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The Lemniscate of Bernoulli

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Introduction

On the evening of Wednesday, May 28, 1969, the Dutch radio network VARA broadcast a piece of music entitled "The Lemniscate of Bernoulli". The announcer, who had to introduce the piece, had rightly considered it necessary to explain the title. He had therefore consulted van Dale's Dutch dictionary and an encyclopedia, and the following announcement was the result of his research:

> The lemniscate is the foot-point curve of the equilateral hyperbola with respect to the centre, and also the inversion of it, with respect to the same point; Bernoulli was a Swiss mathematician.

It was clear from his tone of voice that this explanation did not really clarify matters for him. For me personally, the title of the piece of music was no help for its understanding; it was a very modern piece for percussion instruments. In short: those seven minutes of radio program displayed a complete breakdown of communication between the fields of mathematics, history, and music.

Let us confine ourselves to mathematics and history, and ask how the announcer could have clarified the notion "lemniscate". He might have informed his audience that the lemniscate of Bernoulli belongs to the class of lemniscates of Booth, which, in turn, are a special sort of cissoids; he could have remarked that the lemniscate is a special conchoidal curve, as well as a special Cassini-curve, that the lemniscate belongs to the class of Watt-curves, and that also it is a Lissajous-curve. He should then, of course, add in each case the defining properties of the class of curves mentioned.

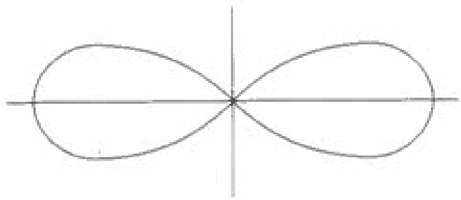

Figure 1. The Lemniscate

We may well suppose that after having considered these possibilities of clarifying the notion "lemniscate", our announcer would conclude that only television could help him; for his colleagues in that medium could solve the problem of explanation merely by putting the equation on the screen

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

together with a sketch of the curve (see Figure 1).

And indeed, such an announcement would be a sufficient clarification. For all those properties of the lemniscate, all those classes to which it belongs, although perhaps of importance in former times, are now almost forgotten. This is not necessarily a matter for regret; mathematics is not a cumulative science, not a science in which one keeps and cherishes all that has once been discovered. For the mathematician it is often just as well to shake the dust of the past off his feet and go on with his researches unencumbered by the past. And thus the lemniscate belongs to the mathematics of the past, to the mathematics which, as the saying goes, is "only of importance to historians".

Now what does "importance to historians" mean? Is it the task of the historian of mathematics to study that mass of mathematics that is already dead and buried? Of course not. The historian must have mastery of the mathematics of the period he studies, but he must not leave it at that. Just as mathematical research is more than doing complicated sums, historical research uses knowledge of earlier mathematics as a tool in the setting and solving of meaningful problems about mathematical activities in the past. What are these meaningful problems? Put more generally: which are the problems to which the discipline of the history of mathematics directs its attention? The history of mathematics is still a young field and, therefore, has not yet achieved a consensus on it goals. Therefore, I can only indicate those sorts of problems which in my opinion should become recognized as the most important.

To put it less formally: in connection with the lemniscate, I can try to raise and partly answer a number of questions which I consider of the sort to be studied in genuine history of mathematics.

The Bernoullis and the Lemniscate

Our radio announcer did well to omit the first name of the Swiss mathematician Bernoulli, for both Jakob and Johann may be considered as the discover of the

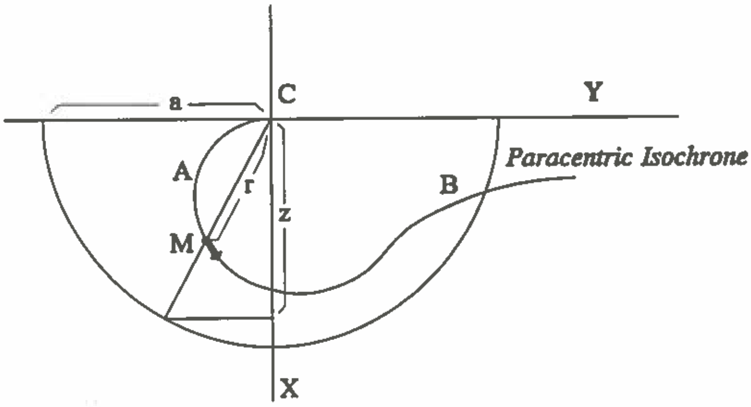

Figure 2. The Paracentric Isochrone

lemniscate. Jakob described this curve in an article which appeared in the *Acta Eruditorum* in September 1694;[1] Johann, Jakob's brother, did the same in the *Acta's* October issue of that year.[2] It can be considered certain that they wrote these articles independently of each other, since neither would allow the other to be in a position to use his unpublished research. Here arises the first question: how is it possible that two mathematicians independently of each other found and studied the same curve at almost the same time? The answer to this question is easy: they found the same curve because they studied the same problem with the same set of tools.

