A finite element discretization for stream-function problems on multiply connected domains

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Abstract

The no-normal-flow condition states that the stream-function is constant at solid boundaries. For multiply connected domains these (unknown) constants differ per boundary and must be determined from integral conditions. This complicates discretization and solution of the problem considerably. In this paper we describe a simple and systematic way for solving this problem within the context of a finite element discretization and apply our ideas to global ocean circulation simulation.

1 Introduction

In 2-d incompressible flow simulations, use is often made of the vorticity-stream function formulation (e.g. [7]). The main advantage is a reduction from three partial differential equations for the primitive variables (pressure and two velocity components) to either one (4th order) partial differential equation for the stream-function or to a set of two (2nd order) equations for the scalar quantities vorticity and stream function. Another advantage is that the continuity equation will be satisfied automatically.

A drawback of the introduction of a stream-function is that the boundary conditions on solid boundaries get somewhat more complicated. The

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condition of no normal flow implies that the stream-function must be constant. On a simply connected domain, with only one boundary, one can simply choose the constant to be zero (or arbitrary). However, if the domain is multiply connected, there are several (internal) boundaries and one has to determine the constants at each of the boundaries. These constants can not be determined from the partial differential equations for stream-function and vorticity but additional integral conditions can be derived to relate stream-functions on the different boundaries to each other. In numerical schemes, these integral conditions may pose serious practical problems if discretized explicitly.

The subject that inspired us for this research is global ocean circulation. It is customary (e.g. [1]) to split the 3-d ocean flow into a depth-averaged barotropic part, which is described by the above equations, and the 3-d baroclinic deviations from it. The barotropic flow component contains amongst others fast surface gravity waves which for many applications are not dynamically important. By assuming a rigid lid, an effectively incompressible flow is obtained which implies that surface waves have an infinite speed. The domain is multiply connected due to the presence of islands and continents. The main-stream discretization method for ocean circulation is the finite difference method. For example, major ocean simulation codes following the Brian-Cox model [1] are finite difference based. In the finite difference method, the integral conditions *must* be discretized explicitly, which in practice e.g. limits the number of islands that are included in the topography.

In this paper we will show that the integral conditions can be discretized implicitly in a finite element setting. This leads to a numerical scheme that is almost as simple as for simply connected domains. We will regard a simplified, though not unrealistic model that suffices to illustrate our ideas.

Two related papers that discuss finite element discretizations of stream-function problems on multiply-connected domains are [10] and [11]. The first paper studies the incompressible Euler equations. For this problem the values on solid boundaries can be determined from the initial distribution of the vorticity. This method is not applicable to viscous flow, a case that is covered by the theory we will present. The second paper studies the homogeneous incompressible Navier-Stokes equations with inhomogeneous viscous boundary conditions. We discuss the inhomogeneous equations with homogeneous boundary conditions, which gives rise to different integral conditions. The approach we propose in Section 4 is to a certain extent analogous to the one suggested in [11].

The outline of this paper is as follows. Section 2 describes a simple mathematical model for ocean circulation, and derives the stream-function-vorticity formulation of the model and integral conditions to determine the stream-function on islands and continents. Section 3 discusses the question of uniqueness of the solution and makes a comparison with other approaches in literature. Section 4 describes the discretization of the problem with the finite element method, using a special choice for the test functions on islands. Section 5 gives computational details and makes some choices specific to ocean circulation. In particular, it describes how the discrete system can be obtained using spherical coordinates. Section 6 explains the extreme simplicity of implementation of the method we propose. Section 7 describes the results of our ideas applied to a more or less realistic ocean circulation problem.

2 Equations of motion

For steady barotropic flow in a homogeneous ocean with constant depth and nearly in geostrophic equilibrium, the momentum and continuity equations can be written as

$$-fv = -g\frac{\partial h}{\partial x} - ru + A\nabla^2 u + F_1 \tag{1}$$

$$+fu = -g\frac{\partial h}{\partial y} - rv + A\nabla^2 v + F_2 \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where u, v are velocities in x, y directions, h is water level, g acceleration due to gravity, r a positive bottom-friction coefficient, A the lateral viscosity, F_1, F_2 an external force and f the Coriolis parameter, which for large-scale geophysical flows will be a function of the North-South coordinate y. In that case, the equations have to be formulated in spherical coordinates (see section 5 below). This system of equations is not essentially different from the Navier-Stokes equations for 2-d incompressible flow, with A playing the role of viscosity and with an added Newtonian friction term $r\mathbf{v}$. The omission of time derivatives and advective terms does not seem to be essential for the method described here.

Boundary conditions state that the normal velocity component on solid boundaries is zero. If the viscosity A is nonzero, the tangential velocity will also be zero (no-slip).

By introducing a stream-function, the continuity equation can be satisfied automatically. A stream-function ψ and a vorticity ζ can be defined such that

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x},\ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{4}$$

The stream-function-vorticity equations then become

$$\begin{aligned} -r\nabla^2\psi - \beta\frac{\partial \psi}{\partial x} - A\nabla^2\zeta = \nabla \times \mathbf{F} \quad \text{in } \Omega, \\ \nabla^2\psi + \zeta = 0 \end{aligned} \tag{5}$$

where $\beta = \frac{\partial f}{\partial y}$. The no-normal-flow boundary conditions state that the stream-function is constant on each continent,

$$\psi = C_k \quad \text{on } \Gamma_k, \quad k = 1, \cdots, n_k, \tag{6}$$

n_k is the number of continents. The values of the constants are not specified. One constant can be picked arbitrarily, e.g. $\psi = 0$ on Γ_1. If the viscosity is nonzero, there is an additional no-slip condition which says that the tangential velocity component is zero. In terms of the stream function, this means:

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma_k, \quad k = 1, \cdots, n_k. \tag{7}$$

There are usually no conditions on the vorticity.

