On some properties of the difference spectrum

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Abstract. We consider problems arising from trying to lift the non-synthesis spectrum of a closed set in a quotient of a locally compact, non-metrizable, abelian group. In particular, we get contrasting results for the difference spectra of linear combinations of elements of a regular Banach function algebra, depending on whether the Gelfand space is metrizable or not.

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1. Conventions and Notation

Let A be a commutative, semisimple, regular Banach algebra with Gelfand space X. We shall identify A with its Gelfand representation, that is, we shall think of A as a Banach algebra of continuous functions (vanishing at infinity) on the locally compact space X. We shall assume that the functions in A with compact supports are dense in A. Our standard example is the Fourier algebra $A = A(G) = \mathcal{F}L^1(\Gamma)$, where G is a locally compact abelian group and Γ is its dual group. In practice we often take G compact (hence Γ discrete). As our emphasis is on local phenomena, this is not really a loss of generality.

We now introduce notation that will be used throughout. Consider a topological space X and a point $x \in X$. We use the symbols $=_x$, $\in_x$, $\subset_x$, ... to denote equality, belonging, inclusion, ..., locally at x. Thus, for functions f and g defined on X, we say that f and g are *locally equal at* x, and we write $f =_x g$, if there is a neighbourhood U of x such that $f|_U = g|_U$; similarly, two subsets E and F of X are called *locally equal* at x (which we denote by $E =_x F$) if $E \cap U = F \cap U$ for some neighbourhood U of x. We observe that $E =_x X$ if (and only if) x is an interior point of E, while $E =_x \emptyset$ if $x \notin \overline{E}$, and $\emptyset \neq_x E \neq_x X$ if $x \in \partial E$. Note that the set of points where two functions, or two subsets, are equal, is always open.

Suppose next that Φ is a collection of functions defined on X and that f is a

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function defined on X. We say that f belongs to Φ, locally at x, and we write $f \in_x \Phi$, if there is a function $g \in \Phi$ such that $f =_x g$. The set

$$\Delta(f, \Phi) := \{x \in X \mid f \notin_x \Phi\}$$

will be called the *spectrum* of f with respect to Φ. The spectrum is always a *closed* set. In particular, the spectrum of f with respect to $O := \{0\}$ is the support of f (in the usual sense); thus

$$\mathrm{supp} f = \Delta(f, O) = \{x \in X \mid f \neq_x 0\}\,.$$

Further, if Φ_1 and Φ_2 are two collections of functions defined on X, then we say that Φ_1 is contained in Φ_2, locally at x, and we write $\Phi_1 \subset_x \Phi_2$, if for each $f \in \Phi_1$ we have $f \in_x \Phi_2$. The two collections are called locally equal at x if $\Phi_1 \subset_x \Phi_2$ and $\Phi_2 \subset_x \Phi_1$.

If the space X is locally compact, then, in the same way, we define symbols $=_\infty$, $\in_\infty$, $\subset_\infty$, ..., to indicate validity outside some compact subset of X. For instance, $f =_\infty 0$ if and only if f has compact support.

We shall have occasion to use a more general form of the above notation. For an arbitrary subset Y of X we shall use symbols $=_Y$, $\in_Y$, $\subset_Y$, ..., to denote validity of the relation concerned in a neighbourhood of Y, and we shall say that the relation holds locally at Y. It is evident that if a relation holds locally at some subset Y of X, then, a fortiori, the same relation holds at any subset of Y. In the opposite direction we have the following: if $Y \subset X$ and if $f =_x g$ for all $x \in Y$, then $f =_Y g$, and similarly for local equality of two sets E and F. However, if $f \in_Y \Phi$ and $f \in_Z \Phi$ (f a function, Φ a collection of functions), it does not follow that $f \in_{Y\cup Z} \Phi$, nor does $\Phi_1 \subset_Y \Phi_2$ and $\Phi_1 \subset_Z \Phi_2$ imply $\Phi_1 \subset_{Y\cup Z} \Phi_2$. For instance, suppose that X consists of two points. We can then identify the real-valued functions on X with $\mathbb{R}^2$. Now take $\Phi_1 = \{(1,0),(0,1)\}$ and $\Phi_2 = \{(0,0),(1,0),(0,1),(1,1)\}$. Then $\Phi_1 =_x \Phi_2$ for all $x \in X$, yet $\Phi_1 =_X \Phi_2$ does not hold.

