A monkey saddle in rigid body dynamics

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A rigid body with three equal moments of inertia is moving in a nonlinear force field with potential z^3. Next to the S^1–symmetry about the vertical axis and a further S^1–symmetry introduced by normalization, there is a discrete symmetry due to a special choice of the mass distribution. The continuous symmetries allow to reduce to a one-degree-of-freedom problem, which exhibits bifurcations related to the elliptic umbilic catastrophe. This bifurcation carries over from the integrable approximation to the original system and further to perturbations that break the S^1–symmetry of the potential.

1. Introduction

The rotational motion of a rigid body with three equal principal moments of inertia, fixed at one point and not subject to external torques or forces, is a three-degrees-of-freedom system where all motions are periodic. Such maximally superintegrable systems are easily analysed. Indeed, as every axis through the fixed point is a principal axis of inertia, any motion will consist in a rotation about such an axis, which is parallel to the (fixed) angular momentum. Similarly, all bounded motions of the spatial Kepler system are periodic. In fact, while the latter can be turned into the geodesic flow on S^3, the flow of the isotropic Euler top is the geodesic flow on $SO(3)$ with respect to the bi-invariant metric.

Placing the fast top in the vertical force field $\vec{G} = -3z^2\vec{e}_z$ amounts to perturbing by the weak potential $\varepsilon \cdot z^3$ with ε inversely proportional to the square of the velocity, cf. [3]. The more integrals an integrable Hamiltonian system possesses, the more difficult it becomes to study perturbations of that system. In the present problem, since the component μ_3 of the angular momentum along the vertical axis $\vec{e}_z$ remains an integral of motion, one can reduce to two degrees of freedom.

The unperturbed reduced system in two degrees of freedom is still super-integrable (this would not be true if the principal moments of inertia were not all equal). The periodic orbits define an S^1–symmetry, and normalization pushes this S^1–symmetry through the Taylor series of the perturbation. The normal form of order one can be computed by simply averaging along the periodic orbits. In this way the total angular momentum $|\mu|$ becomes a further integral of motion.

By construction the normal form can be reduced to one degree of freedom. Here it is important to consider external parameters like ε or the various moments of the mass distribution as fixed constants, while the values of the internal or distinguished parameters μ_3 and $|\mu|$ are given by the initial conditions and thus allowed to vary. Since $\vec{G}$ is a positional force, the latter enter only as the quotient $\frac{\mu_3}{|\mu|}$.

2. The mass distribution

Let us recapitulate the main facts about the Euler top with three equal principal moments of inertia. The reader may find a comprehensive introduction in [1,2,6].

We choose a set of axes $\vec{e}_x, \vec{e}_y, \vec{e}_z$ fixed in space, with $\vec{e}_z$ pointing in the (vertical) direction of the force, and a body set of axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$. The configuration space is the group $SO(3)$ of orientation preserving three-by-three matrices which specify how to transform $\vec{e}_x, \vec{e}_y, \vec{e}_z$ into $\vec{e}_1, \vec{e}_2, \vec{e}_3$. The phase space is the cotangent bundle $T^*SO(3)$, the space of positions and (angular) momenta. An element $\alpha \in T^*SO(3)$ yields the components μ_1, μ_2, μ_3 of the angular momentum with respect to the spatial frame $\vec{e}_x, \vec{e}_y, \vec{e}_z$ and the components ℓ_1, ℓ_2, ℓ_3 with respect to the body set of axes. As the three principal moments of inertia are equal, $I_1 = I_2 = I_3 = 1$, the kinetic energy is given by

$$T \quad = \quad \frac{|\mu|^2}{2} \quad = \quad \frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{2} \quad = \quad \frac{\ell_1^2 + \ell_2^2 + \ell_3^2}{2} \quad .$$

