments, but only algebraic operations forced by regular and uniform steps."

This is clearly seen by Lagrange as an advantage, a way to reach results quickly, and not as a restriction. Are the two volumes of the *Mécanique Analytique*, where all geometry has been weeded out, still of importance? Remarkably enough, the answer is 'yes'. Apart from treating specific problems in mechanics, the general equations of dynamics are derived, the theory of multipliers for extreme values is formulated, the basic ideas of averaging normalization are described in a clear way.

There is no explicit geometry that would have added a synthetic element to the calculations, but the wealth of ideas makes up for this.

The school of Riemann

The successes of analysis in dynamics, in particular in celestial mechanics, had its counterpart in applied mathematics in Germany, but meanwhile geometric thinking went there its autonomous course. This becomes clear in the mathematics of Bernhard Riemann (1826–1866). Poincaré notes in *La valeur de la science* [17]:

Bernhard Riemann (1826–1866)

"Among the German mathematicians of this century, two names are particularly famous; these are the two scientists who have founded the general theory of functions, Weierstrass and Riemann. Weierstrass reduces everything to the consideration of series and their analytical transformations. To express it better, he reduces analysis to a kind of continuation of arithmetic; one can go through all his books without finding a picture. In contrast with this, Riemann calls immediately for the support of geometry, and each of his concepts presents an image that nobody can forget once he has understood its meaning." [17, essay 'L'intuition et la logique en mathématiques']

These delimitations are not meant to suggest one particular way of mathematical thinking. A few lines later, Poincaré writes:

"The two types of spirits are equally necessary for the progress of science. Both the logicians [doing analysis by successive steps] and the intuitives [doing synthetic geometrics] have done great things which the others could not have achieved. Who would dare to say that he would have preferred that Weierstrass had never written a word, or that there had not been a Riemann? So, analysis and synthesis both have their legitimate part to play."

It is interesting to consider Riemann's papers in the light of Poincaré's remarks.

At the occasion of his 'Habilitation' in Göttingen (1854), Riemann lectured on the foundations of geometry [20], see also [19]. Riemann starts with experience and notes that the Euclidean foundations are not necessary, but that they have an acceptable certainty. He formulates a research plan for n-dimensional manifolds and spaces. Weyl [19] links these considerations with later results in geometry, for instance by Klein, and with general relativity.

The collected works [19] starts with a treatise on the foundations of complex function theory, without figures but, as noted by Poincaré, "each of its concepts presenting an image". The interpretation of a complex function in the neighbourhood of a singularity plays a prominent part.

A long article on Abelian functions in [19] is written in the same style, it contains four figures. This long article discusses the problems of many-valued functions in the context of results from the analysis situs of Leibniz.

Comparing the approach of Riemann and Lagrange, it is interesting to consider Lagrange's theory of analytic functions [9]. The book consists of three parts with the first part discussing series expansions for implicitly defined real functions. The second part has a strong geometric flavour; it applies the preceding theory to obtain tangents of curves, curvature calculations and contact problems. There is also an extension to the theory of extreme values and variational calculus. In the third part, finally, the theory of series is applied to problems of mechanics.

In the book of Lagrange on analytic functions we find many calculations to solve a number of particular problems which in the second part are associated with classical geometry. In Riemann's articles, analysis and geometry go hand in hand producing new insights in both fields; an important example is his approach to the many-valuedness of analytic functions near a singularity by introducing a structure of surfaces around it.

Felix Klein

A prominent member of the school of Riemann was Felix Klein (1849–1925). For illustration, we discuss his lectures on ordinary differential equations that started on April 24, 1894 and ended on August 7, 1894, taking place during the so-called 'Sommersemester' in Göttingen. They have been worked out by E. Ritter [7] in 524 handwritten reproduced pages with nice illustrations.

In the preface Klein notes that the present lectures are a natural sequel to his earlier lectures on hypergeometric functions. He also

Felix Klein (1849–1925)

Henri Poincaré, student at the École Polytechnique

mentions that in contrast to other authors he will discuss the global behaviour of solutions, but that this field is so rich that he has to restrict himself to second order linear ordinary differential equations with three singularities. The emphasis in the discussion is on algebraic and transcendental properties of differential equations, oscillation theorems and automorphic functions. The treatment is interesting but was already unusual at that time regarding ordinary differential equations as it is not so much concerned with explicit solutions, but focuses on problems of complex function theory like the role of singularities, Riemann surfaces and uniformisation questions.

