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Parametric excitation in non-linear dynamics

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Abstract

Consider a one-mass system with two degrees of freedom, non-linearly coupled, with parametric excitation in one direction. Assuming the internal resonance 1:2 and parametric resonance 1:2 we derive conditions for stability of the trivial solution by using both the harmonic balance method and the normal form method of averaging. If the trivial solution becomes unstable, a stable periodic solution may emerge, there are also cases where the trivial solution is stable and co-exists with a stable periodic solution; if both the trivial solution and the periodic solution(s) are unstable, we find an attracting torus with large amplitudes by a Neimark–Sacker bifurcation. The results of the harmonic balance method and averaging are compared, as well as the results on the Neimark–Sacker bifurcation obtained by the numerical software package CONTENT and by averaging. In all cases we have good agreement.

Keywords: Dynamical systems; Neimark-Sacker bifurcation; Averaging method and tori

1. Introduction

Non-linear vibrating systems often consist of two—or even more—subsystems, where one of them is excited, the primary system, and the other ones are coupled through non-linear terms; they form the secondary system or excited system. The primary system is an oscillator which can be excited externally, parametrically or by self-excitation, while the secondary system is excited indirectly through the non-linear coupling.

In particular, autoparametric systems [1–3] represent an important example. A typical property of autoparametric systems is the existence of a semitrivial solution of the differential equations of motion in which the primary system oscillates while the secondary system is at rest. In such problems, it is essential to study the boundary limits of the stability region for the trivial solution and to establish whether they are determined by the stability limits of the primary system, i.e., whether they are changed by the action of the secondary (excited) system [3].

In the case where the non-linear coupling terms do not allow for the existence of a semitrivial solution, the secondary system must oscillate when the primary system is oscillating; the system is sometimes called hetero-parametric. In [4], a hetero-parametric example of an externally excited single-mass system having two degrees of freedom was analysed.

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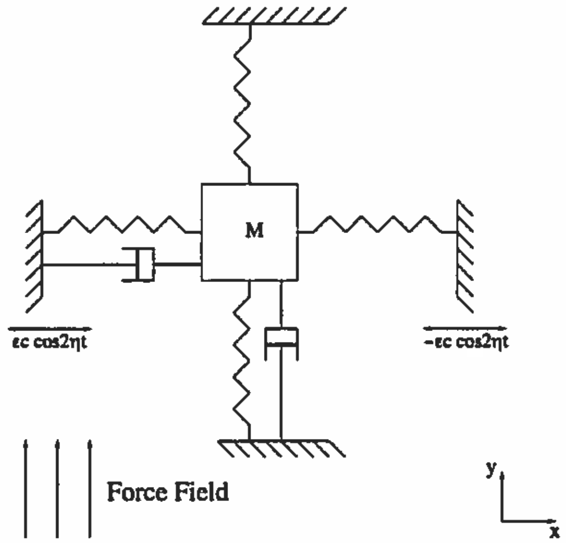

Fig. 1. Two degrees of freedom single-mass system with simply parametric excitation in the presence of a force field parallel to the y-direction.

Also, in the present paper, we shall consider a single-mass system, but with parametric excitation in the primary system and a non-linear coupling expressed by second degree terms in the differential equations. The parametric excitation acts, for example, due to kinematic excitation of the supports through non-linear springs (see Fig. 1). The system is simply parametric when both non-linear springs in the x-direction are identical and two kinematic excitations are simultaneously acting, having the same amplitude and frequency but opposite phase.

The equations of motion are

$$\ddot{x} + \delta x^2\dot{x} + \delta_0\dot{x} + (1 + \varepsilon c\cos 2\eta t)x + \gamma x^3 + axy = 0,$$

$$\ddot{y} + \kappa\dot{y} + q^2 y + bx^2 = 0. \quad (1)$$

For the damping coefficients we have δ, δ_0, $\kappa \geqslant 0$, furthermore $c > 0$, $\gamma \geqslant 0$, ε is a small positive parameter. Apart from linear damping we have assumed the presence of progressive damping to ensure a limited vibration amplitude, even at parametric resonance.

We could drop the assumption of a parallel force field and look at the consequences of much more general forces. We stress here that the non-linear coupling terms chosen here are typical. We can add, for instance, 10 quadratic terms ($x\dot{y}$, $y\dot{y}$, etc.) in the equations for x and y but most of these terms will vanish by averaging normalisation. In particular, for the x-equation all the added potential terms will be of no influence, of the other terms only $\dot{x}y, \dot{x}\dot{y}, x\dot{y}$ will remain after averaging. In the equation for y we have a similar effect at this order of approximation.

In the case of a single kinematic excitation, the system is subjected to a combination of parametric and external excitation (see [5]). A one-mass system with two degrees of freedom with parametric excitation in one direction is treated in [6].

Note that nowadays the parametric excitation can be provided by modern actuators in mechatronic and smart structures, e.g., in actively controlled magnetic bearings. In this way, the elastic mounting with periodically varying stiffness can practically be conceived.

In the next section, we present a harmonic balance calculation which gives us the response and resonance curves for certain sets of parameters. In Section 3, we perform a scaling and first-order averaging to the equations of motion. This leads to explicit results on the stability of the trivial solution. Also, we find families of periodic solutions of which we determine the stability; because of the complexity of the expressions this is quite surprising. A comparison of quantitative results obtained by harmonic balance and averaging calculations are given in Section 4. This is of interest as the harmonic balance method involves no error estimate and no existence of periodic solutions result; see for a discussion [2, Section 9.6]. The averaging method, on the other hand, is a normal form method which produces both quantitative results with error estimates and existence properties; for an introduction and references see also [2, Section 9].

A particular situation arises in Section 3 when both the trivial solution and the periodic solution(s) are unstable. In Section 5, we find that this instability is triggered off by a Neimark–Sacker bifurcation (secondary Hopf bifurcation) of the periodic solution. This results in an attracting torus with fairly large amplitudes.

The Neimark–Sacker bifurcation was first pinpointed by using the numerical bifurcation program CONTENT. An unusual feature is that this bifurcation can also be identified and analysed using the averaging method.

It is remarkable how different this result of an attracting torus is from the dynamics of a related system studied in [7]. In that paper an internal 1:1-resonance is assumed which leads to period-doubling bifurcations and the presence of a strange attractor. For related topics on parametric excitation (see [10,11]).

2. Harmonic balance calculations

The non-linear coupling terms play an important role in determining the frequency tuning so as to produce significant effects. For example, in the considered system, both coupling terms are of second degree and therefore it can be judged that the optimal tuning into internal resonance is of 1:2 type, i.e., the strongest deflection can be expected for $q \simeq 2$.

A periodic solution of Eq. (1) in the internal resonance 1:2 can be approximated using the harmonic balance method; when inserting the term $\cos 2(\eta t - \psi)$ for $\cos 2\eta t$ in order to facilitate the computations, we look for a solution of the form

$$x = R\cos\eta t,$$

$$y = Y_0 + A\cos 2\eta t + B\sin 2\eta t, \tag{2}$$

where ψ is the initial phase, representing the shift of the time origin suitable for simplifying the analytical solution. Equating corresponding terms of the Fourier series yields the following algebraic equations:

$$(1-\eta^2+\tfrac{3}{4}\gamma R^2)R + a(Y_0+\tfrac{1}{2}A)R = -\tfrac{1}{2}\varepsilon c R\cos 2\psi,$$

$$-\tfrac{1}{4}\delta\eta R^3 + (\tfrac{1}{2}aB - \delta_0\eta)R = -\tfrac{1}{2}\varepsilon c R\sin 2\psi,$$

$$(q^2-4\eta^2)A + 2\kappa\eta B + \tfrac{1}{2}bR^2 = 0,$$

$$-2\kappa\eta A + (q^2-4\eta^2)B = 0,$$

$$q^2 Y_0 + \tfrac{1}{2}bR^2 = 0. \tag{3}$$

From the third, fourth and fifth equations of (3), the following relations are obtained:

$$Y_0 = -\frac{1}{2q^2}bR^2,$$

$$A = -\frac{1}{2}bR^2\frac{q^2-4\eta^2}{\Delta},$$

$$B = -bR^2\frac{\kappa\eta}{\Delta}, \tag{4}$$

where

$$\Delta = (q^2-4\eta^2)^2 + 4\kappa^2\eta^2. \tag{5}$$

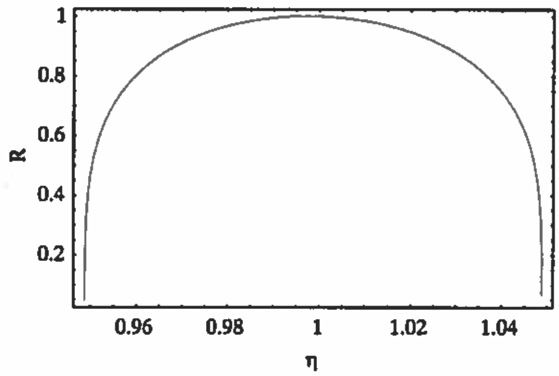

Fig. 2. Resonance curve $R(\eta)$ corresponding to the parameters $q = 2$, $\gamma = 0$ without coupling ($a = b = 0$).

In particular, the vibration amplitude of the secondary (excited) system is given by

$$r = \sqrt{A^2+B^2} = \tfrac{1}{2}bR^2\Delta^{-1/2}. \tag{6}$$

Squaring and adding the first two equations of (3) and using (4), we obtain the frequency response amplitude of the primary system:

$$\left[1-\eta^2+\frac{3}{4}\gamma R^2 - \frac{1}{2}ab\left(\frac{2}{q^2}+\frac{q^2-4\eta^2}{\Delta}\right)R^2\right]^2 + \left(\delta_0\eta + \frac{1}{4}\delta\eta R^2 + \frac{1}{2}ab\frac{\kappa\eta}{\Delta}R^2\right)^2 - \frac{1}{4}\varepsilon^2 c^2 = 0, \tag{7}$$

which can be used for determining R in dependence on η. Then, by means of Eq. (6), the vibration amplitude r in dependence on η can also be determined. It is interesting to see that R depends only on the product ab, i.e. the case when a and b are both positive or negative yields the same result, if the absolute values of a and b are the same.

