Zeeman's monotonicity conjecture

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Abstract
In this paper we prove a conjecture of Zeeman [1] about the monotonicity of the rotation number of a family of diffeomorphisms φ of the first quadrant Q of $\mathbf{R}^2$.

1 The formulation

Consider Zeeman's family of maps

$$\varphi : Q \to Q : (x,y) \to (x',y') = (y, \frac{y+a}{x}). \tag{1}$$

In order to formulate Zeeman's conjecture we first state some simple properties of φ.

1. φ is invertible with inverse $y = x'$ and $x = \dfrac{x'+a}{y'}$.
2. $\varphi = I \circ J$, where I and J are the involutions
$$I : Q \to Q : (x,y) \to (y,x)$$
$$J : Q \to Q : (x,y) \to \Big(\frac{y+a}{x}, y\Big).$$
3. φ preserves the 2-form $\sigma = \dfrac{dx \wedge dy}{xy}$.
4. The function
$$H(x,y) = \frac{(x+1)(y+1)(x+y+a)}{xy} \tag{2}$$
is an integral of φ. In other words, $H(\varphi(x,y)) = H(x,y)$ for every $(x,y) \in Q$. Thus φ maps the level set $H^{-1}(h)$ into itself for every h.

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Now look at the Hamiltonian system on (Q, σ) corresponding to the Hamiltonian function H (2). Let Φ be the flow of the Hamiltonian vector field X_H whose integral curves are the solutions of

$$\begin{aligned} \dot{x} = & \; xy\,\frac{\partial H}{\partial y} = (x+1)\Big(y - \frac{x+a}{y}\Big) \\ \dot{y} = & -xy\,\frac{\partial H}{\partial x} = -(y+1)\Big(x - \frac{y+a}{x}\Big). \end{aligned} \tag{3}$$

Because H is a Morse function with a unique nondegenerate minimum corresponding to the value $h_{\min} = (1+w)^3/w$ with $w = \frac{1}{2}(1+\sqrt{1+4a})$, every level set $H^{-1}(h)$ with $h > h_{\min}$ is diffeomorphic to a circle. Thus every orbit of X_H of energy greater than $h_{\min}$ is periodic of period $T(h)$. As Zeeman shows [1, p.1, 14-16], the map $\varphi|H^{-1}(h)$ is smoothly conjugate to a rotation through an angle which depends smoothly on h. Therefore, the time $\tau(h)$ it takes an integral curve of X_H starting at $(x,y) \in H^{-1}(h)$ to reach $\varphi(x,y) \in H^{-1}(h)$ does not depend on the starting point (x,y) but only on the value h. Hence the rotation number $\rho(h)$ of the map $\varphi|H^{-1}(h)$ is $\tau(h)/T(h)$. In [1, p.7] Zeeman conjectures the following

THEOREM: The function

$$\rho : [h_{\min}, \infty) \to \mathbf{R} : h \to \frac{\tau(h)}{T(h)} \tag{4}$$

is real analytic and is strictly increasing if $0 < a < 1$ and strictly decreasing if $1 < a < \infty$.

In this paper we will prove Zeeman's conjecture.

2 The argument

Our argument goes as follows. More details are given in §3.

STEP 1. First we show that the period $T(h)$ is a real analytic function on $(h_{\min}, \infty)$. Observe that

$$\begin{aligned} T(h) &= \int_{H^{-1}(h)} \frac{1}{xy\,\frac{\partial H}{\partial y}}\,dx = \int_{H^{-1}(h)} \frac{1}{(x+1)\Big(y - \frac{x+a}{y}\Big)}\,dx, \quad \text{using (3)} \\ &= \int_{H^{-1}(h)} \frac{dx}{\frac{\partial f}{\partial y}}, \end{aligned}$$

where

$$f(x,y) = (x+1)(y+1)(x+y+a) - hxy = 0. \tag{5}$$

Thinking of x and y as being complex variables, (5) defines family of affine elliptic curves $\mathcal{E}$. The closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ in complex projective 2-space $\mathbf{C}P^2$ is defined by

