Raakvlak

Gezien: als $z = z(x,y)$ $\boxed{\text{glad}}$ rondom (a,b)

dan is de vgl van het raakvlak

$$z = z_1(a,b)(x-a) + z_2(a,b)(y-b) + z(a,b)$$

Voorbeeld.

- Neem $\boxed{z = \sqrt[3]{xy} = x^{\frac{1}{3}} y^{\frac{1}{3}}}$ dan $z_1 = \frac{\partial z}{\partial x} = \frac{1}{3} x^{-\frac{2}{3}} y^{\frac{1}{3}}$

$$z_2 = \frac{\partial z}{\partial y} = \frac{1}{3} x^{\frac{1}{3}} y^{-\frac{2}{3}}$$

- Neem $(a,b) = (1,-1)$

Invullen: $z_1(1,-1) = -\frac{1}{3}$ en $z_2(1,-1) = +\frac{1}{3}$ $\quad z(1,-1) = -1$

dus $z = -\frac{1}{3}(x-1) + \frac{1}{3}(y+1) - 1$

- Neem $(a,b) = (0,0)$

Invullen $z_1(0,0) = \frac{0}{0}$?!

Part. afgeleides op een andere manier:

$$\frac{\partial z}{\partial x}(0,0) = \lim_{h \to 0} \frac{z(h,0) - z(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

idem $\frac{\partial z}{\partial y}(0,0) = 0$

Dus raakvlak $z = \frac{\partial z}{\partial x}(0,0)(x-0) + \frac{\partial z}{\partial y}(0,0)(y-0) + z(0,0)$

$z = 0$ dwz: het raakvlak is het xy-vlak

OH WACHT. DAT KAN NIET.

Hogere orde afgeleides $z = z(x,y)$ voldoende glad

Hebben al:

$z_1 = \frac{\partial z}{\partial x}$

$z_2 = \frac{\partial z}{\partial y}$

Nog een keer diff

$$z_{11} = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2}$$

DEE TWEEDE ZET DEE IKS KWADRAAT.

$$z_{12} = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Ziet er verwarrend uit.

$$z_{21} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$z_{22} = \frac{\partial}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$$

Zie stelling p. 697

Goed nieuws! Als z glad dan geldt dat

$z_{12} = z_{21}$ oftewel $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Voorbeelden

- $z = x\sin(xy)$: Bereken $z_{11}, z_{12}, z_{21}, z_{22}$.

$z_1 = \sin xy + xy\cos xy$

$z_2 = x^2\cos xy$

$z_{11} = 2y\cos xy \cancel{+ y\cos xy} - xy^2\sin xy$

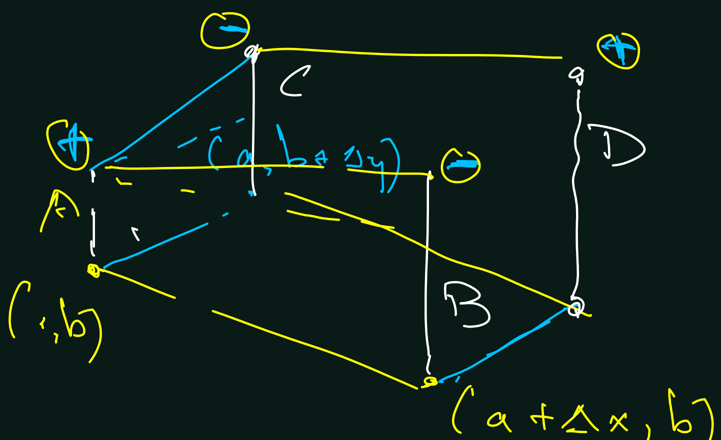
$z_{12} = 2x\cos xy \cancel{+ x\cos xy} - x^2y\sin xy$

$z_{21} = 2x\cos xy - x^2y\sin xy$

z_{22} = etc.

Waarom zijn z_{12} en z_{21} aan elkaar gelijk voor gladde z?

$$z_{21} = \frac{(D-C)-(B-A)}{\Delta x\,\Delta y}$$



$$z_{12} = \frac{(D-B)-(C-A)}{\Delta x\,\Delta y}$$

$$z_1 = \frac{B-A}{\Delta x}$$

$$z_2 = \frac{C-A}{\Delta y}$$

P.D.V. part. diff. vgl.

Laplace-vgl. 2 dim $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ functies $u = u(x,y)$

3 dim $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ $u = u(x,y,z)$

Oplossingen van de laplace-vgl heten harmonisch

Golfvergelijking (1-dim). $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ functie $z = z(x,t)$

Bijv. $z = \cos(x - ct)$ is een fie die hieraan voldoet
stelt voor: een golf die met snelheid c langs de x-as loopt.

Warmte vergelijking.

2 dim: $k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}$ $u = u(x,y,t)$

diffusieprocessen.

Kettingregel in meer dimensies.

In 1 dim: $u = u(v(x))$ $\qquad \frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx}$

Hoger dim:

- Als $z = z(u,v)$ met $u = u(x,y)$ en $v = v(x,y)$

  dan $\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}}$ $\boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}}$

- Als $z = z(u,v)$ met $u = u(x)$ en $v = v(x)$

  dan $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}$ LET OP RECHT/KROM!

Concreet

$z(u,v) = u^2 - v^2$, $\qquad u(x,y) = x+y$, $\qquad v(x,y) = x-y$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$ met $\frac{\partial z}{\partial u} = 2u$ $\qquad \frac{\partial z}{\partial v} = -2v$

invullen $\qquad \frac{\partial u}{\partial x} = 1$ $\qquad \frac{\partial v}{\partial x} = 1$

$$\frac{\partial z}{\partial x} = 2u \cdot 1 - 2v \cdot 1 = 2(x+y) - 2(x-y) = 4y.$$

of Neem $z = uv$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{met } \frac{\partial z}{\partial u} = v \text{ en } \frac{\partial z}{\partial v} = u$$

dus $\frac{\partial z}{\partial x} = v + u = 2x$

Maar kan ook zo:

$$z = uv = (x+y)(x-y) = x^2 - y^2$$

diff: $\frac{\partial z}{\partial x} = 2x$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}} \quad \boxed{\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}}$$

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Matrices!