Characteristic Classes and Hochschild Homology of Group Graded Algebras

A. Bella Baci[1]

Department of Mathematics, University of Utrecht
P.O. Box 80.010, 3508 TA Utrecht
The Netherlands
e-mail: bellabac@math.ruu.nl

Abstract

We define characteristic classes in the Hochschild homology of a G-graded algebra (G a discrete group). These classes determine the differential of the spectral sequence of Lorenz. We give an explicit description for G the fundamental group of a Riemann surface.

Contents

1 Introduction **2**

2 The Picard group and Operation on Hochchild Homology **3**
2.1 The Picard group 3
2.2 Operation on Hochschild Homology 4

3 G-Graded algebras and characteristic classes **6**
3.1 G-Graded Algebras 6
3.2 Characteristic classes 7

[1]-Research supported by the Dutch Science Organisation (NWO).

4 **Hochschild Homology of G-graded algebra** **8**

5 **Example** **11**

1 Introduction

In the past ten years there has been a tremendous surge of activity in the theory of graded rings [NO] and their important special case, namely crossed products. In the topological setting, this kind of algebra is a central object in the studie non-commutative geometry in the work of A. Connes [C].

Let $A = \bigoplus_{g \in G} A_g$, $A_g.A_h \subseteq A_{gh}$ be an algebra that is graded by a group G. Then the Hochschild homology $HH_*(A)$ of A, has canonical decomposition over the conjugacy classes $\langle G \rangle$ of G

$$HH_*(A) = \bigoplus_{[g] \in \langle G \rangle} HH_*(A)_{[g]}$$

In [Lor], M. Lorenz describes the components $HH_*(A)_{[g]}$ of Hochschild homology in the case where A is strongly G-graded. The description is given in terms of a spectral sequence

$$E^2_{pq} = H_p(G^g, HH_q(R, A_g)) \implies HH_p(A)_{[g]}$$

Here $HH_*(R, A_g)$ is the Hochschild homology of the idendity compenent $R = A_e$ of A with coefficients in the R-bimodule A_g and $H_*(G^g, -)$ is the group homlogy of the centralizer G^g of $g \in G$.

If G is a free group (for example, $G = \mathbb{Z}$), the spectral sequence reduces to an exact sequence. The aim of this paper is the study of this spectral sequence when the group G is not nessecarly free (for example, the fundamental groups of Riemann surface)

The group G^g acts in canonical fashion on $H_*(R, A_g)$. For this action we define characteristic classes

$$\Theta_*(A, G^g) \in Ext^2_{G^g}(HH_*(R, A_g), HH_{*+1}(R, A_g)).$$

These classes may be not trivial even if $H_*(R, A_g)$ is trivial G^g-module. Over a field,

$$\Theta_*(A, G^g) \in H^2\Big(G^g, Hom(HH_*(R, A_g), HH_{*+1}(R, A_g))\Big).$$

So these classes are associated to the relations in the presentation of the group.

The contents of the paper is as follows: In the section (4), we prove that the differantial

$$d_2 : H_p(G^g, HH_p(R, A_g) \to H_{p-2}(G^g, HH_{q+1}(R, A_g)$$

is a Yoneda product by the class $\Theta_*(A, G^g)$. In the section (5) we study an example when the homological dimension of the group is equal to two. These classes are defined in the section (3). In section (2) we recall some classical facts about the Picard group, and the operations in Hochschild homology.

Acknowledgment. It is a pleasure for me to thank I. Moerdijk for his kind interest. I am very grateful to Utrecht University for it's hospitality during the academic years 1995/97.

2 The Picard group and Operation on Hochchild Homology

In this section, we give some general fact about the Picard group of an associative algebra A. We describe, in the canonical way, the operation of this group on the Hochschild homology of A. Our main references for this section are [B], [Lor].

