# Quasiperiodic phenomena in the Van der Pol - Mathieu equation

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**Abstract**

The Van der Pol - Mathieu equation, combining self-excitation and parametric excitation, is analysed near and at $1:2$ resonance, using the averaging method. We analytically prove the existence of stable and unstable periodic solutions near the parametric resonance frequency. Above a certain detuning threshold, quasiperiodic solutions arise with basic periods of order 1 and order $1/\varepsilon$ where $\varepsilon$ is the (small) detuning parameter.

## 1 Introduction

In an early but not widely known monograph, Tondl ([1]) formulated the Van der Pol - Mathieu equation to model various engineering problems. The analysis in [1] employs harmonic balance and analogue computer methods. Recently, the Van der Pol - Mathieu equation has played an important role in various other models of dynamical systems with parametric resonance. Momeni et al. ([2]) studied the dynamical behaviour of charged dust grains near parametric resonance, while Pandey, Rand and Zehnder ([3]) use the Van der Pol - Mathieu equation to model MEMS devices. The analysis in [2] however is mathematically deficient and does not describe all the periodic solutions and bifurcations. The analysis in [1] and [3] aims at observing a numer of interesting phenomena without proofs.

In the present paper, we use averaging and the second Bogoliubov theorem to obtain a more complete picture of the dynamics in the case of small self-excitation and parametric excitation. We locate, approximate and prove the existence of stable and unstable periodic solutions for parametric frequency near the 1:2 resonance. Interestingly, we find also stable quasiperiodic (multifrequency) solutions on increasing the detuning of the parametric frequency.

## 2 Averaging

Following [2], we analyse the Van der Pol - Mathieu equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - (\alpha - \beta\, x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2(1 + h\cos\gamma t)x = 0, \tag{1}$$

where we assume $\alpha, \beta, h$ and $\omega_0$ to be nonnegative. Our goal is to analyse this equation for small parameter values. We therefore write $\alpha = \varepsilon\alpha_0$, $\beta = \varepsilon\beta_0$ and $h = \varepsilon h_0$ with $\alpha_0$, $\beta_0$, $h_0 \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Furthermore, we consider the parametric excitation frequency to be $\gamma = 2\omega_0 + 2d\varepsilon$. This way, we introduce a $2:1$-resonance with a small frequency *detuning*, controlled by the detuning parameter $d$. We introduce a new timescale $\tau = (\omega_0 + d\varepsilon)t$, for which equation (1) transforms into

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + x = \frac{\varepsilon}{\omega_0}\left[(\alpha_0 - \beta_0\, x^2)\frac{\mathrm{d}x}{\mathrm{d}\tau} + (2d - h_0\omega_0\cos 2\tau)x\right] + \mathcal{O}(\varepsilon^2) \tag{2}$$

A number of aspects of this equation were discussed in [1]. As usual in averaging we introduce slowly varying quantities by

$$\begin{aligned} x(\tau) &= a(\tau)\cos\tau + b(\tau)\sin\tau \\ \tfrac{\mathrm{d}x}{\mathrm{d}\tau}(\tau) &= -a(\tau)\sin\tau + b(\tau)\cos\tau \end{aligned} \tag{3}$$

with $a$ and $b$ varying slowly in time. This allows us to apply the averaging method discussed in [4] (for a more fundamental treatment see [5]) to obtain

$$\begin{aligned} \tfrac{\mathrm{d}a}{\mathrm{d}\tau} &= \tfrac{\varepsilon}{2\omega_0}\left[\alpha_0\, a - (\tfrac{h_0\omega_0}{2} + 2d)\, b - \tfrac{\beta_0}{4}(a^2 + b^2)\, a\right] + \mathcal{O}(\varepsilon^2) \\ \tfrac{\mathrm{d}b}{\mathrm{d}\tau} &= \tfrac{\varepsilon}{2\omega_0}\left[\alpha_0\, b - (\tfrac{h_0\omega_0}{2} - 2d)\, a - \tfrac{\beta_0}{4}(a^2 + b^2)\, b\right] + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{4}$$

From this point on, we will omit the terms of order $\varepsilon^2$ since treatment up to second order hasn't revealed any new phenomena; only higher precision is achieved.