$$F(x,y,z) = (x+z)(y+z)(x+y+az) - hxyz = 0. \tag{6}$$

$\overline{\mathcal{E}}$ is obtained from $\mathcal{E}$ by adding the three points $P_1 = (1, -1, 0)$, $P_2 = (1, 0, 0)$ and $P_3 = (0, 1, 0)$ at infinity. Because $\overline{\mathcal{E}}$ is nonsingular, it is diffeomorphic to a 2-torus for each fixed value of h. The 1-form $\omega = \frac{dx}{\frac{\partial f}{\partial y}}$ is holomorphic on $\overline{\mathcal{E}}$.

STEP 2. We find an integral expression for the function τ. To do this we need to understand the involutions I and J on $\overline{\mathcal{E}}$. As Zeeman shows [1, p.15], I and J extend to involutions

$$\begin{aligned} \overline{I} &: \mathbf{C}P^2 \to \mathbf{C}P^2 : (x, y, z) \to (y, x, z) \\ \overline{J} &: \mathbf{C}P^2 \to \mathbf{C}P^2 : (x, y, z) \to (z(y + az), xy, xz), \end{aligned} \tag{7}$$

which preserve $\overline{\mathcal{E}}$. By a general theorem (see for instance Walker [2, p.195]), every involution on $\overline{\mathcal{E}}$ is of the form $S_P(q) = P - q$ for some fixed point $P \in \overline{\mathcal{E}}$. Here $\overline{\mathcal{E}}$ is considered to be an abelian group with addition $+$. (For a geometric definition of $+$ see Brieskorn-Knörrer [3, p.307]). Clearly $S_P(0) = P$. The best choice of 0 (which leads to the simplest addition law on $\overline{\mathcal{E}}$) is to take 0 to be an inflection point of $\overline{\mathcal{E}}$ at infinity, namely $0 = P_1$. Since $\overline{I}(0) = 0$, we see that $\overline{I} = S_0$. Similarly, since $\overline{J}(0) = P_3$, we obtain $\overline{J} = S_{P_3}$. Therefore $\varphi(q) = (S_0 \circ S_{P_3})(q) = q - P_3 = q + P_2$, since $P_2 = S_0(P_3) = -P_3$. Let γ_{0P_2} be the negatively oriented curve segment on $\mathcal{E}$ joining 0 to $P_2 = \varphi(0)$. The time it takes an integral curve of X_H starting at 0 to reach P_2 is

$$\tau(h) = \int_{\gamma_{0P_2}} \omega. \tag{8}$$

Thus the rotation number $\rho(h)$ of $\varphi | H^{-1}(h)$ is

$$\rho(h) = \frac{\tau(h)}{T(h)} = \frac{\int_{\gamma_{0P_2}} \omega}{\int_{\gamma} \omega}. \tag{9}$$

Using (9) we have computed $\rho(h)$ numerically and it agrees with the calculations of Zeeman.

In order to bring the integrals for $\tau(h)$ and $T(h)$ into a form which is useful for further computations, we apply the invertible linear change of variables

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} h - a & -2 & 1 \\ h - a & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \\ u \end{pmatrix}, \tag{10}$$

which induces an isomorphism between $\overline{\mathcal{E}}$ and the projective curve $\overline{\mathcal{F}}$ defined by

$$4s^2 u - \Big(u^3 + 2(h - a + 2) r u^2 + ((h - a + 2)^2 + 4h) r^2 u + 4h(h - a + 1) r^3 \Big) = 0. \tag{11}$$

Note that the affine curve $\mathcal{F}$, obtained by taking $u = 1$ in $\overline{\mathcal{F}}$, is a family of elliptic curves in standard form

$$(2s)^2 = \Big(4h(h - a + 1) r^3 + ((h - a + 2)^2 + 4h) r^2 + 2(h - a + 2) r + 1 \Big) = g(r, h). \tag{12}$$

A calculation shows that

1. $\widetilde{L}^*\omega = \frac{dr}{2s}$, where $\widetilde{L}$ is the linear fractional transformation from $\mathcal{F}$ to $\mathcal{E}$

$$x = \frac{1}{2r}\Big((h-a)r - 2s + 1\Big) \qquad y = \frac{1}{2r}\Big((h-a)r + 2s + 1\Big), \tag{13}$$

   which is induced from L (10).