2.1 The Picard group

Let k be a commutative ring, and A, B two k-algebras. An A-B-bimodule P is called invertible, if there exist a B-A-bimodule Q such that $P \otimes_B Q \simeq A$ and $Q \otimes_A P \simeq B$. If $\mathcal{C}$ is k-category we define the Picard group, denoted by $Pic_k(\mathcal{C})$, to be the group of automorphism classes $[T]$ of k-equivalences $T : \mathcal{C} \to \mathcal{C}$. The group law is induced by composition of functors. If A is k-algebra we define $Pic_k(A)$ to be the group of isomorphism classes $[P]$ of invertible left $A \otimes_k A^{op}$-modules, the groupe law $[P].[Q] = [P \otimes_k Q]$ and with $[P]^{-1} = [Hom_{mod-A}(P, A)]$ inverse isomorphisms are

$$Pic_k(A) \xrightarrow{\alpha} Pic_k(mod - A) \quad ; \quad \alpha[P] = [- \otimes_A P]$$

$$Pic_k(mod - A) \xrightarrow{\beta} Pic_k(A) \quad ; \quad \beta[T] = [TA].$$

It's intuitively clear that algebra automorphisms of A should contribute to $Pic_k(mod - A)$. We shall now indicate how they appear in $Pic_k(A)$. For a k-algebra A write $Pic_k(\mathcal{C}_A)$, where $\mathcal{C}_A$ is the category of invertible left $A \otimes_k A^{op}$-modules, and bimodule homomorphisms. Suppose $P \in Pic_k(\mathcal{C}_A)$ and $\alpha, \beta \in Aut_{k-alg}(A)$. Then we define ${}_\alpha P_\beta$ to be the left $A \otimes_k A^{op}$-module whose additive group is P and whose bimodule structure is given by $a.p = \alpha(a)p$, $p.a = p\beta(a) \quad p \in P, a \in A$. For example $P =_1 P_1$,. Moreover, we have

$$ {}_\alpha P_\beta =_\alpha A_1 \otimes_A P \otimes_A 1A_\beta. $$

Suppose that $P, Q \in Pic_k(\mathcal{C}_A)$ and that $f : P \to Q$ is a left A-isomorphism. Since $A = Hom_{A-mod}(P,P)^{op}$, the left A-endomorphism $p \to f^{-1}(f(p)a)$ must be right multiplication by a unique $\alpha(a) \in A$. In other words, $f(p\alpha(a)) = f(p)a$ for $p \in P$ and $a \in A$. Evidently $\alpha \in Aut_{k-alg}(A)$, and this equation therefore can be rephrased by stating that $f :_1 P_\alpha \to Q$ is a bimodule isomorphism. This proves (4) of following

Proposition 2.1.1 *Let A be an k-algebra and let $\alpha, \beta, \gamma \in Aut_{k-alg}$*

1. ${}_\alpha A_\beta =_{\gamma\alpha} A_{\gamma\beta}$ *as bimodules*

2. ${}_1A_\alpha \otimes_A 1A_\beta = 1A_{\alpha\beta}$ *as bimodules.*

3. ${}_1A_\alpha =_1 A_1$ *as bimodules* $\Leftrightarrow \alpha \in InAut(A)$
   Where $InAut(A)$ is the group of inner automorphisms of A.

4. *if $P \in Pic_k(\mathcal{C}_A)$ and if $P \simeq A$ as left A-modules, then $P =_1 A_\alpha$ as bimodules for some $\alpha \in Aut_{k-alg}(A)$. There is a canonical exact sequence*
   $$1 \to InAut_k(A) \to Aut_k(A) \to Pic_k(A)$$
   The last map sends the automorphism α of A to the bimodule ${}_1A_\alpha$.

2.2 Operation on Hochschild Homology

Let ${}_AM_A$ be an A-A-bimodule and put $C_n(A,M) = M \otimes A^{\otimes n}$. Let

$$d_0(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) = (m.a_1 \otimes a_2 \otimes ... \otimes a_n)$$

$$d_i(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) = (m \otimes a_1 \otimes ... \otimes a_i.a_{i+1} \otimes ... \otimes a_n) \qquad 0 < i < n$$

$$d_n(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) = (a_n.m \otimes a_1 \otimes ... \otimes a_i.a_{i+1} \otimes ... \otimes a_{n-1}) \qquad .$$

The boundary map $b_n : C_n(A, M) \to C_{n-1}(A, M)$ is defined by

$$b_n = \sum_{i=0}^{i=n} d_i.$$

Thus $(C_*(A, M), b)$ is the usual Hochschild complex and its homology is the Hochschild homology of A with coefficients in M, denoted by $HH_n(A, M) = H_n(C_*(A, M), b)$. In the special case where $M = A$, we will simply write $C_n(A) = C_n(A, A) = A^{\otimes n+1}$, and $HH_n(A) = HH_n(A, A)$.