The results of this harmonic balance approach are presented in terms of resonance curves showing the oscillation amplitude R (in x-direction), r (in y-direction) and constant deflection Y_0 in dependence of η, the half-frequency of parametric excitation. The following parameter values are used in this section: $\varepsilon c = 0.2$, $\delta = 0.4$, $\kappa = 0.1$, and $\delta_0 = 0$. The relatively high progressive damping coefficient ensures rather limited vibration amplitudes at parametric resonance. For reasons of comparison, in Fig. 2 ($\gamma = 0$) and Fig. 3 ($\gamma = 0.2$), we show the resonance curve $R(\eta)$

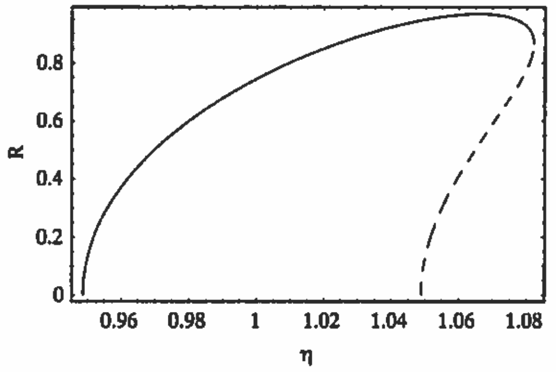

Fig. 3. Resonance curve $R(\eta)$ corresponding to the parameters $q=2$, $\gamma=0.2$ without coupling ($a=b=0$); the dashes refer to the unstable solution.

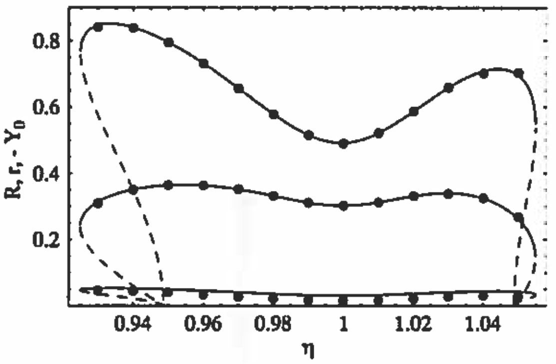

Fig. 4. Resonance curve $R(\eta)$, $r(\eta)$ and constant deflection Y_0 corresponding to the parameters $q=2$, $\gamma=0$ with coupling ($a=b=0.5$); numerical results are indicated by dots.

for the case of zero cross-coupling ($a=b=0$). The full line denotes the stable solution, the dashed line the unstable one.

Fig. 4 shows the analytically predicted amplitudes R, r and the constant deflection Y_0 in dependence on the frequency η for the case $q=2$, $\gamma=0$ and cross-coupling ($a=b=0.5$), as determined by Eqs. (2). We can see that the resonance curves for the motion in the x-direction exhibit two peaks, having maxima smaller than the corresponding maxima for the case where $a=b=0$ (see Fig. 2). In the same figure, the results of the analysis have been supplemented by direct numerical solution of the differential equations (1). We can see that there is a good agreement between analytical and numerical results, also due to the relatively small value of constant deflection.

In a further analysis, the numerical solution was obtained when increasing and decreasing the excitation frequency η by small steps in suitable time intervals. The extreme values (maxima and minima) of the oscillation amplitudes along the x- and y-direction, denoted $[x]$ and $[y]$, were recorded and are shown in Fig. 5. The arrows indicate the direction of jumps in vibration amplitude due to the continuous change of excitation frequency. As a result, one can observe a noticeable bilateral hysteresis effect.

Two sets of similar diagrams show the effect of detuning from internal resonance: $q=1.8$ (Figs. 6 and 7) and $q=2.2$ (Figs. 8 and 9). In both cases, the amplitudes R are higher than for the case $q=2$, while the amplitudes r are smaller than for $q=2$

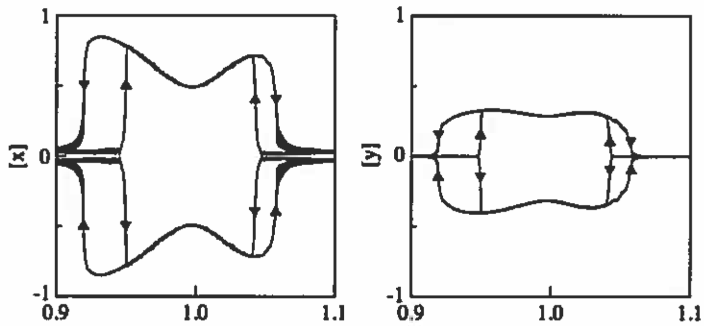

Fig. 5. Numerically computed extreme values $[x]$ and $[y]$ of oscillation amplitudes along the x- and y-direction versus excitation frequency η in the case $\varepsilon c=0.2$, $\delta_0=0$, $\delta=0.4$, $\kappa=0.1$, $a=b=0.5$, $\gamma=0$, $q=2$. The arrows indicate the direction of the jumps in vibration amplitude when increasing or decreasing the frequency.

(see Figs. 4 and 5). The shape of the resonance curves does not exhibit the double peak form anymore: for $q = 1.8$, the resonance curve is bended towards the higher values of η and towards the smaller values of η for the alternative $q = 2.2$. The vibration character is close to a harmonic one because for both $[x]$ and $[y]$ the scatter of the record points is small, although the solution cannot be considered as a periodic solution in the strict sense but rather as a transient vibration with slightly changing frequency. The hysteresis effect is reduced for $q = 1.8$, but becomes more pronounced for $q = 2.2$.

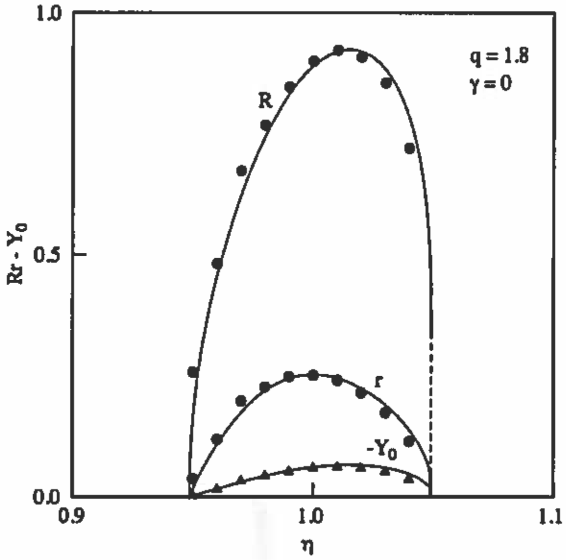

Fig. 6. Oscillation amplitudes R, r and constant deflection Y_0 versus excitation frequency η in the case $\varepsilon c=0.2$, $\delta_0=0$, $\delta=0.4$, $\kappa=0.1$, $a=b=0.5$, $\gamma=0$, $q=1.8$. Comparison between analytical predictions (full line, stable solution; dashed line, unstable solution) and numerical solutions.

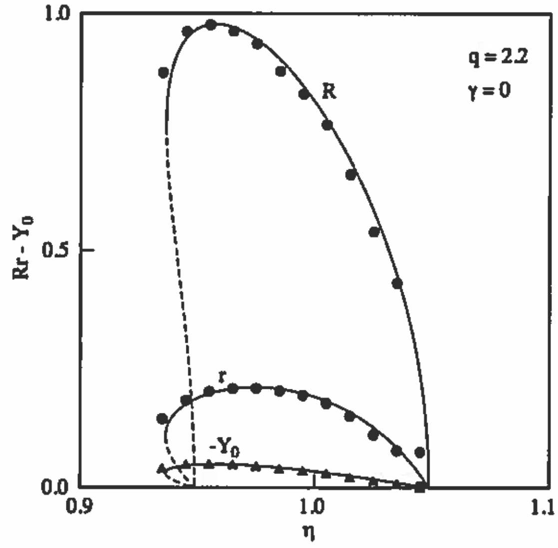

Fig. 8. Oscillation amplitudes R, r and constant deflection Y_0 versus excitation frequency η in the case $\varepsilon c=0.2$, $\delta_0=0$, $\delta=0.4$, $\kappa=0.1$, $a=b=0.5$, $\gamma=0$, $q=2.2$. Comparison between analytical predictions (full line, stable solution; dashed line, unstable solution) and numerical solutions.

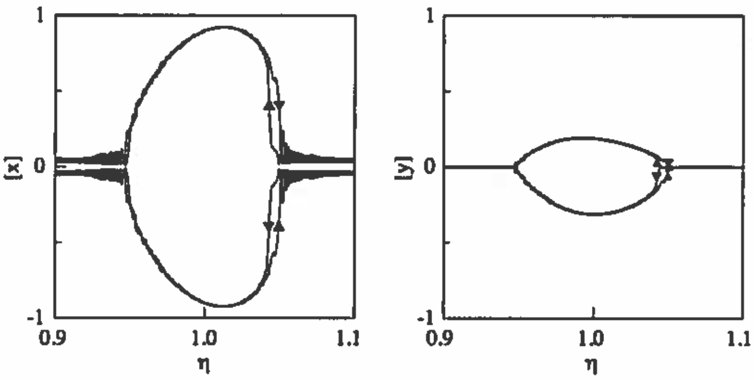

Fig. 7. Numerically computed extreme values $[x]$ and $[y]$ of oscillation amplitudes along the x- and y-direction versus excitation frequency η in the case $\varepsilon c = 0.2$, $\delta_0 = 0$, $\delta = 0.4$, $\kappa = 0.1$, $a = b = 0.5$, $\gamma = 0$, $q = 1.8$. The arrows indicate the direction of the jumps in vibration amplitude when increasing or decreasing the frequency.