The problem concerned the so-called *paracentric isochrones*, a name which Leibniz gave to curves with the following property: A point M (see Figure 2) is supposed to move along the curve $CAMB$ in a vertical plane, as if under the influence of gravity. If the form of the curve $CAMB$ is such that, during the motion of M along it, the radius $r = CM$ varies linearly with the time, then $CAMB$ is a paracentric isochrone.

The problem of determining the paracentric isochrones was publicly proposed by Leibniz in 1689.[3] A long time passed before Johann and Jakob Bernoulli solved it. We know that around 1692, Johann had reduced the problem to the following differential equation:

$$(x\,dx + y\,dy)\sqrt{x} = (x\,dy - y\,dx)\sqrt{a}\,.$$

But this is nothing more than a translation of the conditions of the problem into differentials in rectangular coordinates; the real work was yet to be done. The first step beyond this was Jakob's achievement in choosing new variables which could yield differential equations with separated variables. The first equation was

$$\frac{dr}{\sqrt{ar}} = \frac{a\,dz}{\sqrt{az(a^2 - z^2)}}\,,$$

where r and z are as in Figure 2 (notice that z is not a cartesian coordinate). Jakob transformed this equation into

$$\frac{dr}{\sqrt{ar}} = \frac{2adu}{\sqrt{a^4 - u^4}}$$

by the formal substitution $az = u^2$.

The left-hand sides in (2) and (3) are integrable. You recognise the right-hand sides in both equations as elliptic integrands which cannot be integrated in closed form. We could therefore have expected that, after some further research, Jakob (as well as Johann, who had learned these differential equations from an article by Jakob in the *Acta* of June 1694) would have left it at that, considering the problem in differential equations as having been "reduced to quadratures". This, however, was not the case; Jakob, as well as Johann, sought to interpret the right-hand sides as arc-length differentials of appropriate simple curves. And this research led them to the lemniscate.

Johann considered an appropriate multiple of the right-hand side of (2) and set it equal to an arc-length differential:

$$\frac{a}{\sqrt{2}} \frac{adz}{\sqrt{az(a^2 - z^2)}} = \sqrt{(dU)^2 + (dV)^2}\,.$$

The task is then to discover expressions $U(z)$, $V(z)$ satisfying (4). Johann found that such expressions are

$$U(z) = \sqrt{az + zz} \qquad V(z) = \sqrt{az - zz}\,,$$

as one may verify by differentiation. By eliminating z in (5) one sees that U and V are the coordinates of the algebraic curve

$$(U^2 + V^2)^2 = 2a^2(U^2 - V^2),$$

that is, the lemniscate. Thus the right-hand side of (2) is interpreted as the arc-length differential of an appropriate simple algebraic curve: the lemniscate.

In his article of October 1694 Johann described how the paracentric isochrone can be constructed if one supposes as known the arc-length function for the lemniscate. His article is entitled "Simple construction of the paracentric isochrone by rectification of an algebraic curve".

Jakob's result comes out of his work on a more general problem, namely to find classes of curves with algebraically simple arc-length differentials. He considers curves (U, V) with

$$U = \sqrt{bz^m + cz^n}, \qquad V = \sqrt{bz^m - cz^n}.$$

For $b = c = 1$ and $n = 2m$ these are lemniscates. In that case the arc-length differentials are

$$ds_L = \frac{z^{\frac{m}{2}-1}\,dz}{\sqrt{1 - z^{2m}}}$$

For $m = 2$ one has

$$ds_L = \frac{dz}{\sqrt{1-z^4}},$$

the right-hand side of (3).

In the same way as Johann, Jakob used this result in his article of September 1694, which bears the title "Construction of the paracentric isochrone by way of rectification of an algebraic curve". In this article Jakob gave the lemniscate its name, which means something like "braided band".

Constructions and representations

Apparently both brothers considered it extremely useful to invent constructions for transcendental curves—in this case for the paracentric isochrone—and, in particular, constructions which assume as given an arc-length function of a simpler curve. The question which now arises is: why? Let us first make sure that we have here something more than a special hobby of the Bernoullis: Leibniz also gave a construction of the paracentric isochrone by means of rectification; and Huygens gave a constructions of the catenary by means of the rectification of the parabola. These constructions are very remarkable indeed.