## 3 Equilibrium points

The system of equations (4) has, next to the trivial equilibrium $(a, b) = (0, 0)$, four nontrivial equilibrium points. If the four equilibria are hyperbolic, according to the second Bogoliubov theorem (sometimes called "theorem for periodic solutions by averaging") they correspond with periodic solutions with the same stability characteristics. The theorem can be found in [6], chapter 6. We follow the formulation in [4], theorems 11.5-6, where the proof is based on the implicit function theorem.

**Theorem** Consider the equation

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon) \qquad x \in D \subset \mathbb{R}^n,\ t \geq 0 \tag{5}$$

and suppose that:

a. the vector functions $f$, $g$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial g}{\partial x}$ are defined, continuous and bounded by a constant $M$ (independent of $\varepsilon$) in $[0, \infty) \times D$, $0 \leq \varepsilon \leq \varepsilon_0$;

b. $f(t, x)$ and $g(t, x, \varepsilon)$ are $T$-periodic in $t$ ($T$ independent of $\varepsilon$);

If $p$ is a critical point of the averaged equation

$$\dot{y} = \varepsilon f^0(y) \tag{6}$$

whereas

$$\left| \frac{\partial f^0(y)}{\partial y} \right|_{y=p} \neq 0 \tag{7}$$

then there exists a $T$-periodic solution of $\phi(t,\varepsilon)$ of equation (5) which is close to $p$ such that $\lim_{\varepsilon\to 0} \phi(t,\varepsilon) = p$.
In addition, if the eigenvalues of the critical point $y = p$ of the averaged equation (6) all have negative real parts, the corresponding periodic solution $\phi(t,\varepsilon)$ of equation (5) is asymptotically stable for $\varepsilon$ sufficiently small. If one of the eigenvalues has positive real part, $\phi(t,\varepsilon)$ is unstable.

Notice that, using the transformation (3) we can bring system (2) in the desired form (5).

Returning to the nontrivial equilibrium points, we divide these four points into two pairs, which are labeled *symmetric* and *antisymmetric*:

$$\begin{pmatrix} a \\ b \end{pmatrix}^s_\pm = \pm\sqrt{\frac{4}{\beta_0 h_0 \omega_0}\left(\alpha_0 - \sqrt{\frac{h_0^2\omega_0^2}{4} - 4d^2}\right)} \begin{pmatrix} \sqrt{\frac{h_0\omega_0}{2} + 2d} \\ \sqrt{\frac{h_0\omega_0}{2} - 2d} \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}^a_\pm = \pm\sqrt{\frac{4}{\beta_0 h_0 \omega_0}\left(\alpha_0 + \sqrt{\frac{h_0^2\omega_0^2}{4} - 4d^2}\right)} \begin{pmatrix} \sqrt{\frac{h_0\omega_0}{2} + 2d} \\ -\sqrt{\frac{h_0\omega_0}{2} - 2d} \end{pmatrix} \tag{9}$$

We see that for any nontrivial equilibrium point to exist, the reality condition

$$2|d| \leq \frac{h_0\omega_0}{2} \tag{10}$$

must be satisfied. Furthermore, existence of the symmetric pair demands that

$$\Gamma := \sqrt{\frac{h_0^2\omega_0^2}{4} - 4d^2} \leq \alpha_0 \tag{11}$$

This puts limits on the detuning $d$ of the resonance frequency for the periodic solutions to exist.

# 4 Stability

We determine the stability of the equilibria by computing the eigenvalues at the equilibrium points.

For the trivial equilibrium point $(a,b) = (0,0)$, we find the associated eigenvalues $\lambda^0_\pm$ to be

$$\lambda^0_\pm = \frac{\varepsilon}{2\omega_0}\left[\alpha_0 \pm \Gamma\right] \tag{12}$$

We see that if reality condition (10) is not satisfied, the equilibrium point is an unstable focus. If (10) is satisfied but (11) is not, we obtain a saddle point. If both reality conditions (10) and (11) are satisfied, the equilibrium point is an unstable node.