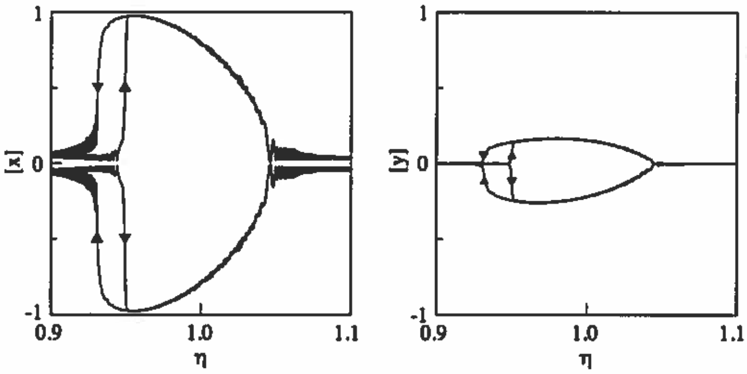

Fig. 9. Numerically computed extreme values [x] and [y] of oscillation amplitudes along the x- and y-direction versus excitation frequency η in the case $\varepsilon c = 0.2$, $\delta_0 = 0$, $\delta = 0.4$, $\kappa = 0.1$, $a = b = 0.5$, $\gamma = 0$, $q = 2.2$. The arrows indicate the direction of the jumps in vibration amplitude when increasing or decreasing the frequency.

3. Scaling and first-order averaging

We shall now use a normal form method (averaging) which enables us to obtain more detailed information about the solutions and for which precise error estimates are known. As before we consider system (1) in the vicinity of the origin. To make this more explicit we introduce the following general scaling:

$$x = \varepsilon^{\nu_x}\bar{x}, \quad y = \varepsilon^{\nu_y}\bar{y}, \quad a = \varepsilon^{\nu_a}\bar{a}, \quad b = \varepsilon^{\nu_b}\bar{b},$$

$$\kappa = \varepsilon^{\nu_\kappa}\bar{\kappa}, \quad \delta = \varepsilon^{\nu_\delta}\bar{\delta},$$

$$\gamma = \varepsilon^{\nu_\gamma}\bar{\gamma}, \quad \text{and} \quad \delta_0 = \varepsilon^{\nu_{\delta_0}}\bar{\delta}_0,$$

where, as before, ε is a small, positive parameter. Balancing the terms in the first equation of system (1) yields

$$\nu_\delta + 2\nu_x = \nu_\gamma + 2\nu_x = \nu_a + \nu_y = \nu_{\delta_0} = 1.$$

Balancing the terms in the second equation of system (1) yields

$$\nu_\kappa = \nu_b + 2\nu_x - \nu_y = 1.$$

We have here six equations with eight unknowns, hence the scaling is not unique. Solving these equations yields

$$\nu_{\delta_0} = \nu_\kappa = 1, \quad \nu_\gamma = \nu_\delta = 1 - 2\nu_x, \quad \nu_a = 1 - \nu_y,$$

$$\nu_b = 1 + \nu_y - 2\nu_x.$$

The third and fourth equations yield

$$0 \leqslant \nu_x \leqslant 1/2, \tag{8}$$

$$0 \leqslant \nu_y \leqslant 1. \tag{9}$$

We can freely choose ν_x as well as ν_y. However, one should first examine the magnitude of the physical parameters involved in system (1) before fixing the values of ν_x and ν_y. Here, we base our choice on the parameters of Section 2. We decided to take the parameter δ to be $O(1)$. This immediately means $\nu_x = 1/2$. The parameter a was taken to be equal to b. This implies $\nu_y = \nu_x = \nu_a = \nu_b = 1/2$ and thus, $x = \varepsilon^{1/2}\bar{x}$, $y = \varepsilon^{1/2}\bar{y}$, $a = \varepsilon^{1/2}\bar{a}$, $b = \varepsilon^{1/2}\bar{b}$, $\kappa = \varepsilon\bar{\kappa}$, $\delta = \bar{\delta}$, $\gamma = \bar{\gamma}$ and $\delta_0 = \varepsilon\bar{\delta}_0$.

Introducing this scaling into system (1) and omitting the bars yields

$$\ddot{x} + (1 + \varepsilon c \cos 2\eta t)x + \varepsilon\delta_0\dot{x} + \varepsilon a x y + \varepsilon(\delta x^2\dot{x} + \gamma x^3) = 0,$$

$$\ddot{y} + q^2 y + \varepsilon\kappa\dot{y} + \varepsilon b x^2 = 0. \tag{10}$$

The next steps are the usual ones in averaging approximations; see for instance [8,12, Chapter 11].

We introduce the following transformation:

$$x(t) = x_1(t)\cos t + x_2(t)\sin t,$$

$$\dot{x}(t) = -x_1(t)\sin t + x_2(t)\cos t,$$

$$y(t) = y_1(t)\cos qt + \frac{1}{q} y_2(t)\sin qt,$$

$$\dot{y}(t) = -q y_1(t)\sin qt + y_2(t)\cos qt.$$

Averaging the resulting system of differential equations for $x_{1,2}, y_{1,2}$ yields

$$\dot{x}_{1a} = \varepsilon\left\{ \left(-\frac{a}{q}\alpha(q)y_{2a} - \frac{1}{2}\delta_0\right)x_{a1} + (a\alpha(q)y_{1a} + c\alpha(2\eta))x_{2a} + O(|X|^3) \right\},$$

$$\dot{x}_{2a} = -\varepsilon\left\{ (-a\alpha(q)y_{1a} - c\alpha(2\eta))x_{1a} + \left(\frac{-a}{q}\alpha(q)y_{2a} + \frac{1}{2}\delta_0\right)x_{2a} + O(|X|^3) \right\},$$

$$\dot{y}_{1a} = \frac{\varepsilon}{q}(-2b\alpha(q)x_{1a}x_{2a} - \tfrac{1}{2}\kappa q y_{1a}),$$

$$\dot{y}_{2a} = -\varepsilon(-b\alpha(q)x_{1a}^2 + b\alpha(q)x_{2a}^2 + \tfrac{1}{2}\kappa y_{2a}), \qquad (11)$$

where the subscript a indicates 'approximation' and

$$\alpha(q) = \begin{cases} -\frac{1}{4} & \text{if } q = \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

3.1. Stability of the trivial solution (the general case)

Linearising system (11) around the origin yields

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\frac{1}{2}\delta_0\varepsilon & c\alpha(2\eta)\varepsilon & 0 & 0 \\ c\alpha(2\eta)\varepsilon & -\frac{1}{2}\delta_0\varepsilon & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\kappa\varepsilon & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\kappa\varepsilon \end{pmatrix}.$$

The eigenvalues of the matrix A are

$$\lambda_1 = (-c\alpha(2\eta) - \tfrac{1}{2}\delta_0)\varepsilon, \quad \lambda_2 = (c\alpha(2\eta) - \tfrac{1}{2}\delta_0)\varepsilon,$$

$$\lambda_3 = \lambda_4 = -\tfrac{1}{2}\kappa\varepsilon.$$

Conclusion. According to this linear analysis we can distinguish between the following cases:

- $\eta \neq \pm 1$: In this case, all the eigenvalues have a negative real part. The averaged trivial solution is asymptotically stable. This result holds also for the original system (10).
- $\eta = \pm 1$ and $\delta_0 > c/2$: This case yields asymptotic stability as well for the averaged system (11). The result also holds for the original system (10). This is remarkable as we have resonance and energy transfer to the system.
- $\eta = \pm 1$ and $\delta_0 < c/2$: One eigenvalue has, in this case, a positive real part. This implies the instability of the averaged trivial solution. The result also holds for the original system (10).
- $\eta = \pm 1$ and $\delta_0 = c/2$: In this case, one of the four eigenvalues of the matrix A is zero, whereas the other three are eigenvalues with negative real parts. Linear analysis is, in this case, not conclusive regarding the stability properties of the averaged trivial solution. We must therefore examine more closely the flow in the centre manifold. See Appendix A [9,13,16].

3.2. The near resonance case $q^2 = 4 + \varepsilon\sigma$, $\eta = 1 + \varepsilon\mu$

Let us assume that the half-frequency of parametric excitation η is not exactly equal to ± 1. We shall instead allow a margin of detuning of magnitude $\varepsilon\mu$ with μ being a constant not dependent on ε. We also allow a margin of detuning in q^2 of magnitude $\varepsilon\sigma$. In this way, system (10) becomes

$$\ddot{x} + (1 + \varepsilon c\cos 2(1+\varepsilon\mu)t)x + \varepsilon\delta_0\dot{x} + \varepsilon a x y + \varepsilon(\delta x^2\dot{x} + \gamma x^3) = 0,$$

$$\ddot{y} + 4y + \varepsilon\kappa\dot{y} + \varepsilon\sigma y + \varepsilon b x^2 = 0. \qquad (12)$$

Transforming $\tau = (1 + \varepsilon\mu)t$ and differentiating with respect to τ yields the following:

$$\ddot{x} + (1 + \varepsilon c\cos 2\tau)x - 2\varepsilon\mu x + \varepsilon\delta_0\dot{x} + \varepsilon a x y + \varepsilon(\delta x^2\dot{x} + \gamma x^3) + O(\varepsilon^2) = 0,$$

$$\ddot{y}+4(1-2\varepsilon\mu+\varepsilon\sigma/4)y+\varepsilon\kappa\dot{y}+\varepsilon bx^2+O(\varepsilon^2)=0. \tag{13}$$

Applying the method of averaging as before produces

$$\begin{aligned}
\dot{x}_{1a}&=\varepsilon\left\{\left(\frac{a}{8}y_{2a}-\frac{\delta_0}{2}-\frac{\delta}{8}(x_{1a}^2+x_{2a}^2)\right)x_{1a}\right.\\
&\quad\left.+\left(-\frac{a}{4}y_{1a}-\frac{c}{4}-\mu+\frac{3\gamma}{8}(x_{1a}^2+x_{2a}^2)\right)x_{2a}\right\},\\
\dot{x}_{2a}&=-\varepsilon\left\{\left(\frac{a}{4}y_{1a}+\frac{c}{4}-\mu+\frac{3\gamma}{8}(x_{1a}^2+x_{2a}^2)\right)x_{1a}\right.\\
&\quad\left.+\left(\frac{a}{8}y_{2a}+\frac{\delta_0}{2}+\frac{\delta}{8}(x_{1a}^2+x_{2a}^2)\right)x_{2a}\right\},\\
\dot{y}_{1a}&=\frac{\varepsilon}{2}\left(\frac{1}{2}bx_{1a}x_{2a}-\kappa y_{1a}+\frac{\sigma}{4}y_{2a}-2\mu y_{2a}\right),\\
\dot{y}_{2a}&=-\varepsilon\left(\frac{b}{4}x_{1a}^2-\frac{b}{4}x_{2a}^2+\frac{\sigma}{2}y_{1a}\right.\\
&\quad\left.+\frac{1}{2}\kappa y_{2a}-4\mu y_{1a}\right).
\end{aligned}\tag{14}$$