For if one is willing to accept an arc-length function as given, why not an area-function? That is to say, why did one not just leave the integral there? Is not $\int f(x)\,dx$ simpler than $\int \sqrt{1+(f'(x))^2}\,dx$? The possibility to leave it at area functions was indeed recognised: it was called "construction by means of quadratures". But, as Leibniz wrote to Johann Bernoulli, it is better to reduce quadratures to rectifications because "the dimension of the line is simpler than the dimension of the plane". And often we read that the reduction of quadratures to rectifications is useful because arc-lengths are more easily measured than areas: one can take a chord and stretch it along the curve, thus measuring its length.

From these quotations we can see that we must state our questions more precisely. Apparently we are dealing with construction methods for curves, primarily transcendental curves. And certain methods of construction are preferred to others. So we must ask: what was the importance of these construction methods? How did they arise? How did they influence mathematical research? Of this influence the case of the lemniscate is an example: its discovery was the result of the commitment of the Bernoullis to constructions by means of rectifications, as a solution of problems involving differential equations. As we have seen, the problem of the paracentric isochrone was only considered solved if a construction of the sought curve was given. And indeed, the importance of constructions in the solution of problems concerning curves lay in its use for the representation of curves. As such, this notion has vanished as a result of the universal acceptance of representation by means of formal expressions.

To gain insight into the role of constructions in seventeenth century mathematics we have, therefore, to go into the methods used in that period for the representation of curves. Up to the beginning of the seventeenth century the class of curves studied by geometers was rather small. Aside from the well-studied conic sections, there were a few special curves, such as the conchoid of Nicomedes and the quadratrix of Dinostratos. The representation of these various curves was accomplished by several means: either by a defining property; or by an imagined mechanical device for their delineation; or by a recipe for the construction of individual points on them; or by a combination of these. This state of affairs was changed by Descartes, who introduced equations into the study of curves. This change, however, was restricted to the class of algebraic curves. The nonalgebraic curves, which after Leibniz we call transcendental, were excluded by Descartes from geometry and relegated to the status of "mechanical".

On the representation of algebraic curves Descartes's innovation had the result that, gradually, the representation of curves by their equations was considered as equivalent—and later even as preferable—to the representation of curves by defining properties, construction recipes, or mechanical delineation devices. This was a slow process and a great deal of the mathematics of the cartesians can be characterised as the exploration of the relations between defining properties, construction recipes, delineation devices, and equations. Descartes's introduction of formulae also had another effect: a considerable increase in the number of algebraic curves that were studied. For the powerful new tool of algebraic techniques created (in the usual fashion) a new set of problems for its exploitation.

In the case of the transcendental curves the development was different. The number of transcendental curves that were studied also increased, but this growth was not caused by the application of an appropriate system of symbols and formulae. In this case it was the study of a certain important class of problems which often led to these curves. These were the so-called inverse tangent problems; they originated, as the name indicates, as the inversion of a well-known, even classical problem, the tangent problem.

The tangent problem involves the construction of lines tangent to a given curve; the inverse tangent problem seeks a curve whose tangent lines satisfy a certain given condition. Such problems were set in geometry, and also in other fields, notably in mechanics, where the study of accelerated motion often led to inverse tangent problems. Translated into modern mathematical language, the inverse tangent problems are first-order differential equations. It is, therefore, not surprising that, even if the given condition of the tangent lines is formulated algebraically, the solution of an inverse tangent problem will often be a transcendental curve. And in the absence of an equation, the seventeenth century mathematician had to represent the solution in that case by some other means, usually a construction.

Two examples

For illustration, let me show you two such constructions. The first example is Leibniz's construction of the tractrix (1693).[4] The tractrix is the curve with constant tangent *a* (see Figure 3); it is the path of a body which is dragged over a resisting horizontal surface by a cord one end of which moves along a straight line.

Leibniz's construction (see Figure 4) proceeds as follows: starting from a pair of perpendicular axes and a circle quadrant with radius *a*, he can construct