After a detailed discussion of algorithmic and synthetic research, Klein states:

"Nowadays one uses everywhere in mathematics again the synthetic method along with the algorithmic method and one can distinguish the problems in the separate disciplines by their treatment according to one or the other. I believe one can weigh the value of both methods against each other: with the algorithmic method, if it can be applied at all, one obtains certainly something, even general comprehensive theorems. This is then not so much the merit of the individual mathematician as he works with the capital of his predecessors, with the supply of ideas which earlier mathematicians have assembled by the creation of the algorithm. It is different with the synthetic method; there everything comes down to having the correct, new thought. There, one does not know whether one finds something, there one has to create one's own path. What one achieves is maybe little, but to a large extent the property of the researcher. The algorithm gives progress in objective respect but not subjectively. One is not so much forced to think independently. The algorithm looks like travelling by train that goes fast and far, but through cultivated landscapes only, the synthetic method is of the settler who with his axe and much trouble penetrates into the jungle and conquers new domains of culture. In any case the second activity must precede the first.

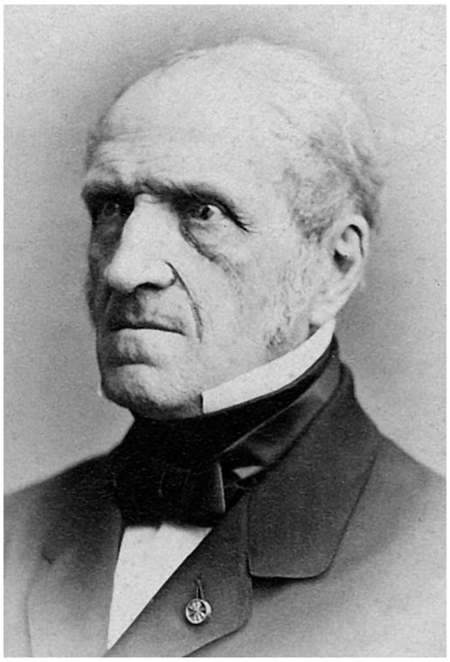
"Klein concludes that in his lectures he will use both algorithmic and synthetic approaches. Algorithmic he considers the treatment of algebraic integrability, including algebraic and transcendental groups, and the theory of Lamé polynomials. Synthetic is the discussion of the oscillation theorem (Sturm–Liouville theory) and the theory of automorphic functions, this last chapter taking nearly hundred pages. It is concerned with the properties of analytic continuation, the relation with the geometry of Riemann surfaces and uniformisation questions.

The mathematical education of Poincaré

In the nineteenth century, geometry in France consisted of the classical Euclidean geometry, supplemented by analytic and projective geometry, including its applications such as descriptive geometry. Henri Poincaré, eighteen years old, enrolled in the special course that would prepare him for the entrance examinations of the 'Grandes Écoles', the École Normale Supérieure and the École Polytechnique. Before he began these studies, he plunged into the mathematics textbooks of the time. According to [1] these included *La Géométrie* by Eugène Rouché, *L'Algèbre* by Joseph Bertrand, *Cours d'Analyse de l'École Polytechnique* by Jean-Marie Duhamel and *La Géométrie Supérieure* by Michel Chasles. The last two books on the list are particularly remarkable. Jean Duhamel's calculus book was a textbook for the École Polytechnique, where Duhamel (1797–1872) himself lectured. The geometry book of Michel Chasles is on advanced geometry while proposing a pronounced philosophy.

Michel Chasles advocates a balance

Michel Chasles's geometry text emphasized the complementary roles of analysis and geometry; it was original, difficult, questioned by a number of colleagues, yet written in an engaging style. Chasles (1793–1880) formulated his approach in his 1846 inaugural lecture for the geometry chair at the Sorbonne as follows (see also [4]):



Michel Chasles (1793–1880)

"If one knows that the subject of geometry is the measure and characteristics of space, then one knows how extended this field is, and that one does not even know where the boundaries of this domain end. For space that one imagines changes shape infinitely often, and the features of each of the forms arising in nature or of those the human spirit can imagine, are themselves extremely numerous, one can even say inexhaustible."

Chasles continued by proposing a drastic change in the way geometry was understood and practised at the beginning of the nineteenth century, for instance under the influence of Lagrange. It emphasized formulas over pictures, analysis over synthesis. Chasles, in contrast, has this to say:

"One can see the respective advantages of Analysis and Geometry: the former leads with the miraculous mechanism of its transformations quickly from the starting point to the point to be reached, but often without revealing the road that was travelled or the significance of the numerous formulas that have been used. Geometry on the other hand derives its inspiration from thoughtful consideration of things and from the ordered arrangement of ideas. It is obliged to discover in a natural way the statements that Analysis could neglect and ignore."

When Chasles's *La Géométrie Supérieure* appeared in 1852, his vision of the importance of geometry, using of course analysis as a support, was rather revolutionary in France.