In the same way that a constant force only acts on the centre of mass (the first moments of the mass distribution), a linear force would only act on the second moments of the mass distribution, *i.e.* not at all since these are equal. Similarly, the force field $\vec{G} = -3\varepsilon z^2 \vec{e}_z$ only "sees" the third moments

$$M_{ijk} \quad = \quad \int \zeta_i \zeta_j \zeta_k \, dm$$

of the mass distribution dm. The freedom of orientation of the "principal axes of inertia" $\vec{e}_1, \vec{e}_2, \vec{e}_3$ may be used to reduce the number of external parameters M_{ijk} from ten to seven, but here we restrict even further to the special case $M_{111} = M_{222} = M_{333} = 1$ and $M_{ijk} = 0$ else. As shown in [9] it is possible to construct corresponding (non-homogeneous) rigid bodies. The centre of mass, principal moments of inertia or higher moments of the mass distribution do not enter the potential energy

$$V \;=\; -\int\int_0^{\zeta} (\vec{G} \mid \mathrm{d}\zeta')\, dm \;=\; \sum_{i=1}^{3} \varepsilon(\vec{e}_z \mid \vec{e}_i) \ . \tag{1}$$

The Hamiltonian function $H = T + V$ not only admits the continuous symmetry of rotations about the vertical axis $\vec{e}_z$, but is furthermore invariant under the discrete symmetry group of all permutations of $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, isomorphic to the dihedral group D_3.

The S^1–symmetry of rotations about $\vec{e}_z$ is the same S^1–symmetry that already appears in the heavy rigid body; its reduction is well-known and goes back to Poisson. In the body representation

$$\begin{array}{ccc} T^*SO(3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ \alpha & \mapsto & (g, \ell) \end{array}$$

the S^1–symmetry amounts to

$$\begin{array}{ccc} S^1 \times (SO(3) \times \mathbb{R}^3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ (\rho, (g, \ell)) & \mapsto & (\exp_\rho \circ g, \ell) \end{array}$$

where $\exp_\rho \in SO(3)$ stands for the rotation by the angle ρ about the third axis. Dividing S^1 out of $SO(3)$ yields a sphere and the isomorphism

$$\begin{array}{ccc} SO(3)\big/_{S^1} & \longrightarrow & S^2 \\ g \ \left(\operatorname{mod} S^1\right) & \mapsto & g^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =: \gamma \end{array}$$

clarifies the geometrical meaning of this sphere. It is the space of possible positions of the vertical axis $\vec{e}_z$ measured in the body set of axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$. In particular the radius of this sphere is 1. The vector γ is called the Poisson vector. The Hamiltonian function $H = T + V$ reads

$$H(\gamma, \ell) \;=\; \frac{1}{2}(\ell_1^2 + \ell_2^2 + \ell_3^2) \;+\; \varepsilon(\gamma_1^3 + \gamma_2^3 + \gamma_3^3)$$

after this reduction to two degrees of freedom.

3. The normal form

The normalization procedure consists in finding a change of co-ordinates that transforms H into its average

$$\bar{H}(\gamma, \ell) \;=\; \frac{1}{\tau(\gamma, \ell)} \int_0^{\tau(\gamma,\ell)} H(\varphi_t(\gamma, \ell))\, \mathrm{d}t \tag{2}$$

along the flow φ_t of the unperturbed system X_T (which is periodic with period $\tau(\gamma, \ell)$ depending on the initial condition) plus higher order terms. Repeating this process yields higher order normal forms, but for our purposes the normal form of order one turns out to be sufficient. As shown in [9] the truncated normal form (2) is given by

$$\bar{H}(\gamma, \ell) \;=\; \frac{1}{2}|\mu|^2 \;+\; \frac{\varepsilon}{2}\frac{\mu_3}{|\mu|}\sum_{i=1}^{3}(5\frac{\mu_3^2}{|\mu|^2} - 3)\frac{\ell_i^3}{|\mu|^3} \;+\; 3(1 - \frac{\mu_3^2}{|\mu|^2})\frac{\ell_i}{|\mu|} \;. \tag{3}$$

By construction $\bar{H}$ is invariant with respect to the S^1–action defined by the flow φ_t of X_T ; the Hamiltonian function no longer depends on γ. The kinetic energy in $\bar{H} = T + \bar{V}$ becomes a "constant" without dynamic meaning and may be omitted after the reduction to one degree of freedom. As $\vec{G}$ is a positional force, the angular momenta $|\mu|$, μ_3 and ℓ_i enter only as quotients $\frac{\mu_3}{|\mu|}$ and $\frac{\ell_i}{|\mu|}$. In particular (3) defines a family of one-degree-of-freedom systems depending on the parameter $\nu := \frac{\mu_3}{|\mu|} \in [-1, 1]$. We also write $\xi_i := \frac{\ell_i}{|\mu|}$ and are left with the Hamiltonian function