2. The map φ becomes the map $\varphi' = L^{-1}\circ\varphi\circ L$ on $\overline{\mathcal{F}}$, which has the same rotation number as φ.

3. The involutions S_0 and S_{P_3} on $\overline{\mathcal{E}}$ become the involutions $S_{0'}$ and $S_{P_3'}$ on $\overline{\mathcal{F}}$, where $0' = L^{-1}(0) = (0,1,0)$ and $P_3' = L^{-1}(P_3) = (0, \frac{1}{2}, 1)$. Therefore for every $q' \in \overline{\mathcal{F}}$, $\varphi'(q') = q' + P_2'$, where $P_2' = (0, -\frac{1}{2}, 1)$.

4. The negatively oriented curves γ and γ_{0P_2} on $\overline{\mathcal{E}}$ become the positively oriented curves γ' and $\gamma_{0'P_2'}$ on $\overline{\mathcal{F}}$.

Thus the rotation number $\rho(h)$ of φ' on $\overline{\mathcal{F}}$ is

$$\rho(h) = \frac{\int_{\gamma'_{0'P_2'}} \Omega}{\int_{\gamma'} \Omega} = \frac{\int_{\gamma'_{P_3'0'}} \Omega}{\int_{\gamma'} \Omega} = \frac{\int_0^\infty \Omega}{\int_\Gamma \Omega}.$$

Here $\Omega = \frac{dr}{2s} = \frac{dr}{\sqrt{g(r,h)}}$ and Γ is taken to be any positively oriented closed curve in the extended complex plane which encloses the largest real root of g and ∞. Because Γ is locally independent of h, $\int_\Gamma \Omega$ is a real analytic function of h. Since $g(r,h)$ is real analytic, $\int_0^\infty \frac{dr}{\sqrt{g(r,h)}}$ is a real analytic. Therefore the rotation number ρ is a real analytic of h.

STEP 3. We now find the Picard-Fuchs operator $\mathcal{L}$ associated to the 1-form Ω on $\overline{\mathcal{F}}$ (for more background, see Brieskorn-Knörrer [3, p. 656]). A computation done using MAPLE shows that

$$\mathcal{L}\,\Omega = \Big(\frac{d^2}{dh^2} + p(h)\frac{d}{dh} + q(h)\Big)\Omega = -d\Big(\frac{v(r,h)}{(2s)^3}\Big). \tag{14}$$

Here p, q are rational functions in h and v is a rational function in r and h given by

$$\begin{aligned}
p(h) &= \Big(3\,a^6 - 18\,h\,a^5 - 30\,a^5 + 123\,a^4 + 42\,h\,a^4 + 12\,h^2\,a^4 - 6\,h\,a^3 - 10\,h^2\,a^3 \\
&\quad + 18\,h^3\,a^3 - 264\,a^3 + 312\,a^2 + 36\,h\,a^2 - 203\,h^2\,a^2 + 126\,h^3\,a^2 \\
&\quad - 15\,h^4\,a^2 - 138\,h\,a + 224\,h^2\,a - 34\,h^3\,a - 192\,a + 48 + 84\,h - 18\,h^2\Big)/\Delta \\
q(h) &= -\Big(3\,a^5 - 12\,a^4 - 9\,h\,a^4 + 21\,a^3 + h^2\,a^3 + 5\,h^3\,a^2 + 51\,h\,a^2 - 28\,h^2\,a^2 \\
&\quad - 30\,a^2 + 30\,a + 12\,h^2\,a - 48\,h\,a - 12 + 6\,h\Big)/\Delta \\
v(r,h) &= -\Big(v_0 + v_1\,r + v_2\,r^2 + v_3 r^3 + v_4\,r^4\Big)/\Delta
\end{aligned}$$