Jean-Gaston Darboux (1842-1917)

It seems more than likely that it became an important influence on Henri's way of thinking, especially considering the fruitful combination of analysis and geometry that is typical of his methods, for instance in the quantitative and qualitative theory of dynamical systems that he would later develop. Jean-Gaston Darboux (1842–1917) wrote his doctoral thesis under the supervision of Michel Chasles, Darboux was one of the supervisors of the doctoral thesis of Henri Poincaré. The view of Chasles is mirrored in [17, essay 'L'analyse et la physique'], where Poincaré writes about the analytic and geometric images evoked by the Laplace equation:

"Thanks to these images, one can see at a glance what pure deduction will show only after successive steps."

Examples of fruitful interaction

We will discuss now a few examples of the interaction of geometry and analysis in the mathematics of Poincaré. More details and other examples can be found in [5] and [21].

Automorphic functions

The integration of differential equations leads more often than not to solutions that are defined implicitly. We are then faced with an inversion problem to find the explicit solution. Consider for instance a simple implicit relation in complex variables: $w = z^2$ with inversion $z = \sqrt{w}$; this leads to the well-known problem that, starting, say on the real axis, and moving on a circle around the origin (the singularity), will produce a different value when arriving again at the real axis. An ingenious solution for the problem of many-valuedness to obtain unique continuation of such a function was proposed by Riemann. Using several surfaces when moving around the singularity and joining them, one obtains the so-called Riemann surface. In the example of the quadratic equation above, one needs two surfaces to be joined. For more general algebraic implicit equations, one needs for such an inversion a finite number of surfaces and so a more complicated Riemann surface.

The German mathematician Fuchs (1833–1902) considered a second order, linear, ordinary differential equation of the form

$$y'' + A(z)y' + B(z)y = 0$$

with A and B holomorphic functions of the complex variable z in a region S. There are two independent solutions $y_1(z)$ and $y_2(z)$ and Fuchs started to consider the ratio $\eta = y_1/y_2$. He was interested in the behaviour of the solutions near singular points of A and B and performed analytic continuation of $y_1(z)$ and $y_2(z)$ along a closed curve around such a singularity and inversion of the function $\eta(z)$. This led him to consider a certain linear transformation of η and, more in general, to look for functions that are invariant under a substitution of the form

$$z \to \frac{az+b}{cz+d},$$

with complex coefficients. So we have

$$f\left(\frac{az+b}{cz+d}\right) = f(z).$$

The ratio of the solutions η should be invariant under these linear substitutions which is a more general property than periodicity. This idea of Fuchs inspired Poincaré to put the idea at a higher level of abstraction. He called the functions with this property Fuchsian, they are now called automorphic. For his subsequent analysis, Poincaré had to distinguish between continuous and discontinuous transformation groups. He understood by a flash of intuition that the continuation of these complex functions, the use of Riemann surfaces and transformations in the complex plane correspond with geometric structures that can be understood only in terms of non-Euclidean geometry. In fact, until Poincaré looked at these problems, non-Euclidean geometry was considered as an artificial playground without much relevance to mathematics in general.

The theory led Poincaré to the formulation of uniformisation problems. The integration of algebraic functions, also obtained from differential equations, and their analytic continuations produce multi-valued analytic functions. Uniformisation of such functions corresponds with obtaining a parametrisation by single-valued meromorphic functions. The development has led to the relation between complex function theory and hyperbolic geometry, and also to many results in the study of quadratic forms and arithmetic surfaces. The theory of uniformisation contains still many fundamental open questions.

Balayage for gravitational potential problems

When considering the Newtonian attraction properties of bodies with a given distribution of mass, one is led to a study of the Laplace and Poisson equations (see *Théorie du potentiel Newtonien* [16]). One of the basic problems is to solve the Laplace equation

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \text{ in } D,$$

with $D \in \mathbb{R}^3$ a bounded domain. If the distribution of mass is given by a function $\rho(x, y, z)$, the corresponding potential is described by the Poisson equation

$$\Delta V = \rho.$$

A twice differentiable function satisfying the Laplace equation $\Delta V = 0$ is called *harmonic*. If we require that on the boundary S of D we have $V = \Phi$ with Φ a known function, this is called the Dirichlet boundary value problem for the Laplace equation. If we would look for solutions on the infinite domain exterior to D with the same boundary condition and certain conditions at infinity, we would have

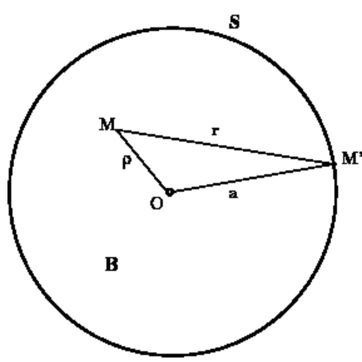


Figure 1 Balayage of a spherical body

the exterior Dirichlet problem for the Laplace equation; this describes the Newtonian gravitational field in the empty space outside a body filling up D; the boundary potential prescribed on S derives from the interior distribution of mass. The exterior Dirichlet problem describes at the same time the electrical force field *outside* a conductor in D with prescribed electrical charge on S.

