A classical particle with spin realized by reduction of a nonlinear nonholonomic constraint

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Abstract[1]
In this paper we describe the motion of a nonlinear nonholonomically constrained system which after reduction realizes a nonrelativistic classical particle with spin.

1 Introduction

The usual physical interpretation of spin is that of internal angular momentum. The main difference between classical spin and angular momentum of a rigid body is that the length of the spin vector is fixed a priori, while the length of the angular momentum vector J is a dynamical variable. This suggests that the dynamics of a particle with spin should be related to that of a rigid body by restricting the phase space to those points where the length of the angular momentum vector is fixed and then reducing the rigid body degrees of freedom. In the present paper we carry out this program and obtain Souriau's formulation of the dynamics of a particle with spin, see Souriau [5].

Since the constraint given by fixing the length of the angular momentum vector is nonlinear in velocities we are lead into the controversial field of dynamics of systems with nonlinear nonholonomic constraints, see Arnol'd [1] or Naimark and Fufaev [4]. The main problem of deciding what the

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dynamics of such a system should be is avoided here, because the postulated constraint is preserved and we know the dynamics of the rigid body. Thus we are left with the problem of decribing the reduction of a system with nonlinear nonholonomic constraints. To solve this problem we use the procedure of Bates and Śniatycki [2]. We have two checks that this approach is correct: (i) our procedure gives the well established Souriau model and (ii) it is equivalent to Marsden-Weinstein reduction.

2 The unconstrained system

Consider a uniformly charged spherically symmetric rigid body with all its principal moments of inertia equal to I having a magnetic moment $q\,J$. Impose a constant magnetic field $b=(b_1,b_2,b_3)$. Mathematically this unconstrained system is described as follows. Its configuration space is the three dimensional rotation group $\mathrm{SO}(3)$ and its phase space (after trivialization by left translation) is $\mathrm{SO}(3)\times \mathrm{so}(3)$, where $\mathrm{so}(3)$ is the Lie algebra of 3×3 skew symmetric matrices. On phase space we have the symplectic form

$$\omega(A,X)\Big((AY_1,Z_1),(AY_2,Z_2)\Big) = Ik(Y_1,Z_2)-Ik(Y_2,Z_1)+Ik(X,[Y_1,Y_2]), \quad (1)$$

where $(A,X)\in \mathrm{SO}(3)\times\mathrm{so}(3)$, $(AY_i,Z_i)\in T_{(A,X)}\Big(\mathrm{SO}(3)\times\mathrm{so}(3)\Big)$ for $i=1,2$, and $k:\mathrm{so}(3)\times\mathrm{so}(3)\to\mathbf{R}:(X,Y)\to -\frac{1}{2}\,\mathrm{tr}\,XY$ is the Killing metric. For more details about this Lie group model for the rigid body, see Cushman and Bates [3]. The Hamiltonian of the unconstrained system is

$$h:\mathrm{SO}(3)\times\mathrm{so}(3)\to\mathbf{R}:(A,X)\to \tfrac{1}{2}\,I\,k(X,X)+qI\,k(\mathrm{Ad}_A X,B), \quad (2)$$

where

$$B=\begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}.$$

The second term in (2) represents the interaction of the charged rigid body with the magnetic field. A straightforward calculation shows that the Hamiltonian vector field X_h associated with the Hamiltonian h has integral curves which satisfy

$$\left\{ \begin{array}{lcl} \dot{A} & = & A(X+q\,\mathrm{Ad}_{A^{-1}}B) \\ \dot{X} & = & 0 \end{array} \right. \quad (3)$$

There are two Hamiltonian actions of SO(3) on our unconstrained system. First, the "body" action, corresponding to the change of frame fixed in the body, which is given by the lift of right multiplication to the tangent bundle $T\,\mathrm{SO}(3)$ of SO(3). After left trivialization the body action becomes

$$\begin{aligned}\Psi : \mathrm{SO}(3) \times (\mathrm{SO}(3) \times \mathrm{so}(3)) &\to \mathrm{SO}(3) \times \mathrm{so}(3) : \\ \big(C, (A, X)\big) &\to (AC^{-1}, \mathrm{Ad}_C X).\end{aligned} \tag{4}$$