For the nontrivial equilibrium points, we observe that both points in each pair exhibit the same stability behaviour, because for each pair, its position and the system of equations (4) is invariant under the double reflection $(a,b) \to (-a,-b)$. The eigenvalues for the symmetric ($\lambda^s_\pm$) and antisymmetric ($\lambda^a_\pm$) pair are

$$\lambda^s_\pm = \frac{\varepsilon}{\omega_0}\left[\frac{-1\pm 1}{2}\alpha_0 + \Gamma\right] \tag{13}$$

$$\lambda^a_\pm = \frac{\varepsilon}{\omega_0}\left[\frac{-1\pm 1}{2}\alpha_0 - \Gamma\right] \tag{14}$$

Both points of the antisymmetric pair are stable nodes, while both points of the symmetric pair are saddle points. The behaviour of the equilibrium points is illustrated in Figure 1. The relevant bifurcation parameter turns out to be $\Gamma$.

Looking at our transformation (3), it is clear that an equilibrium point of the system of equations (4) corresponds with a periodic solution of the original equation (2). Since we have found two stable equilibrium points (the antisymmetric pair), we can translate this into two stable periodic solutions (limit-cycles) of equation (2):

$$x(\tau) = \pm\sqrt{\frac{4(\alpha_0+\Gamma)}{\beta_0 h_0\omega_0}}\left(\sqrt{\frac{h_0\omega_0}{2}+2d}\cos\tau - \sqrt{\frac{h_0\omega_0}{2}-2d}\sin\tau\right) \tag{15}$$

The behaviour of this periodic solution of $x$ is illustrated in Figure 2. Notice that the period is equal to $2\pi$ for our rescaled time $\tau$; for the original time $t$, the period is $\frac{2\pi}{\omega_0+d\varepsilon} = \frac{2\pi}{\omega_0}(1-d\frac{\varepsilon}{\omega_0}) + \mathcal{O}(\varepsilon^2)$. As mentioned before, the rigorous existence of these periodic solutions follows from the second Bogoliubov theorem, see section 3.

# 5 Quasiperiodic behaviour

We look for a stable manifold in the $(a,b)$-plane which is invariant under the flow of the system of equations (4). We assume this manifold to be described by a quadric $Aa^2 + 2Bab + Cb^2 = R$. This assumption turns out to be correct, with

$$\begin{array}{rclrcl} A &=& \alpha_0^2 - 2d(\frac{h_0\omega_0}{2} - 2d) & B &=& \alpha_0\frac{h_0\omega_0}{2} \\ C &=& \alpha_0^2 + 2d(\frac{h_0\omega_0}{2} + 2d) & R &=& \frac{4\alpha_0}{\beta_0}(\alpha_0^2 - (\frac{h_0^2\omega_0^2}{4} - 4d^2) \end{array} \tag{16}$$

Calculating the determinant of the coefficient matrix, we find $AC - B^2 = (\alpha_0^2 + 4d^2)(\alpha_0^2 - (\frac{h_0^2\omega_0^2}{4} - 4d^2))$. We conclude that the quadric is an ellipse when the reality condition (10) is not met, or when both (10) and (11) are satisfied. This is equivalent to $\Gamma \in i\mathbb{R}^+$ resp. $\Gamma < \alpha_0$. When condition (11) is not met ($\Gamma > \alpha_0$), the quadric describes a hyperbola.
Straightforward calculation shows that all nontrivial equilibrium points lie on the quadric. This means that several asymptotic solutions in the $(a,b)$-plane can be identified:

- When both (10) and (11) are satisfied, system (4) has four nontrivial equilibria, all of which lie on the quartic describing an ellipse. This situation is depicted in Figure 5. The ellipse consists of four orbits $(a_e, b_e)$, for which $\lim_{\tau\to-\infty}(a_e,b_e) = (a^s,b^s)_\pm$ and $\lim_{\tau\to\infty}(a_e,b_e) = (a^a,b^a)_\pm$.

- When only (10) is satisfied, system (4) has two nontrivial equilibria (the antisymmetric pair). Each of them is located on a different branch of the quartic describing a hyperbola. Since both nontrivial equilibrium points are stable, they are positive attractors so all orbits converge to one of these two equilibria. The symmetry axis between the two branches of the hyperbola divides the $(a, b)$-plane in two stability regions in each of which one of the equilibrium points is the only attractor. In the caption of Figure 1 we identify the corresponding subcritical pitchfork and saddle-node bifurcations.