Let $Q = P_B^* = Hom_B(P_B, B)$ denote the dual of P_B; note that Q is a B-A-bimodule via:

$$(bq)(p) = bq(p) \quad ; \quad qa(p) = q(ap) \quad \forall a \in A, b \in B, p \in P, q \in Q.$$

Therefore, $Q \otimes_A M \otimes_A P$ is a B-B-bimodule, and we can consider the Hochschild complex $C_*(B, Q \otimes_A M \otimes_A P)$ as above; we will exhibit a chain map

$$\Psi = \Psi^{P,M} : C_*(A, M) \to C_*(B, Q \otimes_A M \otimes_A P)$$

For this choose dual bases for P, which are elements $p_i \in P \quad q_i \in Q = P^* \quad i = 1, 2, ..., r$, with $p = \sum_{i=1}^r p_i q_i(p), \forall p \in P$. Define a k-linear map $\Psi_n : C_n(A, M) \to C_n(B, Q \otimes_A M \otimes_A P)$ by:

$$\Psi_n(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) = \sum_{(i_0, ..., i_n)} (q_{i_0} \otimes_A m \otimes_A p_{i_1}) \otimes q_{i_1}(a_1 p_{i_2}) \otimes ... \otimes q_{i_n}(a_n p_{i_0})$$

where the sum runs over all $(n+1)$-tuples $(i_0, i_1, ..., i_n)$ with $1 \leq i_k \leq r$. It is a straightforward to check that $\Psi_{n-1} b_n = b_n \Psi_n$ holds. Suppose that $\{p'_i\}, \{q'_i\}$ is another choise of dual bases, and let $\Psi' : C_*(A, M) \to C_*(B, Q \otimes_A M \otimes_A \otimes P)$ denote the corresponding chain map. Then a chain homotopy $h_n : C_n(A, M) \to C_{n+1}(B, Q \otimes_A M \otimes_A P)$ with $\Psi_n - \Psi'_n = b_{n+1} h_n + h_{n-1} b_n$ is obtained by defining $h_n = \sum_{t=0}^n h_{n,t}$, where

$$h_{n,t} : C_n(A, M) \to C_{n+1}(B, Q \otimes_A M \otimes_A P),$$

is given by

$$h_{n,t}(m \otimes a_1 \otimes a_2 \otimes ... \otimes a_n) = \sum_{\substack{(i_0, ..., i_{n-1}) \\ (j_0, ..., j_t)}} (q_{i_0} \otimes_A m \otimes_A p'_{j_0}) \otimes q'_{j_0}(a_1 p'_{j_1}) \otimes ... \otimes q_{j'_{t-1}}(a_t p'_{j_t}).$$

We therefore have the following lemma.

Lemma 2.2.1 $\Psi = (\Psi_n) : C(A, M) \to C(B, Q \otimes_A M \otimes_A \otimes P)$ *is a chain map whose homotopy type is independent of the choise of dual bases.*

The above lemma implies in particular that the map on homology that is given by Ψ does not depend on the particular choice of the dual bases for P. Thus we can define $H_*^{A,M} = H_*(\Psi) : H_*(A, M) \to H_*(B, Q \otimes_A M \otimes_A \otimes P)$. In the special case where $M = A$, we have a bimodule map $Q \otimes_A A \otimes_A P \simeq Q \otimes_A P \to B, q \otimes_A p \to q(p)$. This yields a map $HH_*(B, Q \otimes_A A \otimes_A P) \to HH_*(B, B) = HH_*(B)$, and forming the composite with $H_*^{P,A}$, we obtain a map $H_*^P : H_*(A) \to H_*(B)$. Explicitly, H_*^P is induced by the chain map $\Phi = \Phi^P : C_*(A) \to C_*(B)$ which on $C_*(A) = A^{\otimes n+1}$ is given by

$$\Phi_n(a_0 \otimes ... \otimes a_n) = \sum_{i_0,\ldots,i_n} (q_{i_0}(a_0 p_{i_1})) \otimes (q_{i_1}(a_1 p_{i_2})) \otimes ... \otimes (q_{i_n}(a_n p_{i_0}))$$

By the lemma, the homotopy type of Φ is independent of the choice of the dual bases $\{p_i\}, \{q_i\}$ for P. The map H_n^P yield an action of $Pic_k(A)$ on $HH_*(A)$ wich induces the usual action of $Aut_k(A)$.

$$\varphi : Pic_k(A) \times HH_*(A) \to HH_*(A) \tag{1}$$

$$([P], x) \to \varphi([P], x) = H_*^P(x).$$

3 G-Graded algebras and characteristic classes

3.1 G-Graded Algebras

Let A be a k-algebra which is graded by a group G. So

$$A = \bigoplus_{g \in G} A_g$$

is a direct sum of k-submodules A_g with $A_g.A_h \subseteq A_{gh}$ for all $g, h \in G$. Thus the A_g are k-submodule we will put $R = A_e$, (where e is the identity element of G). R is a k-subalgebra of A.