Linearising system (14) around the origin yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}X\\Y\end{pmatrix}=A\begin{pmatrix}X\\Y\end{pmatrix},$$

where

$$A=\begin{pmatrix}
-\frac{\delta_0}{2}\varepsilon & -(\frac{c}{4}+\mu)\varepsilon & 0 & 0\\
-(\frac{c}{4}-\mu)\varepsilon & -\frac{\delta_0}{2}\varepsilon & 0 & 0\\
0 & 0 & -\frac{1}{2}\kappa\varepsilon & (\frac{\sigma}{8}-\mu)\varepsilon\\
0 & 0 & (-\frac{1}{2}\sigma+4\mu)\varepsilon & -\frac{1}{2}\kappa\varepsilon
\end{pmatrix}$$

The eigenvalues of the matrix A are

$$\lambda_1=-\frac{2\delta_0+\sqrt{c^2-16\mu^2}}{4}\varepsilon,$$

$$\lambda_2=\frac{-2\delta_0+\sqrt{c^2-16\mu^2}}{4}\varepsilon,$$

$$\lambda_3=-\frac{\kappa\varepsilon}{2}+\mathrm{i}\frac{(\sigma-8\mu)\varepsilon}{4},$$

$$\lambda_4=-\frac{\kappa\varepsilon}{2}-\mathrm{i}\frac{(\sigma-8\mu)\varepsilon}{4}.$$

Conclusion. We introduce the quantity $z_\mu=\mu/c$ to analyse the eigenvalues of the matrix A. z_μ will play in the sequel an important role in the study of the periodic solutions. This will be the main subject of the next subsection. According to linear analysis, we can distinguish between the following cases:

- $|z_\mu|>\frac{1}{4}$, $c^2-16\mu^2<0$: this case encloses two subcases, namely:
  1. $\delta_0>0$: This case yields asymptotic stability of the averaged trivial solution as all the eigenvalues have negative real part. The result holds also for the original system (13).
  2. $\delta_0=0$: In this case, two of the four eigenvalues are purely imaginary, whereas the other two are eigenvalues with negative real parts. Linear analysis is, in this case, not conclusive regarding the stability of the averaged trivial solution. We therefore must examine more closely the flow in the centre manifold. See Appendix A.
- $|z_\mu|\leqslant\frac{1}{4}$, $c^2-16\mu^2\geqslant 0$: this case encloses three subcases, namely:
  1. $\delta_0>1/2\sqrt{c^2-16\mu^2}$: This case yields asymptotic stability of the averaged trivial solution as all the eigenvalues have negative real part. This result holds also for the original system (13).
  2. $\delta_0<1/2\sqrt{c^2-16\mu^2}$: In this case, $\lambda_2>0$ which immediately implies the instability of the averaged trivial solution. This result holds also for the original system (13).
  3. $\delta_0=1/2\sqrt{c^2-16\mu^2}$: According to whether $|z_\mu|=1/4$ or not, we have ($\lambda_1=\lambda_2=0$) or ($\lambda_2=0$ and $\lambda_1<0$). The rest of the eigenvalues has in both cases negative real parts. Linear analysis is, in this case, not conclusive regarding the stability of the averaged trivial solution. We therefore must examine more closely the flow in the centre manifold. See Appendix A.

Remark. The case ($|z_\mu| > 1/4$ and $\delta_0 = 0$) corresponds to a Hopf-bifurcation, with respect to the parameter δ_0, of the averaged trivial solution as two of the four eigenvalues cross the imaginary axis with non-zero speed. The introduction of the detuning parameter μ has a clear influence on the stability properties of the trivial solution.

3.3. The periodic solutions

We look in this section for periodic solutions of system (13). We introduce, for this purpose, the phase-amplitude transformation

$$x(t) = R_1(t)\cos(t + \phi(t)),$$

$$\dot{x}(t) = -R_1(t)\sin(t + \phi(t)),$$

$$y(t) = R_2(t)\cos(2t + \psi(t)),$$

$$\dot{y}(t) = -2R_2(t)\sin(2t + \psi(t)).$$

Averaging the resulting system yields

$$\dot{R}_{1a} = \varepsilon R_{1a}\left\{\frac{a}{4}R_{2a}\sin(2\phi - \psi) + \frac{c}{4}\sin 2\phi - \frac{\delta}{8}R_{1a}^2 - \frac{\delta_0}{2}\right\},$$

$$\dot{\phi}_a = \varepsilon\left\{\frac{a}{4}R_{2a}\cos(2\phi - \psi) + \frac{c}{4}\cos 2\phi + \frac{3}{8}\gamma R_{1a}^2 - \mu\right\},$$

$$\dot{R}_{2a} = \frac{\varepsilon}{2}R_{2a}\left\{\frac{-bR_{1a}^2}{4R_{2a}}\sin(2\phi - \psi) - \kappa\right\},$$

$$\dot{\psi}_a = \frac{\varepsilon}{2}\left\{\frac{bR_{1a}^2}{4R_{2a}}\cos(2\phi - \psi) + \frac{1}{2}(\sigma - 8\mu)\right\}. \quad (15)$$

Equating the right-hand side of system (15) to zero yields the non-trivial critical points of these equations which correspond with 2π-periodic solutions of the original system (13). Without loss of generality we assume $R_{2a} > 0$. For notational simplicity, we introduce the following quantities:

$$\alpha = \frac{-3\gamma(4\kappa^2 + \sigma_\mu^2) + ab\sigma_\mu}{|b|c\sqrt{4\kappa^2 + \sigma_\mu^2}},$$

$$\beta = \frac{\delta(4\kappa^2 + \sigma_\mu^2) + 2ab\kappa}{|b|c\sqrt{4\kappa^2 + \sigma_\mu^2}},$$

$$z = \frac{\delta_0}{c}, \quad z_\mu = \frac{\mu}{c}, \quad \text{and} \quad \sigma_\mu = \sigma - 8\mu.$$

The results for the non-trivial critical points corresponding with periodic solutions are summarised as follows:

$$R_{1a}^2 = \frac{2\sqrt{4k^2 + \sigma_\mu^2}}{|b|}R_{2a}, \quad (16)$$

$$\cos(2\phi - \psi) = \frac{-\operatorname{sgn}(b)\sigma_\mu}{\sqrt{4\kappa^2 + \sigma_\mu^2}}, \quad (17)$$

$$\sin(2\phi - \psi) = \frac{-2\operatorname{sgn}(b)\kappa}{\sqrt{4\kappa^2 + \sigma_\mu^2}}, \quad (18)$$

$$\cos 2\phi = \beta R_{2a} + 2z, \quad (19)$$

$$\sin 2\phi = \alpha R_{2a} + 4z_\mu, \quad (20)$$

$$R_{2a}^+ = \frac{-(2\beta z + 4\alpha z_\mu) + \sqrt{\alpha^2 + \beta^2 - 4(z\alpha - 2z_\mu\beta)^2}}{\alpha^2 + \beta^2} > 0, \quad (21)$$

$$R_{2a}^- = \frac{-(2\beta z + 4\alpha z_\mu) - \sqrt{\alpha^2 + \beta^2 - 4(z\alpha - 2z_\mu\beta)^2}}{\alpha^2 + \beta^2} > 0. \quad (22)$$

3.4. Stability of the periodic solutions in the case of exact resonance, i.e. $\sigma = 0$, $\mu = 0$

Linearising system (15) around the non-trivial critical points yields

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \varepsilon A \begin{pmatrix} X \\ Y \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\frac{\delta R_{1a}^2}{4} & \frac{-3\gamma R_{1a}^3}{4} & -\frac{aR_{1a}\,\mathrm{sgn}(b)}{4} & 0 \\ \frac{3\gamma R_{1a}}{4} & -\delta_0 - \frac{\delta R_{1a}^2}{4} & 0 & -\frac{aR_{2a}\,\mathrm{sgn}(b)}{4} \\ \frac{|b|R_{1a}}{4} & 0 & -\frac{k}{2} & 0 \\ 0 & \frac{|b|R_{1a}^2}{4R_{2a}} & 0 & -\frac{|b|R_{1a}^2}{8R_{2a}} \end{pmatrix}.$$

The eigenvalues of this matrix are unfortunately too complicated to write down explicitly. In Appendix B we prove the following results for the stability of the periodic solutions:

1. Suppose $ab > 0$, $z < 1/2$, then the periodic solution with amplitude along the y-direction equal to R_{2a}^+ will be stable no matter what the other parameters are.
2. The periodic solution with amplitude along the y-direction equal to R_{2a}^- will, if it exists, always be unstable.
3. Given the parameters c, δ_0, δ, γ, b and κ, suppose that $ab < 0$ and $z < \frac{1}{2}$, then there exists $a_s > 0$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will be stable provided $|a| < a_s$.
4. Given the parameters c, δ, γ, b and κ, suppose that $\beta < 0$, then there exist a_{s1}, a_{s2} both positive and $z_s > \frac{1}{2}$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will be stable for all $1/2 \leqslant z \leqslant \min(1/2\sqrt{1+(\beta/\alpha)^2}, z_s)$, provided $a_{s1} < |a| < a_{s2}$.
5. Given the parameters c, δ_0, δ, γ, b and κ, suppose that $ab < 0$, then there exists $0 < a_u < \infty$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will become unstable provided $|a| \geqslant a_u$.

