Modulated Modulations Approach to the Loss of Stability of Periodic Solutions for the Degenerate Ginzburg–Landau Equation.

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The degenerate Ginzburg–Landau equation gives a description of patterns which arise in the case of weakly unstable PDE's with a unbounded spatial direction when the Landau constant (characterizing the influence of the nonlinearity) is small. This equation possesses a family of periodic solutions, moreover there exists a band of stable periodic solutions among them. We give the full description of the possible behavior of the system just outside this stable band. This is done through derivation of the so–called modulated modulation equations introduced in [Harten, 1994]. We also study some solutions of these equations among which stationary periodic and heteroclinic solutions, moving solitons, cnoidal waves and front–like solutions are found.

1. Introduction.

The amplitude (or modulation) equations are powerful tools in understanding the behavior of complicated systems near a threshold where the patterns exhibit almost periodic behavior. A lot of problems in fluid dynamics [Chossat & Iooss, 1994], [Swinney & Gollub, 1981], [Drazin & Reid, 1981], [Stewartson & Stuart, 1971], combustion, chemical reactions and biology [Kuramoto, 1984], [Brand *et al.*, 1986], [Haken, 1982] lead to the situation where the so–called Ginzburg–Landau (GL) formalism can be applied. The best known example of equations of this type is the complex GL-equation, on the validity of which one can consult [Collet & Eckmann, 1990], [Harten, 1991], [Schneider, 1994], [Bollerman, 1996].

However, there are instances, such as the plane Poiseuille flow [Dhanak, 1983], [Fujimura & Kelly, 1995], [Pekeris & Shkoller, 1967], the Jeffey-Hamel flow in a divergence channels [Eagles, 1973], the Blasius boundary layer [Sen & Vashist, 1989], the Taylor–Couette flow [Chossat & Iooss, 1994], double diffusive convection, in which the Landau constant is small and a bifurcation of higher co–dimension takes place. Then one discovers more terms in the corresponding amplitude equation [Eckhaus & Iooss, 1989]. The resulting equation is called the degenerate GL-equation (dGL) and looks as follows:

$$\frac{\partial U_1}{\partial T} = \check{\mu}_1 U_1 + \mu_1'' \frac{\partial^2 U_1}{\partial X^2} + \check{\beta}_0 |U_1|^2 U_1 + i\,(\check{\beta}_1 U_1 \frac{\partial |U_1|^2}{\partial X} + \check{\beta}_2 U_{-1} \frac{\partial U_1^2}{\partial X}) + \gamma_1 |U_1|^4 U_1. \tag{1.1}$$

In our analysis all constants are taken to be real. With the simple transformation

$$X = \sqrt{\frac{|\mu_1''|}{|\check{\mu}_1|}}x, \qquad T = \frac{1}{|\check{\mu}_1|}t, \qquad U_1 = \left(\frac{|\check{\mu}_1|}{|\gamma_1|}\right)^{\frac{1}{4}} A$$

one can rescale it:

$$\frac{\partial A}{\partial t} = \alpha A + \frac{\partial^2 A}{\partial x^2} + \hat{c}|A|^2 A - |A|^4 A - i\,(b_1 |A|^2 \frac{\partial A}{\partial x} - b_2 A^2 \frac{\partial \bar{A}}{\partial x}). \tag{1.2}$$

where $\alpha = -1$ and $\hat{c} > 0$ corresponds to the sub–critical case and $\alpha = 1$ and $\hat{c} < 0$ to the super–critical case. In [Doelman & Eckhaus, 1991] the last case was studied on the subject of existence and stability of the periodic and quasiperiodic solutions. There the coefficient $\hat{c}$ was taken equal to 0.

The question of validity of the dGL in the systems when the reflection symmetry in the spatial variable is present was considered in [Shepeleva, 1996].

The dGL–equation possesses a lot of interesting solutions such as periodic and quasiperiodic solutions [Eckhaus & Iooss, 1989], [Doelman & Eckhaus, 1991]; singular heteroclinic orbits [Kapitula, 1995]; also plane waves and their stability were studied in [Kapitula, 1992]. There is a list of papers investigating solutions of the equation (1.2) without gradient terms $b_1 = b_2 = 0$ (for the overview one can consult [Saarloos & Hohenberg, 1992]).

We consider the well known class of periodic solutions of the dGL. In some situations they are stable. We are interested in the mechanism of the stability loss of these solutions, i.e. when one is just outside the Eckhaus threshold. Typically, the Eckhaus instability restricts the range of allowed wave numbers of the pattern, i.e. its nonlinear evolution eventually brings the system back to the stable range of wave numbers. This instability was extensively studied both theoretically and practically (for a survey and relevant references, one can consult [Golovin *et al.*, 1997]).

To our knowledge, the first answer to the question "what happens just outside the Eckhaus stable band" was given by Kramer and Zimmerman [Kramer & Zimmermann, 1984]: for the nonlinear diffusion equation they derived the Kuramoto–Sivashinsky equation as the description of the slow modulation of the original periodic solutions. In a similar context the perturbed KdV equation was derived by Bernoff in [Bernoff, 1988]. Later the notion of modulated modulation equation (MME) was introduced by Van Harten [Harten, 1994] and at the same time he gave a complete analysis of the MME's arising for the complex GL–equation (cGL). He discovered that depending on the parameters of the cGL–equation the dynamics could be described by the Kuramoto–Sivashinsky, perturbed KdV or Burger's equation coupled to a GL–equation.

We expand this method to get similar results for the dGL–equation. The fact that this can be achieved is not trivial: apart from the difficulty that the basic solution is not translation invariant in the spatial direction we have to deal with a more complicated stability condition for the periodic solutions which makes calculations (which are already complicated in the GL–case) quite long. For the most general case of the Eckhaus instability for the dGL–equation we write down the MME with coefficients explicitly expressed in the original parameters. This equation of the Kuramoto–Sivashinsky(or Cahn–Hilliard) type similar to the one discovered by Kramer–Zimmerman [Kramer & Zimmermann, 1984] and Van Harten [Harten, 1994], which is not really surprising as far as it describes dynamics happening near the "classical" Eckhaus threshold (which is the only possible instability for the GL–equation) and it shows that the MME's are universal objects i.e. not depending on the perturbation parameter.

A new and interesting phenomenon is that in the dGL–case the MME's can come from the fact that the "trivial instability"† plays a non–trivial role, i.e. the band of stable solutions can be bounded by this trivial instability (which was not the case for GL) resulting in a new type of MME's. These MME's have as solutions solitons and cnoidal waves (for the original dGL it corresponds to a slight defect traveling through the periodic solution).

Moreover we will also demonstrate that for some parameter values the simultaneous occurrence of the "trivial" and "classical" stability loss may take place which immediately leads to the new classes of MME's (of the reaction–diffusion type). In this case we demonstrate the appearance of a slowly moving front through the original periodic pattern and leaving behind a periodic pattern with a slightly changed amplitude and different in phase. This is similar to the effect described in [Eckmann & Gallay, 1993].

We finish the introduction by sketching the structure of this paper. In the second section we will discuss the super–critical case: we review the results of [Doelman & Eckhaus, 1991] and introduce the control parameter we will be using later. In the third section the linear analysis of the arising equations will be given. The forth section deals with the nonlinear analysis and the derivation of MME's. After this we will consider some interesting examples of the solutions of the equations we have derived. We conclude this paper by comparing the situations for super– and sub–critical cases.

2. Supercritical Case

Let us overview some results of [Doelman & Eckhaus, 1991] which will be useful in this study. They considered equation (1.2) with $\hat{c} = 0$ and $\alpha = -1$. Among other things they have shown that the band of periodic solutions $A = R_0 e^{i(k_0 x + \omega_0 t)}$ exists with the following property:

$$R_0^4 - bR_0^2 k_0 + k_0^2 = 1 \tag{2.1}$$

with $\omega_0 = 0$ and $R_0, k_0 \in \mathbb{R}$. Following [Doelman & Eckhaus, 1991], in our paper instead of b_1 and b_2 the following coefficients will be used: $b = b_1 + b_2$ and $h = (b_1 - b_2)/4$. We will take $b > 0$ since for $b < 0$ exactly the same analysis can be done. By a linear stability analysis it can be shown that these periodic solutions are stable if:

$$R_0^4 - k_0^2 + 1 > 0 \tag{2.2}$$

$$-1 + R_0^4(2 - bh) + k_0 R_0^2(2h - b) > 0 \tag{2.3}$$

† by this we mean that from two periodic solutions with the same wave number, only the solution with larger amplitude can be stable

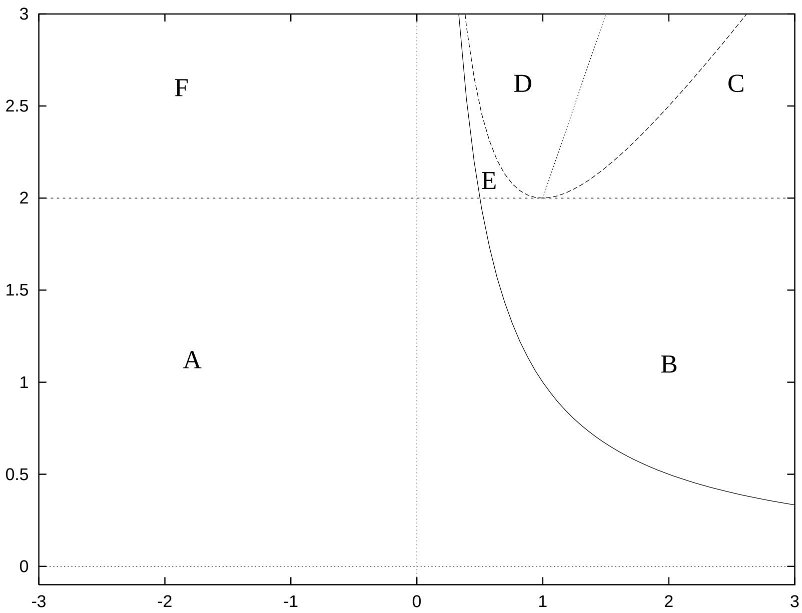


FIGURE 1. (h, b) – parameter space.

