PROOF OF A CONJECTURE OF A. PITTS

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1. Introduction. In this paper we prove a conjecture of Andrew Pitts [12], which states that the Beck-Chevalley condition holds for lax pullbacks (or "comma squares") of coherent toposes (see Theorem 2 below).

Pitts' conjecture was put forward as a way towards the lax descent theorem for coherent toposes (Theorem 1 below). The latter entails a dual version for pretoposes, which was eventually established by Zawadowski [13] in the setting of Makkai's elaborate theory of Stone duality [8,9,10] . Our results therefore furnish a proof of the lax descent theorem for pretoposes along the lines originally conceived by Pitts. As explained in Zawadowski's paper, this theorem can be interpreted as a very general definability result for coherent logic.

Perhaps surprisingly, our proof of Pitts' conjecture needs only simple properties of inverse limits and localization of coherent toposes, which are all (at least implicitly) contained in [1]. We have tried to give an accesssible presentation of these properties in the first sections of this paper. Moreover, our arguments are completely constructive, and valid over an arbitrary base topos.

We would like to point out that, independently, yet another proof has recently been given of the descent theorem for pretoposes by David Ballard and Bill Boshuck [3]. This elegant proof also uses methods of model theory, and seems unrelated to our approach.

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§1 Coherent toposes and statement of the main theorem

2. Preliminaries on coherent toposes. We begin by briefly recalling the basic definitions concerning coherent toposes and morphisms ([1], see also [4,11,7]).

A topos $\mathcal{E}$ is *coherent* if $\mathcal{E}$ is (equivalent to) the category of sheaves on a *finitary* site, i.e. a site with finite limits all of whose covering families are finite. Given a coherent topos $\mathcal{E}$, there is always a canonical such site, viz. the full subcategory (pretopos) of *coherent* ([4, 7.3.1]) objects with the evident topology of finite epimorphic families. (Recall also that any pretopos arises in this way, as the category of coherent objects in a coherent topos.) Coherent toposes are exactly those toposes which arise as classifying toposes of finitary geometric logic [11].

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A morphism $f\colon \mathcal{F} \to \mathcal{E}$ between coherent toposes is said to be *coherent* if f^* sends coherent objects to coherent objects. This is the case if and only if f is induced by a morphism of finitary sites. For such an f, the direct image f_* commutes with filtered colimits. If $f\colon \mathcal{F} \to \mathcal{E}$ is *surjective* then f^* also *reflects* coherence, in the sense that an object E in $\mathcal{E}$ is coherent whenever $f^*(E)$ is coherent in $\mathcal{F}$. Recall also that if $\mathcal{F} \to \mathcal{E}$ and $\mathcal{G} \to \mathcal{E}$ are coherent morphisms, then the pullback $\mathcal{F} \times_{\mathcal{E}} \mathcal{G}$ is a coherent topos and the projections are coherent morphisms.

3. Lax pullbacks. The lax pullback or "comma square" of two topos morphisms $f\colon \mathcal{F} \to \mathcal{E}$ and $g\colon \mathcal{G} \to \mathcal{E}$ is a universal square

$$\begin{tikzcd} \mathcal{H} \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{E} \end{tikzcd},$$

which commutes up to a (not necessarily invertible) 2-cell $\tau\colon gv \Rightarrow fu$ (i.e., a natural transformation $\tau\colon v^*g^* \Rightarrow u^*f^*$). Such lax pullbacks always exist and are unique up to equivalence. We shall denote the lax pullback by "$\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}$" (suppressing f and g from the notation). If f and g are coherent morphisms between coherent toposes, then the lax pullback $\mathcal{H}$ and the morphisms u and v are again coherent (as is evident from any of the wellknown constructions of $\mathcal{H}$, e.g. in terms of classifying toposes).