The original equations (1, 2) cannot be reconstructed unambiguously from (5). By adding an arbitrary function of x to the equation (1), or an arbitrary function of y to (2), we can still get the same equation (5). We therefore have to derive an extra condition from the original (primitive) equations (1, 2). The extra condition we impose is that the water-level should be continuous around any contour Γ_k.

$$\oint_{\Gamma_k} g\frac{\partial h}{\partial s}ds = -\oint_{\Gamma_k}(f\mathbf{v}\cdot\mathbf{n} + r\mathbf{v}\cdot\mathbf{s} + A\nabla^2\mathbf{v}\cdot\mathbf{s} + \mathbf{F}\cdot\mathbf{s})ds = 0 \tag{8}$$

which for a continent boundary becomes

$$\oint_{\Gamma_k} r\frac{\partial \psi}{\partial n} + A\frac{\partial \zeta}{\partial n}\ ds = -\oint_{\Gamma_k}\mathbf{F}\cdot\mathbf{s}ds \tag{9}$$

Note that the Coriolis parameter f does not occur in this condition. Whether the equations (5-7) plus the circulation condition (9) have a unique solution is a question that will be addressed in the next section.

3 Uniqueness of solution

For the case without lateral friction ($A = 0$), Kamenkovich [6] presented an argument that the above mentioned problem has a unique solution. The argument is approximately as follows. The solution can be split into components

$$\psi = \psi_0 + \sum_{k=1}^{n_k} Q_k \psi_k \tag{10}$$

and similarly for u, v, h, ζ with constants Q_k to be determined. Here, ψ_0 satisfies the inhomogeneous equations ($\mathbf{F} \neq 0$) with homogeneous boundary conditions (all $C_k = 0$), whereas ψ_k satisfies the homogeneous equations ($\mathbf{F} = 0$) with boundary conditions $\psi_k = \delta_{ki}$ on $\Gamma_i, i = 1, n_k$. All components are zero on Γ_1.

Introducing (10) into (9) gives a set of equations from which Q_k can be solved, provided its determinant is nonzero, or, in other words, the solution is unique.

Uniqueness can be proved more generally by showing that an unforced solution ($F = 0$) would be exactly zero. First of all, the maximum principle [2] states that such a solution could not take a maximum anywhere inside the region. Therefore, if a nontrivial solution exists, it must take its maximum on one of the boundaries, say Γ_N and be constant there. The maximum principle, again states that then $\frac{\partial \psi}{\partial n} < 0$ on this contour, such that, assuming $r > 0$,

$$\oint_{\Gamma_k} r \frac{\partial \psi}{\partial n} ds < 0 \tag{11}$$

However, this is in contradiction with (9) for $\mathbf{F} = 0$. Therefore, the homogeneous problem has a zero solution only and the solution (10) must be unique.

3.1 Comparison with other approaches

Bryan and Cox [1], in their GFDL (or MOM) model, use the Kamenkovich theory in a straightforward way. The contributions $\psi_k, k = 1, n_k$ in (10) are evaluated once and for all. The component ψ_0 is determined in each time step using the actual forcing and the coefficients Q_k are determined by solving the circulation conditions at each time step.

Godfrey [4] and Wajsowicz [14] use a specialized approach using typical properties of oceanic flows. They choose a contour not along continent boundaries but containing open-sea parts. Then (8) applies. Choosing a contour PQRS such that QR and SP are on Eastern boundaries, where boundary layers do not occur and the normal velocity is zero, and PQ and RS along latitude circles, they obtain explicit expressions for the flow

$$\psi_P - \psi_Q = \psi_S - \psi_R = \frac{\oint \mathbf{F} \cdot \mathbf{s} ds}{f_{PQ} - f_{RS}} \tag{12}$$

where friction has been neglected away from Western boundaries. This can be used to relate the values of the stream-function at different continents. The method is not generally applicable (particularly to non-rotating flows).

3.2 Case with lateral friction

If lateral friction is taken into account ($A \neq 0$), the construction of the solution can be done in the same way as (10). The number of unknown constants C_k is the same as before. The boundary conditions for component ψ_k are as before, but with added Neumann condition $\frac{\partial \psi}{\partial n} = 0$ on each solid boundary. Eq.(9) then becomes

$$\oint_{\Gamma_k} A \frac{\partial \zeta}{\partial n} ds = -\oint_{\Gamma_k} \mathbf{F} \cdot \mathbf{s} ds \tag{13}$$

It is less easy to show that the solution constructed this way is unique.

In [3], Glowinski and Pironneau also studied uniqueness of two-dimensional flow problems. Their set of equations is different from ours. Moreover, their technique to prove uniqueness is not applicable to a non-self-adjoint problem, which we have due to the presence of Coriolis terms.