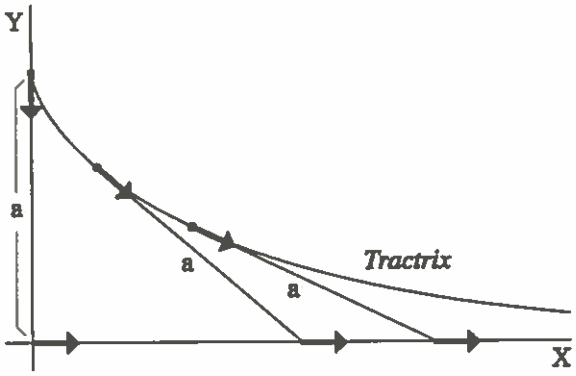

Figure 3. The Tractrix

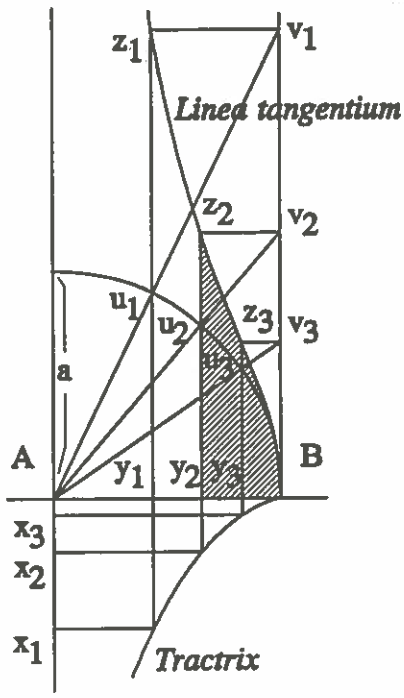

Figure 4. Leibniz's construction of the Tractrix

to every abscissa y a point z via the points u and v. These points z lie on a curve which Leibniz calls the *linea tangentium*. The construction of the tractrix itself is now performed by means of the quadrature of the *linea tangentium*: the tractrix ordinate x corresponding to abcissa y satisfies $ax =$ area yBz, and is therefore "constructible" if the area yBz is known, that is, if the quadrature of the *linea tangentium* is known. Leibniz asserts that this quadrature is dependent on the quadrature of the hyperbola; and so considers the tractrix problem as adequately solved.

The second example concerns Huygens's construction of the catenary,[5] the form of a chain suspended from two points. The construction dates from 1693. With respect to a pair of perpendicular axes as in Figure 5 Huygens draws a parabola with vertex A and focus F. To determine the ordinate y of the catenary corresponding to abscissa x he constructs a point u on the horizontal axis such that $Fx = Fu$; he draws the tangent uv to the parabola; and finally he determines the difference between the arc-length Av and the length of the tangent uv. This difference $y = Av - uv$ is the ordinate of the catenary corresponding to the abscissa x.

This construction proceeds by means of a rectification. Huygens knew well that the rectification of the parabola in turn depends on the quadrature of the hyperbola. He could, therefore, have expanded this construction to a construction by means of the quadrature of the hyperbola. From the fact that he did not do so, we may conclude that he thought the representation of the catenary by means of the arc-length of the parabola as good as, or perhaps even better than, the representation by means of the quadrature of the hyperbola.

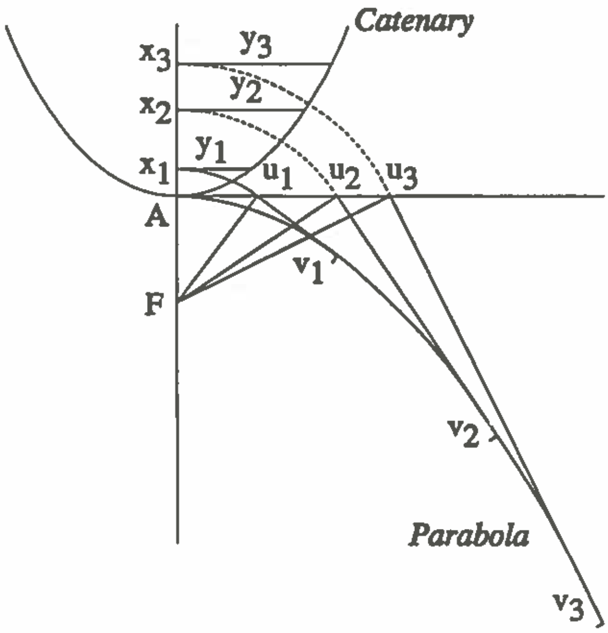

Figure 5. Huygens's construction of the Catenary

Criteria of adequacy

Of course there are often several different constructions possible for the same curve. In these circumstances it was necessary to apply certain criteria of adequacy for assessing the merits of particular sorts of constructions. These criteria, rarely stated explicitly and never as absolute conditions, nonetheless strongly influenced the methods of research; and thus they are a rewarding field of historical study. Through them, we can see which relations were thought self-evident, and which difficult; which problems were considered important; and how the mathematical objects themselves were conceived. The criteria underwent changes in the course of time, and from these changes we can learn about the changes in mathematical concepts and ways of mathematical thinking.