where

$$
\begin{aligned}
v_0 &= a\Big(-2-2\,h\,a^2+7\,a+h+h\,a-7\,a^2+2\,a^3\Big)\\
v_1 &= -\Big(84\,a^3-12+42\,a-48\,h\,a+6\,h-78\,a^2+27\,h\,a^2+9\,h^2\,a\\
&\quad -45\,a^4-31\,h^2\,a^2+36\,h\,a^3+5\,h^3\,a^2+7\,h^2\,a^3-21\,h\,a^4+9\,a^5\Big)\\
v_2 &= \Big(12\,a^6-39\,h\,a^5-75\,a^5+159\,h\,a^4+37\,h^2\,a^4+180\,a^4-133\,h^2\,a^3\\
&\quad -123\,h\,a^3-222\,a^3-5\,h^3\,a^3-180\,h\,a^2+177\,a^2-15\,h^3\,a^2-5\,h^4\,a^2\\
&\quad +177\,h^2\,a^2-10\,h^3\,a+225\,h\,a-40\,h^2\,a-108\,a-42\,h+36-6\,h^2\Big)\\
v_3 &= -\Big(-295\,a^3+24-164\,a-344\,h\,a+54\,h^2+96\,h\\
&\quad +346\,a^2+244\,h\,a^2-39\,h^2\,a+66\,a^4-240\,h^2\,a^2+232\,h\,a^3\\
&\quad +90\,h^3\,a-90\,h^3\,a^2+397\,h^2\,a^3-356\,h\,a^4+49\,a^5+45\,h^4\,a^2\\
&\quad +60\,h^3\,a^3-222\,h^2\,a^4+148\,h\,a^5-31\,a^6-20\,h^3\,a^4+5\,h^4\,a^3\\
&\quad +30\,h^2\,a^5-20\,a^6\,h+5\,a^7\Big)\\
v_4 &= 2\Big(576\,a^3-48+240\,a+129\,h\,a-60\,h^2-54\,h-504\,a^2-45\,h\,a^2\\
&\quad +101\,h^2\,a-387\,a^4-21\,h^2\,a^2-123\,h\,a^3-100\,h^3\,a+135\,h^3\,a^2\\
&\quad -88\,h^2\,a^3+144\,h\,a^4+153\,a^5-50\,h^4\,a^2-90\,h^3\,a^3+105\,h^2\,a^4-60\,h\,a^5\\
&\quad -33\,a^6+35\,h^3\,a^4-10\,h^4\,a^3-37\,h^2\,a^5+9\,a^6\,h+3\,a^7\Big)
\end{aligned}
$$

and

$$\Delta = h(-h-1+a)\,(5\,h\,a+6-9\,a+3\,a^2)(a^3-2\,h\,a^2-6\,a^2+h^2\,a+12\,a-10\,h\,a-8+h).$$

Note that $v/(2s)^3$ vanishes at $r=\infty$, because as polynomials in r the degree of v^2 is less than the degree of $(2s)^6$. Using Picard-Fuchs operator $\mathcal{L}$ (14) we obtain

$$\mathcal{L}\,T(h) = \int_{\gamma'} \mathcal{L}\,\Omega \;=\; 0, \tag{15}$$

because γ' is a closed curve. In addition,

$$\mathcal{L}\,\tau(h) = \int_0^\infty \Omega \;=\; -\int_0^\infty d\left(\frac{v(r,h)}{(2s)^3}\right) \;=\; v(0,h). \tag{16}$$