The algebra A is said to be strongly G-graded if $A_g.A_h = A_{gh}$ holds for $\forall g, h \in G$. In this case the multiplication of A gives R-R-bimodule isomorphisms $A_g \otimes_R A_h \simeq A_{gh}$, for $\forall g, h \in G$, see [NO].

We note that a twisted crossed product algebra is a special case of a group graded algebra. Let A be a unital algebra over k upon wich G acts by α as a group of automorphisms. Let σ be a 2-cocycle on G with values in the unit circle S^1. Then σ satisfies the equality

$$\sigma(g,h)\sigma(gh,k) = \sigma(g,hk)\sigma(h,k), \quad \forall g,h,k \in G.$$

The twisted algebraic crossed product of A by G is denoted by $A \rtimes_{\alpha.\sigma} G$. Elements in $A \rtimes_{\alpha.\sigma} G$ are of the form $\sum a_g[g]$, where $a_g \in G$ and the sum is finite. The product is the usual one with the rule that, $a[g].b[h] = a.b^g[g][h] = ab^g\sigma(g,h)[gh]$, where $a, b \in A, g, h \in G$ and $b^g = gbg^{-1} = \alpha_g(b)$. If we note by $B = A \rtimes_{\alpha\sigma} G$ then $B_g = A.[g]$. When $A = k$, we have $k(G,\sigma)$, the twisted group algebra.

Given a G-graded algebra A, each homogeneous component A_g is thus an invertible R-R-bimodule and as such is finitely generated projective as R-bimodule, on either side. Moreover $(A_g)^*_R \simeq_R (A_{g^{-1}})$. Thus we have a group homomorphism $G \longrightarrow Pic_k(R)$, $g \longrightarrow [A_g]$. So from (1), the group G acts on the modules $Z_*(R) = Ker(b)$, $Z'_*(R) = coker(b)$, $B_* = Im(b)$ and $HH_*(R)$. $b : C_*(R) \longrightarrow C_{*-1}(R)$ is the Hochschild bondary.

3.2 Characteristic classes

We consider now the following short exact sequence of G-modules

$$e_*(R) : 0 \longrightarrow HH_{*+1}(R) \longrightarrow Z'_{*+1}(R) \longrightarrow B_*(R) \longrightarrow 0$$

$$e'_*(R) : 0 \longrightarrow B_*(R) \longrightarrow Z_*(R) \longrightarrow HH_*(R) \longrightarrow 0.$$

These extensions define the classes

$$[e_*(R)] \in Ext^1_G(HH_{*+1}(R), B_*)$$

$$[e'_*(R)] \in Ext^1_G(B_*, HH_*(R)).$$

Definition 3.2.1 *The secondary caracteristic class, associated to the G-graded algebra A, is the class*

$$\Theta_*(A,G) = [e'_*(R)] \cup [e_*(R)] \in Ext^2_G(HH_*(R), HH_{*+1}(R))$$

the Yoneda product of the classes $[e_*(R)]$ and $[e'_*(R)]$.

For each q, $\Theta_*(A,G)$ determine a class :

$$\Theta_q(A,G) = [e'_*(R)] \cup [e_*(R)] \in Ext^2_G(HH_q(R), HH_{q+1}(R))$$

These groups fits in a following long exact sequence, as a particular case of the one given in [CE].