In the nineteenth century one could solve such problems for simple geometries like spheres and circular cylinders. Existence, uniqueness and actual construction of solutions for more general domains D was a different matter. Today, variational methods like the Dirichlet principle, are useful in this respect, but around 1890, the validity of the variational principles was not yet proved, indeed Weierstrass had thrown doubt on it, so scientists were looking for alternative solution methods. Poincaré invented the so-called balayage or sweeping method, a didactical presentation is given in [16]. In what follows, we describe his theory.

Consider to start with a ball B with spherical surface S, centre O and radius a in three-dimensional space. A point M is located inside the sphere, distance to O is ρ; a surface element $d\omega'$ has centre of gravity M'; the distance from M to M' is r, see Figure 1.

Inside and outside the sphere we have a distribution of mass; at the centre of gravity of a volume element $d\tau$, the density is μ. The gravitational potential V at a point is written as

$$V = V_1 + V_2,$$

with V_1 the potential due to the mass interior to the sphere, V_2 the potential due to the mass exterior to the sphere. As we know from potential theory, the potential V_1 will not change when we replace each mass $\mu d\tau$ in the ball B by a mass layer on the surface S with density in M':

$$\mu' = \frac{a^2 - \rho^2}{4\pi a r^3}\mu d\tau.$$

The mass distribution on the surface S is called the *equivalent layer*. With this procedure we perform a sweeping (balayage) of all the mass in the interior, producing in M' on S the density

$$\mu'' = \int_B \mu \frac{a^2 - \rho^2}{4\pi a r^3} d\tau.$$

The integration is taken over the interior of the

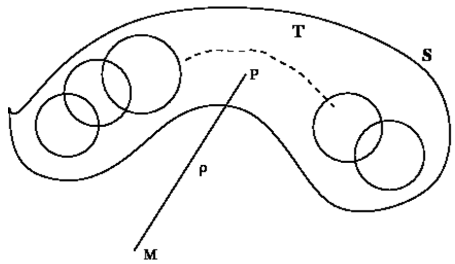


Figure 2 Balayage of a bounded, connected domain T

sphere. Clearly $\mu'' > 0$. We put

$$U_1 = \int_S \frac{\mu''}{r} d\omega'.$$

The potential U_1 equals V_1 outside S, $U_1 \leq V_1$ inside S; U_1 is called subharmonic.

Consider now, more in general, a bounded, connected domain T with smooth boundary surface S, see Figure 2. P is a point in T where a mass $m = 1$ is located. A point M in space at distance ρ undergoes a Newtonian attraction force from m determined by the potential

$$V = \frac{m}{\rho}.$$

There exists a Green's function G with respect to the domain T with the properties

$$G = \frac{1}{\rho} - H,$$

where H is harmonic and determined by

$$\Delta H = 0 \text{ in } T \text{ and } H = \frac{1}{\rho} \text{ on } S.$$

Consider now a function V that coincides with H in domain T and equals $1/\rho$ outside the boundary S; V is harmonic in T and outside S, continuous in S and regular at infinity. For a surface element $d\omega'$ as formulated above, the theory of Green's functions produces the potential

$$V = -\frac{1}{4\pi} \int_S \frac{\partial G}{\partial n} \frac{d\omega'}{r}.$$

So we can consider V as the potential of a material layer spread out over S with density

$$-\frac{1}{4\pi} \frac{\partial G}{\partial n}.$$

For a point outside S, the potential from a mass in P equals the potential of the layer on S.

To extend the balayage of a sphere to domain T we use a covering of T by a denumerable set of balls and a corresponding sequence of harmonic functions. In this sequence, each potential U_{n+1} is obtained from the preceding one U_n by balayage. It requires a subtle reasoning for which we refer to [16] and [10].

In modern potential theory one considers domains in $\mathbb{R}^n$ with Borel measures (instead of mass distributions) on sets of a general nature where the balayage produces another suitable measure.

Differential equations and dynamical systems
Poincaré has written extensively about differential equations and dynamical systems, writings that contain a huge amount of material on the interaction of analysis and geometry. We will discuss some examples from two major contributions, the *Mémoire* of 1882 and the *Méthodes Nouvelles de la Mécanique Célèste*.

The Mémoire. Poincaré's first appointment was to mining engineer in Vesoul. This was soon followed by an assignment in mathematics at the university of Caen where he wrote a Mémoire [13] that represented a completely new approach to the study of ordinary differential equations (ODEs). Although the Mémoire is restricted to autonomous 2nd order equations, the research programme sketched by Poincaré for ODEs is very general and at present the programme still dominates research. He writes in the beginning:

"Unfortunately it is evident that in general these equations [ODEs] can not be integrated using known functions, for instance using functions defined by quadrature. So, if we would restrict ourselves to the cases that we could study with definite or indefinite integrals, the extent of our research would be remarkably diminished and the vast majority of questions that present themselves in applications would remain unsolved."