The last equality above follows because $v/(2s)^3$ vanishes at $r=\infty$ and $2s(0,h)=1$. Let $v(h)=v(0,h)$. From the explicit expression for $v(r,h)$ it follows that $v(h)$ is the rational function

$$-\Big(a(a-1)\,(2\,a^2-5\,a-2\,h\,a-h+2)\Big)/\Delta. \tag{17}$$

Step 4. We now show that the rotation number ρ (4) is monotonic. Since

$$\frac{d\rho}{dh} = \frac{d}{dh}\left(\frac{\tau(h)}{T(h)}\right) \;=\; \frac{\ell(h)}{(T(h))^2},$$

where $\ell(h) = \det \begin{pmatrix} \frac{d\,\tau(h)}{dh} & \frac{d\,T(h)}{dh} \\ \tau(h) & T(h) \end{pmatrix}$, we need only determine the sign of $\ell(h)$. Note that for every $h > h_{\min}$ the value $T(h)$ is positive, since it is the period of the closed orbit $H^{-1}(h)$ of the Hamiltonian vector field X_H. Using (15) and (16), a straightforward calculation shows that ℓ satisfies the differential equation

$$\frac{d\ell}{dh} = -p(h)\,\ell + v(h)\,T(h). \tag{18}$$

Let Γ and $\widetilde{\Gamma}$ be a homology basis for the Riemann surface of $\overline{\mathcal{F}}$. Then $y_1(h) = \int_\Gamma \Omega$ and $y_2(h) = \int_{\widetilde{\Gamma}} \Omega$ form a fundamental system of solutions of the homogeneous Picard-Fuchs equation $\mathcal{L}y = 0$. Their Wronskian $W = \det \begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dh} & \frac{dy_2}{dh} \end{pmatrix}$ satisfies the differential equation

$$\frac{dW}{dh} = -p(h)\,W. \tag{19}$$

W can be computed up to a multiplicative nonvanishing function $C = C(a)$ by integrating (19) using the partial fraction decomposition of $p(h)$. A calculation gives

$$W(h) = C\,\frac{(5\,h\,a + 3\,a^2 - 9\,a + 6)}{h\,(h + 1 - a)\Big((a^3 - 6\,a^2 + 12\,a - 8) - (2\,a^2 + 10\,a - 1)h + ah^2\Big)} \tag{20}$$

Note that $h_{\min}$ is a root of the third factor in the denominator of W. From (18) and (20) we obtain

$$\frac{d}{dh}\left(\frac{\ell}{W}\right) = \frac{v(h)}{W(h)}\,T(h). \tag{21}$$

Observe that (21) is also valid for $\ell/(\frac{1}{C}W)$. Thus we may assume that $C = 1$ in (20). Integrating (21) from $h_{\min}$ to h gives

$$\frac{\ell(h)}{W(h)} = \frac{\ell(h_{\min})}{W(h_{\min})} + \int_{h_{\min}}^{h} \frac{v(h)}{W(h)}\,T(h)\,dh. \tag{22}$$

For all h near $h_{\min}$, the function ℓ is real analytic, while at $h_{\min}$ the Wronskian W is infinite. Hence the first term on the right hand side of (22) is zero. Using (17) and (20), a calculation shows that

$$\frac{v(h)}{W(h)} = -\frac{a(a-1)(2a+1)\Big(h - \frac{(2a^2-5a-5)}{2a+1}\Big)}{(3\,a^2 - 9\,a + 6 + 5\,h\,a)^2}.$$

But $h_{\min} > \dfrac{(2a^2 - 5a - 5)}{2a+1}$ for every $a > 0$. Therefore, for $h > h_{\min}$ the function $v(h)/W(h)$ is positive when $0 < a < 1$ and negative when $a > 1$. Since $T(h)$ and $W(h)$ are positive when $h > h_{\min}$ and $a > 0$, we deduce that for $h > h_{\min}$ the function $\ell(h)$ is positive when $0 < a < 1$ and negative when $a > 1$. This proves Zeeman's conjecture about the rotation number of the map φ.

3 More details

Here we provide most of the missing details for the argument given in §2.