Domain A	$b < 2$, $bh < 1$	ellipse	instabilities caused by (2.3)
Domain B	$b < 2$, $bh > 1$	ellipse	instab. caused by (2.2) and (2.3)
Domain C	$b > 2$, $2h - b > 0$	hyperb	no stable periodic solutions
Domain D	$b > 2$, $h^2 - bh + 1 < 0$	hyperb	no stable periodic solutions
Domain E,F	$\{b > 2\} \setminus \{C\} \setminus \{D\}$	hyperb	instability caused by (2.3)

Here "ellipse" and "hyperbole" says which curve in (k_0, R_0^2)–plane the condition (2.1) gives and by "instability caused by (2.2) or (2.3)" we mean that the band of the stable modes is bounded by the intersection of with (2.2) or (2.3) in the (k_0, R_0^2)-plan (see Fig. 2)

Therefore summarizing these results in the parameter space, one gets the following picture (see figure 1).

We want to investigate the solutions of (1.2) just outside this stable band. Let $s = R_0^2/k_0$ be the parameter responsible for the stability loss and let us consider solutions of the following form:

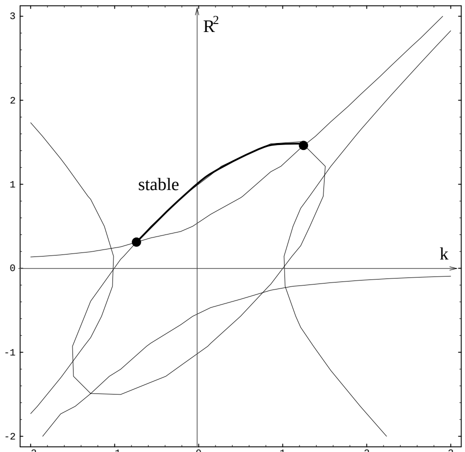
$$A(x,t) = R_0(1 + r(x,t))e^{i(k_0 x + f(x,t))}$$

Substituting it in (1.2) and changing coordinates $t \to k_0^2 t$, $x \to k_0(x - ct)$, one gets

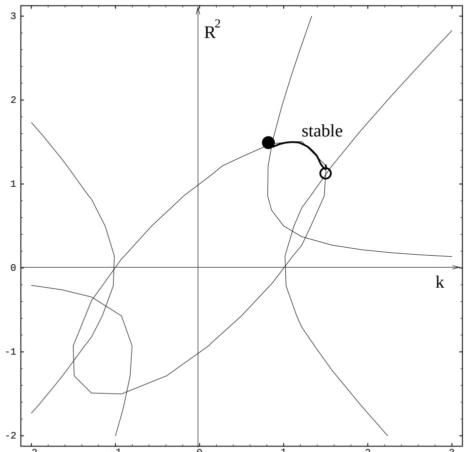
$$\begin{pmatrix} r \\ f \end{pmatrix}_t = \begin{pmatrix} 2s(b-2s) + c\partial_x + \partial_{xx} & (-2+bs)\partial_x \\ (2-4hs)\partial_x & \partial_{xx} + c\partial_x \end{pmatrix} \begin{pmatrix} r \\ f \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{2.4}$$

with

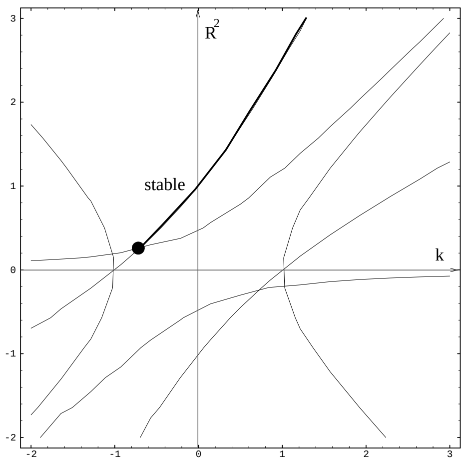
$$\begin{aligned} F_1 &= r^2(3bs - 10s^2) + r^3(bs - 10s^2) - 5s^2r^4 - s^2r^5 \\ &\quad + rf_x(-2+3bs) + r^2 f_x(3bs + bsr) - f_x^2 - rf_x^2 \\ F_2 &= \frac{2r_x(f_x - r)}{1+r} - 4hsrr_x \end{aligned} \tag{2.5}$$



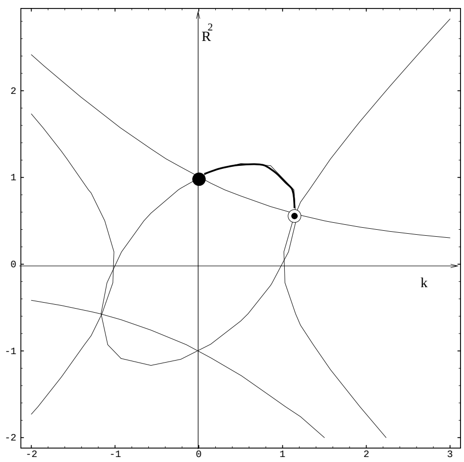
(a) Domain A.



(b) Domain B.



(c) Domain F.



(d) **case 4**

FIGURE 2. In these pictures • denotes points where the instability is described by **case 2**, at ∘ the instabilities as studied in **case 1** occur and ⊙ denotes the simultaneous occurrence of both instabilities described by **case 4**.

3. Linear Analysis

Let us consider the linear part of (2.4). After the Fourier transform:

$$[\mathcal{F}g](l) = \int g(x)e^{-ilx}\,dx$$

it looks as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{r}\\ \hat{f}\end{pmatrix} = \begin{pmatrix}2s(b-2s)-l(l+ic) & il(bs-2)\\ 2il(1-2hs) & -l(l-ic)\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{f}\end{pmatrix} \tag{3.1}$$

where $\hat{r}$ and $\hat{f}$ denote $\mathcal{F}r$ and $\mathcal{F}f$ respectively. Now the eigenvalues of the evolution operator are given by

$$\lambda_{1,2} = -l(l+ic) + s(b-2s) \mp d^{\frac{1}{2}}$$

with

$$d = 2l^2(bs-2)(2hs-1) + (sb-2s^2)^2 \tag{3.2}$$

Let us note that $d \in \mathbb{R}$ which is different from [Harten, 1994]. First, two cases will be distinguished: when d is positive and the eigenvalues are real and then when d is negative. Surprisingly, this simple separation will lead us to the two completely different situations.

- **case 1**: Let $d < 0$, then the eigenvalues are complex conjugated (if $c = 0$) with the real part $\mathrm{Re}(\lambda_{1,2}) = -l^2 + s(b-2s)$. The stability loss obviously may occur at $s^* = 0$ or $s^* = b/2$. If $s = 0$ then $d = 4l^2$ which is always positive and does not fit in this case. For $s^* = b/2$ we find $d(s^*) = d^* = (-4+b^2)(-1+bh)l^2$. Indeed it is negative in domains B and F (see figure 1).

Let us consider $s = s^* - \epsilon^2 = b/2 - \epsilon^2$. Then

$$d = d^* + \epsilon^2 2l^2(b+4h-2b^2h) + \epsilon^4 b(b+4hl^2)$$

Let us note that we consider the case when $d^* < 0$ of order one. The case $d^* = 0$ will be studied separately. Let us now rescale $l = \epsilon m$, then

$$\lambda_{1,2} = \epsilon^2(b-m^2) - 2\epsilon^4 - i\epsilon m\left(c \mp |(4-b^2)(1-bh)|^{\frac{1}{2}}\right)$$

- **case 2**: Let now $d > 0$, then the eigenvalues are real and one of the eigenvalues is dominating, assume $\lambda_1 < \lambda_2$. The stability condition in this case reads as $\lambda_2 \leq 0$

$$-l^2 + s(b-2s) + \sqrt{d} \leq 0$$

or inserting d from (3.2)

$$l^4 - 2l^2\left[s(b-2s) + (bs-2)(2hs-1)\right] \geq 0 \tag{3.3}$$

We are looking now for the s^*, the critical value of the parameter, responsible for the stability loss in this case. Let us check when the coefficient of l^2 in (3.3) is zero:

$$s^2(bh-1) - 2hs + 1 = 0$$

Defining $\mathcal{D} = h^2 - bh + 1$ and noticing that when $\mathcal{D} < 0$, one can not have $l^4 - |C_+|l^2 \geq 0$ for $\forall l$ and hence there is no stable periodic solution and we rediscover region D (see Fig. 1).