4 Discretization

The main difficulties in discretizing the equations (5-7), (9) are posed by the boundary condition (6) and the integral condition (9). In this section we will show that these conditions can be imposed in a natural way in the finite element method.

The weak form of (5) is given by

$$\int_\Omega r \nabla \chi \cdot \nabla \psi - \beta \chi \frac{\partial \psi}{\partial x} + A \nabla \chi \cdot \nabla \zeta d\Omega = \int_\Omega \chi \nabla \times \mathbf{F} d\Omega + \sum_{k=1}^{n_k} \oint_{\Gamma_k} r \chi \frac{\partial \psi}{\partial n} + A \chi \frac{\partial \zeta}{\partial n} ds$$

$$\int_\Omega \nabla \xi \cdot \nabla \psi + \xi \zeta d\Omega = \sum_{k=1}^{n_k} \oint_{\Gamma_k} \xi \frac{\partial \psi}{\partial n} ds \tag{14}$$

with χ and ξ test functions. The test functions are chosen such that the boundary conditions can be satisfied.

We define a mesh on Ω of n nodal points, and choose a set of independent basis functions λ_i so that

$$\lambda_i = \delta_{ij} \text{ in node } j. \tag{15}$$

Approximate solutions $\tilde{\psi}$ and $\tilde{\zeta}$ are now sought as linear combinations of the basis functions:

$$\tilde{\psi} = \sum_{j=1}^{n} \tilde{\psi}_j \ \lambda_j, \quad \tilde{\zeta} = \sum_{j=1}^{n} \tilde{\zeta}_j \ \lambda_j \quad . \tag{16}$$

In these sums the coefficients $\tilde{\psi}_j$ and $\tilde{\zeta}_j$ are unknown and have to be determined. Since $\lambda_i = \delta_{ij}$ in node j the coefficients $\tilde{\psi}_j$ and $\tilde{\zeta}_j$ are the numerical approximations for ψ and ζ in the grid points. Because of this the $\tilde{\psi}_j$ should satisfy (6). We can immediately substitute these conditions into the first sequence in (16). Let μ_k be defined by

$$\mu_k = \sum_{j \in \Gamma_k} \lambda_j \quad . \tag{17}$$

Substituting yields

$$\tilde{\psi} = \sum_{j \in \bar{\Omega}} \tilde{\psi}_j \ \lambda_j + \sum_{k=1}^{n_k} \tilde{\psi}_k \mu_k \tag{18}$$

in which $\tilde{\psi}_k$ is the value of $\tilde{\psi}$ in the grid points in continent Γ_k. By the notation $j \in \bar{\Omega}$ we denote the indices that correspond to sea points. The "sea" $\bar{\Omega}$ is the interior of the domain:

$$\bar{\Omega} = \Omega \setminus \bigcup_{k=1}^{n_k} \Gamma_k. \tag{19}$$

We now have different sets of basis functions for approximating ψ and for approximating ζ. For ψ we have the set

$$\lambda_j, \ \forall j \in \bar{\Omega} \ \bigcup \ \mu_k, k = 1, 2, \cdots, n_k \tag{20}$$

and for ζ we have the original set

$$\lambda_j, \ j = 1, 2, \cdots, n. \tag{21}$$

Note that

$$\mu_k = 1 \text{ on } \Gamma_k \tag{22}$$

if the λ_j are piecewise polynomial, which is the standard choice in the finite element method. We apply the finite element method to (14). For the test functions χ we take the basis functions (20) and for the test functions ξ we take (21) and substitute these in (14). By this choice the boundary conditions can be satisfied in a natural way. This would not be possible if we would take (21) for the test functions χ and (20) for the test functions ξ. The sequences (18) and the integral condition (9) are also substituted into (14). By also taking into account the properties of the basis functions, we obtain the following system of linear equations:

$$\sum_{j\in\bar{\Omega}} \int r\nabla\lambda_i \cdot \nabla\lambda_j - \beta\lambda_i \frac{\partial \lambda_j}{\partial x} d\Omega \ \tilde{\psi}_j + \sum_{k=1}^{n_k} \int r\nabla\lambda_i \cdot \nabla\mu_k - \beta\lambda_i \frac{\partial \mu_k}{\partial x} d\Omega \ \tilde{\psi}_k$$

$$+ \sum_{j=1}^{n} \int A\nabla\lambda_i \cdot \nabla\lambda_j d\Omega \ \tilde{\zeta}_j = \int \lambda_i \nabla \times \mathbf{F} d\Omega \ \ \forall i \in \bar{\Omega}, \tag{23}$$

$$\sum_{j\in\bar{\Omega}} \int r\nabla\mu_i \cdot \nabla\lambda_j - \beta\mu_i \frac{\partial \lambda_j}{\partial x} d\Omega \ \tilde{\psi}_j + \sum_{k=1}^{n_k} \int r\nabla\mu_i \cdot \nabla\mu_k - \beta\mu_i \frac{\partial \mu_k}{\partial x} d\Omega \ \tilde{\psi}_k +$$

$$\sum_{j=1}^{n} \int A\nabla\mu_i \cdot \nabla\lambda_j d\Omega \ \tilde{\zeta}_j = \int \mu_i \nabla \times \mathbf{F} d\Omega - \oint_{\Gamma_i} \mathbf{F} \cdot \mathbf{s} ds \ \ i = 1, 2, \cdots, n_k, \tag{24}$$

and

$$\sum_{j\in\bar{\Omega}} \int \nabla\lambda_i \cdot \nabla\lambda_j d\Omega \ \tilde{\psi}_j + \sum_{k=1}^{n_k} \int \nabla\lambda_i \cdot \nabla\mu_k d\Omega \ \tilde{\psi}_k + \sum_{j=1}^{n} \int \lambda_i \lambda_j d\Omega \ \tilde{\zeta}_j$$

$$= \sum_{k=1}^{n_k} \oint_{\Gamma_k} \lambda_i \frac{\partial \psi}{\partial n} ds \quad i = 1, 2, \cdots, n. \tag{25}$$