3.5. Stability of the periodic solutions in the case $\delta_0 = 0$, $\gamma = 0$

Linearising system (15) around the non-trivial critical points yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ Y \end{pmatrix} = \varepsilon A \begin{pmatrix} X \\ Y \end{pmatrix},$$

where

$$A = \begin{pmatrix} -\frac{\delta R_{1a}^2}{4} & \frac{2\mu R_{1a}}{4} & \frac{-akR_{1a}\,\mathrm{sgn}(b)}{2D} & \frac{aR_{1a}R_{2a}\sigma_\mu\,\mathrm{sgn}(b)}{4D} \\ 0 & -\frac{\delta R_{1a}^2}{4} & \frac{-a\sigma_\mu\,\mathrm{sgn}(b)}{4D} & \frac{-a\kappa R_{2a}\,\mathrm{sgn}(b)}{2D} \\ \frac{\kappa R_{1a}|b|}{2D} & \frac{R_{1a}^2\sigma_\mu|b|}{4D} & -\frac{k}{2} & \frac{-R_{1a}^2\sigma_\mu|b|}{8D} \\ \frac{-R_{1a}\sigma_\mu|b|}{4DR_{2a}} & \frac{\kappa R_{1a}^2|b|}{2DR_{2a}} & \frac{R_{1a}^2\sigma_\mu|b|}{8DR_{2a}^2} & \frac{-\kappa R_{1a}^2|b|}{4DR_{2a}} \end{pmatrix},$$

$$D = \sqrt{4\kappa^2 + \sigma_\mu^2}.$$

The analysis in Appendix C of the Routh–Hurwitz system of conditions (see [14,15]) for the stability of the periodic solutions yields:

1. Suppose $ab > 0$ and $|\sigma| < 4\sqrt{2}\kappa$, then the periodic solution with amplitude along the y-direction equal to R_{2a}^+ will, if it exists, be stable no matter what the other parameters are.
2. Suppose $ab > 0$ and $|\mu| < 1/2\kappa$, then the periodic solution with amplitude along the y-direction equal to R_{2a}^+ will, if it exists, be stable no matter what the other parameters are.
3. The periodic solution with amplitude along the y-direction equal to R_{2a}^- will, if it exists, always be unstable.
4. Given the parameters c, μ, δ, σ, b and κ, suppose that $ab < 0$ and $|z_\mu| < \frac{1}{4}$, then there exists $a_s > 0$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will be stable provided $|a| < a_s$.
5. Given the parameters c, μ, δ, σ, b and κ, suppose that $ab > 0$, $|z_\mu| < \frac{1}{4}$ and $|\mu| > 1/2\kappa$, then there exist σ_1, σ_2 and $a_u > 0$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will become unstable provided $|a| \geqslant a_u$ and $\sigma_1 < \sigma < \sigma_2$.
6. Given the parameters c, μ, δ, σ, b and κ, suppose that $ab < 0$, $|z_\mu| < \frac{1}{4}$ and $|\mu| < 1/2\kappa$, then there exists $a_u > 0$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will become unstable provided $|a| \geqslant a_u$.
7. Given the parameters c, μ, δ, σ, b and κ, suppose that $ab < 0$, $|z_\mu| < \frac{1}{4}$ and $|\sigma| < 4\sqrt{2}\kappa$, then there exists $a_u > 0$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will become unstable provided $|a| \geqslant a_u$.

4. Harmonic balance versus averaging method

The harmonic balance method, which is essentially a Fourier projection, and the averaging method are classical methods but they are seldom compared; see also the discussion in [2, Chapter 9].

In Figs. 10 and 11, we have a superposition of the results obtained by both methods in terms of resonance curves showing the oscillation amplitudes in x- and y-direction. There is good agreement between the results of both methods. Making ε smaller and keeping the parameters a and b constant in the averaged system does improve this agreement, as expected. See Figs. 12 and 13.

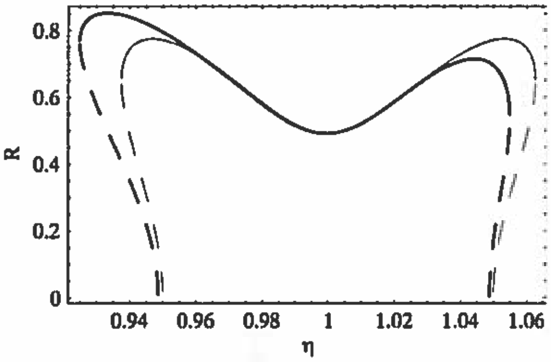

Fig. 10. Oscillation amplitude R versus excitation frequency η by superposition of the results from the harmonic balance (bold line) and the first-order averaging method corresponding to the parameters $\varepsilon=0.1$, $c=2$, $a=b=0.5/\sqrt{\varepsilon}$, $\delta_0=0$, $\delta=0.4$, $\kappa=1$, $\gamma=0$, $\sigma=0$ $(q=2)$. The dashes refer to the unstable solutions.

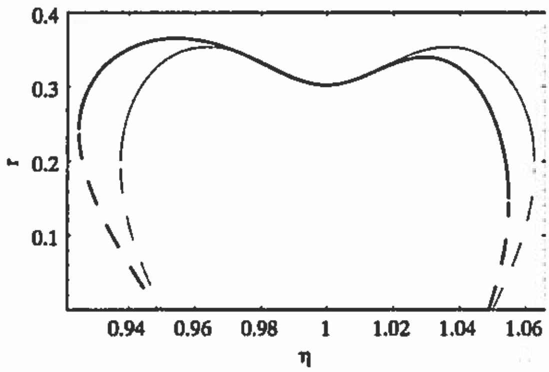

Fig. 11. Oscillation amplitude r versus excitation frequency η by superposition of the results from the harmonic balance (bold line) and the first-order averaging method corresponding to the parameters $\varepsilon=0.1$, $c=2$, $a=b=0.5/\sqrt{\varepsilon}$, $\delta_0=0$, $\delta=0.4$, $\kappa=1$, $\gamma=0$, $\sigma=0$ $(q=2)$. The dashes refer to the unstable solutions.

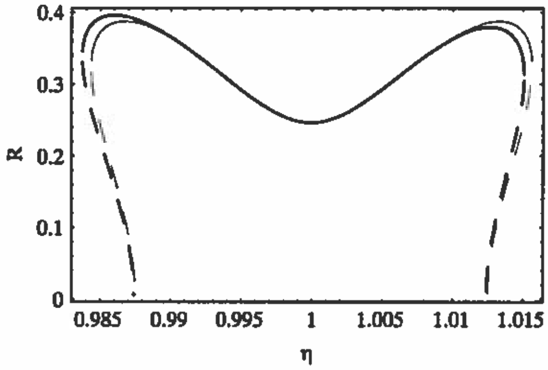

Fig. 12. Oscillation amplitude R versus excitation frequency η by superposition of the results from the harmonic balance (bold line) and the first-order averaging method corresponding to the parameters $\varepsilon = 0.025$, $c = 2$, $a = b = 0.25/\sqrt{\varepsilon}$, $\delta_0 = 0$, $\delta = 0.4$, $\kappa = 1$, $\gamma = 0$, $\sigma = 0$ $(q = 2)$. The dashes refer to the unstable solutions.

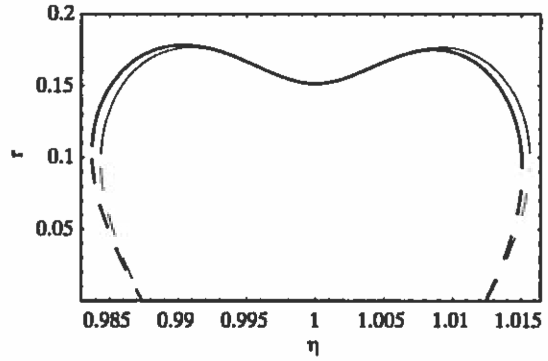

Fig. 13. Oscillation amplitude R versus excitation frequency η by superposition of the results from the harmonic balance (bold line) and the first-order averaging method corresponding to the parameters $\varepsilon = 0.025$, $c = 2$, $a = b = 0.25/\sqrt{\varepsilon}$, $\delta_0 = 0$, $\delta = 0.4$, $\kappa = 1$, $\gamma = 0$, $\sigma = 0$ $(q = 2)$. The dashes refer to the unstable solutions.

5. The Neimark–Sacker bifurcation

We know from the propositions above that the periodic solution with amplitude along the y-direction equal to R^{+}_{2a} becomes at some point unstable. This is an interesting situation as also the trivial solution is unstable. The solutions show then a very different behaviour as they are attracted to a torus on which we have quasi-periodic dynamics. One of the possible causes for this behaviour can be a Neimark–Sacker bifurcation, also known as a secondary Hopf bifurcation. We have made use of the special software CONTENT to pinpoint this bifurcation; see [9], Appendix C. We first found a periodic solution when $ab < 0$ with $b < 0$ and small. When running CONTENT, we used the method of continuation with, this time, b as a control parameter and monitored the multipliers of the periodic solution. CONTENT's results are presented below.