If $\mathcal{D} \geq 0$ then one finds oneself in the situation with two critical values of s:

$$s^*_\pm = \frac{h \pm \sqrt{\mathcal{D}}}{bh-1} \tag{3.4}$$

One immediately sees that the situation with $bh = 1$ differs from the rest; it will be considered separately.

Notice that both values of the critical parameter are acceptable. Really, substituting it in the expression for d one concludes: $d(s = s^*)$ is always nonnegative which means that we satisfy the restriction of case 2. We want now to surpass slightly the critical value $s^*_\pm$:

$$s = s^*_\pm \pm \epsilon^2 = \frac{h \pm \sqrt{\mathcal{D}}}{bh-1} \pm \epsilon^2$$

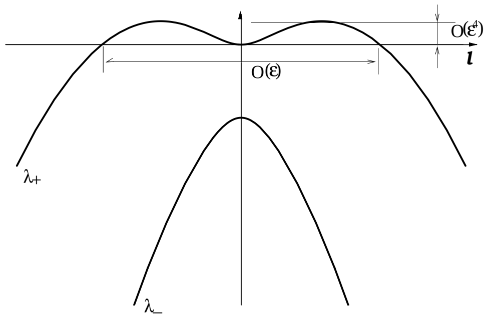


FIGURE 3. The eigenvalues in case 2 and 3b: one eigenvalue is slightly positive, the other one is negative and of order one.

and rescale $l = \epsilon m$. After some elementary calculations, one finds:

$$\begin{aligned} \lambda_1 &= \frac{-2\mathcal{L}}{(-1+bh)^2} + \mathcal{O}(\epsilon^2) \\ \lambda_1 &= \frac{\epsilon^4 m^2(-1+bh)^2}{2\mathcal{L}}(-m^2 + 8\sqrt{\mathcal{D}}) + \mathcal{O}(\epsilon^6) \end{aligned} \tag{3.5}$$

with $\mathcal{L} = h(b + 2h - b^2 h) + \sqrt{\mathcal{D}}(b + 4h - b^2 h) + 2\mathcal{D}$ which is positive everywhere except in the region C where no stable solution exists.
The conclusion of the linear analysis in this case is *there is one dominating eigenvalue which becomes slightly positive after the control parameter s is passing through the critical value $s^*_\pm = \frac{h \pm \sqrt{\mathcal{D}}}{bh-1}$* which is shown in figure 3.
Before we continue the linear analysis for some special cases, let us take a closer look on the critical values of s^* we just found. They should obviously represent the intersection points of (2.1) with (2.2) and (2.3). Combining (2.1) to (2.2) gives us

$$\frac{R_0^2}{k} = \frac{b}{2} = s^*_{\text{case 1}}$$

Analogously, combining (2.1) to (2.3) gives us

$$\begin{aligned} \left(\frac{R^2}{k}\right)^2 (1 - bh) + \frac{R^2}{k} 2h &= 1 \\ \frac{R^2}{k} = \frac{h \pm \sqrt{h^2 - bh + 1}}{bh - 1} &= s^*_{\text{case 2}} \end{aligned} \tag{3.6}$$

From our linear stability analysis we can now make the important conclusion that *the stability mechanism differs essentially if instability arises due to condition (2.2) or (2.3).*
To conclude the linear analysis, let us analyze the special sub–cases:

- **case 3**: If $d(b/2) = 0$ or $(b^2 - 4)(bh - 1) = 0$:
  (a) if $b = 2$ and $h \geq 1$: no stable periodic solutions exist.
  (b) if $b = 2$ and $h < 1/2$, then $d(b/2 - \epsilon^2) = 4\epsilon^2 l^2 (1 - 2h) + 4\epsilon^2 > 0$ and we have to turn to the second case. Therefore, there is a stable branch (see figure 2(b)),

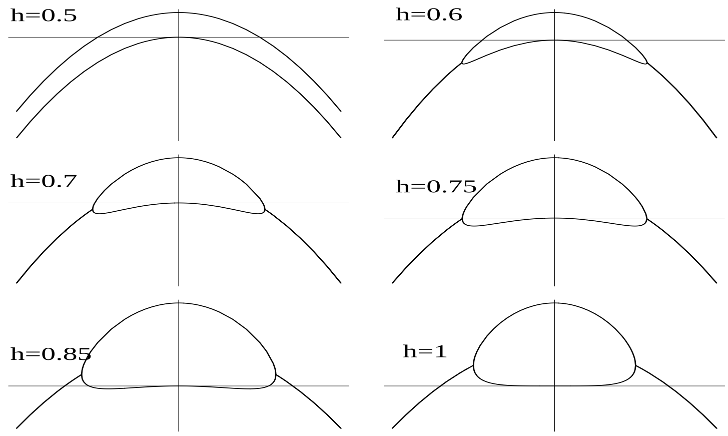


FIGURE 4. Real part of the eigenvalues in case **3b** and **5**.

$s_+^* = \frac{1}{2h-1}$ and the eigenvalues are

$$\begin{aligned}\lambda_1 &= -\frac{8(1-h)}{(2h-1)^2} + \mathcal{O}(\epsilon^2)\\ \lambda_2 &= \epsilon^4 \frac{(-1+2h)^2}{1-h} m^2 \left((1-h) - \frac{m^2}{8}\right) + \mathcal{O}(\epsilon^6)\end{aligned} \tag{3.7}$$

which is similar to the **case 2** with eigenvectors

$$e_1 = \begin{pmatrix} 1 \\ im\epsilon(2h-1)/(4h-4) \end{pmatrix}; \quad e_2 = \begin{pmatrix} \epsilon\, i\, m\,(2h-1)/2 \\ 1 \end{pmatrix}$$

(c) if $b = 2$ and $s = 1 - \epsilon^2$, then

$$\lambda_{1,2} = -\epsilon m(\epsilon m + ic) + 2\epsilon^2 \pm 2\epsilon^2\sqrt{1 + m^2(1-2h)}$$

with eigenvectors

$$e_1 = \begin{pmatrix} \epsilon(1 - \sqrt{1+m^2(1-2h)}) \\ im(1-2h) \end{pmatrix}; \quad e_2 = \begin{pmatrix} \epsilon(1 + \sqrt{1+m^2(1-2h)}) \\ im(1-2h) \end{pmatrix}$$

For $h \leq \frac{1}{2}$ the real parts of the eigenvalues are shown in figure 4

- **case 4**: If $bh = 1$ then $\lambda_2 \leq 0$ iff

$$l^4 - 2l^2(-2hs + 1) \geq 0$$

which gives $s^* = 1/(2h)$. Perturbing it with $s = s^* - \epsilon^2$, one has

$$\lambda_{1,2} = \frac{\epsilon^2}{h}\left[1 - hm^2 \mp (1 + 2hm^2(4h^2-1))^{\frac{1}{2}}\right]$$

with eigenvectors

$$e_1 = \begin{pmatrix} \frac{1}{4}\left[1 - \left(1 + 2hm^2(4h^2-1)\right)^{\frac{1}{2}}\right] \\ \epsilon(im)h^2 \end{pmatrix}; \quad e_2 = \begin{pmatrix} \frac{1}{4}\left[1 + \left(1 + 2hm^2(4h^2-1)\right)^{\frac{1}{2}}\right] \\ \epsilon(im)h^2 \end{pmatrix}$$

One can distinguish two cases: when $1-4h^2 > 0$, the eigenvalues are real and one slightly dominates the other, if $1 - 4h^2 < 0$ the situation is similar to the one described in case **3c**. The real parts of the eigenvalues look similar to the ones shown in figure 4.

• **case 5**: If $b=2$ and $h=1/2$. This is in some sense a combination of a degeneration of cases 3 and 4. Now $s=1-\epsilon^2$ and the eigenvalues read

$$\lambda_{1,2}=2\epsilon^2-m^2\epsilon^2\pm 2\epsilon^2(1-2\epsilon^2+\epsilon^4+l^2)^{\frac{1}{2}}$$

or, after rescaling $l=\epsilon m$

$$\begin{aligned}\lambda_+ &= (4-m^2)\left(\epsilon^2+\frac{\epsilon^6 m^2}{4}\right)+\frac{\epsilon^8 m^2}{8}(8-6m^2+m^4)=\epsilon^2(4-m^2)+\dots\\ \lambda_- &= -m^2\epsilon^2+4\epsilon^6 m^2(m^2-4)-\frac{\epsilon^8 m^2}{8}(8-6m^2+m^4)=-\epsilon^2 m^2+\dots\end{aligned} \tag{3.8}$$

4. Nonlinear Analysis

In this section we are going to derive the modulation equations based on our linear analysis, the form of the nonlinearity and the fact that the solutions are reflection symmetric. Again, the different cases as mentioned in section 3 will be distinguished.