Finally, we shall use $\neq_x$, $\notin_x$, $\not\subset_x$, ..., to denote that the relation in question does *not* hold, locally at x, and similarly for a subset Y instead of a point x.

2. The difference spectrum

Now let A be a Banach algebra as described in Section 1. In [2] we gave the following definition: the *difference spectrum* of two (not necessarily closed) ideals I_1 and I_2 in A is the set $\Delta(I_1, I_2)$ of all $x \in X$ where I_1 and I_2 are locally different; thus

$$\Delta(I_1, I_2) = \{x \in X \mid I_1 \neq_x I_2\}\,.$$

If the locally compact space X is not compact, we also define the *extended difference spectrum* $\Delta_\infty(I_1, I_2)$ to be the set

$$\Delta_\infty(I_1, I_2) = \{x \in X \cup \{\infty\} \mid I_1 \neq_x I_2\}\,.$$

A partition-of-unity argument gives $I_1 = I_2$ if (and only if) $\Delta_\infty(I_1, I_2) = \emptyset$ ([1, p.19]). When I_1 and I_2 are closed, we already have $I_1 = I_2$ if $\Delta(I_1, I_2) = \emptyset$ (since the functions in A with compact supports are dense, by assumption).

In a completely analogous way the following result is proved. (Note that the assumption that I be an ideal is essential, in view of the remarks towards the end of Section 1.)

Proposition 2.1. *Let A and X be as before, and let Y be a compact subset of X. Let I be an ideal in A, and let f be a function, defined on X. If $f \in_x I$ for all $x \in Y$, then $f \in_Y I$.*

Let I be an ideal in A. Then the *hull* of I (also called the *zero set* or the *cospectrum* of I) is the set $\mathrm{hull}(I) = \Delta(I, A) = \{x \in X \mid I \neq_x A\}$.

Let E be a closed subset of X. The *kernel* of E is the ideal

$$I_E = \{f \in A \mid f|_E = 0\}.$$

It is the largest ideal with hull E. The smallest ideal with hull E is the ideal

$$j_E = \{f \in A \mid f =_{E \cup \{\infty\}} 0\}.$$

The closure of j_E is denoted by J_E; it is the smallest *closed* ideal with hull E. As is well-known, E is called a *set of (spectral) synthesis* if $J_E = I_E$; otherwise E is called a *set of non-synthesis*. We denote the difference spectrum $\Delta(J_E, I_E)$ by Δ_E, and we call this set the *spectrum of non-synthesis* of E.

Note that $\Delta_E \subset \partial E$ and that $\Delta_E = \emptyset$ if and only if E is a set of synthesis.

We also use the following terminology: the *spectrum* and the *cospectrum* of an ideal I are defined as the difference spectra of I and A and of I and $O = \{0\}$, respectively; thus

$$\mathrm{cosp}(I) = \Delta(I, A) = \{x \in X \mid I \neq_x A\},$$

$$\mathrm{spec}(I) = \Delta(O, I) = \{x \in X \mid O \neq_x I\},$$

In [1, p.125] the spectrum of an ideal is defined as the *closure* of the above set $\mathrm{spec}(I)$; our definition above seems more natural. For instance, the spectrum of j_E is the open set $X \setminus E$.

In the special case that X is metrizable, the following holds (cf. [2, p. 532]): if I_1 is closed, then $\{x \in X \mid I_1 \subset_x I_2)$ is open. As a consequence we have the following result.