Proposition 3.2.1 *Over an heriditary ring, we have a long exact sequence*

$$.. \longrightarrow H^{n-2}(G, Ext(HH_*(R), HH_{*+1}(R)) \longrightarrow H^n(G, Hom(HH_*(R), HH_{*+1}(R)) \longrightarrow$$

$$Ext^n_G(HH_*(R), HH_{*+1}(R)) \longrightarrow H^{n-1}(G, Ext(HH_*(R), HH_{*+1}(R)) \longrightarrow ...$$

Corollary 3.2.1 *Over a field we have. For all q*

$$\Theta_q(A, G) \in H^2(G, Hom(HH_q(R), HH_{q+1}(R))$$

These characteristic classes act by products in the following sens. Let J_*, N_* be a left G-module graded by $\mathbb{Z}$, and C_* be a right G-module graded by $\mathbb{Z}$, projective as a G-module. The natural morphism

$$Hom^*_G(J_*, I_*) \otimes (C_* \otimes_G J_*) \longrightarrow C_* \otimes_G I_*$$

(where I_* is an G-injective resolution of N_*), gives the product

$$\cap : Ext^*_G(J_*, N_*) \otimes H_*(C_* \otimes_G J_*) \longrightarrow H_*(C_* \otimes_G N_*). \tag{2}$$

4 Hochschild Homology of G-graded algebra

In this section we give the main theorem of this paper, We start by describing the spectral sequence of Lorenz [Lor], which is related to the Hochschild homology of G-graded algebra and their complicated structure.

G will denoted a discrete group and $\langle G\rangle$ the set of conjugacy classes of G. Given a conjugacy class $[x]$, let h be an element in $[x]$. The subgroup $G^h = \{g \in G | gh = hg\}$ of G (the centralizer) contains a central cyclic subgroup $(h) = h^{\mathbb{Z}}$ generated by h. The quotient $G^h/(h)$ is denoted by N^h. Obviously, G^h and N^h do not depend on the element h chosen up to group isomorphism. Let A be a G-graded algebra (not necessarily strongly graded), and let V be an A-A-bimodule such that

$$V = \bigoplus_{g\in G} V_g \quad , \quad A_g.V_h \subseteq V_{gh} \quad \forall g, h \in G.$$

For each class $[g] \in \langle G\rangle$, let

$$C_*(A, V)_{[g]} = \{v_{g_0} \otimes a_{g_1} \otimes a_{g_2} \otimes ... \otimes a_{g_n} / v_{g_0} \in V_{g_0}; a_{g_i} \in A_{g_i}; g_o.g_1...g_n \in [g]\}.$$

In the special case where $V = A$, we write $C_n(A)_{[g]} = C_n(A, A)_{[g]}$. The maps d_i send $C_n(A)_{[g]}$ to $C_{n-1}(A)_{[g]}$; this is clear for $i < n$ and for $i = n$ it follows from the fact that $[g_n.g_0...g_{n-1}] = [g_o.g_1...g_n]$. Therefore, the Hochschild complexes $C_*(A,V)$ decompose :

$$C_*(A,V) = \bigoplus_{[g]\in\langle G\rangle} C_*(A,V)_{[g]},$$

and similary for $C_*(A)$. Consequently, we obtain corresponding decomposition of Hochschild. homology

Lemma 4.0.1

$$HH_*(A,V) = \bigoplus_{[g]\in\langle G\rangle} HH_*(A,V)_{[g]};$$

$$HH_*(A) = \bigoplus_{[g]\in\langle G\rangle} HH_*(A)_{[g]}.$$

Theorem 4.0.1 [Lor] *Let $A = \bigoplus_{g\in G} A_g$ be a strangly graded algebra and let V be an A-A-bimodule. There exists a first quadrant spectral sequence*

$$E^2_{pq} = H_p(G, HH_q(R,V)) \quad \overset{\text{CV}}{\Longrightarrow} \quad H_{p+q}(A,V).$$

Suppose V has a decomposition $V = \bigoplus_{g\in G} V_g$ with $S_g V_h \subseteq V_{gh}$ and $V_g S_h \subseteq V_{gh}$ for $g, h \in G$. Then for each $[g] \in \langle G\rangle$, there is a spectral sequence

$$E^2_{pq} = H_p(G^g, HH_q(R,V_g)) \quad \overset{\text{CV}}{\Longrightarrow} \quad H_{p+q}(A,V)_{[g]}.$$

It's easy to see from the construction of this spectral sequence given in [Lor], (see also [FT], in the special case of crossed product) that we have a quasi-isomorphism

$$\bigoplus_{[g]\in\langle G\rangle} B(G^g) \otimes_{G^g} C_*(R,V_g) \simeq C_*(A,V)_{[g]}$$

where $B(G^g)$ is the reduced bar complex of G^g, [M].