In addition, if the reality condition (10) is not met (no nontrivial equilibrium points exist), the quadric (which is an ellipse in this case) contains a periodic orbit. This follows from the fact that, for these parameter values, the origin is an unstable focus, while for $a, b \gg 1$ (outside the ellipse) the direction field points inwards. Applying the Poincaré-Bendixson theorem to the averaged system (4) yields the existence of a periodic orbit on the ellipse. This situation is depicted in Figure 6. The periodic orbit corresponds with a torus in the original system (2) as follows from [5], appendix C.

The period of this orbit can be found if we write the system of equations (4) in polar coordinates. Choosing $a(\tau) = r(\tau)\cos\theta(\tau)$ and $b(\tau) = r(\tau)\sin\theta(\tau)$, we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\varepsilon}{2\omega_0}\left[\left(\alpha_0 - \frac{h_0\omega_0}{2}\sin 2\theta\right) r - \frac{\beta_0}{4} r^3\right] \tag{17}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{\varepsilon}{2\omega_0}\left[2d - \frac{h_0\omega_0}{2}\cos 2\theta\right] \tag{18}$$

Substituting $y(\tau) = \tan\theta(\tau)$, we can solve equation (18), yielding

$$\sqrt{\frac{2d + \frac{h_0\omega_0}{2}}{2d - \frac{h_0\omega_0}{2}}}\tan\theta(\tau) = \tan\left(\frac{\varepsilon}{2\omega_0}|\Gamma|(\tau - \tau_0) + \phi_0\right) \tag{19}$$

with $\tan\phi_0 = \sqrt{\frac{2d+\frac{h_0\omega_0}{2}}{2d-\frac{h_0\omega_0}{2}}}\tan\theta(\tau_0)$. Notice that, since (10) is not satisfied, $|\Gamma| = \sqrt{4d^2 - \frac{h_0^2\omega_0^2}{4}}$. From (19) we infer that the frequency $\omega_\varepsilon$ of this orbit is $\omega_\varepsilon = \frac{\varepsilon}{\omega_0}|\Gamma|$, so the period of the orbit is $\frac{2\pi}{\omega_\varepsilon} = \frac{2\pi\omega_0}{\varepsilon|\Gamma|}$.
This means that $x(\tau)$ exhibits quasiperiodic behaviour. We can distinguish two frequencies: the first one equal to unity for our time scale $\tau$, the second one of order $\varepsilon$, equal to $\frac{\varepsilon}{\omega_0}|\Gamma|$. This behaviour is illustrated in Figure 3; a Poincaré section is depicted in Figure 4. Notice that this behaviour only occurs when no nontrivial equilibrium points in the $(a, b)$-plane exist. This is equivalent to the situation that the reality condition (10) is not satisfied, so that the detuning $|d|$ is above the threshold $\frac{h_0\omega_0}{4}$. For our original time scale $t$, the new period becomes $\frac{2\pi\omega_0}{\varepsilon(\omega_0+d\varepsilon)|\Gamma|} = \frac{2\pi}{\varepsilon|\Gamma|}(1 - d\frac{\varepsilon}{\omega_0}) + \mathcal{O}(\varepsilon)$.

## 6 Conclusion

We studied equation (1) for small (order $\varepsilon$) values of $\alpha, \beta$ and $h$. As in [2] we considered the main resonance frequency $\gamma = 2\omega_0 + 2d\varepsilon$, which means that we are perturbing around the 1:2 resonance. As new features with respect to [2], we found stable periodic and stable quasi-periodic solutions. These more complicated modulations, quasiperiodic solutions, arise if the detuning crosses a certain threshold.