4. Lax descent. For a morphism $f\colon \mathcal{F} \to \mathcal{E}$ of toposes, one can construct iterated pullbacks to form a "universal diagram"

$$\begin{tikzcd} \mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & \mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathcal{F} \arrow[r, "f"] & \mathcal{E} \end{tikzcd},$$

with 2-cells $\tau\colon fd_0 \Rightarrow fd_1$, etc. *Lax descent data* on an object $F \in \mathcal{F}$ consists of a morphism $\theta\colon {d_0}^*(F) \to {d_1}^*(F)$ satisfying the obvious unit and cocycle conditions (analogous to the "non-lax" case — see [6]). The natural transformation $\tau\colon fd_0 \Rightarrow fd_1$ defines a functor from $\mathcal{E}$ to the category $\mathrm{LDes}(f)$ of objects of $\mathcal{F}$ equipped with such descent data. If this functor is an equivalence of categories, one says that f is of *lax effective descent.* A primary consequence of Pitts' conjecture is

Theorem 1. *Any coherent surjection between coherent toposes is of lax effective descent.*

Since, as said, coherent morphisms "reflect" coherence of objects, the descent property implies that f^* restricts to an equivalence of pretoposes, from the category $\mathrm{Coh}(\mathcal{E})$ of coherent objects of $\mathcal{E}$ to the category of objects in $\mathrm{Coh}(\mathcal{F})$ equipped with descent data. In other words, Theorem 1 restricts to a theorem about pretoposes. It is this latter result which was originally proved by Zawadowski [13].

5. Tripleability and descent. The main result to be proved in this paper is

Theorem 2. *Consider a lax pullback of coherent toposes and coherent morphisms*

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E} .
\end{tikzcd}$$

(i) *("Beck-Chevalley condition") The transformation $\hat{\tau}: g^* f_* \Rightarrow {d_0}_* {d_1}^*$ induced by τ is an isomorphism. Moreover*

(ii) *The morphism d_0 renders $\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}$ a coherent topos relative to $\mathcal{G}$ (see 6 below).*

Theorem 1 is an immediate consequence of Theorem 2 (i) for the case where $g = f$, by a well-known standard argument. Indeed, if $f: \mathcal{F} \to \mathcal{E}$ is a surjection, then it follows immediately from Beck's tripleability theorem that $\mathcal{E}$ is (equivalent to) the category of coalgebras for the comonad $f^* f_*$ on $\mathcal{F}$ ([4, 4.15(ii)], [7, VII Proposition 4.4]). Furthermore, by a classical result due to Bénabou and Roubaud [2], if the Beck-Chevalley condition holds then coalgebra structures $F \to f^* f_*(F)$ translate via the isomorphism $\hat{\tau}: f^* f_* \cong {d_0}_* {d_1}^*$ and the adjunction between ${d_0}_*$ and ${d_0}^*$ to descent data $\theta: {d_0}^*(F) \to {d_1}^*(F)$. Theorem 1 thus follows by composing these two well-known equivalences $\mathcal{E} \cong$ Coalgebras and Coalgebras $\cong$ LDes(f).

§2 Relative coherence

6. Relative coherence. The definitions concerning coherence obviously make sense over an arbitrary base topos $\mathcal{S}$. Thus, an $\mathcal{S}$-topos $\mathcal{E} \to \mathcal{S}$ is said to be coherent *over* (or, *relative to*) $\mathcal{S}$ if $\mathcal{E}$ is (equivalent to) the category of $\mathcal{S}$-internal sheaves on a finitary site in $\mathcal{S}$. Similarly, the definition of "coherent morphism" can be relativized to morphisms of $\mathcal{S}$-toposes. We remark that a morphism $f: \mathcal{F} \to \mathcal{E}$ between coherent toposes (over Sets) is coherent whenever $\mathcal{F}$ is coherent as an $\mathcal{E}$-topos, but not conversely.