Second, the "space" action, corresponding to rotation in physical space, given by left multiplication on $T\,\mathrm{SO}(3)$. After left trivialization the space action becomes

$$\begin{aligned}\mathrm{SO}(3) \times (\mathrm{SO}(3) \times \mathrm{so}(3)) &\to \mathrm{SO}(3) \times \mathrm{so}(3) : \\ \big(C, (A, X)\big) &\to (CA, X).\end{aligned} \tag{5}$$

The right action (4) is a symmetry of the unconstrained system whereas the left action (5) is not, due to the presence of the magnetic field. The momentum of the left action

$$J : \mathrm{SO}(3) \times \mathrm{so}(3) \to \mathrm{so}(3) : (A, X) \to I\mathrm{Ad}_A X,$$

(which is physically the angular momentum of the body), is not conserved by the unconstrained system.

3 The constrained system

Now constrain the Hamiltonian system $(h, \mathrm{SO}(3) \times \mathrm{so}(3), \omega)$ to the submanifold

$$M = \Big\{(A, X) \in \mathrm{SO}(3) \times \mathrm{so}(3) \big| k(X, X) = 1\Big\}. \tag{6}$$

This corresponds to requiring that the magnitude of the angular momentum of the rigid body is I. (In a similar way we can look at any nonzero magnitude of the angular momentum.) Because the magnitude of the angular momentum is conserved, the unconstrained vector field X_h is tangent to M.

By its very definition M is a *nonlinear nonholonomic constraint.* M determines a constraint 1-form φ on $\mathrm{SO}(3) \times \mathrm{so}(3)$ given by

$$\varphi(A, X)(AY, Z) = k(X, Y) \tag{7}$$

Because M is not a subbundle of TM, the constrained system $(M, X_h|M, \varphi)$ is not linear nonholonomic and therefore falls outside the theory of Bates and Śniatycki . The 1-form φ and the tangent bundle TM of M determine a constraint distribution H on $\mathrm{SO}(3) \times \mathrm{so}(3)$ defined by

$$\begin{aligned} H_{(A,X)} &= \ker \varphi(A,X) \cap T_{(A,X)}M \\ &= \Big\{ (AY, Z) \in T_{(A,X)}\Big(\mathrm{SO}(3) \times \mathrm{so}(3)\Big) \Big| k(X,Y) = k(X,Z) = 0 \Big\}. \end{aligned} \tag{8}$$

Observe that the constrained vector field $X_h|M$ does *not* lie in the constraint distribution H. To remedy this, we note that $H_{(A,X)}$ is a symplectic subspace of $\Big(T_{(A,X)}(\mathrm{SO}(3) \times \mathrm{so}(3)), \omega(A,X)\Big)$. Therefore we may write

$$T_{(A,X)}(\mathrm{SO}(3) \times \mathrm{so}(3)) = H_{(A,X)} \oplus H^{\perp}_{(A,X)}$$

for every $(A,X) \in (\mathrm{SO}(3) \times \mathrm{so}(3))$. Here $H^{\perp}_{(A,X)}$ is the symplectic perpendicular of $H_{(A,X)}$.

We can decompose $X_h|M$ into its components on $H_{(A,X)}$ and $H^{\perp}_{(A,X)}$:

$$X_h(A,X) = X_h^H(A,X) + X_h^{H^{\perp}}(A,X).$$

A calculation shows that for every $(A,X) \in M$,

$$X_h^H(A,X) = \Big(A(q \,\mathrm{Ad}_{A^{-1}} B - q\, k(X, \mathrm{Ad}_{A^{-1}} B) X), 0 \Big) \tag{9}$$

and

$$X_h^{H^{\perp}}(A,X) = \Big(A(X + q\, k(X, \mathrm{Ad}_{A^{-1}} B) X), 0 \Big). \tag{10}$$

Looking forward to the next section, (see the proof of (14)), we observe that $X_h^{H^{\perp}}$ is killed by the reduction process.