• **case 1**: This is the case of imaginary eigenvalues. From the preceding analysis it follows that we have to perturb around $s^*=b/2$, which is relevant in region B. We will show that the result in this particular situation depends on the choice of the constant c, the speed of the frame we are moving with.
As we have found in the previous section, the eigenvalues in this case are

$$\lambda_{1,2}=\epsilon^2(b-m^2)-i\epsilon m(c\mp\sqrt{\mu})+\text{h.o.t.}$$

with $\mu=(4-b^2)(bh-1)\geq 0$ in region B. Dropping the higher order term in ϵ, results in the following system:

$$\begin{pmatrix} r\\ f\end{pmatrix}_t=\begin{pmatrix}\partial_{xx}+c\partial_x+2b\epsilon^2 & -\frac{1}{2}(4-b^2+3b\epsilon^2)\partial_x\\ -2(bh-1-2h\epsilon^2)\partial_x & \partial_{xx}+c\partial_x\end{pmatrix}\begin{pmatrix} r\\ f\end{pmatrix}+\begin{pmatrix} G_1\\ G_2\end{pmatrix} \tag{4.1}$$

with

$$\begin{aligned} G_1(r,f) &= -b^2r^2(1+2r+\frac{5}{4}r^2+\dots)+rf_x\left[-\frac{1}{2}(4-3b^2)+\frac{b^2r}{2}(3+r)\right]\\ &\quad -f_x^2(1+r)+\epsilon^2 br\left[r(7+9r+\dots)-f_x(3+3r+\dots)\right]+\mathcal{O}(\epsilon^4)\\ G_2(r,f) &= -2rr_x(1+bh-r+\dots)+2f_xr_x(1-r+\dots)+\epsilon^2 4hrr_x+\mathcal{O}(\epsilon^4)\end{aligned} \tag{4.2}$$

Note that we will be looking for a small solution, so we can drop higher order terms in r, which we denoted as "..." in the nonlinearity (4.2). Taking $c=0$ and

$$\begin{aligned} r&=\epsilon R(\xi,\tau)\\ f&=\epsilon F(\xi,\tau)\end{aligned} \tag{4.3}$$

with $\xi=\epsilon x$ and $\tau=\epsilon t$, one gets the following system:

$$\begin{pmatrix} R\\ F\end{pmatrix}_\tau=\begin{pmatrix} -\frac{1}{2}(4-b^2)F_\xi-b^2R^2+\epsilon\left(R_{\xi\xi}+2bR-\frac{1}{2}(4-3b^2)RF_\xi-2b^2R^3\right)\\ -2(bh-1)R_\xi+\epsilon\left(F_{\xi\xi}-2RR_\xi(1+bh)\right)\end{pmatrix} \tag{4.4}$$

Alternatively, one can from the beginning choose $c = \sqrt{\mu}$ and introduce the following transformation (in the Fourier space)

$$\begin{pmatrix} \hat{r} \\ \hat{f} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\sqrt{\mu}}{2(1-bh)} \\ \frac{2(bh-1)}{\sqrt{\mu}} & 1 \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{f} \end{pmatrix} \tag{4.5}$$

Then one has to investigate the following system

$$\begin{pmatrix} \tilde{r} \\ \tilde{f} \end{pmatrix}_t = \begin{pmatrix} \epsilon^2(b-m^2) & 0 \\ 0 & 2im\epsilon\sqrt{\mu} + \epsilon^2(b-m^2) \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{f} \end{pmatrix} + \begin{pmatrix} F_1(\tilde{r},\tilde{f}) \\ F_2(\tilde{r},\tilde{f}) \end{pmatrix} \tag{4.6}$$

with

$$\begin{aligned} F_1(\tilde{r},\tilde{f}) &= \frac{b^2}{256(1-bh)^5}\left(-\sqrt{\mu}\tilde{f} + 2\tilde{r}(bh-1)\right)^2 \left(2(bh-1) - \sqrt{\mu}\tilde{f} + 2\tilde{r}(bh-1)\right) \times \\ &\quad \times \left(4(bh-1) - \sqrt{\mu}\tilde{f} + 2\tilde{r}(bh-1)\right)^2 + \mathcal{O}(\epsilon) \\ F_2(\tilde{r},\tilde{f}) &= \frac{b^2}{128(1-bh)^4\sqrt{\mu}}\left(-\sqrt{\mu}\tilde{f} + 2\tilde{r}(bh-1)\right)^2 \left(2(bh-1) - \sqrt{\mu}\tilde{f} + 2\tilde{r}(bh-1)\right) \times \\ &\quad \times \left(4(bh-1) - \sqrt{\mu}\tilde{f} + 2\tilde{r}(bh-1)\right)^2 + \mathcal{O}(\epsilon) \end{aligned} \tag{4.7}$$

The form of the nonlinearity suggests the construction of the solution in the following form

$$\begin{aligned} \tilde{r} &= \epsilon^2\hat{R}(l/\epsilon, \epsilon^2 t) + \ldots = \epsilon^2\hat{R}(m, \tau_R) + \ldots \\ \tilde{f} &= \epsilon^2\hat{F}(l/\epsilon, \epsilon t) + \ldots = \epsilon^2\hat{F}(m, \tau_F) + \ldots \end{aligned} \tag{4.8}$$

Substituting (4.8) in (4.6) and using the inverse Fourier transformation, provides us with

$$\begin{aligned} R_{\tau_R} &= bR + R_{\xi\xi} - \frac{1}{8(1-bh)^2}\left(-\sqrt{\mu}F + 2(bh-1)R\right)^2 + \mathcal{O}(\epsilon^2) \\ F_{\tau_F} &= 2\sqrt{\mu}F_\xi + \epsilon\left[\frac{b^2}{4(bh-1)\sqrt{\mu}}\left(-\sqrt{\mu}F + 2(bh-1)R\right)^2 + bF + F_{\xi\xi}\right] + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.9}$$

Or, if instead of $c = \sqrt{\mu}$ one takes $c = -\sqrt{\mu}$, the result is the same except that the linear parts for $\tilde{r}$ and $\tilde{f}$ are interchanged. Further analysis of this system will be given in the next section, where will consider some solutions of (4.9).

• **case 2**: Let us now concentrate on the second case. When modulated, the equation will be the result of the slight perturbation of our control parameter s around the value $s^*_\pm = (h \pm \sqrt{\mathcal{D}})/(bh-1)$. Let us take for example one of the possible cases:

$$s = \frac{h + \sqrt{\mathcal{D}}}{bh-1} + \epsilon^2$$

The other situation can be treated in the same way. This kind of instabilities are observed in most regions: A, B, E and F (see figure 1).

To derive the modulation equation, we prefer to work in Fourier coordinates, which is just a matter of taste — one can proceed in the original spaces and get the modulation equation as a result of applying the Fredholm alternative.

We will stick to the Fourier space method here as far as it makes some steps computationally easier.

First of all, we will transform our linear part to diagonal form by

$$\begin{pmatrix} \hat{r} \\ \hat{f} \end{pmatrix} = \mathcal{S}\begin{pmatrix} \tilde{r} \\ \tilde{f} \end{pmatrix}$$

with $\mathcal{S}$ the transformation matrix build from the eigenvectors

$$e_1 = \begin{pmatrix} 1 \\ \frac{i(bh-1)\epsilon m}{\mathcal{L}}(1 - bh + 2h^2 + 2h\sqrt{\mathcal{D}}) \end{pmatrix}; \quad e_2 = \begin{pmatrix} \frac{i}{2}\frac{\epsilon m}{\mathcal{L}}(bh-1)(2 - bh + b\sqrt{\mathcal{D}}) \\ 1 \end{pmatrix}$$

Then our system will look like

$$\begin{pmatrix} \tilde{r} \\ \tilde{f} \end{pmatrix}_t = \begin{pmatrix} -\frac{2\mathcal{L}}{(bh-1)^2} & 0 \\ 0 & \epsilon^4 \frac{m^2(bh-1)^2}{2\mathcal{L}}(-m^2 + 8\sqrt{\mathcal{D}}) \end{pmatrix} \begin{pmatrix} \tilde{r} \\ \tilde{f} \end{pmatrix} + \begin{pmatrix} \tilde{\mathcal{N}}_1(\tilde{r}, \tilde{f}) \\ \tilde{\mathcal{N}}_2(\tilde{r}, \tilde{f}) \end{pmatrix} \tag{4.10}$$

with

$$\begin{pmatrix} \tilde{\mathcal{N}}_1(\tilde{r}, \tilde{f}) \\ \tilde{\mathcal{N}}_2(\tilde{r}, \tilde{f}) \end{pmatrix} = \mathcal{S}^{-1} \begin{pmatrix} F_1(\tilde{r}, \tilde{f}) \\ F_2(\tilde{r}, \tilde{f}) \end{pmatrix}$$

where F_1 and F_2 are given in (2.5) with $s = s^*_{\pm} \pm \epsilon^2$ and

$$\mathcal{S}^{-1} = \begin{pmatrix} 1 & -\frac{i}{2\mathcal{L}}\epsilon(-1 + bh)m(2 - bh + b\sqrt{\mathcal{D}}) \\ -\frac{i}{\mathcal{L}}\epsilon(-1 + bh)m(1 - bh + 2h^2 + 2h\sqrt{\mathcal{D}}) & 1 \end{pmatrix}$$