The derivation of this linear system is outlined in more detail in the appendix. In the above equations there are three different types of summations:

$\sum_{j\in\bar{\Omega}}$, a summation over the sea-grid points,

$\sum_{i=1}^{n_k}$, a summation over the continents, and

$\sum_{j=1}^{n}$, a summation over all grid points.

Note that the right-hand-side of the last equation is equal to zero in case of lateral friction. If there is no lateral friction, the values for the stream-function can be determined from (23, 24) only.

5 Evaluation of the entries of the system matrix and of the right-hand-side vector.

To evaluate the coefficients of the system we have to make an explicit choice for the basis functions. Once the basis functions are explicitly known, the integrals can be evaluated using a numerical integration rule.

We want to solve equations (5) on a global domain, with a realistic topography. Since our domain is spherical it is convenient to use spherical coordinates. We first recall some definitions. The derivatives in the latitudinal direction α and in the longitudinal direction θ are given by

$$\frac{\partial}{\partial x} = \frac{1}{R\cos\theta}\frac{\partial}{\partial\alpha}, \qquad \frac{\partial}{\partial y} = \frac{1}{R}\frac{\partial}{\partial\theta} \tag{26}$$

with R the earth radius (assumed constant). The determinant of the Jacobian J of the coordinate transformation is

$$|J| = R^2\cos\theta. \tag{27}$$

The divergence of a vector field is defined by

$$\nabla\cdot\mathbf{F} = \frac{1}{R\cos\theta}\left(\frac{\partial}{\partial\alpha}F_1 + \frac{\partial}{\partial\theta}F_2\cos\theta\right), \tag{28}$$

and the curl of a vector field $\nabla\times\mathbf{F}$ by

$$\nabla\times\mathbf{F} = \frac{1}{R\cos\theta}\left(\frac{\partial F_2}{\partial\alpha} - \frac{\partial F_1\cos\theta}{\partial\theta}\right). \tag{29}$$

The above relations can be substituted into (23 - 25) to get the linear system for the coordinate system we are interested in. E.g., substituting yields

$$\int\nabla\lambda_i\cdot\nabla\lambda_j d\Omega = \int\frac{\partial\lambda_i}{\partial\alpha}\frac{1}{\cos\theta}\frac{\partial\lambda_j}{\partial\alpha} + \frac{\partial\lambda_i}{\partial\theta}\cos\theta\frac{\partial\lambda_j}{\partial\theta}d\Omega. \tag{30}$$

All but one other terms can be rewritten in an equally straightforward way.

The original spherical domain is mapped onto a rectangular domain by the change to spherical coordinates. This domain is decomposed into triangles,

$$\Omega = \bigcup_{e=1}^{n_e}\Omega_e \tag{31}$$

in which Ω_e are the triangular sub domains, the "elements", and n_e is the number of elements. The nodal points of the mesh are in the corners of the triangles. The basis functions λ_i are now chosen to be piecewise linear, i.e. linear inside each element. Since $\lambda_i = 0$ in all nodes, except in node i, it is nonzero only inside the six elements that have node i as a corner. Note that the basis functions are piecewise linear in the coordinates (α, θ), not in (x, y). The condition $\lambda_i = \delta_{ij}$ in node j yields three linear equations for the coefficients of each λ_i inside an element. Since λ_i is linear inside an element three unknown coefficients must be determined. The three linear equations give sufficient conditions for this.

The integrals can be evaluated by evaluating them per element and adding the results together:

$$\int int\ d\Omega = \sum_{e=1}^{n_e} \int int\ d\Omega_e \tag{32}$$

with int an unspecified integrand. The element integrals are computed by a simple Newton-Cotes integration rule:

$$\int int\ d\Omega_e \approx \frac{Area}{3}(int_1 + int_2 + int_3) \tag{33}$$

with int_1, int_2, and int_3 the values of the integrand in the corners of the element.