We begin by giving the derivation of the change of variables L (10). First we rotate and stretch the x-y coordinate axes by

$$\begin{cases} x &= \xi - \eta \\ y &= \xi + \eta. \end{cases} \tag{23}$$

The elliptic curve $\mathcal{E}$ defined by (5) is now defined by

$$(2\xi + a - h)\eta^2 = (2\xi + 1)(\xi + 1)^2 - h\xi^2. \tag{24}$$

Let $\zeta = 2\xi + a - h$. Then (24) becomes

$$4\zeta\eta^2 = (\zeta + h)(\zeta + h - a + 2)^2 - h(\zeta + h - a). \tag{25}$$

Finally, setting

$$r = 1/\zeta \quad \text{and} \quad s = \eta/\zeta, \tag{26}$$

equation (25) becomes the standard form (12). Combining all these variable changes together gives (13) from which we obtain formula (10) for L.

Next we discuss how we obtained the Picard-Fuchs operator $\mathcal{L}$ (14). Let $g = g(r, h)$ be the polynomial of degree three in r which is the right hand side of (12). We want to find a rational function $v(r, h)$ and rational functions $p(h)$ and $q(h)$ such that

$$\left(\frac{d^2}{dh^2} + p(h)\frac{d}{dh} + q(h)\right)\left(\frac{dr}{g^{1/2}}\right) = -d\left(\frac{v(r,h)}{g^{3/2}}\right). \tag{27}$$

Carrying out the differentiations in (27) (with $' = \frac{d}{dh}$) we obtain

$$\tfrac{3}{4}(g')^2 - \tfrac{1}{2}g\,g'' - \tfrac{1}{2}p\,gg' + q\,g^2 - \tfrac{5}{2}g'v + \frac{dv}{dr}g = 0, \tag{28}$$

after multiplying by $g^{5/2}$. We assume that $v(r, h)$ is a polynomial of degree 4 in r and treat its coefficients as unknowns together with p and q. Because g is a polynomial of degree 3 in r, the left hand side of (28) is a polynomial of degree 6 in r with seven coefficients which depend linearly on seven unknowns. Using MAPLE we solve these linear equations to obtain the Picard-Fuchs operator $\mathcal{L}$.

We now verify the inequalities

$$h_{\min} > \frac{(2a^2 - 5a - 5)}{2a + 1} \tag{29}$$

and

$$W(h) > 0 \quad \text{when } h > h_{\min}, \tag{30}$$

which we used in §2 to prove the monotonicity of the rotation number. First recall that $a > 0$. We now show that

$$h_{\min} > a + 2. \tag{31}$$

Using $h_{\min} = \dfrac{(3+\sqrt{1+4a})^3}{4(1+\sqrt{1+4a})}$, a straightforward calculation shows that

$$4(1+\sqrt{1+4a})(h_{\min} - a - 2) = 28 + 32a + 20\sqrt{1+4a} > 0.$$

This proves (31). To prove (29) note that

$$\begin{aligned} (2a+1)h_{\min} &> 2a^2 + 5a + 2, \quad \text{using (31)} \\ &> 2a^2 - 5a - 5. \end{aligned}$$

To show that

$$W(h) = \frac{5ha + 3a^2 - 9a + 6}{h(h+1-a)\Big((a^3 - 6a^2 + 12a - 8) - (2a^2 + 10a - 1)h + ah^2\Big)}$$

is positive when $h > h_{\min}$, we verify that each factor in the numerator and denominator of $W(h)$ is positive when $h > h_{\min}$. The positivity of the numerator follows because

$$\begin{aligned} 5ah + 3a^2 - 9a + 6 &> 5ah_{\min} + 3a^2 - 9a + 6 \\ &> 8a^2 + a + 6, \quad \text{using (31)} \\ &> 0. \end{aligned}$$

Second, the positivity the denominator of $W(h)$ follows because for the first factor we have $h > h_{\min} > a + 2$; while for the second factor we have $h + 1 - a > h_{\min} + 1 - a > 3$, using (31); the third factor can be written as $a(h - h_+)(h - h_-)$ where

$$h_\pm = \Big((2a^2 - 10a - 1) \pm (4a+1)^{3/2}\Big)/(2a).$$

But $h_{\min} = h_+ > h_-$. Therefore the third factor is positive when $h > h_{\min}$. Hence $W(h)$ is positive when $h > h_{\min}$.

4 Acknowledgements

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