From the above decomposition, it's easy to see [Lor] that $H_*(R,V_g)$ inherites a G^g-module structure for each $g \in G$, and as in the last section we define a characteristic class for each conjugacy class

$$\Theta_q(A,G^g) \in Ext^2_{G^g}((HH_q(R,V_g), HH_{q+1}(R,V_g)).$$

Now in the product (2), we take, for each q, $J_q = HH_q(A, V_g)$, $N_q = HH_{q+1}(A, V_g)$ and $C_q = B_q(G^g)$. So the product takes the form

$$Ext^l_{G^g}(HH_q(A, V_g), HH_{q+1}(A, V_g)) \otimes H_p(G^g, HH_q(A, V_g))$$

$$\longrightarrow H_{p-l}(G^g, HH_{q+1}(A, V_g))$$

Let $d_2 : E^2_{pq} \longrightarrow E^2_{p-2,q+1}$ be the differential of the above spectral sequence.

Theorem 4.0.2 *For G, A, V as above*

1. $d_2 = \Theta_q(A, V) \cap -$
2. *If V has a decomposition $V = \bigoplus_{g\in G} V_g$ with $S_g V_h \subseteq V_{gh}$ and $V_g S_h \subseteq V_{gh}$ for $\forall g, h \in G$, then*

   $$\Theta_q(A, G) = \bigoplus_{[g]\in\langle G\rangle} \Theta_q(A, G^g)_{[g]}$$

   and $d_2 : H_p(G^g, HH_q(R, V_g)) \longrightarrow H_{p-2}(G^g, HH_{q+1)}(R, V_g))$ is given by: $d_2(x) = \Theta_q(A, V)_{[g]} \cap x$, $\forall x \in H_p(G^g, HH_q(R, V_g))$

Before the proof of this theorem, we introduce another definition of the characteristic classes associated to the G-complex $(C_*(R), b)$. We give the proof for G; the same method works for the other conjugacy classes. Let I^*_* be a G-injective resolution of $HH_*(R)$. The first term of the spectral sequence associated to the filtration F^* of $Hom^*_G(C_*(R), I^*_*)$ by the degree of the resolution is

$$\underline{E}^{p,q}_2 = Ext^p_G(HH_*(R),\ HH_{*-q}(R)).$$

In particular:

$$\underline{E}^{0,0}_2 = Hom_G(HH_*(R),\ HH_*(R)).$$

The image of $1 \in \underline{E}^{00}_2$ by the differential ∂_2 of this spectral sequence is a class

$$\Omega_*(R, G) = \partial_2[1] \in Ext^2_G(HH_*(R),\ HH_{*+1}(R)).$$

In the same way as in [L], it's easy to show that $\Omega_*(M, W)$ is the opposite of the class $\Theta_*(R, G)$.

Proof of the theorem.

Let $\underline{F}_*$ the filtration of $B_*(G) \otimes_G I^*_*$ given by:

$$\underline{F}_*(B_*(G)\otimes_G I^*_*) = \sum_{i-j=k} B_i(G)\otimes_G I^j_*.$$

The spectral sequence with first term ${}^2\underline{E}_{*,*}$ associated to the above filtration collapses, and the differantial $\underline{d}_2$ is trivial. Let σ the morphism

$$\sigma : Hom^*_G(C_*(R), I^*_*)\otimes(B_*(G)\otimes_G C_*(R)) \rightarrow B_*(G)\otimes_G I^*_*$$

given by :

$$\sigma(f\otimes c\otimes m) = (-1)^{deg(c).deg(f)} c\otimes f(m)$$

where $f\in Hom^*_G((C_*(R), I^*_*)$, $c\in B_*(G)$ and $m\in C_*(R)$.

Let now F_* be the filtration of $B_*(G)\otimes_G C_*(R)$ by the degree of $B_*(G)$. The morphism σ induces a product on the level of the spectrals sequences:

$$\cap : \underline{E}_2^{s,t}\otimes E^2_{p,q} \rightarrow^2 \underline{E}^2_{p-s,q+t}.$$

Now the diagram

$$\begin{array}{ccc}
\underline{E}_2^{*,*}\otimes E^2_{*,*} & \longrightarrow & {}^2\underline{E}_{*,*} \\
\Big\downarrow D_2 & & \Big\downarrow \underline{d}_2 \\
\underline{E}_2^{*,*}\otimes E^2_{*,*} & \longrightarrow & {}^2\underline{E}_{*,*}
\end{array}$$

is commutative, with $D_2 = \partial_2\otimes 1 + 1\otimes d_2$. Since the differential of the spectral sequence ${}^2\underline{E}_{*,*}$ is trivial, we have for x in $H_p(G, HH_q(R))$:

$$0 = \underline{d}_2(1\cap x) = \partial_2[1]\cap x + 1\cap d_2(x).$$

The theorem is now proved by the fact that the class $\Omega_q(R,G)$ is the opposite of the characteristic class $\Theta_q(R,G)$.