In actual computations it is not necessary to explicitly evaluate integrals involving the basis functions μ_k. All integrals can be computed by evaluating integrals with the functions λ_i only, by making use of the definition (17) of μ_k. For example

$$\int r\nabla\lambda_i \cdot \nabla\mu_k - \beta\lambda_i \frac{\partial\mu_k}{\partial x} d\Omega = \sum_{j\in\Gamma_k} \int r\nabla\lambda_i \cdot \nabla\lambda_j - \beta\lambda_i \frac{\partial\lambda_j}{\partial x} d\Omega.$$

The only term that requires some more thought is $\oint_{\Gamma_k} \mathbf{F} \cdot \mathbf{s} ds$. The integral can be evaluated as it is, but both the coordinate transformation and the directional integration are not trivial to do and therefore error prone. To avoid errors we apply Stokes theorem to turn the contour integral into an area integral:

$$\oint_{\Gamma_k} \mathbf{F} \cdot \mathbf{s} ds = -\int \nabla \times \mathbf{F} d\Gamma_k, \tag{34}$$

with $\int int\ d\Gamma_k$ we denote integration over the area enclosed by Γ_k, hence integration over continent k. To evaluate these integrals we can extend the grid over the continents as well. We denote the extended domain, i.e. sea, coasts and the interior of the continents, by $\hat{\Omega}$. Linear basis functions are defined on the continents in the same way as in the seas. The definition (17) of μ_k now also includes summation of the λ_j in the interior of Γ_k. The right-hand-side in (24) can now be rewritten as follows.

$$\int \mu_k \nabla \times \mathbf{F} d\Omega - \oint_{\Gamma_k} \mathbf{F} \cdot \mathbf{s} ds = \int \mu_k \nabla \times \mathbf{F} d\Omega + \int_{\Gamma_k} \nabla \times \mathbf{F} d\Gamma_k = \int \mu_k \nabla \times \mathbf{F} d\hat{\Omega} \tag{35}$$

Here we have made use of the fact that $\mu_k = \delta_{ik}$. Substituting (17) yields

$$\int \mu_k \nabla \times \mathbf{F} d\hat{\Omega} = \sum_{j \in \Gamma_k} \int \lambda_j \nabla \times \mathbf{F} d\hat{\Omega}. \tag{36}$$

Here $j \in \Gamma_k$ also includes nodes in the interior of Γ_k. Integrals of the type $\int \lambda_j \nabla \times \mathbf{F} d\hat{\Omega}$ already had to be evaluated for nodes j that are in the sea. The only difference between land and sea nodes is that the discrete nodal values of the right-hand-side of nodes that are inside the same continent must be added together to get the discrete right-hand-side for the complete continent.

For all other integrals it is of no consequence whether μ_k is defined on the edge of a continent or on the whole continent. The integrands contain either derivatives of μ_k, which are equal to zero on a continent, or products with basis functions λ_j which are zero on the continent. Hence, the interior of a continent gives no contribution to any of the integrals other than for the right-hand-side.

At first sight there seems little advantage in including land points in the grid. The evaluation of a contour integral is replaced by the evaluation of an area integral, which is more expensive. Integrals are evaluated in the interior of continents where there is no contribution. However, hardly any distinction needs to be made any more between land and sea points. This simplifies implementation. And there is one other big advantage. By including the interior of the continents in the grid it is possible to keep the matrix structured, which makes operations with the matrix less expensive. This will be explained in the next section.

6 Implementation.

Continents give rise to a large bandwidth of the matrix. The reason is that the basis functions μ_k have large support. This property makes the solution of the linear system expensive if solved with a direct solution method like Gaussian elimination. Therefore, iterative solution methods, which do not suffer from a large bandwidth, are the preferred choice. These methods only address the matrix for performing matrix-vector multiplications. There are good reasons to try the keep the matrix structured, see [13]. The most important of these is that indirect addressing can be avoided if the matrix has only a few diagonals with nonzero elements. A structured grid gives rise to such a matrix. The continents destroy the structure in the matrix, but in this section we will show how the linear system can be solved iteratively, by performing matrix-vector products with a *structured* matrix, despite the continents.

As was explained in the previous section, the grid can be extended to include also land points. Moreover, integrals involving μ_k's can be computed by evaluating integrals with λ_j's only, by making use of the definition of μ_k. If we would ignore the continents we would obtain a matrix $\mathbf{K}$ of the following (block)-structure:

$$\mathbf{K} = \begin{pmatrix} r\mathbf{L} - \mathbf{C} & A\mathbf{L} \\ -\mathbf{L} & \mathbf{M} \end{pmatrix} \tag{37}$$

In this expression we can recognize the discrete counterparts of the various differential operators:

$$\nabla^2 \rightarrow \mathbf{L}, \quad \beta\frac{\partial}{\partial x} \rightarrow \mathbf{C}, \quad 1 \rightarrow \mathbf{M} \ . \tag{38}$$

These matrices are of size $n \times n$, with n the number of nodes. We know that, because of (6), the nodal values of the stream-function on each continent must be equal to each other. On each continent we define a "master" node, hence the nodal values of the stream-function must be equal to value of the stream-function in the master node. Let the vector $\bar{\psi}$ contain only the values of the stream-function in the master nodes (including values in sea nodes), and let the vector ψ contain the values of the stream-function in all nodes of the grid, including all land points. These two vectors are related by

$$\mathbf{P}\bar{\psi} = \psi \tag{39}$$

in which the matrix $\mathbf{P}$ is defined as follows. The columns that correspond to sea-points are just basis-vectors:

$$\mathbf{P_j} = \mathbf{e_j} \quad \forall j \in \bar{\Omega}. \tag{40}$$

And a column that correspond to the "master" unknown on a continent is the sum of the basis vectors corresponding to the unknowns on the continent:

$$\mathbf{P_k} = \sum_{\mathbf{j} \in \mathbf{\Gamma_k}} \mathbf{e_j}. \tag{41}$$