7. Internal sheaves. Let $\mathbb{C}$ be a finitary site in a base topos $\mathcal{S}$. Then for any morphism ("base extension") $a: \mathcal{S}' \to \mathcal{S}$, the structure $a^*(\mathbb{C})$ is again a finitary site. (It is at this point that the finiteness of the covers makes such finitary internal sites easy to handle: for general sites, $a^*(\mathbb{C})$ does not satisfy the transitivity axiom for Grothendieck topologies, and is only a "basis" for a topology.) Moreover, again by finiteness of the covers, the notion of an internal *sheaf* E on $\mathbb{C}$ can be expressed by finite limits hence by geometric formulas. In particular, if E is a sheaf, so is $a^*(E)$. This can be expressed more explicitly as follows. Write $\mathrm{Sh}_{\mathcal{S}}(\mathbb{C})$ for the $\mathcal{S}$-topos of internal sheaves, and similarly $\mathrm{Sh}_{\mathcal{S}'}(a^*(\mathbb{C}))$, so as to get a pullback diagram

$$\begin{tikzcd}
\mathrm{Sh}_{\mathcal{S}'}(a^*(\mathbb{C})) \arrow[r, "b"] \arrow[d, "\gamma'"'] & \mathrm{Sh}_{\mathcal{S}}(\mathbb{C}) \arrow[d, "\gamma"] \\
\mathcal{S}' \arrow[r, "a"] & \mathcal{S}\,.
\end{tikzcd}$$

Then $b^*(E)$ is simply constructed by applying a^* to E and its structure maps ($E \to \mathbb{C}_0$ and $E \times_{\mathbb{C}_0} \mathbb{C}_1 \to E$), and no sheafification is needed. In particular, if $x \in \mathbb{C}_0$ is a (generalised) element of $\mathbb{C}_0$ in $\mathcal{S}$, then the sections of $b^*(E)$ over $a^*(x) \in a^*(\mathbb{C}_0)$ are described by

$$b^*(E)(a^*(x)) = a^*(E(x)). \tag{1}$$

The following lemma is an immediate consequence of this observation.

Lemma 1 ("change of base"). *Consider a pullback diagram*

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "f'"] \arrow[d, "c"'] & \mathcal{E}' \arrow[r, "\gamma'"] \arrow[d, "b"] & \mathcal{S}' \arrow[d, "a"] \\
\mathcal{F} \arrow[r, "f"] & \mathcal{E} \arrow[r, "\gamma"] & \mathcal{S}\,.
\end{tikzcd}$$

If f is a coherent morphism between coherent toposes $\mathcal{E}$ and $\mathcal{F}$ over $\mathcal{S}$, then the same is true for f', $\mathcal{E}'$ and $\mathcal{F}'$ relative to $\mathcal{S}'$. Moreover, the squares satisfy the Beck-Chevalley condition (e.g., ${f'}_* c^* = b^* f_*$ for the left-hand one).

PROOF. If f is induced by a morphism $T\colon \mathbb{C} \to \mathbb{D}$ between finitary internal (in $\mathcal{S}$) sites $\mathbb{C}$ and $\mathbb{D}$ for $\mathcal{E}$ and $\mathcal{F}$, then f' is induced by the morphism $T' = a^*(T)\colon a^*(\mathbb{C}) \to a^*(\mathbb{D})$. Thus, the first assertion is evident. The Beck-Chevalley condition follows immediately from (1). Indeed, if $F \in \mathcal{F}$ is any sheaf on $\mathbb{D}$ and $x \in \mathbb{C}_0$ is any (generalised) element of $\mathbb{C}_0$, then

$$\begin{aligned}
b^* f_* F(a^* x) &= a^*((f_*(F))(x)) && \text{(by (1))}\\
&= a^*(F(Tx)) && \text{(definition of } f_*)\\
&= (c^*F)(a^*(Tx)) && \text{(by (1))}\\
&= (c^*F)(T'(a^*x) && \text{(definition of } T')\\
&= {f'}_*(c^*F)(a^*x) && \text{(definition of } {f'}_*).
\end{aligned}$$

Since this holds for any such x, we find that the canonical map $b^* f_*(F) \to {f'}_* c^*(F)$ is an isomorphism. ∎

§3 Inverse limits and localization

We review some essentially known facts concerning the notions in the title of this section.