5.1. Numerical data generated by CONTENT

The following parameters, with respect to system (10), have been used in our numerical analysis. $\varepsilon = 0.1$, $c = 1$, $a = 0.5/\sqrt{\varepsilon}$, $\eta = 1$, $\delta_0 = 0$, $\kappa = 1$, $\delta = 0.4$, $\gamma = 0.2$, and $\sigma = 0.8$ ($q = 2.02$). System (10) has to be made autonomous to be able to spot the bifurcation with CONTENT. Its dimension becomes therefore of the sixth order:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -(1 + \varepsilon c z_1)x_1 - \varepsilon a x_1 y_1 - \varepsilon(\delta x_1^2 x_2 + \gamma x_1^3),$$

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = -(4 + \varepsilon\sigma)y_1 - \varepsilon\kappa y_2 - \varepsilon b x_1^2,$$

$$\dot{z}_1 = z_1 - 2z_2 - z_1(z_1^2 + z_2^2),$$

$$\dot{z}_2 = z_2 + 2z_1 - z_2(z_1^2 + z_2^2). \tag{23}$$

A Neimark–Sacker bifurcation has been found at the critical value $b_c = -0.179576/\sqrt{\varepsilon}$. *Note that, because of the scaling introduced in Section 3, to obtain the original values of b corresponding to system (1), we have to multiply with $\sqrt{\varepsilon}$.*

The corresponding multipliers computed by CONTENT, presented in the modulus-argument form, are as follows:

$$\rho_1 = 1, \quad \phi_1 = 0.207607,$$

$$\rho_2 = 1, \quad \phi_2 = -0.207607,$$

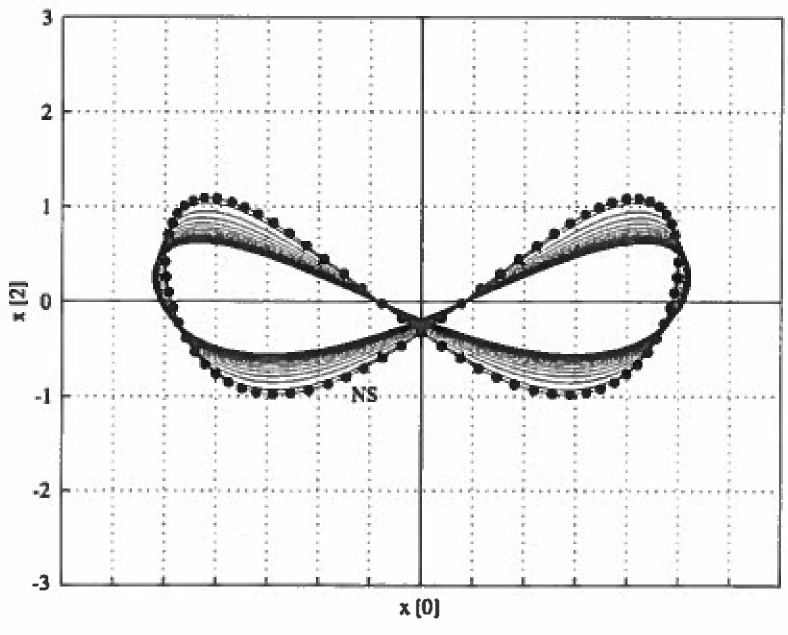

Fig. 14. Figure generated by CONTENT; for the case under consideration see Section 5.1. The cycles (prior to the bifurcation) are projected on the $(x/\sqrt{\varepsilon}, y/\sqrt{\varepsilon})$ plane. The outer cycle with dots is the one that bifurcates. NS stands for Neimark–Sacker bifurcation.

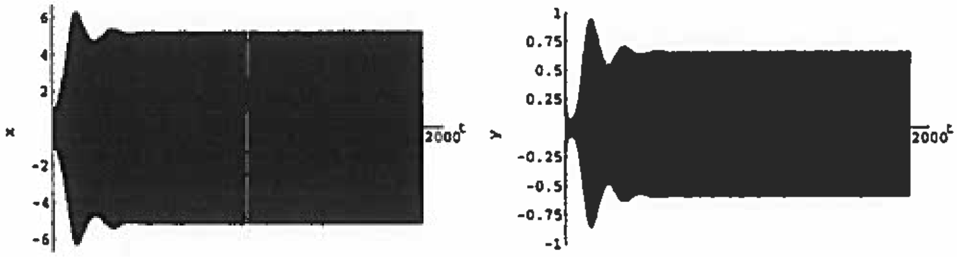

Fig. 15. Case $b > b_c$ with vibration records showing attraction to a periodic solution of $x(t)$ (left) and $y(t)$ (right) corresponding to the parameter values $\varepsilon = 0.1$, $c = 1$, $a = 0.5/\sqrt{\varepsilon}$, $b = -0.1/\sqrt{\varepsilon}$, $\delta_0 = 0$, $\eta = 1$, $\delta = 0.4$, $\gamma = 0.2$, $\kappa = 1$, $\sigma = 0.8$ ($q = 2.02$).

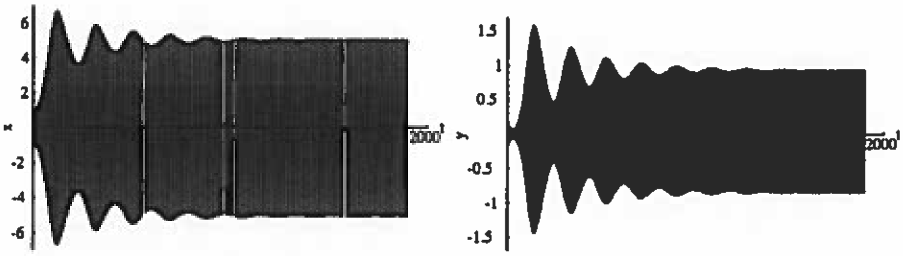

Fig. 16. Case where b is just above b_c with vibration records showing 'slow' attraction to a periodic solution (just before bifurcation) of $x(t)$ (left) and $y(t)$ (right) corresponding to the parameter values $\varepsilon = 0.1$, $c = 1$, $a = 0.5/\sqrt{\varepsilon}$, $b = -0.15/\sqrt{\varepsilon}$, $\delta_0 = 0$, $\eta = 1$, $\delta = 0.4$, $\gamma = 0.2$, $\kappa = 1$, $\sigma = 0.8$ ($q = 2.02$).

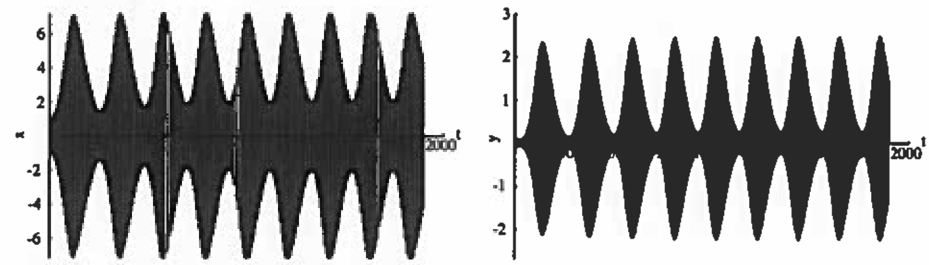

Fig. 17. Case $b < b_c$ with vibration records of attraction to a torus of $x(t)$ (left) and $y(t)$ (right) corresponding to the parameter values $\varepsilon = 0.2$, $c = 1$, $a = 0.5/\sqrt{\varepsilon}$, $b = -0.2/\sqrt{\varepsilon}$, $\delta_0 = 0$, $\eta = 1$, $\delta = 0.4$, $\gamma = 0.2$, $\kappa = 1$, $\sigma = 0.8$ ($q = 2.02$).

$$\rho_3 = 1, \quad \phi_3 = 0,$$

$$\rho_4 = 0.623983, \quad \phi_4 = -0.163293,$$

$$\rho_5 = 0.623983, \quad \phi_5 = 0.163293,$$

$$\rho_6 = 3.4873 \times 10^{-6}, \quad \phi_6 = 0.$$

We conclude that when the control parameter b goes below the threshold value $b_c = -0.179576/\sqrt{\varepsilon}$ a Neimark–Sacker bifurcation takes place (see Fig. 14). Numerical results shown in Figs. 15–17 confirm this very clearly. We can see from these figures that as the parameter b decreases, it takes longer for the periodic solution to stabilise. When b drops below b_c, the periodic solution looses its stability in the bifurcation.

5.2. Averaging method results

Remarkably enough, one can also track down the Neimark–Sacker bifurcation by looking for a Hopf bifurcation of the non-trivial critical points of the averaged system (15). This way, we will be able to locate this bifurcation without use of sophisticated software like CONTENT. We have done this in the case $\delta_0=0$ and compared the averaged results with the more accurate data obtained by CONTENT. Starting at $b=-0.1/\sqrt{\varepsilon}$, the averaged system has a non-trivial critical point which undergoes a Hopf bifurcation at the critical value $\overline{b_c} \simeq -0.171/\sqrt{\varepsilon}$. This is a rather good first-order approximation of the more accurate value $b_c = -0.179576/\sqrt{\varepsilon}$ computed by CONTENT. The relative error in b is as follows:

$$\left|\frac{b_c-\overline{b_c}}{b_c}\right| \simeq 5\%.$$

The averaging method predicts with satisfactory precision the bifurcation point b_c. This precision will improve if we take ε to be smaller.

6. Conclusions

1. The instability intervals in parameter space of the trivial solution of the parametrically excited primary system are not changed much by the presence of the non-linear coupling to the secondary oscillator.
2. In the examined system we chose the prominent resonance 1:2 both for the internal and the parametric resonance. For other resonances, the effect of non-linear coupling to the stability of the trivial solution is expected to be even weaker.
3. Non-trivial solutions become important when the trivial solution is unstable. In particular, the attracting torus which we found and which is characterised by fairly large amplitudes can be undesirable from an engineering point of view.
4. There are not many comparisons of the harmonic balance method and other analytic approximation methods in the literature. We have found good agreement between the periodic solution results of harmonic balance and averaging.
5. In most cases Neimark–Sacker bifurcations are studied by numerical means. Interestingly, we can also analyse this bifurcation in our problem by using the normal form method of averaging. The results are in good agreement with the numerics.

Appendix A. Study of the centre manifold

When the trivial solution is non-hyperbolic, the problem of stability becomes much more complicated as the stability of the trivial solution in the centre manifold of the averaged system does not have to imply the stability in the original system. There is not much known on the relation between centre manifolds in (averaging) normal forms and the original system. In spite of this, we shall study the flow in the averaged centre manifold and check numerically whether this suggests that the result holds for the original system. In the following, we present the results of Section 3 with details for one interesting case. For the background see [9] and the references therein:

- *The case* $\eta=\pm 1,\ \delta_0=c/2$: In this case, one of the four eigenvalues was equal to zero. Study of the averaged centre manifold reveals that the flow is governed by the following differential equation:

  $$\dot{u} = -\frac{\varepsilon ab\alpha(q)^2}{\kappa q}u^3 + O(u^4).$$

  The averaged trivial solution is stable provided $q=\pm 2$ and $ab>0$. Using numerical data, we conjecture that these results hold for the original system as well.
- *The case* $|z_\mu|<1/4,\ \delta_0 = 1/2\sqrt{c^2-16\mu^2}$: In this case, one of the four eigenvalues equals zero. Study of the averaged centre manifold reveals that the flow is governed by the following differential equation:

  $$\dot{u} = -\frac{\varepsilon abc(\delta_0\kappa + \mu(\sigma-8\mu))}{2\delta_0(4\kappa^2+(\sigma-8\mu)^2)}u^3 + O(u^5).$$

  We see from this equation that the trivial solution can be both stable and unstable depending on the parameters involved in the system. Numerical analysis suggests this holds for the original system as well.
- *The case* $|z_\mu| = 1/4,\ \delta_0 = 0$: In this case, we are dealing with a two-dimensional centre manifold. This case is of such complexity that even the restriction of the flow to the averaged centre manifold does not yield a differential equation which can be studied easily. We shall therefore draw our conclusion from numerical analysis only. We find the trivial solution in this case to be unstable.