The suitable shape of approximation in this case is as follows:

$$\begin{aligned} \tilde{r} &= \epsilon^4 \hat{R}(m, \epsilon^4 t) + \mathcal{O}(\epsilon^6) \\ \tilde{f} &= \epsilon \hat{F}(m, \epsilon^4 t) + \mathcal{O}(\epsilon^3) \end{aligned} \tag{4.11}$$

with the rescaled original variables $l = \epsilon m$ and $\tau = \epsilon^4 t$. After some lengthy calculations and inverse Fourier transformation, one ends up with an equation of the following form

$$F_\tau = -\mu_1 \frac{\partial^2}{\partial \xi^2}\left(\mu_2 + \frac{\partial^2}{\partial \xi^2}\right) F + \beta_1 (F)_\xi (F)_{\xi\xi} \tag{4.12}$$

and with R slaved to F

$$R = -\frac{\alpha_1}{\lambda_1} F_\xi^2 \tag{4.13}$$

with λ_1 as given in (3.5) and with the coefficients given by

$$\begin{aligned} \alpha_1 = & \frac{1}{4\mathcal{L}^2}\left(-80 + 4b^2 + 56bh + 10b^3h - 352h^2 + 168b^2h^2 - 32b^4h^2\right. \\ & + 256bh^3 - 128b^3h^3 + 18b^5h^3 - 256h^4 + 128b^2h^4 - 16b^4h^4 \\ & + 2\sqrt{\mathcal{D}}(-28b + 3b^3 - 112h + 68b^2h - 6b^4h + 64bh^2 - 32b^3h^2 + 3b^5h^2 \\ & \left.-128h^3 + 64b^2h^3 - 8b^4h^3)\right) \\ \mu_1 = & \frac{(1-bh)^2}{2\mathcal{L}}, \qquad \mu_2 = 8\sqrt{\mathcal{D}} \\ \beta_1 = & \frac{8(-1 + 2bh - h^2 - b^2h^2 + bh^3)}{\mathcal{L}^3}\left(4 + b^2 + 4bh - 3b^3h + 32h^2 - 18b^2h^2\right. \\ & + 3b^4h^2 - 16bh^3 + 8b^3h^3 - b^5h^3 + 32h^4 - 16b^2h^4 + 2b^4h^4 \\ & \left.+\sqrt{\mathcal{D}}(4b + 16h - 6b^2h + 32h^3 - 16b^2h^3 + 2b^4h^3)\right) \end{aligned} \tag{4.14}$$

This situation, though more complicated, is similar to the one discovered by Aart van Harten in [Harten, 1994] for the Ginzburg Landau equation:

$$A_t = (1 + ia)A_{xx} + A - (1 + ib)|A|^2 A \tag{4.15}$$

in the case when $a = b$ he gets that

$$A = R_0 e^{ik_0 x - i\alpha t}\left(1 - \frac{\epsilon^3}{2}\frac{\partial \Psi}{\partial \xi}\right) e^{i\epsilon^2 \Psi}$$

with

$$\Psi_\tau = -(1 + a^2)\left(\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{1}{4}\frac{\partial^4 \Psi}{\partial \xi^4}\right) - \frac{3a}{2}\left(\frac{\partial \Psi}{\partial \xi}\right)^2$$

The equations of this type (with the dissipative part of the 4th order) are often called the extended Fisher–Kolmogorov, Kuramoto–Sivashinsky, Swift–Hohenberg or Cahn–Hilliard type of equations.

To finish the classification of possible approximated equations arising from the dGL–equation, we will discuss the degenerate cases where the parameters are chosen on some particular line in the parameter space. It leads, as in the case considered by [Harten, 1994] which we mentioned above, to some quite interesting situations.

• **case 3b**: Let $b = 2$ and $h < 1/2$, then the situation is almost the same as in case 2. Perturbing s as $(1/(2h-1) + \epsilon^2)$ which corresponds to the $+$ sign in (3.6) and looking for solutions in the form (4.11), one gets (4.12) and (4.13) with

$$\begin{aligned} &\alpha_1 = h - 1, \quad \mu_1 = \frac{(1-2h)^2}{8(1-h)}, \quad \mu_2 = 8(1-h) \\ &\beta_1 = 2(-1 + 3h - 2h^2) \end{aligned} \tag{4.16}$$

• **case 3c**: Consideration of $s = 1 - \epsilon^2$ (when instabilities of the type $\circ$ and $\bullet$ coincide, which happens on the line separating Domains A and F) leads us to the following scaling of the solutions:

$$\begin{aligned} \hat{R} &= \epsilon \hat{R}(m, \epsilon^2 t) + \dots \\ \hat{F} &= \epsilon \hat{F}(m, \epsilon^2 t) + \dots \end{aligned} \tag{4.17}$$

From the form of eigenvalues and the transformation we have to apply in this situation one would expect to get the final equation containing pseudo–differential operators. After substituting (4.17) into the system which was first transformed to diagonal form, one ends up with the following set of equations:

$$\begin{aligned} \Upsilon_1 \begin{pmatrix} R \\ F \end{pmatrix}_\tau &= \begin{pmatrix} 2 + \partial_{\xi\xi} - 2\Upsilon_1 & 0 \\ 0 & 2 + \partial_{\xi\xi} + 2\Upsilon_1 \end{pmatrix} \begin{pmatrix} \Upsilon_1 R \\ \Upsilon_1 F \end{pmatrix} \\ &+ \frac{1}{2}\left(2(F+R) + 2\Upsilon_1(F-R) + (2h-1)(F+R)_{\xi\xi}\right)^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \tag{4.18}$$

where Υ_1 denotes the following pseudo–differential operator

$$\Upsilon_1 = \sqrt{1 + (2h-1)\partial_{\xi\xi}}$$

which is well–defined as far as $h < \frac{1}{2}$.
Using the fact that instability which occurs in this case is a sideband instability, one can proceed in an alternative way using the following approximation for the eigensystem:

$$\lambda_- = 2\epsilon^2(h-1)m^2 + \dots \qquad \lambda_+ = 2\epsilon^2(2 - hm^2) + \dots$$

and

$$e_1 = \begin{pmatrix} -\frac{\epsilon}{2}m^2(1-2h) \\ (im)(1-2h) \end{pmatrix}; \quad e_2 = \begin{pmatrix} 2\epsilon + \frac{\epsilon}{2}m^2(1-2h) \\ (im)(1-2h) \end{pmatrix}$$

Now denoting $\tilde{\Delta}_1 = (-2 + (1-2h)\partial_{\xi\xi})$,one will get the following system of modulation equations:

$$\tilde{\Delta}_1 \begin{pmatrix} R \\ F \end{pmatrix}_\tau = \begin{pmatrix} 2(1-h)\partial_{\xi\xi} & 0 \\ 0 & 2(2+h\partial_{\xi\xi}) \end{pmatrix} \begin{pmatrix} \tilde{\Delta}_1 R \\ \tilde{\Delta}_1 F \end{pmatrix} + 4(2F - (1-2h)F_{\xi\xi})^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.19}$$

Let us remark that if $h > \frac{1}{2}$ one can formally repeat the derivation of the modulation equation (4.18), but one should be careful with the definition of Υ_1 in this case.

• **case 4**: In this case we have to distinguish between several situations again, though in all of them the appropriate scaling will be

$$\begin{aligned} \hat{R} &= \epsilon^2 \hat{R}(m, \epsilon^2 t) + \dots \\ \hat{F} &= \epsilon^2 \hat{F}(m, \epsilon^2 t) + \dots \end{aligned} \tag{4.20}$$

Let us first take $1 - 4h^2 < 0$, then denoting by

$$\Upsilon_2 = \sqrt{1 - 2\alpha\partial_{\xi\xi}}; \qquad \alpha = h\left(4h^2 - 1\right) > 0$$

(which is well defined in this case), we derive:

$$\begin{aligned} \Upsilon_2 \begin{pmatrix} R \\ F \end{pmatrix}_\tau &= \frac{1}{h} \begin{pmatrix} 1 + h\partial_{\xi\xi} - \Upsilon_2 & 0 \\ 0 & 1 + h\partial_{\xi\xi} + \Upsilon_2 \end{pmatrix} \begin{pmatrix} \Upsilon_2 R \\ \Upsilon_2 F \end{pmatrix} \\ &+ \frac{1}{16h^2} \begin{pmatrix} [(F+R) + \Upsilon_2(F-R)]\left[(F+R) - \Upsilon_2(F+3R) - 2\left(\Upsilon_2\right)^2(F-R)\right] \\ -[(F+R) + \Upsilon_2(F-R)]\left[(F+R) + \Upsilon_2(F+3R) + 2\left(\Upsilon_2\right)^2(F-R)\right] \end{pmatrix} \end{aligned} \tag{4.21}$$

which reminds us of **3c**. Analogously to **3c**, exploring the fact of the sideband instability, one derives:

$$\tilde{\Delta}_2 \begin{pmatrix} R \\ F \end{pmatrix}_\tau = \tilde{\Delta}_2 \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} R \\ F \end{pmatrix} - \frac{1}{16h^2} \begin{pmatrix} \tilde{\Delta}_2 - 1 \\ \tilde{\Delta}_2 + 1 \end{pmatrix} \mathcal{H}^2 \tag{4.22}$$

where

$$\begin{aligned} \tilde{\Delta}_2 &= 1 - \alpha\partial_{\xi\xi} \\ \mathcal{H} &= 2F - \alpha(F-R)_{\xi\xi} \end{aligned}$$

and

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} 4h^2\partial_{\xi\xi} + \frac{\alpha^2}{2h}\partial_{IV} + \frac{\alpha^3}{2h}\partial_{VI} \\ \frac{2}{h} + 2(1-2h^2)\partial_{\xi\xi} - \frac{\alpha^2}{2h}\partial_{IV} \end{pmatrix}$$

If now $1 - 4h^2 > 0$, one has to define more precisely the meaning of the symbol Υ_2.