4 Symmetry and its reduction

We now show that the nonholonomic system $(M, X_h^H|M, \varphi)$ has a symmetry. Consider the action Ψ given by (4). Since the constraint manifold M is invariant under Ψ, the induced action $\widetilde{\Psi} = \Psi|(\mathrm{SO}(3) \times M)$ is defined. Because the 1-form φ is $\widetilde{\Psi}$-invariant, it follows that the constraint distribution H is also $\widetilde{\Psi}$-invariant. Consequently, $\widetilde{\Psi}$ is a symmetry of the nonholonomic system $(M, X_h^H|M, \varphi)$, that is,

$$T_{(A,X)}\Psi_C\Big(X_h^H(A,X)\Big) = X_h^H(\Psi_C(A,X))$$

for every $C \in \mathrm{SO}(3)$ and every $(A,X) \in M$.

To gain a some insight into the dynamical meaning of $(M, X_h^H|M, \varphi)$ we remove its SO(3) symmetry following the procedure of Bates and Śniatycki [2]. (We also use their notation.) The infinitesimal generator of $\widetilde{\Psi}$ in the direction $Y \in \mathrm{so}(3)$ is given by the vector field $X^Y(A,X) = (-AY, -[X,Y])$. Thus the SO(3) symmetry distribution V on M is

$$V_{(A,X)} = \Big\{ X^Y(A,X) \in T_{(A,X)}M \,\big|\, Y \in \mathrm{so}(3) \Big\}.$$

A calculation shows that the distribution $V \cap H$ on M is given by

$$(V \cap H)_{(A,X)} = \Big\{ (-AY, -[X,Y]) \in T_{(A,X)}M \,\big|\, k(X,Y) = 0 \Big\}.$$

Consequently, the distribution

$$U = \{ u \in H \,|\, \omega_H(u,v) = 0 \quad \text{for every } v \in V \cap H \}$$

on M is

$$U_{(A,X)} = \Big\{ (AY, 0) \in T_{(A,X)}M \,\big|\, k(X,Y) = 0 \Big\}. \tag{11}$$

Note that $X_h^H(A,X) \in U_{(A,X)}$ for every $(A,X) \in M$, since

$$k(q\, \mathrm{Ad}_{A^{-1}}B - q\, k(X, \mathrm{Ad}_{A^{-1}}B)X, X) = 0.$$

Now consider the map

$$\rho : M \to \mathrm{so}(3)^* : (A,X) \to -Ik^\sharp(\mathrm{Ad}_A X) = \nu. \tag{12}$$

Because ρ is constant on $\widetilde{\Psi}$ orbits, it induces a map

$$\widetilde{\rho} : M/\,\mathrm{SO}(3) \to \mathrm{so}(3)^*$$

on the orbit space $M/\,\mathrm{SO}(3)$. Since the adjoint action of SO(3) is transitive on $S = \{X \in \mathrm{so}(3) \,|\, k(X,X) = 1\}$, the range of ρ (and hence of $\widetilde{\rho}$) is the SO(3) coadjoint orbit

$$\mathcal{O}_\mu = \Big\{ \mathrm{Ad}^t_{A^{-1}}\mu \,\big|\, A \in \mathrm{so}(3) \Big\},$$

where $\mu = -Ik^\sharp(X_0)$ for some $X_0 \in S$. From the fact that fiber $\rho^{-1}(\nu)$ is a single $\widetilde{\Psi}$ orbit, it follows that the map $\widetilde{\rho}$ is a diffeomorphism of $M/\mathrm{SO}(3)$ onto $\mathcal{O}_\mu$. Therefore ρ is the reduction map of the SO(3) symmetry of $(M, X_h^H|M, \varphi)$.