As was argued in the previous section, given the fact that the μ_k's are the sum of all λ_j's on the corresponding continent, the nodal values of discretized curl of the external force field, which we will denote by $\mathbf{f}$, should be added together on each continent. With the above definition of $\mathbf{P}$ this can be denoted by

$$\bar{\mathbf{f}} = \mathbf{P^T f}. \tag{42}$$

The same arguments hold for the discrete differential operators, e.g.

$$\bar{\mathbf{L}} = \mathbf{P^T L}. \tag{43}$$

Hence the system we actually have to solve is:

$$\begin{pmatrix} \mathbf{P^T} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} r\mathbf{L} - \mathbf{C} & A\mathbf{L} \\ -\mathbf{L} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \bar{\psi} \\ \bar{\zeta} \end{pmatrix} = \begin{pmatrix} \mathbf{P^T} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} \tag{44}$$

For iterative solution methods one only has to multiply with the system matrix. This operation can be performed in three steps.

- Give all "slave" nodes the value of the "master" node: $\psi = \mathbf{P}\bar{\psi}$
- Multiply $\begin{pmatrix} \psi \\ \zeta \end{pmatrix}$ by (37). (Structured matrix-vector product).
- Add up the values of the stream-function on each continent: $\bar{\psi} = \mathbf{P^T}\psi$

This approach makes the implementation of the boundary conditions for continents extremely simple. One can first discretize equation (14) without taking the boundary condition (6) into account. The boundary condition (6) is taken care of in the matrix-vector multiplications by means of two extremely simple operations on vectors.

In global ocean circulation models one of course has the periodic conditions

$$\psi(-\pi, \theta) = \psi(\pi, \theta), \quad \zeta(-\pi, \theta) = \zeta(\pi, \theta). \tag{45}$$

These conditions can be taken into account in the same way as (6), means of a master-slave hierarchy between the nodes. In fact, we have implemented the periodic condition in this manner.

We want to stress the extreme simplicity and elegance of the method described above. The "unnatural" evaluation of wind-stress *on land* results in an easily and straightforwardly implementable method. Moreover, a structured grid results in a structured matrix (if the nodes are properly ordered).

7 Example

In this section we will describe the results of two simulations of global ocean circulation.

7.1 Choice of parameters

For realistic simulations we need the following information and data:

- Topography information,
- The external force field,
- Earth radius R,
- Coriolis parameter β,
- Lateral viscosity parameter A.
- Bottom friction coefficient r,

We have extracted the topography information from a datafile provided by NCAR[1]. This datafile gives depth-information with a resolution of one degree. We have used a resolution of two degrees. Our grid ranges from 89^o South to 89^o North, thus circumventing the singularity at the poles due to the use of spherical coordinates. From the depth-information we determined whether our grid points are either land or water by linear interpolation. Next, we determine which grid points belong to the same continent by a simple region growing algorithm, see e.g. [9].

The external force field relates to the wind stress by

$$\mathbf{F} = \frac{1}{H\rho}\tau. \tag{46}$$

We have used a the long term averaged data of the wind stress τ in January, collected by Hellerman and Rosenstein [5]. Figure 1 shows the wind field and the different islands and continents. There is a total number of 26 islands and continents.

The water density ρ is $1000 kg/m^2$, and we have taken for the average depth of the ocean $H = 500m$.

The Earth radius and Coriolis parameter are known to be $R = 6.4 \cdot 10^6 m$ and $\beta = \cos\theta \cdot 2\Omega/R = 2.3 \cdot 10^{-11} \cos\theta (ms)^{-1}$, with Ω the angular velocity of the Earth.

[1] http://www.scd.ucar.edu/dss/datasets/ds750.1.html

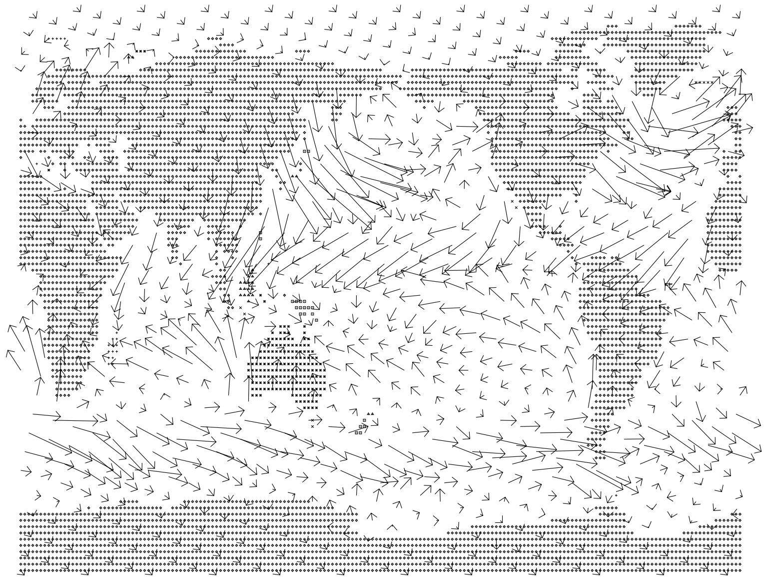

Figure 1: Average wind field in January.

The lateral viscosity A is not well know. Estimations range from $10 - 10^5 m^2/s$. We have neglected lateral viscosity in the first experiment and taken $A = 500 m^2/s$ in the second experiment. The latter value has been taken from [8].