• **case 5**: This case turns out to be surprisingly easy to handle. The transformation matrix in this case is

$$\mathcal{S} = \begin{pmatrix} \frac{i\epsilon m}{2} & 1 \\ 1 & \frac{i\epsilon m}{2} \end{pmatrix}$$

and its inverse reads as

$$\mathcal{S}^{-1} = \begin{pmatrix} -\frac{i\epsilon m}{2} & 1 \\ 1 & -\frac{i\epsilon m}{2} \end{pmatrix}$$

The suitable shape of approximation is as follows:

$$\begin{aligned} \tilde{r} &= \epsilon^5 \hat{R}(m,\tau) + \mathcal{O}(\epsilon^7) \\ \tilde{f} &= \epsilon^2 \hat{F}(m,\tau) + \mathcal{O}(\epsilon^4) \end{aligned} \tag{4.23}$$

with the rescaled original variables $l = \epsilon m$ and $\tau = \epsilon^2 t$. After the inverse Fourier transformation, one ends up with an equation of the following form

$$R_\tau = R_{\xi\xi} + F\left(-12F_\xi + 14FF_\xi - F_{\xi\xi\xi}\right) \tag{4.24}$$

and

$$F_\tau = (4 + \partial_{\xi\xi})F - 4F^2 \tag{4.25}$$

The second equation of this set is known as Fisher or Kolmogorov–Petrovski–Piskunov equation. Further analysis of this set will be given in the next section.

5. Some solutions of the modulated-modulation equations.

5.1. *Case 1.*

In this section we will focus our attention on some classes of solutions of the equations we just derived. First we will discuss the case when two eigenvalues of the perturbed problem are complex conjugated with a small real part, which as we already mentioned, is possible in the domains B and F (see figure 1). The equations that describe the bifurcation in this case read:

$$\begin{aligned} R_\tau &= \epsilon\Big(bR + R_{\xi\xi} - \frac{1}{8(1-bh)^2}\left(-\sqrt{\mu}F + 2(bh-1)R\right)^2\Big) + \mathcal{O}(\epsilon^3) \\ F_\tau &= 2\sqrt{\mu}F_\xi + \epsilon\left[\frac{b^2}{4(bh-1)\sqrt{\mu}}\left(-\sqrt{\mu}F + 2(bh-1)R\right)^2 + bF + F_{\xi\xi}\right] + \mathcal{O}(\epsilon^2) \end{aligned} \tag{5.1}$$

with the following rescaling of the original variables: $\xi = \epsilon x$, $\tau = \epsilon^2 t$ and

$$\begin{pmatrix} r \\ f \end{pmatrix} = \epsilon^2 \begin{pmatrix} 1 & \frac{\sqrt{\mu}}{2(1-bh)} \\ \frac{2(bh-1)}{\sqrt{\mu}} & 1 \end{pmatrix} \begin{pmatrix} R \\ F \end{pmatrix} \tag{5.2}$$

We will consider a traveling wave solution of the form

$$\begin{aligned} F(\xi,\tau) &= F(\xi + 2\sqrt{\mu}\tau) = F(\eta) = f_0(\eta) + \epsilon f_1(\eta) + \dots \\ R(\xi,\tau) &= R(\xi + 2\sqrt{\mu}\tau) = R(\eta) = r_0(\eta) + \epsilon r_1(\eta) + \dots \end{aligned} \tag{5.3}$$

If $\epsilon = 0$, one immediately gets that $r_0'(\eta) = 0$ and consequently that r_0 is constant. The second equation is trivially satisfied for any $f_0(\eta)$. Considering the next terms in ϵ one has:

$$\begin{aligned} f_0'' &= -bf_0 - \frac{b^2}{4(bh-1)\sqrt{\mu}}\left(-\sqrt{\mu}f_0 + 2(bh-1)r_0\right)^2 \\ 2\mu r_1' &= br_0 - \frac{1}{8(1-bh)^2}\left(-\sqrt{\mu}f_0 + 2(bh-1)r_0\right)^2 \end{aligned} \tag{5.4}$$

Integrating the first equation we get

$$\frac{1}{2}(f_0')^2 = -\frac{b}{2}f_0^2 + \frac{b^2}{12(bh-1)\mu}\left(-\sqrt{\mu}f_0 + 2(bh-1)r_0\right)^3 + C_0 \tag{5.5}$$

which can be rewritten as

$$\frac{1}{2}(f_0')^2 = \alpha_0 + \alpha_1 f_0 + \alpha_2 f_0^2 + \alpha_3 f_0^3 = \mathcal{F}(f_0) \tag{5.6}$$

where α_j denote constants, depending on b, h, r_0 and C_0. The coefficient $\alpha_3 = \frac{b^2\sqrt{\mu}}{12(1-bh)}$ is negative in the domain B and positive in F (other domains are not covered by this case). Using the standard argument (see [Drazin, 1986] or [Abramowitz & Stegun, 1965] we conclude that depending on parameters cnoidal and soliton-like solutions may exist. If for instance $\mathcal{F}(f_0)$ can be rewritten as $\mathcal{F}(f_0) = -|\alpha_3|(f_0 - A)^2(f_0 - B)$ with $A < B$ then soliton of the form:

$$f_0(\xi, \tau) = A + (B - A)\mathrm{sech}^2\left\{\sqrt{\frac{|alpha_3|(B-A)}{2}}\,(\xi + \sqrt{\mu}\tau)\right\} \tag{5.7}$$

exists. Otherwise if the function $\mathcal{F}(f_0)$ has a positive bounded branch and $\mathcal{F}(f_0) = -|\alpha_3|(f_0 - A)(f_0 - B)(f_0 - C)$ with $A < B < C$ then the cnoidal waves can be discovered:

$$f_0(\xi, \tau) = B + (C - B)\mathrm{cn}^2\left\{\sqrt{\frac{|\alpha_3|(C-A)}{2}}\,(\xi + \sqrt{\mu}\tau)\,; m\right\} \quad ; \quad m = \frac{C - B}{C - A} \tag{5.8}$$

with $0 < m < 1$ and $cn(z; m)$ is the Jacobian elliptic function.
Assume that we have found solutions of the type (5.7) or (5.8) then we have to take care of the r_1-term of the second equation in (5.4). We have to make sure that this term remains bounded as η grows. Hence for the soliton–like solutions the following equality should hold:

$$br_0 - \frac{1}{8(1 - bh)^2}\left(-\sqrt{\mu}A + 2(bh - 1)r_0\right)^2 = 0 \tag{5.9}$$

And for the cnoidal waves we have to restrict ourself to such r_0 that

$$\int_0^P \left(br_0 - \frac{1}{8(1 - bh)^2}\right)\left(-\sqrt{\mu}f_0(\eta) + 2(bh - 1)r_0\right)^2\, d\eta = 0 \tag{5.10}$$

where $f_0(\eta)$ is given by (5.8) and P is the period of the Jacobian elliptic function $cn(z; m)$. Let us formulate now the following result.

THEOREM 1. *If $b^2 > 1$ and $1 - bh < 0$ then there exist soliton-like solution of (5.4) of the form (5.7) with*

$$A = \frac{4b^3(1 - bh)}{(1 + b^2)^4\sqrt{\mu}}\ ;\ B = \frac{2(-3 + b^4)(bh - 1)}{b(1 + b^2)^4\sqrt{\mu}}\ ;\ r_0 = \frac{2b}{(1 + b^2)^2}$$

Proof.

To proof this statement one has to check that $D \equiv q^2 + p^3 = 0$ and (5.9) is satisfied; where $q \equiv \frac{\alpha_2^3}{27\alpha_3^3} - \frac{\alpha_2\alpha_1}{6\alpha_3^2} + \frac{\alpha_0}{2\alpha_3} \neq 0$ and $p \equiv \frac{3\alpha_1\alpha_3 - \alpha_2^2}{9\alpha_3^2} \neq 0$.

Remark. Note that the restriction $1 - bh < 0$ comes only from the need to stay in the domain B (as far as we have the instability of the first type "$\circ$" there) and it is not necessary for existence of the soliton-like solution with A and B as given in the theorem. However, this restriction a plays role in showing that there is no other (with different A,

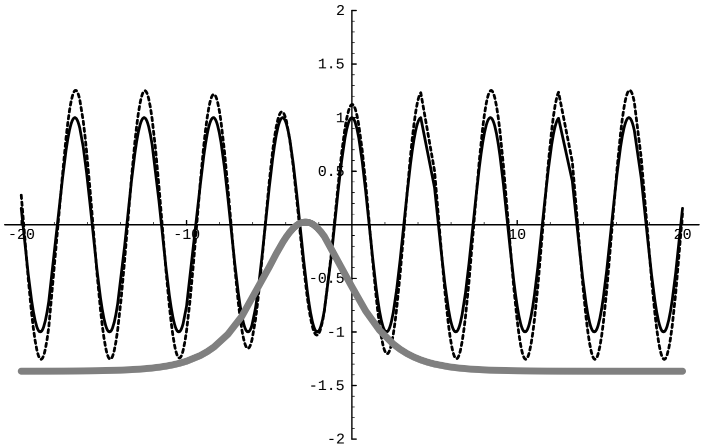


FIGURE 5. The gray thick line denotes the soliton running to the left with the speed $\mu^{\frac{1}{2}}$; the solid line denotes the original periodic state and the dashed line the modulated solution; the parameters used are $b = 2$ and $h = 1/4$.