$$\mathcal{H}_\nu \;:=\; \frac{\varepsilon\nu}{2}\sum_{i=1}^{3}(5\nu^2 - 3)\xi_i^3 \;+\; 3(1 - \nu^2)\xi_i \tag{4}$$

on the sphere with radius 1. The Poisson bracket is given by $\{\xi_i, \xi_j\} = -\epsilon_{ijk}\,\xi_k$, where the alternating Levi-Civita symbol ϵ_{ijk} denotes the sign of the permutation $\binom{1\,2\,3}{i\,j\,k}$ and vanishes if two of the i, j, k are equal.

From the perturbed rigid body $\mathcal{H}_\nu$ inherits a D_3–invariance. However, the Poisson bracket is not D_3–invariant; for $\sigma \in D_3$ we have

$$\begin{aligned} \{\sigma(\xi_i), \sigma(\xi_j)\} \;&=\; \{\xi_{\sigma(i)}, \xi_{\sigma(j)}\} \;=\; -\epsilon_{\sigma(i)\sigma(j)\sigma(k)}\,\xi_{\sigma(k)} \\ &=\; -\operatorname{sgn}\sigma\,\epsilon_{ijk}\,\sigma(\xi_k) \;=\; \operatorname{sgn}\sigma\cdot\sigma(\{\xi_i, \xi_j\}) \;. \end{aligned}$$

While even permutations, *i.e.* rotations about the axis along $\begin{pmatrix}1\\1\\1\end{pmatrix}$ with an angle $\frac{2\pi k}{3}$, $k \in \mathbb{Z}_3$, are therefore symmetries of the Hamiltonian system $X_{\mathcal{H}_\nu}$, the transpositions only lead to time-reversing symmetries.

Remark 3.1. The Hamiltonian system defined by (4) has another reversing symmetry, the reflection

$$\psi \;:\; \xi \;\mapsto\; -\xi \tag{5}$$

about the origin. While the Poisson bracket is invariant under ψ, the Hamiltonian (4) satisfies $\mathcal{H}_\nu \circ \psi = -\mathcal{H}_\nu$. In particular, the composition $\psi \circ \tau$ with a transposition $\tau \in D_3$ is a (non-reversing) symmetry of the Hamiltonian system that multiplies both $\mathcal{H}_\nu$ and the Poisson bracket by -1.

The symmetry (5) is induced by the time-reversing reflection about the origin

$$(\vec{e}_1, \vec{e}_2, \vec{e}_3) \;\mapsto\; (-\vec{e}_1, -\vec{e}_2, -\vec{e}_3)$$

of the perturbing potential (1). Similarly, the reflection $\vec{e}_z \mapsto -\vec{e}_z$ induces the time-reversing symmetry $\nu \mapsto -\nu$ of the family $X_{\mathcal{H}_\nu}$. At $\nu = 0$ this latter symmetry enforces all points to be equilibria.

For $\nu = \pm\sqrt{\frac{3}{5}}$ the equilibria fulfill $\xi_1 = \xi_2 = \xi_3$, *i.e.* $\xi = \frac{\pm 1}{\sqrt{3}}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$. These two equilibria are fully D_3–symmetric, and they exist for all values of ν. Further occurring equilibria break the $\mathbb{Z}_3$–symmetry and taking furthermore the reversing symmetry (5) into account these equilibria form sextuples. All equilibria in the family $(X_{\mathcal{H}_\nu})_{\nu\in[-1,1]}$ retain a remaining reversing symmetry defined by one of the three transpositions.

Proposition 3.2. *For $\nu \neq 0$ the equilibria of $X_{\mathcal{H}_\nu}$ all lie in the union*

$$\left\{\, \xi \in S^2 \;\;|\;\; \xi_1 = \xi_2 \;\; or \;\; \xi_1 = \xi_3 \;\; or \;\; \xi_2 = \xi_3 \,\right\}$$

of three great circles.