And a few sentences on:

"The complete study of a function [solution of an ODE] consists of two parts:

1. Qualitative part (to call it like this), or geometric study of the curve defined by the function.
2. Quantitative part, or numerical calculation of the values of the function."

Consider the two-dimensional system $dx/dt = X(x,y)$, $dy/dt = Y(x,y)$ with orbits in the (x,y)-phaseplane. For the analysis of the system, Poincaré uses gnomonic projection; this is a cartographic projection of a plane onto a sphere (in cartography of course the other way around). The plane is tangent to the sphere and each point of the plane is projected through the centre of the sphere, producing two points on the spherical surface, one on the Northern hemisphere, one on the Southern. The equatorial plane separates the two hemispheres. Each straight line in the plane projects onto a great circle. So a tangent to an orbit in the plane projects onto a great circle that has one point in common with the projection of the orbit on the sphere. Such a point will be called a *contact*. A point on the great circle in the equatorial plane corresponds with infinity.

The advantage of this projection is that the plane is projected on a compact set which makes it much more tractable. The prize we pay for this is of course that we have to consider with special attention the equatorial great circle which corresponds with the points at infinity of the plane. A bounded set in the plane is projected on two sets, symmetric with respect to the centre of the sphere and located in the two hemispheres.

If in a point (x_0, y_0) we have not simultaneously $X = Y = 0$, (x_0, y_0) is a regular point of the system and we can obtain a power series expansion of the solution near (x_0, y_0).

If in a point (x_0, y_0) we have simultaneously $X = Y = 0$, (x_0, y_0) is a singular point. Under certain nondegeneracy conditions Poincaré finds four types for which he introduces the nowadays well-known names saddle, node, focus and centre. These are called singularities of first type. In the case of certain degeneracies we have singularities of the second type. Points on the equatorial great circle may correspond with singularities at infinity and can be investigated by simple transformations. For instance, if the point is not on the great circle $x = 0$, we transform

$$x = \frac{1}{z},\ y = \frac{u}{z}$$

and consider the transformed equation in z and u. If a point on the great circle $x = 0$ is investigated we transform

$$x = \frac{u}{z},\ y = \frac{1}{z}.$$

The next section of the Mémoire discusses the distribution and the number of singular points. Assuming that X and Y are polynomial and of the same degree and if X_m, Y_m indicate the terms of the highest degree, while we have *not* $xY_m - yX_m = 0$, then the number of singular points is at least 2 (if the curves described by $X = 0$ and $Y = 0$ do not intersect on the two hemispheres after projection, there must be an intersection on the equatorial circle). In addition it is shown that a singular point on the equator has to be a node or a saddle, in the plane one cannot spiral to or from infinity. An important concept to be introduced is *index*. Consider a closed curve, a cycle, located on one of the hemispheres. Taking one tour of the cycle in the positive sense, the expression Y/X jumps h times from $-\infty$ to $+\infty$, it jumps k times from $+\infty$ to $-\infty$. We call i with

$$i = \frac{h-k}{2}$$

the index of the cycle. It is then relatively easy to see that for cycles consisting of regular points one has:

- A cycle with no singular point in its interior has index 0.
- A cycle with exactly one singular point in its interior has index +1 if it is a saddle, index −1 if it is a node or a focus.
- If N is the number of nodes within a cycle, F the number of foci, C the number of saddles, the index of the cycle is $C - N - F$.
- If the number of nodes on the equator is $2N'$, the number of saddles $2C'$, the index of the equator is $N' - C' - 1$.
- The total number of singular points on the sphere is $2 + 4n,\ n = 0, 1, \ldots$.

A solution of the ODE may touch a curve or cycle in a point. Such a point is called a *contact*; in a contact the orbit and the curve have a common tangent. An algebraic curve or cycle has only a finite number of contacts with an orbit. Counting the number of contacts and the number of intersections for a given curve contains information about the geometry of the orbits.

A useful tool is the 'théorie des conséquents', what is now called the theory of Poincaré maps. We start with an algebraic curve parametrised by t so that $(x, y) = (\phi(t), \psi(t))$ with $\phi(t), \psi(t)$ algebraic functions; the endpoints A and B of the curve are given by $t = \alpha$ and $t = \beta$. Assume that the curve AB has no contacts and so has only intersections with the orbits. Starting on point M_1 with a semi-orbit, we may end up again on the curve AB in point M_1 which is the 'conséquent' of M_0. Nowadays we would call M_1 the Poincaré map of M_0 under the phase-flow of the ODE. Of course the semi-orbit may fail to return to AB, for instance because it will swirl around a focus far away or because it ends up at a node. It is also possible to choose the semi-orbit that moves in the opposite direction and return to the curve AB in M'; this point is called the 'antécédent' of M_0. If $M_0 = M_1$, the orbit is a cycle and Poincaré argues that returning maps correspond with either a cycle or a spiralling orbit. It is possible to discuss various possibilities with regards to the existence of cycles in which the presence or absence of singular points plays a part.