B and r_0) soliton–like solutions.
Looking back through all transformations (2.4), (5.2), (5.3) one can interpret the existence of the soliton–like solution (5.3) as a slowly traveling slight localized defect in the amplitude and space through original periodic state (figure 5):

$$A(x,t) = R_0\left[1+\epsilon^2(r_0+c_1f_0(\eta))\right]e^{k_0x+\epsilon^2(c2r_0+f_0(\eta))}\ ;\ \eta = \epsilon(\frac{x}{k_0}+\sqrt{\mu}t)+2\mu\epsilon^2\frac{t}{k_0^2}$$

Let us now formulate the condition on the existence of periodic solutions of (5.4).

THEOREM 2. *If $D < 0$ (defined above) and condition (5.10) is satisfied then there exists a cnoidal-wave solution of (5.4).*

5.2. *Case 2.*

Let us now focus on the second case and case 3b. We have taken $A(x,t) = R_0(1 + r(x,t))e^{i(k_0x+f(x,t))}$ and after the transformation:

$$\begin{aligned} r(\xi,\tau) &= \epsilon^2 d_1 F_\xi(\xi,\tau)+\epsilon^4 R(\xi,\tau) \\ f(\xi,\tau) &= \epsilon F(\xi,\tau)+\epsilon^5 d_2 R_\xi(\xi,\tau) \end{aligned} \tag{5.11}$$

with $\xi = \epsilon x$; $\tau = \epsilon^4\tau$ and d_1, d_2 constants depending on the parameters b and h. We derived the following modulation equations

$$F_\tau = -\mu_1\frac{\partial^2}{\partial\xi^2}\left(\mu_2+\frac{\partial^2}{\partial\xi^2}\right)F+\beta_1(F)_\xi(F)_{\xi\xi}\ ;\ R = -\frac{\alpha_1}{\lambda_1}F_\xi^2 \tag{5.12}$$

Stationary solutions of this system are given by

$$-\mu_1\frac{\partial^2}{\partial\xi^2}\left(\mu_2+\frac{\partial^2}{\partial\xi^2}\right)F+\beta_1(F)_\xi(F)_{\xi\xi} = 0 \tag{5.13}$$

Integrating this equation gives:

$$-\mu_1(\mu_2 F_\xi + F_{\xi\xi\xi}) + \frac{\beta_1}{2} F_\xi^2 = C_0 \tag{5.14}$$

which after multiplication with $\mathcal{F}_\xi$ (defined as $\mathcal{F} = F_\xi$) and integration becomes

$$\frac{1}{2}(\mathcal{F}_\xi)^2 = -\frac{\mu_2}{2}\mathcal{F}^2 + \frac{\beta_1}{6\mu_1}\mathcal{F}^3 + C_1\mathcal{F} + C_2 \tag{5.15}$$

Now we can repeat the arguments of the previous case and conclude that *there exist stationary periodic and heteroclinic solutions of (5.12).*

Let us note that though we have found a large class of solutions of the modulation-modulated equations in the 1st and 2nd cases the question whether they are stable or not still remains open. The periodic solutions of the MME correspond to the quasiperiodic solutions of the original modulation equation. However in case of the Ginzburg-Landau equation it was shown that these solutions are unstable (for more details, see [Doelman *et al.*, 1995]).

5.3. *Case 3 and 4.*

Let us turn now to the modulation equations of cases 3 and 4. We will discuss in more details one (simpler) "representative" of equations of this type, namely equation (4.19). First let us denote

$$\tilde{\Delta}_1 R(\xi,\tau) = R_1(\xi,\tau)\ ;\ \tilde{\Delta}_1 F(\xi,\tau) = F_1(\xi,\tau)\ ;\ \tilde{\Delta}_1 = (-2 + (1-2h)\partial_{\xi\xi})$$

Then the system (4.19) can be rewritten as the following reaction-diffusion system:

$$\begin{array}{rcl} (F_1)_\tau & = & 2(2 + h\partial_{\xi\xi})F_1 - 4F_1^2 \\ (R_1)_\tau & = & 2(1-h)\partial_{\xi\xi}R_1 + 4F_1^2 \end{array} \tag{5.16}$$

The first equation can be brought by the transformation $t = 4\tau$; $x = \sqrt{\frac{2}{h}}\xi$ to the standard Fisher or Kolmogorov–Petrovskii–Piskunov (KPP) equation:

$$(F_1)_t - (F_1)_{xx} = F_1(1 - F_1^2) \tag{5.17}$$

Let us summarize some properties of the solutions of this equation. It is natural to look for the traveling solutions of the form $F_1(x,\tau) = \mathcal{F}(\eta) = \mathcal{F}(x - ct)$. Then (5.17) reduces to the 2nd order ordinary differential equation:

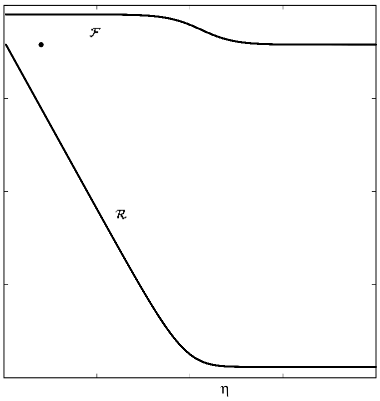
$$\mathcal{F}'' + c\mathcal{F}' + \mathcal{F}(1 - \mathcal{F}) = 0 \tag{5.18}$$

The analysis of the $(\mathcal{F}, \mathcal{F}')$ phase–plane of this equation gives that there exist two critical points: $(0,0)$ – which is a stable node for $|c| > 2$ and is a stable spiral for $|c| < 2$; and $(1,0)$ – which is a saddle. The separatrix connecting these critical points is a front-type traveling wave of the Fisher equation. In the case $c > 2$, using perturbation technique see e.g. [Logan, 1994] one can write down the approximate solution:

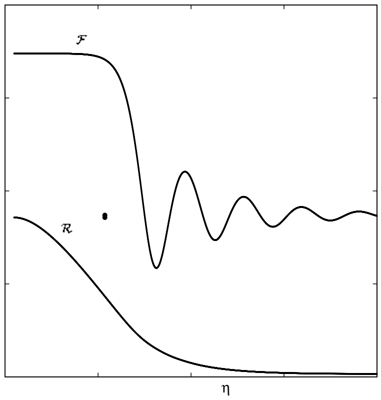
$$\mathcal{F}(\tau) = \frac{1}{1 + e^{\eta/c}} + \frac{1}{c^2} e^{\eta/c}(1 + e^{\eta/c})^{-2} ln \frac{4e^{\eta/c}}{(1 + e^{\eta/c})^2} + O(\frac{1}{c^4}) \tag{5.19}$$

Now $\mathcal{R}(\eta) = R_1(x - ct)$ can be easily found as a solution of the non-autonomous ODE:

$$\mathcal{R}'' = \frac{h}{h-1}\left\{c\mathcal{R}' + 4\mathcal{F}^2\right\} \tag{5.20}$$



(a) $c > 2$.



(b) $c < 2$.

FIGURE 6. Front-type traveling wave solution.

see figure 6. To find the solutions $R(\eta)$ and $F(\eta)$ of the original equations one has to solve (keeping in mind that solutions we are looking for should remain bounded as $\eta \to \pm\infty$) the following system of ODEs:

$$\begin{aligned} R'' &= \tfrac{h}{1-2h}(R + \tfrac{1}{2}\mathcal{R}) \\ F'' &= \tfrac{h}{1-2h}(F + \tfrac{1}{2}\mathcal{F}) \end{aligned} \tag{5.21}$$

with $\mathcal{R}$ and $\mathcal{F}$ known functions (see figure 6). Then one recovers

$$\begin{aligned} r &= \epsilon^2(R + F + \tfrac{1}{2}(\mathcal{R} - \mathcal{F})) \\ f &= \epsilon\sqrt{\tfrac{2}{h}}(1-2h)(R - F)_\eta \end{aligned} \tag{5.22}$$

This way we discover the following solutions of the original problem:

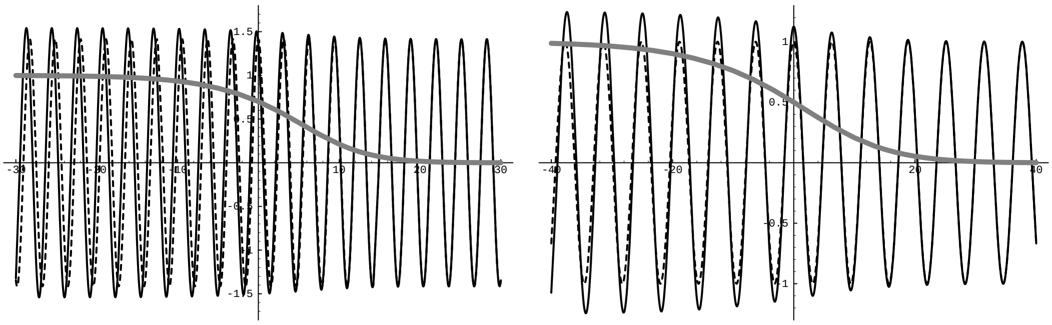
$$A(x,t) = R_0(1 + \epsilon^2\mathcal{F}(\eta))e^{i(k_0x + \epsilon\phi_1(\eta))}$$

with $\phi_1(\eta) = \frac{1-2h}{\sqrt{2h}}(4\mathcal{F}^2 - \mathcal{F}')$ – a traveling front, and $\eta = \epsilon\frac{x}{k_0}\sqrt{\frac{2}{h}} - \frac{4\epsilon^2 ct}{k_0^2}$. This solution is slowly moving trough the original periodic solution leaving behind new periodic solution with slightly changed amplitude and shifted phase (see figure 7).