Proof. When $\nu \neq \pm\sqrt{\frac{3}{5}}$ the last factors in

$$\begin{aligned}
\dot{\xi}_1 &= \{\xi_1, \mathcal{H}_\nu\} = \tfrac{3}{2}\varepsilon\nu(\xi_2 - \xi_3)\left((5\nu^2 - 3)\xi_2\xi_3 + \nu^2 - 1\right)\\
\dot{\xi}_2 &= \{\xi_2, \mathcal{H}_\nu\} = \tfrac{3}{2}\varepsilon\nu(\xi_3 - \xi_1)\left((5\nu^2 - 3)\xi_1\xi_3 + \nu^2 - 1\right)\\
\dot{\xi}_3 &= \{\xi_3, \mathcal{H}_\nu\} = \tfrac{3}{2}\varepsilon\nu(\xi_1 - \xi_2)\left((5\nu^2 - 3)\xi_1\xi_2 + \nu^2 - 1\right)
\end{aligned}$$

can only vanish simultaneously if $\xi_2\xi_3 = \xi_1\xi_3 = \xi_1\xi_2$. $\square$

The detailed analysis in [9] reveals that the family of Hamiltonian systems $(\mathcal{H}_\nu)_{\nu\in[-1,1]}$ undergoes seven bifurcations. Next to the degenerate vector field for $\nu = 0$ there are centre-saddle bifurcations at the four parameter values

$$\pm\nu_b \;=\; \pm\sqrt{\frac{3+\sqrt{8}}{5+\sqrt{8}}} \qquad \text{and} \qquad \pm\,\nu_c \;=\; \pm\sqrt{\frac{3-\sqrt{8}}{5-\sqrt{8}}}\;.$$

Furthermore the two centres $\frac{\pm 1}{\sqrt{3}}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$ each undergo two $\mathbb{Z}_3$–equivariant bifurcations at $\pm\nu_a = \pm\frac{1}{2}\sqrt{3}$ where the Hamiltonian displays a monkey saddle, the singularity D_4^-. The family (4) is structurally stable with respect to small perturbations that respect all discrete symmetries.

The close numerical values $\nu_a \approx 0.866$ and $\nu_b \approx 0.8629$ suggest to treat the two bifurcations as one sequence, see figures 1 and 2. This is reminiscent of a periodic orbit passing through a 1:3 resonance, cf. [7,8,10]. Following

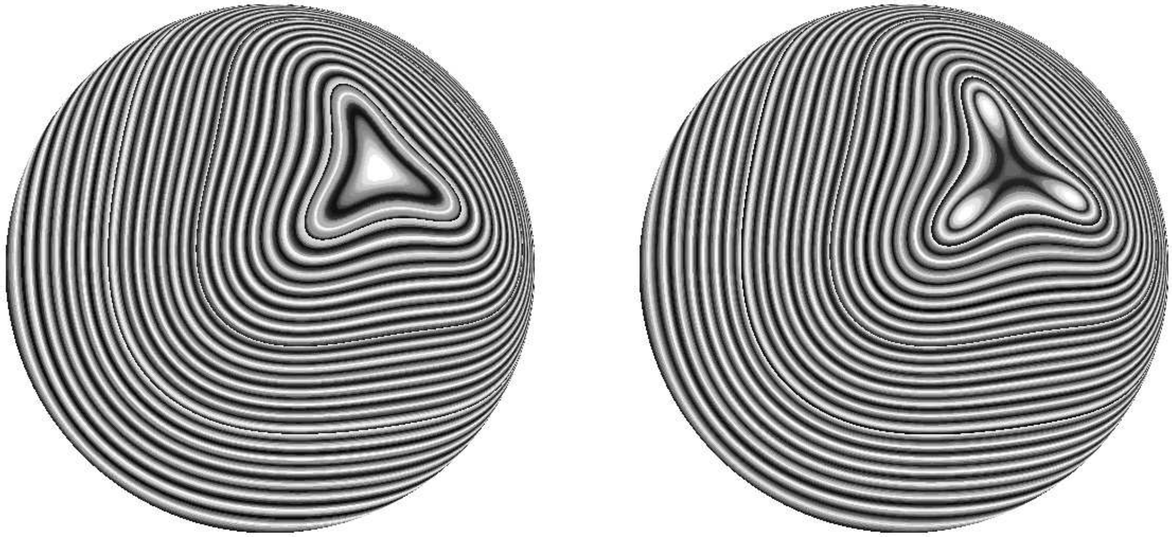

Fig. 1. Phase portraits [12] for parameter values $\nu = 0.8629$ and $\nu = 0.864$.