- *The case* $|z_\mu| < 1/4$, $\delta_0 = 1/2\sqrt{c^2-16\mu^2}$: This case will be studied in detail below.

A.1. Study of the centre manifold $|z_\mu| > \frac{1}{4}$, $\delta_0 = 0$

We first apply the following transformation (see [13]:

$$\tilde{x}_{1a} = \frac{1}{2(c-4\mu)} x_{2a},$$

$$\tilde{x}_{2a} = \frac{-1}{2\sqrt{16\mu^2-c^2}} x_{1a}.$$

After which, we restrict the flow to the centre manifold and introduce the complex coordinates

$$y = \tilde{x}_{1a} + i\tilde{x}_{2a}, \quad \bar{y} = \tilde{x}_{1a} - i\tilde{x}_{2a}.$$

The resulting system is then normalised using the complex transformation

$$y = z + h_2(z),$$

where $h_2(z)$ denotes a polynomial in z of degree $m \geqslant 2$. This normalisation eliminates all third-order terms except $z^2\bar{z}$. The system can then be transformed back into Cartesian coordinates. Omitting the tildes yields

$$\begin{aligned}\dot{x}_{1a} &= \mathrm{Re}(\lambda_1)x_{1a} - \mathrm{Im}(\lambda_1)x_{2a}\\ &\quad + (\alpha(\delta_0)x_{1a} - \beta(\delta_0)x_{2a})(x_{1a}^2 + x_{2a}^2)\\ &\quad + O(|x_{a1}|^5, |x_{2a}|^5),\\ \dot{x}_{2a} &= \mathrm{Im}(\lambda_1)x_{1a} + \mathrm{Re}(\lambda_1)x_{2a}\\ &\quad + (\alpha(\delta_0)x_{2a} + \beta(\delta_0)x_{1a})(x_{1a}^2 + x_{2a}^2)\\ &\quad + O(|x_{a1}|^5, |x_{2a}|^5). \end{aligned} \tag{24}$$

The parameters $\alpha(\delta_0)$ and $\beta(\delta_0)$ will be specified later. We find it more convenient to work in polar coordinates. The system transforms to

$$\begin{aligned}\dot{r} &= \mathrm{Re}(\lambda_1)r + \alpha(\delta_0)r^3 + O(r^5),\\ \dot{\theta} &= \mathrm{Im}(\lambda_1) + \beta(\delta_0)r^2 + O(r^4).\end{aligned} \tag{25}$$

We are especially interested in the dynamics of the flow near δ_0=0. It is therefore natural to Taylor expand the coefficient around $\delta_0 = 0$ so that:

$$\begin{aligned}\dot{r} &= -\frac{\delta_0}{2} r + \alpha(0)r^3 + O(\delta_0 r^3, r^5),\\ \dot{\theta} &= \frac{\sqrt{16\mu^2-c^2}}{4} + \beta(0)r^2 + O(\delta_0 r^2, r^4).\end{aligned} \tag{26}$$

Taking $\delta_0 = 0$ and neglecting the higher-order terms in the system produces

$$\begin{aligned}\dot{r} &= \alpha(0)r^3,\\ \dot{\theta} &= \frac{\sqrt{16\mu^2-c^2}}{4} + \beta(0)r^2.\end{aligned} \tag{27}$$

where

$$\alpha(0) = \frac{2ab\kappa(c-4\mu)(c^2(\sigma-16\mu)+2\mu(4\kappa^2+(\sigma-16\mu)^2))}{(4c^2-(4\kappa^2-\sigma(\sigma-16\mu)))^2+16\kappa^2(\sigma-8\mu)^2},$$

$$\beta(0) = \frac{-ab(c-4\mu)\Sigma}{4\sqrt{16\mu^2-c^2}(16c^4-8c^2(4\kappa^2-\sigma(\sigma-16\mu))} + \frac{-ab(c-4\mu)\Sigma}{(4\kappa^2+\sigma^2)(4\kappa^2+(\sigma-16\mu)^2))} \tag{28}$$

with

$$\begin{aligned}\Sigma &= (32c^6(8\mu-\sigma) - 4c^4(1536\mu^3 + 4\kappa^2(56\mu-5\sigma)\\ &\quad + 64\mu^2\sigma - 56\mu\sigma^2 + 3\sigma^3)\\ &\quad -32\mu^2\sigma(16\kappa^4 + (128\mu^2 - 24\mu\sigma + \sigma^2)^2\\ &\quad + 8\kappa^2(160\mu^2 - 24\mu\sigma + \sigma^2))\\ &\quad + c^2(131072\mu^5 + 16\kappa^4(24\mu-\sigma) - 16384\mu^4\sigma\\ &\quad -512\mu^3\sigma^2 - 64\mu^2\sigma^3 + 24\mu\sigma^4 - \sigma^5\\ &\quad + 8\kappa^2(3328\mu^3 - 416\mu^2\sigma + 24\mu\sigma^2 - \sigma^3)))\\ &\quad /(4\kappa^2 + (\sigma-16\mu)^2).\end{aligned} \tag{29}$$

The sign of $\alpha(0)$ is decisive for the stability of the trivial solution. If $\alpha(0) < 0$, the trivial solution will be stable, if $\alpha(0) > 0$, the trivial solution will be unstable. After a detailed study of the numerator we came to the following conclusion. The sign of $\alpha(0)$ depends strongly on all the parameters involved in this system. Some cases will yield stability and others will give instability. This result holds of course for the averaged system. Numerical analysis suggests the stability results hold for the original system as well.

Remark. From the study above, we can state that when δ_0 is near zero and $\alpha(0)$ is positive, the averaged system has an unstable cycle. The 'averaged' trivial solution is in this case asymptotically stable. If, for some values of the parameters, the flow in the centre manifold is stable or unstable, then reversing the sign of the parameter a or b will make the flow respectively unstable or stable. Numerical results suggest this holds also for the original system.

Appendix B. Stability of the periodic solutions in the case of exact resonance, i.e. □ = 0; □ = 0

The results of Section 3.4 can be obtained using the Routh–Hurwitz criterion to study the stability of the periodic solutions. A necessary and sufficient condition for the stability of the periodic solution translates in our case, after some simplifications, to the following:

$$\delta_0+\kappa+\frac{2\delta\kappa R_{2a}}{|b|}>0,$$

$$\begin{aligned}&8\delta(\delta^2+9\gamma^2)k^2R_{2a}^3+4k(9\delta_0\gamma^2+\delta^2(3\delta_0+4k))R_{2a}^2|b|\\&\quad+2\delta R_{2a}|b|^2(2(\delta_0^2+4\delta_0k+2k^2)+akR_{2a}\,\mathrm{sgn}(b))\\&\quad+|b|^3((2\delta_0+k)^2+a(\delta_0+k)R_{2a}\,\mathrm{sgn}(b))>0,\end{aligned}$$

$$\begin{aligned}&8\delta(\delta^2+9\gamma^2)k^3R_{2a}^3+4k^2R_{2a}^2|b|(9\delta_0\gamma^2+\delta^2(3\delta_0+2k)\\&\quad+a\delta^2R_{2a}\,\mathrm{sgn}(b))+2\delta kR_{2a}|b|^2(2\delta_0^2+4\delta_0k\\&\quad+k^2+2a(\delta_0+k)R_{2a}\,\mathrm{sgn}(b))\\&\quad+|b|^3(\delta_0k(2\delta_0+k)+a(\delta_0+k)^2R_{2a}\,\mathrm{sgn}(b))\\&\quad>0,\end{aligned}$$

$$\begin{aligned}&4(\delta^2+9\gamma^2)k^2R_{2a}+4\delta k|b|(\delta_0+aR_{2a}\,\mathrm{sgn}(b))\\&\quad+a|b|^2\,\mathrm{sgn}(b)(2\delta_0+aR_{2a}\,\mathrm{sgn}(b))>0. \qquad (30)\end{aligned}$$

Proposition 1. *Suppose $ab>0$, $z<1/2$, then the periodic solution with amplitude along the y-direction equal to R_{2a}^+ will be stable no matter what the other parameters are.*

Proof. This can easily be seen from the Routh–Hurwitz equations. The condition $z<\frac{1}{2}$ guarantees the existence of the periodic solution as β is positive. □

Proposition 2. *The periodic solution with amplitude along the y-direction equal to R_{2a}^- will, if it exists, always be unstable.*

Proof. The fourth equation of the Routh–Hurwitz system can be rewritten as follows:

$$(\alpha^2+\beta^2)b^2c^2R_{2a}+4\delta\kappa|b|\delta_0+2a|b|b\delta_0>0. \qquad (31)$$

Having done this, it is now easy to see that R_{2a}^- will always yield a negative number. □

Remark. From Eq. (31) we see that R_{2a}^+ will always satisfy the fourth inequality of the Routh–Hurwitz system.