## References


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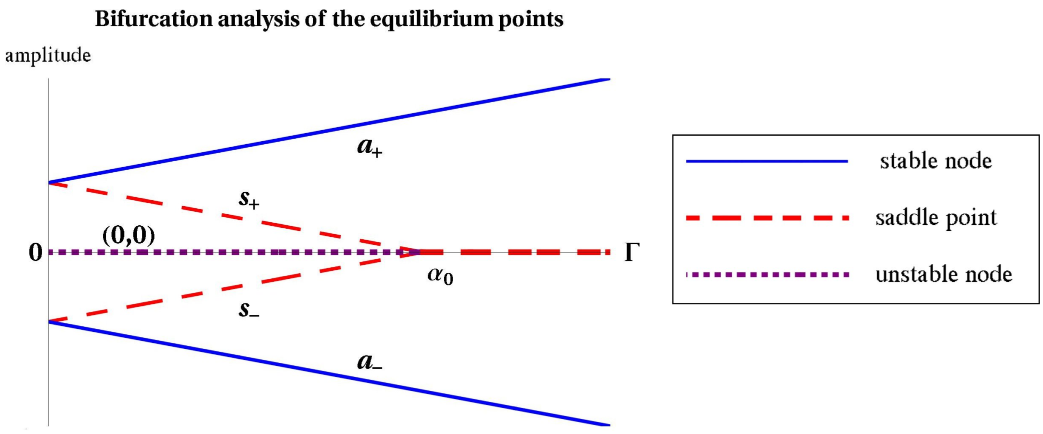


Figure 1: Bifurcation analysis of the equilibrium points of the system of equations (4). The points of the symmetric pair are indicated by $s_\pm$ and those of the antisymmetric pair by $a_\pm$. The relevant bifurcation parameter is $\Gamma = \sqrt{\frac{h_0^2\omega_0^2}{4} - 4d^2}$. We see that if $\Gamma < \alpha_0$, four nontrivial equilibrium points exist while the origin is an unstable node. For $\Gamma > \alpha_0$, only two nontrivial equilibrium points exist; the origin has turned into a saddle point. We can therefore identify a subcritical pitchfork bifurcation at $\Gamma = \alpha_0$ and two simultaneous saddle-node bifurcations at $\Gamma = 0$.

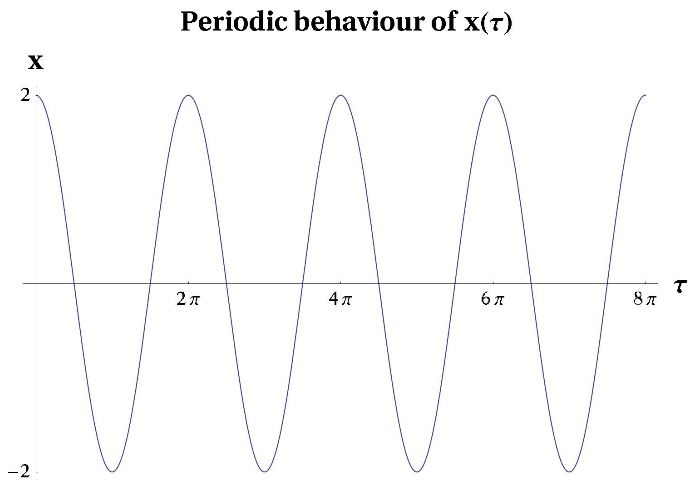


Figure 2: The periodic behaviour of $x(\tau)$, as described by equation (15). The function is plotted for $\alpha_0 = 1$, $\beta_0 = 1$, $h_0 = 2$, $\omega_0 = 2$ and $d = 1$. The sign in (15) is chosen to be positive.

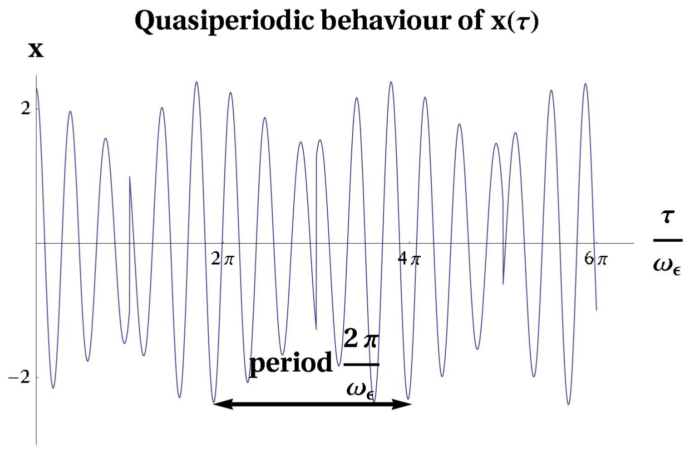


Figure 3: The quasiperiodic behaviour of $x(\tau)$. The function is plotted for $\alpha_0 = 1$, $\beta_0 = 1$, $h_0 = 2$, $\omega_0 = 2$ and $d = 1$, while $\varepsilon = 0.1$. Since $2|d| > \frac{h_0\omega_0}{2}$, $a$ and $b$ exhibit periodic behaviour. This period of order $\frac{1}{\varepsilon}$ is indicated in the figure. The initial frequency of order 1 is also visible.