The bottom friction arises from Ekman boundary layers at the bottom, and is related to the vertical eddy-viscosity coefficient A_v, which is again not very well known ($10^{-1} - 10^{-5} m^2/s$):

$$r = \frac{\sqrt{A_v f}}{2H}. \tag{47}$$

Because A_v is not well known, the bottom friction r is usually approximated by a parameter that does not depend on latitude or longitude, despite the dependency on the Coriolis parameter ($f = 2\Omega \sin\theta$). The parameter r plays the part of a diffusion coefficient in (5) and occurs therefore in the mesh-Peclet number defined as

$$Pe = \frac{h\beta R \cos\theta}{r} \tag{48}$$

with h the mesh size, for our grid $h = \pi/90$. We have chosen $r = 5 \cdot 10^{-6}s^{-1}$. This value is physically realistic and yields a maximal mesh-Peclet number slightly greater than one, which is sufficiently small to avoid spatial oscillation ('wiggles').

We have assumed a rather shallow ocean with an average depth of 500 m. If we would have assumed a deeper ocean, the bottom friction would be correspondingly smaller and it would be harder to attain numerical stability. The velocities we obtain for our choice of parameters are of the correct order of magnitude, i.e. centimeters per second.

The stream-function should be prescribed on one island or continent. We have prescribed the stream-function to be zero on the largest continent, i.e. America, Eurasia and Africa. Our grid is too coarse to resolve Bering strait. The results are not influenced by the choice on which continent the stream-function is prescribed. Prescribing on the largest continent has the advantage that the linear system becomes better conditioned and hence easier to solve by an iterative method.

The iterative solution method we used is GMRESR [12], combined with diagonal scaling.

7.2 Results

The model we have used as our test problem is too simplified to expect very realistic results. E.g. bottom topography, nonlinear effects and three dimensional effects are all neglected. Our model mainly gives a balance between wind stress, Coriolis force and bottom friction. The effect of lateral viscosity is neglected in the first experiment and small in the second experiment. The surface currents are wind driven and should therefore more or less follow the wind field. The currents near the equator deviate most prominently from this global patter.

Figure 2 shows a contour plot of the stream-function. The qualitative result is good, all main surface currents are present.

The velocities can be computed from the values of the stream-function by numerical differentiation. This result is show in Figure 3.

Some important currents and phenomena that are well captured are:

- Western boundary currents, due to the Coriolis force,
- The Antarctic Circumpolar Current, the strongest current,
- The Gulf stream, the most important current for the climate in Northern Europe,

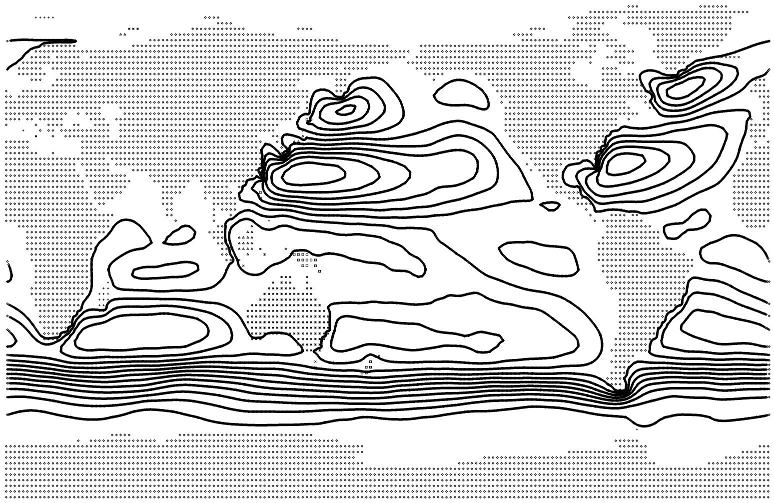

Figure 2: Stream-function, contour plot.

- Back-flow around the tip of Africa.

In the second experiment we also included the effect of lateral viscosity. As was remarked before, this effect is small, and the graphical results are not significantly different from the results of the first experiment.

8 Concluding remarks

In this paper we have described a method for discretizing and solving stream-function problems on multiply connected domains. We have applied this method to a simple global ocean circulation model.

8.1 Application to more complicated models

An important question is: Can the approach we have taken to implement the boundary conditions for the stream-function also be applied to more complicated models? The answer will depend on the model of course. We have also implemented our ideas in a model that describes unsteady barotropic

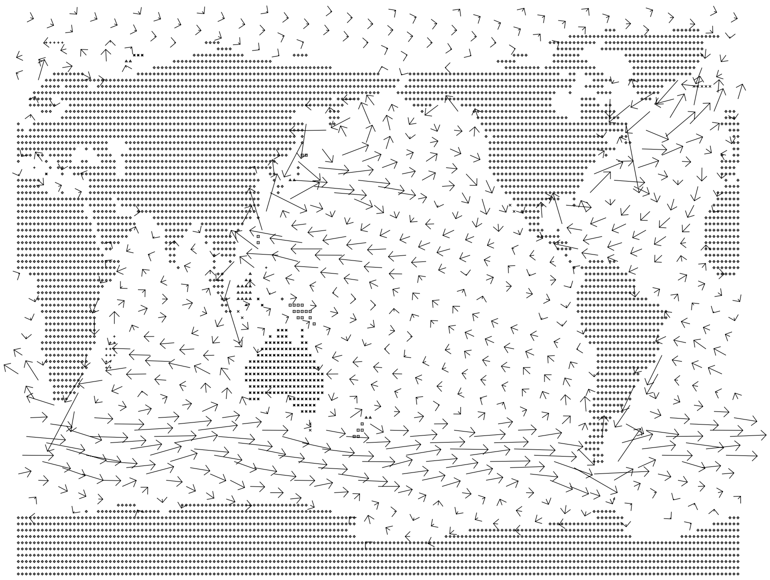

Figure 3: Velocity field.

flow and that includes (nonlinear) advective terms. In this model, the no-slip condition can be implemented in exactly the same way as described in this paper.