We know that the distribution U pushes down under ρ to the reduced distribution $\overline{H}$ on $\mathcal{O}_\mu$. A calculation shows that

$$\begin{aligned} \overline{H}_\nu &= \Big\{ -\mathrm{ad}^t_{\mathrm{Ad}_A Y}\nu \,\big|\, k(X,Y)=0,\ Y \in \mathrm{so}(3) \Big\} \\ &= \Big\{ -\mathrm{ad}^t_{\mathrm{Ad}_A Y}\nu \,\big|\, Y \in \mathrm{so}(3) \Big\} \;=\; T_\nu \mathcal{O}_\mu . \end{aligned}$$

We also know that the 2-form ω_H on H pushes down under ρ to a symplectic form $\omega_{\overline{H}}$ on $\mathcal{O}_\mu$. Again a calculation shows that

$$\omega_{\overline{H}}(\nu)\Big(-\mathrm{ad}^t_{\mathrm{Ad}_A Y_1}\nu, -\mathrm{ad}^t_{\mathrm{Ad}_A Y_2}\nu \Big) = -\mathrm{Ad}^t_A \nu\Big([Y_1, Y_2] \Big).$$

Thus the reduced vector field is

$$\begin{aligned} X_{\overline{H}}(\nu) &= T_{(A,X)}\rho\Big(X_h^H(A,X) \Big) \\ &= -q\,\mathrm{ad}^t_{\overline{B}}\nu, \quad \text{where } \overline{B} = B - k(B, \mathrm{Ad}_A X)\mathrm{Ad}_A X \\ &\qquad\qquad\qquad \text{and } \nu \text{ is given by (12)} \\ &= -q\,\mathrm{ad}^t_B \nu. \end{aligned} \tag{13}$$

We now show that we obtain the same result using Marsden-Weinstein reduction. Below we show that

$$X_h^{H^\perp}(A,X) \in \ker T_{(A,X)}\rho \tag{14}$$

for every $(A,X) \in M$. For the moment assume (14). Applying Marsden-Weinstein reduction on M to remove the spatial SO(3) symmetry from the vector field $X_h|M$, we find that the reduced vector field is

$$T_{(A,X)}\rho\Big(X_h(A,X) \Big) = T_{(A,X)}\rho\Big(X_h^H(A,X) \Big) \;=\; X_{\overline{H}}(\rho(A,X)).$$

This is the same as (13). We now prove (14). First, note that

$$T_{(A,X)}\rho : T_{(A,X)}M \to T_\nu \mathcal{O}_\mu : (AY, Z) \to -\mathrm{ad}^t_{\mathrm{Ad}_A Y}\nu - \mathrm{Ad}^t_{A^{-1}}(Ik^\sharp(Z)). \tag{15}$$

Using (15) and formula (10) for $X_h^{H^\perp}$, we obtain

$$\begin{aligned} T_{(A,X)}\rho\Big(X_h^{H^\perp}(A,X)\Big) &= (1+q\,k(X,\mathrm{Ad}_{A^{-1}}B))T_{(A,X)}\rho(AX,0) \\ &= (1+q\,k(X,\mathrm{Ad}_{A^{-1}}B))\,\mathrm{ad}^t_{\mathrm{Ad}_A X}\Big(Ik^\sharp(\mathrm{Ad}_A X)\Big). \end{aligned}$$

But for every $Z \in \mathrm{so}(3)$ we see that

$$\begin{aligned} \mathrm{ad}^t_{\mathrm{Ad}_A X}\Big(Ik^\sharp(\mathrm{Ad}_A X)\Big)Z &= I\,k(\mathrm{Ad}_A X, [\mathrm{Ad}_A X, Z]) \\ &= I\,k([\mathrm{Ad}_A X, \mathrm{Ad}_A X]\,, Z) \,=\, 0. \end{aligned}$$

Thus (14) follows.

The reduced vector field $X_{\overline{H}}$ (13) is Hamiltonian on $(\mathcal{O}_\mu, \omega_{\overline{H}})$ with Hamiltonian

$$\overline{h} : \mathcal{O}_\mu \subseteq \mathrm{so}(3)^* \to \mathbf{R} : \nu \to \tfrac{1}{2}\,\mu(k^\flat(\mu)) - q\,\nu(B). \tag{16}$$

Physically, the reduced system $(\overline{h}, \mathcal{O}_\mu, \omega_{\overline{H}})$ is a classical nonrelativistic particle with spin as defined by Souriau. Thus intrinsic spin in classical mechanics is realized by reducing a nonlinear nonholonomically constrained system.

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