This analysis has important consequences for the theory of limit cycles. Semi-orbits will be a cycle, a semi-spiral not ending at a singular point, or a semi-orbit going to a singular point. Interior and exterior to a limit cycle there has always to be at least one focus or one node. Of the various possibilities considered it is natural to select annular domains, not containing singular points and bounded by cycles without contact. Such annular domains are often used to prove the existence of one or more limit cycles (Poincaré–Bendixson theory).

We note that Poincaré generalized the 'théorie des conséquents' later for higher-dimensional systems.

Non-integrability of Hamiltonian systems. In the eighteenth and nineteenth centuries, the integrability of conservative dynamics, later formulated as Lagrangian or Hamiltonian systems, was not in doubt among scientists. The only remaining problem, one thought, was to find the integrals in explicit physical models. In that period one could solve the gravitational two-body problem, one also made advances in the theories of rotating rigid bodies and rotating fluid masses and it seemed a matter of time and ingenuity to solve other cases like the gravitational three-body problem. In 1890, Poincaré, in his prize-winning essay [14] for the birthday of the Swedish king Oscar II, overturned this philosophy. Hamiltonian systems with two or more degrees of freedom are in general non-integrable; to put it differently, they are integrable only under additional assumptions. It took a very long time for scientists to understand and accept this result. Partly this was due to the philosophical impact of this non-existence theorem, but this can not be the full explanation as it was not simply a destructive result; it was accompanied by an extensive collection of new qualitative and quantitative concepts and tools to study dynamical systems. It is difficult to understand why so few scientists continued this line of research, perhaps it was the requirement of both analytic and geometric insight; people who followed it up in the first half of the twentieth century were Birkhoff, Denjoy, Siegel and Kolmogorov.

In [15], the fundamental theorem is formulated and proved in the case of the time-independent $2n$-dimensional Hamiltonian equations of motion

$$\dot{x} = \frac{\partial F}{\partial y}, \; \dot{y} = -\frac{\partial F}{\partial x}$$

with small parameter μ and the convergent expansion $F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$; F_0 depends on x only and the Jacobian is non-singular, $|\partial F_0/\partial x| \neq 0$. Suppose $F = F(x, y)$ is analytic and periodic in y in a domain D; $\Phi(x, y)$ is analytic in x, y in D, analytic in μ and periodic in y:

$$\Phi(x, y) = \Phi_0(x, y) + \mu\Phi_1(x, y) + \mu^2\Phi_2(x, y) + \cdots$$

The statement is then that with these assumptions, $\Phi(x, y)$ can not be an independent first integral of the Hamiltonian equations of motion unless we impose further conditions.

In the prize essay [14] and in the *Méthodes Nouvelles* [15] the approach to non-integrability is carried out in different ways. In the prize essay of 1890, the explicit theorem of non-integrability is formulated for two degrees of freedom systems with as application the planar, restricted, circular three-body problem. The starting point in the prize essay [14] is an unstable T-periodic solution with two zero characteristic exponents, one positive and one negative. The last two exponents correspond with an unstable and a stable invariant manifold emanating from the periodic solution; Poincaré calls these invariant manifolds 'surfaces asymptotiques'. Using series expansions for the solutions on the manifolds, he finds an infinite number of intersections instead of merging of the manifolds. This precludes the existence of homoclinic manifolds that would indicate the presence of a second integral. In the prize essay, the description of the geometry of the dynamics is tied in with the non-integrability results.

In the *Méthodes Nouvelles* [15], chapter 5 of volume 1, the technique is first analytic: a second integral should Poisson commute with and be independent of the Hamiltonian; expanding the second integral with respect to a suitable small parameter and applying these conditions leads to a contradiction unless additional assumptions are made (see also [21]). The dynamics and its geometry is later extensively studied in volume 3.

The proof is in both publications inspired by the actual Hamiltonian dynamics of stable and unstable manifolds. In chapter 32 of [15] we find the famous description of chaotic dynamical behaviour when considering the Poincaré section of an unstable periodic solution in a two degrees of freedom Hamiltonian system:

"If one tries to represent the figure formed by these two curves with an infinite number of intersections whereas each one corresponds with a double asymptotic solution, these intersections are forming a kind of lattice-work, a tissue, a network of infinite closely packed meshes. Each of the two curves must not cut itself but it must fold onto itself in a very complex way to be able to cut an infinite number of times through each mesh of the network.