Proposition 3. *Given the parameters c, δ_0, δ, γ, b and κ, suppose that $ab<0$ and $z<\frac{1}{2}$, then there exists $a_s>0$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will be stable provided $|a|<a_s$.*

Proof. When a tends to zero, the parameter β becomes positive. One can easily see that the periodic solution with amplitude R_{2a}^+ still exists because $z<\frac{1}{2}$. Taking $a=0$ in the Routh–Hurwitz system of conditions, we can see that the positivity condition is met. As R_{2a}^+ and the Routh–Hurwitz system of conditions depend continuously on the parameter a, we can use the continuity principle to prove the existence of $a_s>0$ such that the positivity conditions will still be met provided $|a|<a_s$. □

Proposition 4. *Given the parameters c, δ, γ, b and κ, suppose that $\beta<0$, then there exist a_{s1}, a_{s2} both positive and $z_s>\frac{1}{2}$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will be stable for all $1/2\leqslant z\leqslant\min(1/2\sqrt{1+(\beta/\alpha)^2},z_s)$, provided $a_{s1}<|a|<a_{s2}$.*

Proof. Define a_{s1} as follows:

$$\lim_{|a|\to a_{s1}}\beta=0.$$

Taking $z=\frac{1}{2}$, we find

$$\lim_{|a|\to a_{s1}} R_{2a}=0.$$

Putting $|a|=a_{s1}$ and $z=\frac{1}{2}$ in the Routh–Hurwitz system, we can see that the positivity condition is met. Using the continuity principle, we can prove the existence of an open region U around $(\frac{1}{2},|a_{s1}|)$ in the za-plane such that the positivity condition will still be satisfied provided the point (z,a) lies in this region U. Defining the parameters a_{s2} and z_s such that

$$\left[\tfrac{1}{2},z_s\right]\times(a_{s1},a_{s2})\subset U$$

concludes the proof. The condition $z\leqslant \min(1/2\sqrt{1+(\beta/\alpha)^2},z_s)$ guarantees the existence of the periodic solution. □

Proposition 5. *Given the parameters c, δ_0, δ, γ, b and κ, suppose that $ab<0$, then there exists $0<a_u<\infty$ such that the periodic solution with amplitude R_{2a}^+ along the y-direction will become unstable provided $|a|\geqslant a_u$.*

Proof. When $|a|$ tends to infinity (keeping the other parameters constant), the parameter β will tend to $-\infty$. The parameter α is not affected by a (exact resonance). The threshold value $z=1/2\sqrt{1+(\beta/\alpha)^2}$ will hence never be exceeded. The periodic solution with amplitude R_{2a}^+ will consequently still exist. Its magnitude is as follows:

$$\lim_{|a|\to\infty} aR_{2a}^+=(2\delta_0+c)\,\mathrm{sgn}(a).$$

In other words R_{2a}^+ becomes insignificant when $|a|$ increases. Bearing this in mind and looking at the third condition of the Routh–Hurwitz system, we find that as $|a|$ increases, the last term of this equation will become decisive for its sign. As

$$\begin{aligned}&\lim_{|a|\to\infty} |b|^3(\delta_0 k(2\delta_0+k)+a(\delta_0+k)^2R_{2a}\,\mathrm{sgn}(b))\\&\quad=-|b|^3(c(\delta_0+\kappa)^2+\delta_0(2\delta_0^2+2\delta_0\kappa+\kappa^2))\\&\quad<0,\end{aligned}$$

we can prove, using the continuity principle, the existence of $0<a_u<\infty$ such that the third expression of the Routh–Hurwitz system will remain negative or become zero as $|a|\geqslant a_u$. □

Appendix C. Stability of the periodic solutions in the case □$_0$ = 0, □= 0

After some simplifications, the Routh–Hurwitz system of conditions based on Section 3.5 becomes in this case

$$\begin{aligned}&\kappa+\frac{\delta DR_{2a}}{|b|}>0,\\[4pt]
&4\delta^3D^2R_{2a}^3+16\delta^2D\kappa R_{2a}^2|b|+\kappa|b|^3\\
&\quad\times(D+2aR_{2a}\,\mathrm{sgn}(b))\\
&\quad+2\delta R_{2a}|b|^2(8\kappa^2+aDR_{2a}\,\mathrm{sgn}(b))>0,\\[4pt]
&16\delta^5D^4\kappa R_{2a}^4+2a\kappa^2(D^2-8\mu\sigma_\mu)|b|^5\,\mathrm{sgn}(b)\\
&\quad+8\delta^4D^3R_{2a}^3|b|(8\kappa^2+aDR_{2a}\,\mathrm{sgn}(b))\\
&\quad+8\delta^3D^2\kappa R_{2a}^2|b|^2(D^2+8\kappa^2+4aDR_{2a}\,\mathrm{sgn}(b))\\
&\quad+\delta D\kappa|b|^4(D^3+4aR_{2a}(D^2+4\kappa^2-8\mu\sigma_\mu)\mathrm{sgn}(b))\\
&\quad+2\delta^2D^2R_{2a}|b|^3(8D\kappa^2+aR_{2a}(D^2+20\kappa^2\\
&\quad-8\mu\sigma_\mu)\mathrm{sgn}(b))>0,\\[4pt]
&\delta^2R_{2a}D^3+4ab\delta kR_{2a}D+a|b|^2\,\mathrm{sgn}(b)\\
&\quad\times(4\mu\sigma_\mu+aR_{2a}D\,\mathrm{sgn}(b))>0.\end{aligned}\tag{32}$$

Proposition 6. *Suppose $ab>0$ and $|\sigma|<4\sqrt{2}\kappa$, then the periodic solution with amplitude along the y-direction equal to R_{2a}^+ will, if it exists, be stable no matter what the other parameters are.*

Proof. The condition $|\sigma|<4\sqrt{2}\kappa$ implies $D^2-8\mu\sigma_\mu>0$. It is now easy to see that under this condition, the Routh–Hurwitz system of conditions will be satisfied. □

Proposition 7. *Suppose $ab>0$ and $|\mu|<1/2\kappa$, then the periodic solution with amplitude along the y-direction equal to R_{2a}^+ will, if it exists, be stable no matter what the other parameters are.*

Proof. The condition $|\mu|<1/2\kappa$ implies $D^2-8\mu\sigma_\mu>0$. It is now easy to see that under this condition, the Routh–Hurwitz system of conditions will be satisfied. □

Proposition 8. *The periodic solution with amplitude along the y-direction equal to R_{2a}^{-} will, if it exists, always be unstable.*

Proof. The fourth equation of the Routh–Hurwitz system can be rewritten as follows:

$$(\alpha^2+\beta^2)b^2c^2DR_{2a}+4\mu\sigma_\mu ab^2\,\mathrm{sgn}(b). \tag{33}$$

Having done this, it is now easy to see that R_{2a}^{-} will always yield a negative number. □

Remark. From Eq. (33) we easily see that R_{2a}^{+} will always satisfy the fourth equation of the Routh–Hurwitz system of conditions.

Proposition 9. *Given the parameters c, μ, δ, σ, b and κ, suppose that $ab<0$ and $|z_\mu|<1/4$, then there exists $a_s>0$ such that the periodic solution with amplitude R_{2a}^{+} along the y-direction will be stable provided $|a|<a_s$.*

Proof. When a tends to zero, the parameter α becomes zero. One can easily see that the periodic solution with amplitude R_{2a}^{+} still exists because $|z_\mu|<\frac{1}{4}$. Taking $a=0$ in the Routh–Hurwitz system of conditions, we can see that the positivity condition is met. As R_{2a}^{+} and the Routh–Hurwitz system of conditions depend continuously on the parameter a, we can use the continuity principle to prove the existence of $a_s>0$ such that the positivity conditions will still be met provided $|a|<a_s$. □

Proposition 10. *Given the parameters c, μ, δ, σ, b and κ, suppose that $ab>0$, $|z_\mu|<\frac{1}{4}$ and $|\mu|>1/2\kappa$, then there exist σ_1, σ_2 and $a_u>0$ such that the periodic solution with amplitude R_{2a}^{+} along the y-direction will become unstable provided $|a|\geqslant a_u$ and $\sigma_1<\sigma<\sigma_2$.*

Proof. When $|\mu|>1/2\kappa$ then there are σ_1 and σ_2 such that

$$D^2-8\mu\sigma_\mu<0 \quad \forall\sigma\in(\sigma_1,\sigma_2).$$

When $|a|$ tends to infinity, the periodic solution with amplitude R_{2a}^{+} will still exist as $|z_\mu|<\frac{1}{4}$. For its amplitude we have

$$\lim_{|a|\to\infty}|a|R_{2a}^{+}=\text{positive constant}.$$

R_{2a}^{+} becomes consequently insignificant when $|a|$ increases. Bearing this in mind, we can see that the third condition of the Routh–Hurwitz system becomes negative as $|a|$ tends to infinity. We can now prove, using the continuity principle, the existence of $0<a_u<\infty$ such that the third equation of system (25) will become negative or zero provided $|a|\geqslant a_u$. □

Proposition 11. *Given the parameters c, μ, δ, σ, b and κ, suppose that $ab<0$, $|z_\mu|<\frac{1}{4}$ and $|\mu|<1/2\kappa$, then there exists $a_u>0$ such that the periodic solution with amplitude R_{2a}^{+} along the y-direction will become unstable provided $|a|\geqslant a_u$.*

Proof. When $|\mu|<1/2\kappa$, then $D^2-8\mu\sigma_\mu$ will always be positive. When $|a|$ tends to infinity, the periodic solution with amplitude R_{2a}^{+} will still exist as $|z_\mu|<\frac{1}{4}$. For its amplitude we have

$$\lim_{|a|\to\infty}|a|R_{2a}^{+}=\text{positive constant}.$$

R_{2a}^{+} becomes consequently insignificant when $|a|$ increases. Bearing this in mind, we can easily see that the left-hand side of the third condition of the Routh–Hurwitz system becomes negative as $|a|$ tends to infinity and $ab<0$. We can now prove, using the continuity principle, the existence of $0<a_u<\infty$ such that the third equation of system (25) will become negative or zero provided $|a|\geqslant a_u$. □

Proposition 12. *Given the parameters c, μ, δ, σ, b and κ, suppose that $ab<0$, $|z_\mu|<\frac{1}{4}$ and $|\sigma|<4\sqrt{2}\kappa$, then there exists $a_u>0$ such that the periodic solution with amplitude R_{2a}^{+} along the y-direction will become unstable provided $|a|\geqslant a_u$.*

Proof. The proof runs similar to the previous one. □

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