8.2 Application to the finite difference method

The finite difference method is widely used for simulating ocean circulation. The question whether our ideas can be incorporated into the finite difference method is therefore of particular interest. Looking at the final algorithm, described in Section 6, the obvious analog for the finite difference method would be:

- Discretize the equations with the finite difference method without taking the islands into account. This gives a matrix similar to (37).
- Take the islands into account by making a "master - slave" hierarchy between the nodes.

- Solve system (44).

A sound mathematical basis for this procedure is completely lacking, and more research in this direction should be carried out.

8.3 Conclusions

Our method has the following features and advantages:

- Integral conditions are implicitly taken into account in the solution phase, by making a "master - slave" node hierarchy per island or continent. It is not necessary to determine explicit paths of integration.
- The method allows arbitrarily many islands or continents, including extra islands only changes the "master - slave" relations between nodes.
- No distinction needs to be made between "land" and "sea" grid points in the discretization.
- A structured grid yields a structured matrix (if the nodes are properly ordered).

Finally, we want to stress the elegance and simplicity of the method.

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A Detailed derivation of the linear system of equations.

Choosing $\chi = \lambda_i$, with $i \in \bar{\Omega}$ and substituting this in the first equation of (14) we obtain the equations

$$\sum_{j\in\bar{\Omega}} \int r\nabla\lambda_i \cdot \nabla\lambda_j - \beta\lambda_i \frac{\partial \lambda_j}{\partial x} d\Omega \ \tilde{\psi}_j + \sum_{k=1}^{n_k} \int r\nabla\lambda_i \cdot \nabla\mu_k - \beta\lambda_i \frac{\partial \mu_k}{\partial x} d\Omega \ \tilde{\psi}_k$$

$$+ \sum_{j=1}^{n} \int A\nabla\lambda_i \cdot \nabla\lambda_j d\Omega \ \tilde{\zeta}_j$$

$$= \int \lambda_i \nabla \times \mathbf{F} d\Omega + \sum_{k=1}^{n_k} \oint_{\Gamma_k} r\lambda_i \frac{\partial \psi}{\partial n} + A\lambda_i \frac{\partial \zeta}{\partial n} ds \quad \forall i \in \bar{\Omega}. \tag{49}$$

Choosing $\chi = \mu_i, i = 1, 2, \cdots, n_k$ and substituting these into the first equation of (14) yields

$$\sum_{j\in\bar{\Omega}} \int r\nabla\mu_i \cdot \nabla\lambda_j - \beta\mu_i \frac{\partial \lambda_j}{\partial x} d\Omega \ \tilde{\psi}_j + \sum_{k=1}^{n_k} \int r\nabla\mu_i \cdot \nabla\mu_k - \beta\mu_i \frac{\partial \mu_k}{\partial x} d\Omega \ \tilde{\psi}_k$$

$$+ \sum_{j=1}^{n} \int A\nabla\mu_i \cdot \nabla\lambda_j d\Omega \ \tilde{\zeta}_j$$

$$= \int \mu_i \nabla \times \mathbf{F} d\Omega + \sum_{k=1}^{n_k} \oint_{\Gamma_k} r\mu_i \frac{\partial \psi}{\partial n} + A\mu_i \frac{\partial \zeta}{\partial n} ds \quad i = 1, 2, \cdots, n_k. \tag{50}$$

Finally, choosing $\xi = \lambda_i, i = 1, 2, \cdots, n$ and substituting these into the second equation of (14) yields

$$\sum_{j\in\bar{\Omega}} \int \nabla\lambda_i \cdot \nabla\lambda_j d\Omega \ \tilde{\psi}_j + \sum_{k=1}^{n_k} \int \nabla\lambda_i \cdot \nabla\mu_k d\Omega \ \tilde{\psi}_k + \sum_{j=1}^{n} \int \lambda_i\lambda_j d\Omega \ \tilde{\zeta}_j$$

$$= \sum_{k=1}^{n_k} \oint_{\Gamma_k} \lambda_i \frac{\partial \psi}{\partial n} ds \quad i = 1, 2, \cdots n. \tag{51}$$

Note that

$$\lambda_i = 0 \quad \text{on } \Gamma_k, k = 1, \cdots, n_k \forall i \in \bar{\Omega} \tag{52}$$

and

$$\mu_k = \delta_{ik} \quad \text{on } \Gamma_i, \ i = 1, \cdots, n_k \tag{53}$$

Substituting (52), (53) and the circulation condition (9) into (49), (50), and (51) yields the resulting system of linear equations (23 - 25).