the semi-gobal approach of [4] we choose a local chart performing a planar projection along $\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$. The resulting co-ordinates (q, p) are not canonical but satisfy $\{q, p\} = -\sqrt{1 - p^2 - q^2}$. Omitting the constant term and dividing by $3\sqrt{3}(2\nu^3 - \nu)\varepsilon$ the 4–jet reads

$$j^4(\mathcal{H}_\nu)(p, q) \;=\; -\frac{1}{2}\left(\frac{p^2 + q^2}{2}\right)^2 \;+\; A\left(\frac{p^2 q}{2} - \frac{q^3}{3}\right) \;+\; \lambda(\nu)\,\frac{p^2 + q^2}{2}$$

with coefficient $A = \frac{-1}{3\sqrt{2}}\frac{5\nu^2-3}{2\nu^2-1}$ ($A \leq \frac{-4}{3+3\sqrt{8}}$ for $\nu \geq \nu_b$) and reparametrisation $\lambda(\nu) = \frac{1}{3}\frac{4\nu^2-3}{2\nu^2-1}$ satisfying $\lambda(\nu_a) = 0$ and $\lambda'(\nu_a) = 8 > 0$. The analysis of $j^4(\mathcal{H}_\nu)$ in [4,8] predicts a triplet of centre-saddle bifurcations at $\lambda = -\frac{1}{8}A^2$ which is smaller than the correct value $\lambda(\nu_b) = -\frac{3-\sqrt{8}}{3+3\sqrt{8}}$. As the centre-saddle bifurcation takes place at $\frac{1}{2}\left(\begin{smallmatrix}1\\1\\\sqrt{2}\end{smallmatrix}\right)$ (and its symmetric counterparts) the higher order terms $\mathcal{H}_\nu - j^4(\mathcal{H}_\nu)$ cannot be made arbitrarily small.

4. Perturbation analysis

Reconstructing the flow to two degrees of freedom amounts to attaching a 1–torus (*i.e.* an S^1) to every point on the sphere $\{\mu_3\} \times S^2_{|\mu|}$. In this way the periodic orbits of $X_{\mathcal{H}_\nu}$ give rise to invariant 2–tori, and the equilibria lead to periodic orbits of $X_{\bar{H}}$. The normal behaviour of the latter is induced from the linearization of the corresponding equilibrium. Thus, most periodic orbits are elliptic or hyperbolic, those at the (periodic) centre-saddle bifurcations are parabolic, and the (normal) linearization of the ones undergoing the $\mathbb{Z}_3$–symmetric bifurcations at $\frac{\mu_3}{|\mu|} = \pm\frac{\sqrt{3}}{2}$ vanishes. When $\mu_3 = \pm|\mu|$

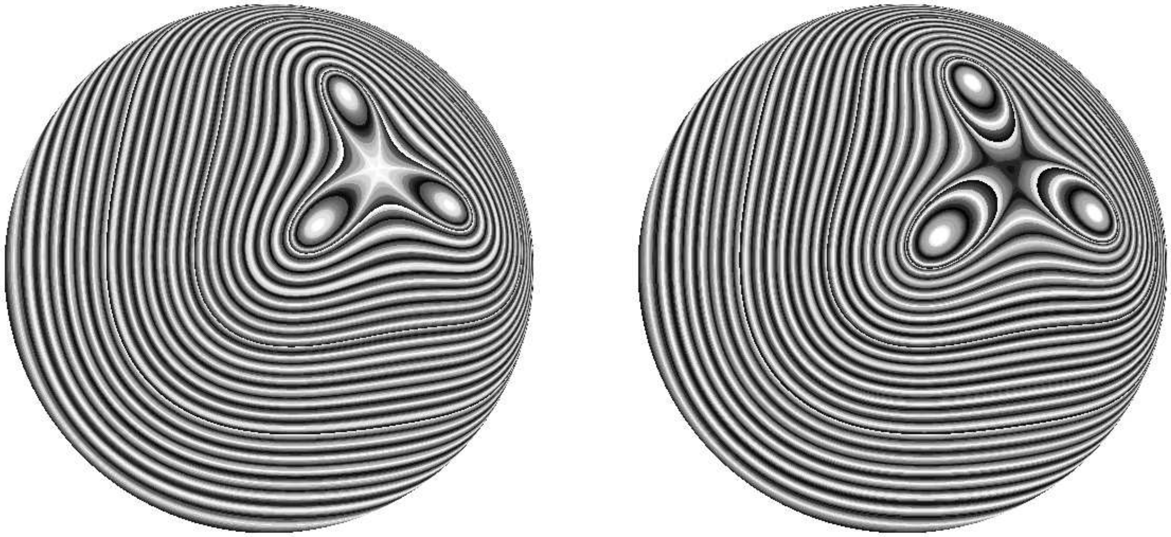

Fig. 2. Phase portraits [12] for parameter values $\nu = 0.866$ and $\nu = 0.868$.