5.4. *Case 5.*

Case 5 is similar to the one we just considered. One has first to look at (4.25) which is again Fisher equation and then to solve (4.24) as an non-autonomous ODE with known F. The final answer in this case is:

$$\begin{aligned} r(\eta) &= \epsilon^2\mathcal{F} + \epsilon^6\mathcal{R}_\eta \\ f(\eta) &= \epsilon^3\mathcal{F}_\eta + \epsilon^5\mathcal{R} \end{aligned} \tag{5.23}$$



(a) Case 3: $h = 2, b = 1.5$. (b) Case 5: $h = \frac{1}{2}, b = 2$.

FIGURE 7. The dashed line denotes the original periodic state; the thick gray line denotes the moving to the right front as solution of the modulated modulation equation and the thin solid line shows the solutions in the slightly Eckhaus unstable region.

with $\eta = \epsilon \frac{x}{2k_0} - \epsilon^2 \frac{ct}{4k_0^2}$ and $\mathcal{F}(\eta)$ is a moving front (see figure 6) and $\mathcal{R}(\eta)$ is a solution of the rescaled version of 4.24. Notice that unlike the previous case, the change in amplitude is of lower order in ϵ then the change of the phase. (see figure 7)

Remark Let us mention that unlike the traveling pulse solutions which we discovered in case 1 and quasiperiodic solutions of case 2 where nothing is known about stability of these special solutions, in the situations reduced to the Fisher equation when traveling front exists it is well known (see for instance [Logan, 1994], [Friedman, 1969] and references therein) that the front-like solutions are stable to small localized perturbations imposed in the moving coordinate frame of the wave and a lot is known about the basin of attraction of them (see i.e. [Aronson & Weinberger, 1978], [Gallay, 1993] and [Bramson, 1983]).

Remark The mechanism of the traveling front solutions is somehow similar to the amazing phenomenon described in [Eckmann & Gallay, 1993]. There it was proven for the Ginzburg–Landau equation (4.15) with $a = b = 0$ that there exist "solutions which look like a fixed envelope moving to the right with some constant velocity $c > 0$, while leaving behind a periodic pattern $\left[1 - k_1^2\right]^{1/2} e^{ik_1 x + i\phi}$ and destroying another one $\left[1 - k_2^2\right]^{1/2} e^{ik_2 x + i\phi}$ in front". Let us underline some differences between our and their result: first of all they work with the Ginzburg–Landau equation with real coefficient (which is just a very special case of our original equation (1.2)) where modulated modulation equations of the form described by cases 1 and 5 do not appear; our solutions are slowly moving and the speed of solutions considered there are of order one; the change in amplitudes in our case is just slight and instead of changing the wave number a phase shift takes place.

6. Discussions. Sub–critical case.

Let us now briefly discuss the periodic solutions and their stability for the sub–critical case, i.e. we consider now equation (1.2) with $\alpha = -1$ and $c > 0$. This case differs from

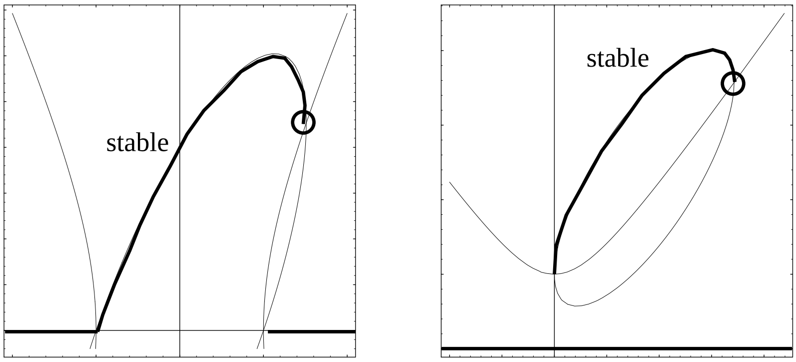


FIGURE 8. Trivial stability condition for super– and sub–critical cases.

the super–critical one. To make it easier to compare our results we will be using the same parameters as before: $b = b_1 + b_2$ and $h = \frac{1}{4}(b_1 - b_2)$. The condition for periodic solutions $A(x,t) = R_0 e^{i(k_0 x + \omega t)}$ to exist reads as follows:

$$R_0^4 - bR_0^2 k_0 + k_0^2 - \hat{c}R_0^2 = -1 \; ; \; \omega = 0 \tag{6.1}$$

To obtain the result analogous to (2.2) and (2.3) we linearize around the periodic solution. And by controlling when the real parts of the eigenvalues of the linear system are negative one recovers:

$$R_0^4 - k_0^2 - 1 > 0 \tag{6.2}$$

$$2 - 3\hat{c}R_0^2 + 2R_0^4(2 - bh) + 2k_0 R_0^2(2h - b) > 0 \tag{6.3}$$

Let us notice that the condition (6.1) in this case corresponds to the ellipse in the (k_0, R_0^2)–plane if $4 - \hat{c} < b^2 < 4$. That is why instead of $\hat{c} = 0$ (as it was done before) in order not to loose any interesting cases but to simplify our calculations, we choose $c = 2$. Similar to the super–critical case (6.2) and (6.3) define hyperbolas in (k_0, R_0^2)–plane and (6.2) is so called *trivial stability condition* ([Eckhaus & Iooss, 1989], [Golovin *et al.*, 1997]) which should be understood as follows: if there are two periodic solutions with the same wave number and different amplitudes then only the solution with the large amplitude can be stable (see figure 8). Not going into detailed calculations we give, analogously to figure 1, a summary of the stability results in the (h, b)–plane (see figure 9).

We will not write down explicitly the modulated modulation equations for the sub–critical case† but using the results of the previous calculations and the argument that the essence of the arising MME mainly depends on the type of instability, we are able to describe possible MME's in this case and their solutions – as a matter of fact the same names of the various domains in the parameter plane were given for this reason. Hence

† As a matter of fact this is a bit more difficult because the curve given by (6.1) is not centered in 0; one therefore has some difficulties perturbing $s = R_0^2/k_0$. But this can be easily corrected by an obvious transformation and then one can proceed analogously to the section2.

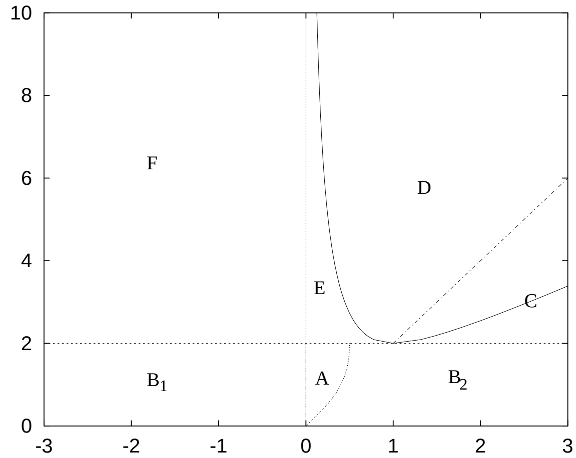


FIGURE 9. Parameter space for the sub–critical case.

Domain A	$b < 2$, $0 < h < \frac{2b}{4+b^2}$	ellipse	instabilities caused by (6.3)
Domain B_1	$b < 2$, $h < 0$	ellipse	inst. caused by (6.2) and (6.3)
Domain B_2	$b < 2$, $h > \frac{2b}{4+b^2}$	ellipse	inst. caused by (6.2) and (6.3)
Domain C	$b > 2$, $2h - b > 0$	hyperb	no stable periodic solutions
Domain D	$b > 2$, $4b^2(h^2 - bh + 1) < 4 - b^2$	hyperb	no stable periodic solutions
Domain E,F	$\{b > 2\} \setminus \{C\} \setminus \{D\}$	hyperb	inst. caused by (6.2) and(or) (6.3)

Here we separated the domains B_1 and B_2 by the following principle: in B_1 the band of the stable modes is bounded by $\circ$ from the left and by $\bullet$ from the right and in B_2 it is the other way round. These two similar regions are separated by the domain A and on the boundaries between B_j and A the degenerate instabilities of the type $\odot$ occur.

the solutions of the MME's in different regions in the sub–critical case are similar to the solutions in the corresponding regions of the super–critical case.

Remark. In the section devoted to the solutions of MME's we consider the simplest examples of the solutions, there are many more of them, it is a separate topic to study. Let us also mention that the MME approach can be used as a powerful tool for investigation of the stability of new solutions of the original dGL equation.

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