One will be struck by the complexity of this picture that I do not even dare to sketch. Nothing is more appropriate to give us an idea of the intricateness of the three-body problem and in general all problems of dynamics where one has not a uniform integral and where the Bohlin series are divergent."

The double asymptotic solutions are the remaining homoclinic solutions that are produced by the intersections. The Bohlin series are formal series obtained by Bohlin for periodic solutions in celestial mechanics.

The Poincaré–Birkhoff theorem

A number of famous proofs developed by Poincaré to prove the existence and approximation of periodic solutions have been based on the implicit function theorem. In these cases one starts with a known periodic solution, for instance in the case of the three-body problem the limit case of one of the masses being zero, thus reducing the system to the known gravitational two-body problem. Starting with the known periodic solutions of this limiting problem one can try to obtain periodic solutions by continuation when increasing the vanishing mass from zero to a small value. Poincaré was bothered by the restriction of obtaining solutions in this way that are always close to known solutions, as the global view of the dynamics is still missing. He was able to formulate an important theorem regarding this question. In 1912, not long before his death, he wrote in the *Rendiconti del Circolo Mathematica di Palermo* [18]:

"I have never made public a work that is so unfinished; so I believe it is necessary to explain in a few words the reasons that have induced me to publish it and to start with the reasons that brought me to undertake this. Already a long time ago, I have shown the existence of periodic solutions of the three-body problem. However, the result is not quite satisfactory, for if the existence of each type of solution had been established for small values of the masses, one did not see what would happen for much larger values, which of the solutions would persist and in which order they would vanish. Thinking about this question I became convinced that the answer would depend on a certain geometric theorem being correct or false, a theorem of which the formulation is very simple, at least in the case of the restricted problem and of dynamics problems that have not more than two degrees of freedom."

George David Birkhoff (1884–1944)

Poincaré adds that during two years he had tried to prove the theorem but without success. However, he was absolutely convinced that the theorem was correct. What to do? Leave the matter rest?

"It seems that under these conditions, I would have to abstain of all publication of which I had not solved the problems. After all my fruitless efforts of long months, it seemed to me the wisest road to let the problem ripen and put it off my mind for a few years. That would have been very good if I had been certain that I could retake it some time, but at my age I could not say so. Also the importance of the matter is too great and the amount of results obtained already too considerable ..."

He had already been suffering from serious prostate problems for several years and it seemed to be a waste to keep all these ideas

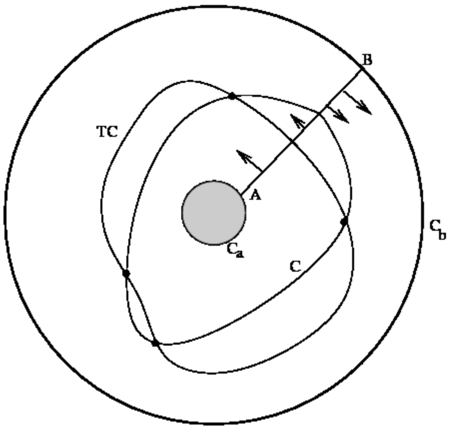


Figure 3 Twist map T of a ring shaped domain R

fruitless. As it turned out he was right, it is a beautiful fixed point theorem that combines geometric thinking with dynamics. In [12] it is classified under geometry, but it belongs as well to mechanics or differential equations. The theorem can be formulated as follows (see also Figure 3):

Poincaré–Birkhoff theorem. *Consider in $\mathbb{R}^2$ the ring R bounded by the smooth closed curves C_a and C_b. The map $T : R \mapsto R$ is continuous, one-to-one and area preserving. Applying T to R the points of C_a move in the negative sense, the points of C_b move in the positive sense (T is a 'twist' map). Then T has at least two fixed points.*

The theorem was proved by Birkhoff [2] (see also [21]) in a relatively simple way. It is difficult to understand why Poincaré did not produce a proof like this. Looking at the 39 pages of [18], one has the feeling that Poincaré just for once saw too many small difficulties, that he got bogged down in details.

The applications Poincaré had in mind can be indicated as follows. Consider a dynamical system derived from a time-independent Hamiltonian with two degrees of freedom. When studying the flow on a bounded energy manifold, one can make a Poincaré section of the flow in a neighbourhood of a periodic solution. This periodic solution is represented as a fixed point of the Poincaré map; if it is stable, the fixed point will be surrounded by closed curves corresponding with invariant tori around the periodic solution. The Poincaré map is area preserving, so that the application of the geometric theorem is possible if the twist condition has been satisfied. This can be checked by considering the rotation properties of the map on the closed curves.

An interesting aspect is that if one is able to apply the theorem for a Hamiltonian system, one finds not only two fixed points corresponding with two periodic solutions, but an infinite number. This is caused by the presence of an infinite number of the tori enabling us to construct an infinite number of rings. If the tori are close, the twist will usually be 'small' and in this case the period of the periodic solutions will be large. A strong twist of the map produces short-periodic solutions, a small twist long-periodic solutions.