8. Inverse limits. We recall the construction of filtered inverse limits [1]. If $\{\mathcal{E}_i\}$ is a filtered inverse system of coherent toposes with bonding maps $f_{ij}\colon \mathcal{E}_i \to \mathcal{E}_j$, its inverse limit $\mathcal{E} = \varprojlim \mathcal{E}_i$ is again a coherent topos and the projections are coherent morphisms. This is immediate from the construction of $\mathcal{E}$: the inverse image functors ${f_{ij}}^*$ restrict to pretopos morphisms $F_{ij}\colon \mathrm{Coh}(\mathcal{E}_j) \to \mathrm{Coh}(\mathcal{E}_i)$. Let $\mathcal{C}$ be the (pseudo-)colimit of this directed system of pretoposes. Then $\mathcal{C}$ is again a pretopos, and $\mathcal{E} = \mathrm{Sh}(\mathcal{C})$.

Lemma 2. *Let $\mathcal{E} = \varprojlim_I \mathcal{E}_i$ be as above.*

(i) *For any object E_i in $\mathcal{E}_i$, the canonical map*

$$\varinjlim_k {f_{ki}}_* {f_{ki}}^*(E_i) \to {\pi_i}_* {\pi_i}^*(E_i)$$

is an isomorphism.

(ii) *For any object $E \in \mathcal{E}$, the canonical map*

$$\varinjlim_i {\pi_i}^* {\pi_i}_*(E) \to E$$

is an isomorphism.

Proof. Let $\mathcal{C}_i = \mathrm{Coh}(\mathcal{E}_i)$ and $\mathcal{C} = \varinjlim \mathcal{C}_i$ be the finitary sites of coherent objects for $\mathcal{E}_i$ and $\mathcal{E}$ respectively. Then $\pi_i\colon \mathcal{E} \to \mathcal{E}_i$ is induced by the canonical morphism of sites $\eta_i\colon \mathcal{C}_i \to \mathcal{C}$ in the standard way ([7, VII Theorem 10.2]). In particular, ${\pi_i}^*$ is "compose with η_i" while ${\pi_i}^*$ is given by

$$ {\pi_i}^*(E_i)(\eta_k(C_k)) = \varinjlim_{j \geq i,k} {f_{ji}}^*(E)({f_{jk}}^*(C_k)) \tag{1}$$

(for any object $\eta_k(C_k)$ of $\mathcal{C}$ — here and below we use "$j \geq i, j$" to indicate that j ranges over the "double comma category" $I/i,k$ with objects of the form $i \leftarrow j \to k$). Property (i) is immediate from (1), while (ii) follows by an easy calculation:

$$\begin{aligned}
(\varinjlim_i {\pi_i}^*{\pi_i}_*(E))(\eta_k(C_k)) &= \varinjlim_i ({\pi_i}^*{\pi_i}_*(E)(\eta_k(C_k))) \\
&= \varinjlim_i \varinjlim_{j \geq i,k} {f_{ji}}^*{\pi_i}_*(E)({f_{jk}}^*(C_k)) \quad \text{(by (1))} \\
&= \varinjlim_{i \geq k} {\pi_i}_*(E)({f_{ik}}^*(C_k)) \\
&\qquad \text{("$i = j$ is cofinal" by directedness)} \\
&= \varinjlim_{i \geq k} E(\eta_i {f_{ik}}^*(C_k)) \\
&= \varinjlim_{i \geq k} E(\eta_k(C_k)) \\
&= E(\eta_k(C_k)). \quad \blacksquare
\end{aligned}$$