the S^1 we "attach" has shrunk to a point, so the flow on $\{\pm|\mu|\} \times S^2_{|\mu|}$ carries over directly, leading to several equilibria and further families of elliptic periodic orbits.

The occurring equilibria, hyperbolic and elliptic periodic orbits persist the passage from the normal form approximation $X_{\bar{H}}$ back to X_H by means of the implicit mapping theorem. As shown in [8,11] the periodic centre-saddle bifurcations and the $\mathbb{Z}_3$–equivariant bifurcations persist as well. To ensure the persistence of a large Cantor set of 2–tori on each energy shell we need that the determinant

$$\det \begin{pmatrix} D^2\bar{H}_\varepsilon & D\bar{H}_\varepsilon \\ D\bar{H}_\varepsilon & 0 \end{pmatrix} \quad = \quad -|\mu|^2 \frac{\partial^2 \bar{H}_\varepsilon}{\partial \mathcal{I}^2} \; + \; \mathcal{O}(\varepsilon^2) \tag{6}$$

is bounded away from zero, cf. [2]. Here $\bar{H}_\varepsilon = T + \bar{V}$ is an analytic function in the actions $(|\mu|, \mathcal{I})$ and extracting some neighbourhood of the isolated zeros of $\frac{\partial^2 \bar{H}}{\partial \mathcal{I}^2}$ we get the necessary bound on the determinant (6). The persistent invariant 2–tori divide the 3–dimensional energy shells, whence the elliptic equilibria and periodic orbits are stable in the sense of Lyapunov.

Attaching an S^1 to every point reconstructs the motion of the rigid body in $T^*SO(3)$. Equilibria thereby turn into periodic orbits, while periodic orbits become invariant 2–tori on $T^*SO(3)$, under preservation of the normal behaviour. The Cantor family of quasi-periodic 2–tori yields invariant 3–tori that may be resonant, but do not foliate into periodic orbits. All elliptic periodic orbits and normally elliptic invariant 2–tori are stable in the sense of Lyapunov. For $\mu_3 = 0$ the motion is the periodic motion of the unperturbed system.

It is instructive to consider a small perturbation $H_\delta = T + V + P$ with $\|P\| < \delta \leq \varepsilon^2$ that breaks the S^1–symmetry of the potential but leaves the discrete symmetries intact. This perturbation analysis has to take place in 3 degrees of freedom. A lower bound $\varepsilon^2 \kappa$ on the leading term $|\mu|^2 \left| \frac{\partial^2 \bar{H}_\varepsilon}{\partial \mathcal{I}^2} \frac{\partial^2 \bar{H}_\varepsilon}{\partial \mu_3^2} \right|$ of the determinant corresponding to (6) yields again a Cantor family of maximal tori, but in three degrees of freedom these 3–dimensional tori do not divide the 5–dimensional energy shells. Hence, elliptic equilibria that are not extrema of $\bar{H}$ may become unstable. The families of persistent 2–tori become Cantorised as well, see [5] for the hyperbolic and elliptic 2–tori and [8] for quasi-periodic 2–tori involved in a bifurcation. Note that the persistence proof in [8] is compatible with the approach at the end of Section 3 to consider the 4–jet of the 3–determined singularity D_4^- and treat the bifurcations at ν_a and ν_b simultaneously.

Breaking the $\mathbb{Z}_3$–symmetry makes the monkey saddle a phenomenon of co-dimension 2, and of co-dimension 3 if the D_3–symmetry is broken. Such symmetry breaking may come from a more general mass distribution, and as shown in [9] the necessary unfolding parameters may be provided by the differences $M_{333} - M_{222} - M_{111}$ and $M_{333} - M_{222}$ of the "diagonal" third moments of the mass distribution.

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