5 Example

In this section, we study the case when G is the fundamental group of a Riemann surface Σ of genus g. Recall that G has the following presentation

$$G = \{\alpha_1, \alpha_2, ..., \alpha_g : \prod_{j=1}^{g}[\alpha_{2j-1}, \alpha_{2j}]\}$$

where $(\alpha_i)_i$ are the generators, and $\prod_{j=1}^{g}[\alpha_{2j-1},\alpha_{2j}]$ is the relation. The surface groups are torsion free, and have homological dimension equal to two. This appires in to $\mathbb{Z}^2$ the fundamental group of the 2-torus. This example give an extension of that given in [Lor] for the case of an infinite cyclic group $\mathbb{Z}$. The characteristic classes explain the contribution of the relation in the presentation of G, when studying the Hochschild homology of a G-graded algebra. Let $F =< \alpha_1,\alpha_2,...,\alpha_g >$ be the free group with the same generators as G, and A_F the associated F-graded algebra.

Proposition 5.0.2 *Let A be a G-graded algebra, where G is the fundamental group of a Riemann surface. Then we have an exact sequence*

$$... \to H_2(G,HH_{n-1}(R)) \xrightarrow{\partial} HH_n(A_F) \to HH_n(A) \to H_2(G,HH_{n-2}(R)) \to ...$$

where the bondary operators is given by the characteristic classes

$$\Theta_q(A,G) \in Ext^2_G(HH_q(R),HH_{q+1}(R)\Big) \simeq H^2(G,Hom(HH_q(R),HH_{q+1}(R))\Big)$$

Proof

Let $E^2_{*,*}(G)$ be the spectral sequence of the theorem (4.2) associated to G. Because the homological dimension of the group G is equal to two, the spectral sequence $E^2_{*,*}(G)$ is concentrated in three column $p = 0,1,2$. The first two columns correspond to the spectral sequence associated to A_F, which we are noted by $E^2_{*,*}(F)$. So we have an exact sequence of complexes

$$0 \to (E^2_{*,*}(F), d_2 = 0) \to (E^2_{*,*}(G), d_2) \to (E^2_{*,*}(G)[2], d_2 = 0) \to 0.$$

We denote by $\Im^*(G)$ (resp, $\Im^*(F)$) the filtration on $HH_*(A)$. (resp $HH_*(A_F)$). First, we have the following short exact sequence deduced from the convergence:

$$(1) \qquad 0 \to \Im^0(F) \to \Im^1(F) \to \Im^1(F)/\Im^0(F) \to 0$$

$$(2) \qquad 0 \to \Im^0(G) \to \Im^1(G) \to \Im^1(G)/\Im^0(G) \to 0$$

$$(3). \qquad 0 \to \Im^1(G) \to \Im^2(G) \to \Im^2(G)/\Im^1(G) \to 0$$

We also have the following identifications:

$$\Im^2(G) = HH_n(A) \quad ; \quad \Im^1(F) = HH_n(A_F)$$

$$\Im^0(G) = E^\infty_{0,n}(G) = E^3_{0,n}(G) = Cokerd_2$$
$$\Im^0(F) = E^\infty_{0,n}(F) = E^2_{0,n}(F) = E^2_{0,n}(G)$$
$$\Im^2(G)/\Im^1(G) = E^\infty_{2,n-2}(G) = E^3_{2,n-2}(G) = kerd_2$$
$$\Im^1(G)/\Im^0(G) = E^\infty_{1,n-1}(G) = E^3_{1,n-1}(G) = E^2_{1,n-1}(G)$$
$$\Im^1(F)/\Im^0(F) = E^\infty_{1,n-1}(F) = E^3_{1,n-1}(F) = E^2_{1,n-1}(G)$$

these give the following exact sequences:

$((1) \Rightarrow (1'))$ $$0 \to E^2_{0,n}(G) \to HH_n(A_F) \to E^2_{1,n-1}(G) \to 0$$