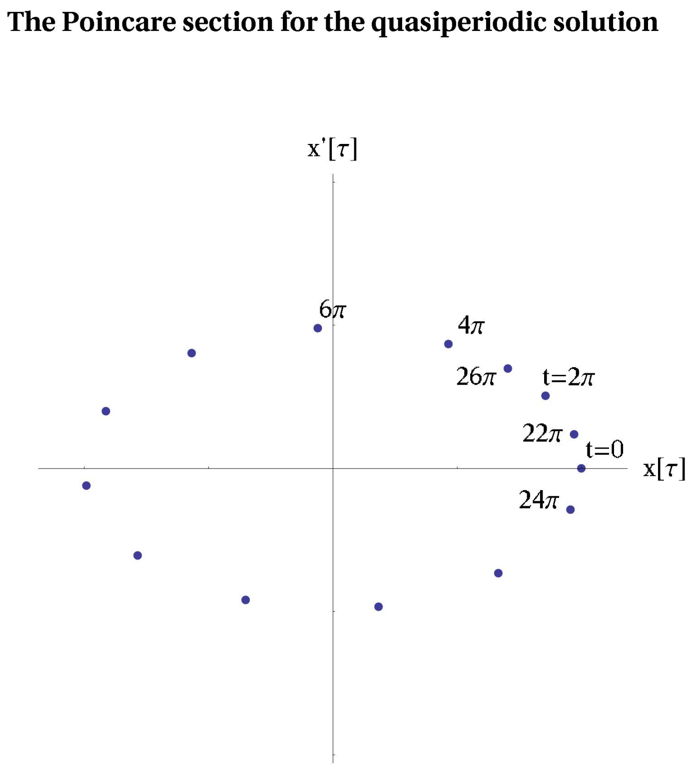


Figure 4: The Poincare section (also known as the "time=$2\pi$"-map or stroboscopic map) based on the original system (1), of the quasiperiodic solution of $x(\tau)$ in phase space $(x(\tau), x'(\tau))$. The first 13 iterations are shown. In this plot, $\omega_0 = 1$.