The criteria of adequacy for constructions were twofold. On the one hand one required the transcendental step in the construction (in our examples the quadrature and the rectification) to be performed on a simple standard curve. Hence the endeavour to reduce constructions to quadratures of the hyperbola or the circle; or, as we would now say, to logarithmic or inverse trigonometric functions. On the other hand certain practical considerations play a curious role. Such practical considerations led Johann and Jakob Bernoulli to the lemniscate; they considered construction by rectification better in practice than construction by quadratures. As an example of these practical considerations there is a passage in which Jakob describes a method of construction which he even considers better than the reduction to rectification. This method uses curves "given by nature", such as the catenary. Jakob remarks[6] that before the draughtsman has put the first lines of the construction of the hyperbola on paper, nature has drawn a catenary: one needs only to suspend a small chain along a vertical piece of paper.

The catenary is known to be constructible from the quadrature of the hyperbola. Conversely, therefore, the quadrature of the hyperbola is constructible from the catenary, and thus all other curves which depend on the quadrature of the hyperbola can be constructed by means of the catenary. Therefore it is, according to Bernoulli, useful to reduce constructions—if possible—to the catenary rather than to the quadrature of the hyperbola.

Questions

Apparently the mathematician of the seventeenth century saw his mathematical objects and operations—curves and constructions—quite differently from ourselves. In the development of operations and procedures the idea of practical applicability played a role, and in this context curves were either drawn lines, stretched cords, or suspended chains. In representing curves the seventeenth century mathematician applied neatness requirements which originated in geometrical, almost mechanical, considerations, and which therefore were quite different from the neatness requirements which the mathematicians of later times made in the representation of curves.

And this different context, this different way of conceiving mathematical objects and operations, had a considerable influence on the direction of mathematical investigation in the later seventeenth century.

Now a large number of questions arise. Did one actually perform these constructions mechanically? Or was this way of thinking a meaningless remnant from some former time? If so, where did this way of thinking originate? Can we study and understand the process in which these ways of thinking died out and were replaced by a more abstract and formalistic style? Can we pass judgment on the advantageous and disadvantageous side effects of this process of transition?

History

I will confine myself to raising these questions. This is not only for lack of time, but also because I do not know the answers; I could only give you more illustrations of these processes. And after all I did not promise you more than to raise questions which fascinate me and which I consider genuinely historical questions. I hope to have convinced you that the study of these questions is necessary for the understanding of the mathematics of the period under consideration. If so, then I have convinced you of the importance of these questions as historical questions. The history of mathematics belongs to the discipline of general historical study. It is not an auxiliary science for mathematics. One should not pursue historical studies on mathematics only in order to dig out old theorems which provide diverting literature for the modern mathematician or which might even be handy in modern theories.

If we grant that the study of the history of mathematics should be concerned with the many different and ever changing conceptions of mathematical objects and operations, and that it should search for understanding and explanation of the directions of the earlier development of mathematics, we must then ask whether such a history would be of any use for the mathematics of the present day?

I think it would. For the processes of change and evolution in the objects and methods of mathematics, whereby problems and theories arise, flourish and decline into oblivion, are present now as much as at any other time in history. And perhaps the study of the history of mathematics can contribute to the understanding of these processes, by which the mathematics of the future is shaped.

Notes

1. Jakob Bernoulli, "Constructio curvae accessus et recessus aequabilis", *Acta Eruditorum*, 1694 (Sept.); in Jakob Bernoulli, *Opera*, Geneva, 1744 (reprint Brussels (Culture et Civilisation), 1967), pp. 608–612.

2. Johann Bernoulli, "Constructio facilis curvae accessus et recessus aequabilis", *Acta Eruditorum*, 1694 (Oct.); in Johann Bernoulli, *Opera omnia*, Lausanne/Geneva, 1742 (reprint Hildesheim (Olms) 1968) vol. 1, pp. 119–122.

3. G.W. Leibniz, "De linea isochrona", *Acta Eruditorum*, April 1689, in G.W. Leibniz, *Mathematische Schriften* (7 vols, ed. C.I. Gerhardt, Berlin-Halle, 1849–1863 (reprint Hildesheim (Olms) 1961–1962)), vol. 5, pp. 234–237.

4. Leiniz G.W., "Supplementum geometriae dimensoriae", *Acta Eruditorum* September, 1693 in Leibniz, *Mathematische Schriften* (cf. note 3), vol. 5, pp. 294–301.

5. Huygens, C., *Oeuvres Complètes*, (22 vols, The Hague 1888–1950), vol. 10, pp. 414 sqq.

6. Bernoulli, Jakob, "Constructio" (cf. note 1) p. 608.