Lemma 3. *Let $\{\mathcal{E}_i, f_{ij}\}$ and $\{\mathcal{F}_i, g_{ij}\}$ be inverse systems as above, and let $\tau_i\colon \mathcal{F}_i \to \mathcal{E}_i$ be a natural system of coherent maps, inducing a coherent morphism $\tau\colon \mathcal{F} \to \mathcal{E}$. If each* *of the left-hand squares below satisfies the Beck-Chevalley condition ($f_{ij}{}^*\tau_{j*} = \tau_{i*}g_{ij}{}^*$), then so does each limit square on the right ($\pi_i{}^*\tau_{i*} = \tau_*\rho_i{}^*$):*

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "g_{ij}"] \arrow[d, "\tau_i"'] & \mathcal{F}_j \arrow[d, "\tau_j"] \\ \mathcal{E}_i \arrow[r, "f_{ij}"] & \mathcal{E}_j \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F} \arrow[r, "\rho_i"] \arrow[d, "\tau"'] & \mathcal{F}_i \arrow[d, "\tau_i"] \\ \mathcal{E} \arrow[r, "\pi_i"] & \mathcal{E}_i \end{tikzcd} .$$

Proof. Fix i, and again write $j \geq i$ to indicate that j ranges over I/i. By Lemma 2 (ii), it suffices to show that for any $j \geq i$,

$$\pi_{j*}\pi_i{}^*\tau_{i*} = \pi_{j*}\tau_*\rho_i{}^*.$$

But

$$\begin{aligned} \pi_{j*}\pi_i{}^*\tau_{i*} &= \pi_{j*}\pi_i{}^* f_{ji}{}^*\tau_{i*} & \\ &= \varinjlim_k f_{kj*}f_{kj}{}^* f_{ji}{}^*\tau_{i*} & \text{(Lemma 2(i))} \\ &= \varinjlim_k \tau_{j*}g_{kj*}g_{kj}{}^*g_{ji}{}^* & \text{(assumption)} \\ &= \tau_{j*}(\varinjlim_k g_{kj*}g_{kj}{}^*g_{ji}{}^*) & (\tau_j \text{ coherent}) \\ &= \tau_{j*}\rho_{j*}\rho_i{}^* & \text{(Lemma 2(i))} \\ &= \pi_{j*}\tau_{j*}\rho_i{}^*. \quad \blacksquare & \end{aligned}$$

9. Localization (see [1,5]). Recall that for a coherent topos $\mathcal{E}$ and a point p of $\mathcal{E}$, a *neighbourhood* of p is pair (U, x) where $U \in \mathcal{E}$ and $x \in p^*(U)$. We write $\mathcal{N}(p)$ for the category of these neighbourhoods. The full subcategory given by pairs (U, x) where U is coherent is cofinal, and will also simply be denoted by $\mathcal{N}(p)$. The *localization* of $\mathcal{E}$ at p is the inverse limit

$$\mathrm{Loc}_p(\mathcal{E}) = \varprojlim_{(U,x)\in\mathcal{N}(p)} \mathcal{E}/U.$$

(Note that if U is coherent then $\mathcal{E}/U$ is again a coherent topos.) Clearly $\mathrm{Loc}_p(\mathcal{E})$ is again a coherent topos, and the projections $\pi_{(U,x)}\colon \mathrm{Loc}_p(\mathcal{E}) \to \mathcal{E}/U$ are coherent morphisms.

Lemma 4. *Let $f\colon \mathcal{F} \to \mathcal{E}$ be a coherent map between coherent toposes. Then in the pullback square*

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\rho"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathrm{Loc}_p(\mathcal{E}) \arrow[r, "\pi"] & \mathcal{E} \end{tikzcd}$$

all toposes and maps are coherent, and the Beck-Chevalley $\pi^*f_* = g_*\rho^*$ holds.