Other applications can be found in problems of three-dimensional divergence-free flow and conservative billiard dynamics, see [3].

Conclusion

Henri Poincaré is sometimes described as 'the last universal scientist'. He stressed the importance of interaction between different fields of knowledge as a source of innovation in science. We restricted ourselves to the interaction of geometry and analysis in his work, for Poincaré there were many more fields of interaction from astronomy and astrophysics, physics, engineering and the humanities.

Acknowledgements

Comments upon the manuscript of this paper by Henk Broer and Steven Wepster are gratefully acknowledged.

Referenties

1 André Bellivier, *Henri Poincaré, ou la vocation souveraine*, Paris, Gallimard, 1956.

2 G.D. Birkhoff, Proof of Poincaré's geometric theorem, *Trans. AMS* 14, pp. 14–22 , 1913.

3 G.D. Birkhoff, On the periodic motions of dynamical systems, *Acta Mathematica* 50, pp. 359–379, 1927.

4 Michel Chasles, *Discours d'Inauguration du Cours de Géométrie de Faculté des Sciences, Paris*, 22 dec. 1846. Also included in *Traité de Géométrie Supérieure*, 2e ed. 1880, Paris, Gauthier-Villars.

5 J. Gray, *The soul of the fact: a scientific biography of Henri Poincaré*, Princeton University Press, 2012.

6 Christiaan Huygens, *Traité de la lumière*, Publ. by P. van der Aa, Leiden, 1690 (Dutch transl. D. Eringa, *Epsilon uitgaven*, 1990)

7 F. Klein, Ueber lineare Differentialgleichungen der zweiten Ordnung, lecture at Sommersemester 1894, elaborated by E. Ritter, Göttingen, 1894.

8 J.-L. Lagrange, *Mécanique Analytique*, 2 vols., Paris, 1788 (reprint ed. Albert Blanchard, Paris, 1965, with notes by various scientists).

9 J.-L. Lagrange, *Théorie des Functions Analytiques*, Bachelier, Paris, 1847 (1st ed. 1797, 2d extended ed. 1806), with a note by J.-A. Serret.

10 J. Mawhin, Henri Poincaré and the partial differential equations of mathematical physics, in *The scientific legacy of Poincaré* (E. Charpentier, E. Ghys and A. Lesne, eds.), AMS, History of mathematics 36, pp. 373–391, 2010.

11 Isaac Newton, *Philosophiae Naturalis Principia Mathematica*, London, 1688 (transl. by Andrew Motte, rev. by F. Cajori, University of California Press, 1934, new transl. by I.B. Cohen and A. Whitman assisted by J. Budenz, University of California Press, 1999).

12 Henri Poincaré, *Oeuvres de Henri Poincaré publiées sous les auspices de l'Académie des Sciences*, Vol. 1–12, Gauthier-Villars, Paris, 1916-1954.

13 Henri Poincaré, Mémoire sur les courbes définies par une équation différentielle, *J. de Mathématiques*, 3e série, vol. 7, pp. 375–422 (1881), vol. 8, p. 251–296 (1882). *Annalen*, vol. 19, pp. 553–564 (1882), also in [12] vol. 2, p. 92–105.

14 Henri Poincaré, Sur le probléme des trois corps et les équations de la dynamique, *Acta Mathematica*, 13, pp. 1–270 (1890), also in [12] vol. 7, pp. 262–479.

15 Henri Poincaré, *Les Méthodes Nouvelles de la Mécanique Célèste*, 3 vols. Gauthier-Villars, Paris, 1892, 1893, 1899.

16 Henri Poincaré, *Théorie du potentiel Newtonien*, 366 pp., Gauthier-Villars, Paris, 1899 (Eds. E. le Roy and G. Vincent), 2nd ed. Gauthier-Villars, Paris, 1912.

17 Henri Poincaré, *La Valeur de la Science*, Flammarion, Paris, 1905.

18 Henri Poincaré, Sur un théorème de géometrie, *Rend. Circolo Mat. Palermo* 33, pp. 375–407, 1912; also in [12], vol. 6, pp. 499–538.

19 B. Riemann, *Gesammelte Mathematische Werke und wissenschaftlicher Nachlass* (Complete Works), ed. H. Weber with cooperation of R. Dedekind, B.G. Teubner, Leipzig, 1876.

20 B. Riemann, *Über die Hypothesen, welche die Geometrie zu Grunde liegen*, Göttingen, June 1854, ed. and annotated by Hermann Weyl, Berlin, Julius Springer (1921), also in [19].

21 F. Verhulst, *Henri Poincaré, impatient genius*, Springer, 2012.