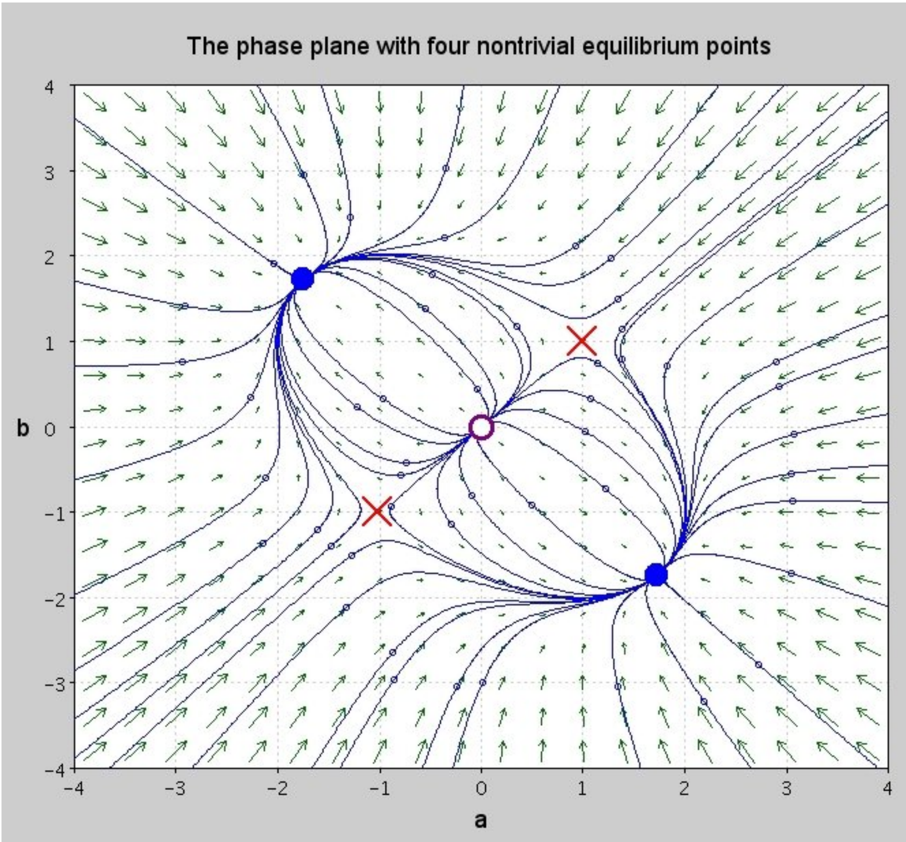


Figure 5: The phase plane of the system (4) has been drawn for $\alpha_0 = 1$, $\beta_0 = 1$, $\omega_0 = 1$, $h_0 = 1$ and $d = 0$. In this case, $\Gamma < \alpha_0$ so all four nontrivial equilibrium points exist. The quadric, which is an ellipse for these parameter values, is visible. The four nontrivial equilibrium points lie on the ellipse. For these parameter values the origin is an unstable node, both points of the symmetric pair are saddle points and both points of the antisymmetric pair are stable nodes.

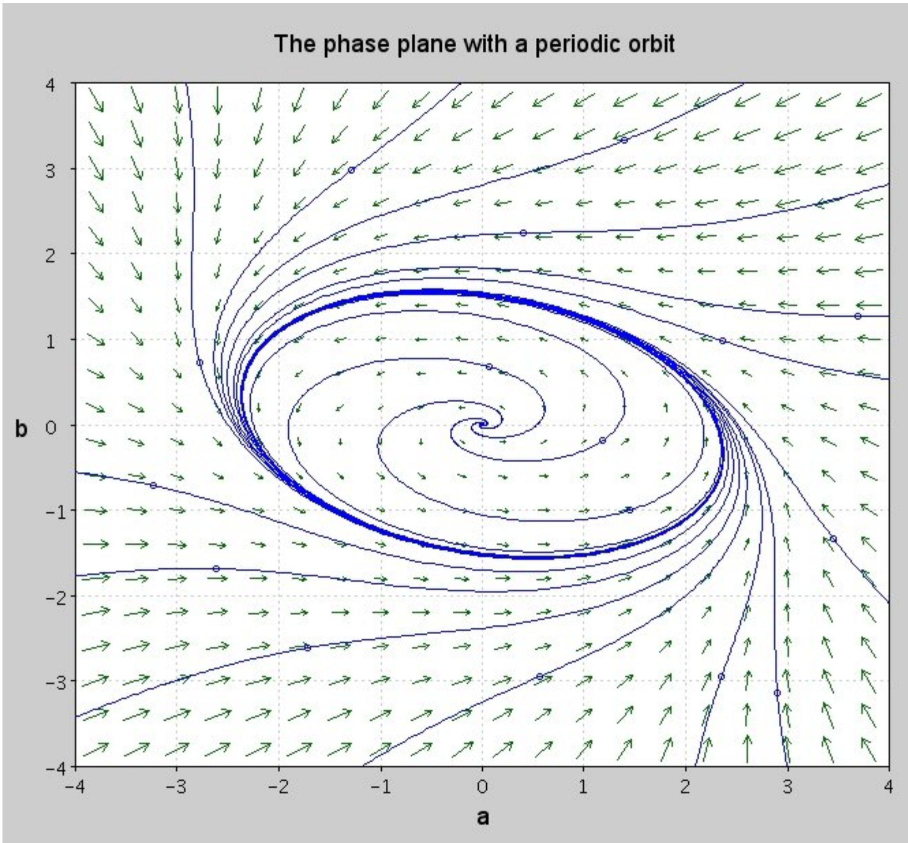


Figure 6: The phase plane of the system (4) has been drawn for $\alpha_0 = 1$, $\beta_0 = 1$, $\omega_0 = 1$, $h_0 = 2$ and $d = 1$. In this case, $2|d| > \frac{h_0\omega_0}{2}$ so no nontrivial equilibrium points exist.. The quadric, which is an ellipse for these parameter values, is clearly visible. All drawn solutions converge to the periodic elliptical orbit.