Proof. Immediate from Lemma 3 and the fact that the Beck-Chevalley condition always holds for the pullback along a slice map $\mathcal{E}/U \to \mathcal{E}$. $\blacksquare$

Using the notions of relative coherence from §2, it is clear that these properties of inverse limits and localization hold over and arbitrary base topos $\mathcal{S}$. As a particular case, we mention localization at the generic point:

10. Universal localization (see [5, p296]). Any localization is the pullback of the "universal" localization at the generic point. To be more explicit, consider any $\mathcal{S}$-topos $\mathcal{E}$. After change of base along $\mathcal{E} \to \mathcal{S}$ itself, the $\mathcal{E}$-topos $\pi_1 \colon \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \to \mathcal{E}$ has a point, viz. the diagonal δ. The localization $\mathrm{Loc}_\delta(\mathcal{E} \times_{\mathcal{S}} \mathcal{E} \to \mathcal{E}) = \mathcal{L}$ with its two maps $d_0, d_1 \colon \mathcal{L} \rightrightarrows \mathcal{E}$ is the cotensor $\mathbf{2} \pitchfork \mathcal{E}$. That is, the square

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{E} \arrow[d, "\mathrm{id}"] \\
\mathcal{E} \arrow[r, "\mathrm{id}"] & \mathcal{E}
\end{tikzcd}$$

is a lax pullback. Moreover, this lax pullback satisfies the Beck-Chevalley condition (${d_0}_* {d_1}^* \cong \mathrm{id}$ in this case) because ${d_0}_* = \Delta^*$ where $\Delta \colon \mathcal{E} \to \mathcal{L}$ is the "diagonal".

§4 Conclusion

We shall now collect the previous auxiliary results together and derive Theorem 2 in a completely formal way.

11. Proof of Theorem 2. First observe that the lax pullback of Theorem 2 (like any lax pullback) can be constructed in stages, as in the diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[r] \arrow[dd, "u"'] \arrow[ddr, phantom, "(4)"] & \mathcal{K} \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "(3)"] & \mathcal{E} \times \mathcal{F} \arrow[r] \arrow[d, "\mathrm{id} \times f"] \arrow[dr, phantom, "(2)"] & \mathcal{F} \arrow[d, "f"] \\
 & \mathcal{L} \arrow[r, "{(d_0, d_1)}"] \arrow[d, "d_0"'] & \mathcal{E} \times \mathcal{E} \arrow[r] \arrow[dl, dashed, "\pi_1"] \arrow[dr, phantom, "(1)"] & \mathcal{E} \arrow[d, "\mathrm{id}"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E} \arrow[rr, "\mathrm{id}"] & & \mathcal{E} \, .
\end{tikzcd}$$

Here the rectangle (1) (ignoring the dotted arrow) is a lax pullback (see 10) while (2), (3) and (4) are pullbacks.

We first consider coherence. To begin with, $d_0 \colon \mathcal{L} \to \mathcal{E}$ is coherent relative to $\mathcal{E}$, because $\mathcal{L}$ (as an $\mathcal{E}$-topos via d_0) is the localization of the coherent $\mathcal{E}$-topos ($\pi_1 \colon \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \to \mathcal{E}$) as explained in 10. Next, for square (2), note that $\mathrm{id} \times f$ is coherent over $\mathcal{E}$ since $f \colon \mathcal{F} \to \mathcal{E}$ is coherent over the base topos $\mathcal{S}$ (Lemma 1). Now pullback (3) is an instance of Lemma 4 (over the base $\mathcal{E}$ by the dotted arrow), so $\mathcal{K}$ and f' are coherent over $\mathcal{E}$.

But then the composite $d_0 f'$ is again coherent over $\mathcal{E}$, and hence (Lemma 1 again) its pullback u is coherent over $\mathcal{G}$.

Next, to see that the outer square satisfies the Beck-Chevalley condition, it is sufficient to prove that each square does so separately. For square (1) this was observed in paragraph 10, for (2) it is an instance of Lemma 2, for (3) of Lemma 3 and, finally, for (4) it is again an instance of Lemma 1. ∎

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