$((2) \Rightarrow (2'))$ $$0 \to Cokerd_2 \to \Im^1(G) \to E^2_{1,n-1}(G) \to 0$$

$((3) \Rightarrow (3'))$. $$0 \to \Im^1(G) \to HH_n(A) \to kerd_2 \to 0$$

The exact sequences (1') and (2) give the following commutative diagram

$$\begin{tikzcd}
 & E^2_{0,n}(G) \arrow[d, "d_2"] & \\
0 \to E^2_{0,n}(G) \arrow[r] \arrow[d] & HH_n(A_F) \arrow[r] \arrow[d] & E^2_{1,n-1}(G) \to 0 \arrow[d, "\wr"] \\
0 \to cokerneld_2 \arrow[r] & \Im^1(G) \arrow[r] \arrow[d] & E^2_{1,n-1}(G) \to 0 \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

Then therefore have an exact sequence

(4). $$0 \to Imd_2 \to HH_n(A_F) \to \Im^1(G) \to 0$$

Now the long exact sequence :

$$\to H_2(G, HH_{n-1}(R)) \to HH_n(A_F) \to HH_n(A) \to H_2(G, HH_{n-2}(R)) \to$$

is given by the sequences (1'), (3') and (4). The last part of the proposition results from the fact that ∂ is the differantial d_2, which was already calculated in the theorem (4.0.2).

The Hochschild homology of A_F is given by the following sequence, which generalises the example given in [Lor]:

Proposition 5.0.3 *There is an exact sequence*

$$... \to HH_{*-1}(A_F) \to \oplus_{i\in I} HH_*(A) \xrightarrow{\partial} HH_*(A) \to HH_*(A_F)) \to ..$$

With $\partial(\oplus_{i\in I} c_i) = \oplus_{i\in I}(c_i - \alpha_{*i}(c_i))$ *which we may identifiated with the characteristic class* $\Theta_*(A,F) \in H^1(F, Ext(HH_*(R), HH_{*+1}(R)))$

Proof.

We give the proof for $F = \mathbb{Z}$, and is easily extented to the general case. The spectral sequence of theorem (4.0.1), takes the following form

$$E^2_{*,*} = H_*(\mathbb{Z}, HH_*(R)) \Longrightarrow HH_*(A_Z).$$

In this case we have just two columns $p = 0, 1$ the only no trivial terms being:

$$E^2_{0,*} = H_0(\mathbb{Z}, HH_*(R) = HH_*(R)_{\mathbb{Z}}$$

$$E^2_{1,*} = H_1(\mathbb{Z}, HH_*(R)) = HH_*(R)^{\mathbb{Z}}$$

and $E^2 = E^\infty$. The isomorphism

$$Ext^2_{\mathbb{Z}}(HH_*(R), HH_{*+1}(R)) \cong H^1(\mathbb{Z}, Ext(HH_*(R), HH_{*+1}(R)))$$

is given by the proposition (3.2.1).

References

[B] **H. Bass** *Algebraic K-theory* Benjamin 1968.

[C] **A. Connes** *Non-commutative Geometry* Academic Press 1995.

[CE] **H. Cartan-S. Eilenberg.** *Homological Algebra* Princeton U. P Princeton (1956).

[Cor] **J. Cornick.** *On the Homology of group graded algebra.* Journal of Algebras 174, (1995) pp.999-1023

[FT] **B.L. Feigin-B.L. Tsygan .** Additive K-theory L.N.M 1289 (1987).

[F] **E. Fieux .** *Classes caractéristiques dune π-algbre et suite spectrale en K-theorie bivariante.* K-Theory 5 , pp: 71-96 (1991).

[L] **A. Legrand** *Caractérisation des opérations d'algèbres sur les modules differentiels*,Compositio Math 66 (1988) pp. 23-36.

[Lo] **J. L. Loday** *Cyclic Homology.* Springer Verlag (1992)

[Lor] **M. Lorenz** *On the homology of graded algebra* 20(2), (1992) pp 489-507

[M] **S. Maclane.** *Homology.* Berlin-Gottingen-Heidelberg Springer (1963).

[NO] **C. Nastasescu and F. Van Oystaeyen.** *Graded Ring Theory.* Math. Library 28, North Holland, Amsterdam. (1982)

[Y] **N. Yoneda .** *On the homotopy theory of modules.* Jour. Fac. Sci. Uni Tokyo (sect. 1) 7 pp:193-227 (1954).