Proposition 2.2. *Let A and X be as before. Suppose that X is metrizable. Then:*

(i) *if I_1, I_2 are closed ideals of A, then $\Delta(I_1, I_2)$ is closed;*
(ii) *the cospectrum (the hull) of any ideal is closed;*
(iii) *the spectrum of any closed ideal is closed.*

Proof. $\Delta(I_1, I_2)$ is the complement of the intersection of two open sets. $\Delta(I, A)$ and $\Delta(O, I)$ are the complements of $\{x \in X \mid A \subset_x I\}$ and $\{x \in X \mid I \subset_x O\}$, respectively. $\square$

Now consider the case where X is non-metrizable. In [2] we have proved the following:

Theorem 2.3. *Let G be a non-metrizable compact abelian group (thus Γ is a discrete uncountable group).*

(i) *There is a set of non-synthesis E such that its spectrum of non-synthesis Δ_E is not closed.*

(ii) *The set $\Delta(O, I)$ (the spectrum of I) is closed whenever I is a closed ideal.*

From Section 1 we recall that the spectrum $\Delta(f, I)$ of an individual function $f \in A$ with respect to an arbitrary ideal I of A is closed, even if X is non-metrizable.

3. Linear combinations of functions

Let A and X be as before. Consider a compact non-synthesis subset E of X. For $f \in I_E$ we shall denote the spectrum $\Delta(f, J_E)$ of f with respect to J_E, by Δ_f, or by Δ_f^E, if it is necessary to specify the set E. For two functions $f, g \in I_E$ it is obvious that $\Delta_{f+g} \subset \Delta_f \cup \Delta_g$. It is also clear that equality need not hold in general: it may happen that part of the 'bad behaviour' of f is neutralized by that of g. If this happens, we can avoid it by adding to f, not g itself, but a suitable multiple of g. The multiple has to be chosen in such a way that no 'neutralization of bad behaviour' occurs at some other part of the spectrum. The following lemma shows that suitable multiples are always available when X is metrizable.

Lemma 3.1. *Suppose that the Gelfand space X of the Banach algebra A is metrizable. Let E be a compact set of non-synthesis in X. Consider two functions $f, g \in I_E \setminus J_E$. Then there are at most countably many $\alpha \in \mathbb{C}$ such that $\Delta_{f+\alpha g} \neq \Delta_f \cup \Delta_g$.*

Proof. Clearly $\Delta_{f+\alpha g} \subset \Delta_f \cup \Delta_g$, any $\alpha \in \mathbb{C}$.

Consider the sets U_α, $\alpha \in \mathbb{C}$, defined by $U_\alpha = (\Delta_f \cup \Delta_g) \setminus \Delta_{f+\alpha g}$. Suppose that $x \in U_\alpha \cap U_\beta$. Then $f + \alpha g \in_x J_E$ and $f + \beta g \in_x J_E$, hence (subtract!) $(\alpha - \beta)g \in_x J_E$ and, similarly, $(\beta - \alpha)f \in_x J_E$. If $\alpha \neq \beta$, this implies that $x \notin \Delta_f \cup \Delta_g$, which gives a contradiction. We conclude that the sets U_α ($\alpha \in \mathbb{C}$) are disjoint.

It now follows from the metrizability of X that there can be at most countably many α's for which $U_\alpha \neq \emptyset$. Indeed, we can fix a metric d giving the topology on X. The sets U_α are relatively open in $\Delta_f \cup \Delta_g$, hence for a point $x_\alpha \in U_\alpha$ there exists $\delta_\alpha > 0$ such that, for $y \in \Delta_f \cup \Delta_g$, we have $d(x_\alpha, y) < \delta_\alpha \Rightarrow y \in U_\alpha$. If there were uncountably many non-empty U_α, there would be a $\delta > 0$ such that $\delta \leq \delta_\alpha$ for infinitely many distinct values of α, giving an infinite set of points x_α with no accumulation point in the compact set $\Delta_f \cup \Delta_g$, which is impossible. □

Our aim is to prove that in the case of a metrizable Gelfand space the result of the lemma can be extended to infinite linear combinations. But first we need another lemma.

Lemma 3.2. *Let A and X be as before. Suppose that X is metrizable. Let E be a compact subset of X of non-synthesis with spectrum of non-synthesis Δ_E. Then there exists a sequence of functions $(\tau_k)_{k\geq 1}$ in A such that for every $x \in \Delta_E$, every neighbourhood U of x and every $n \geq 1$ there is a $k > n$ such that $\tau_k =_x 1$ and $\tau_k =_y 0$ for every $y \notin U$.*

Proof. Since $\Delta_E \subset E$ is metrizable and compact, we may select a countable collection of open sets U_n, $n \in \mathbb{N}$, in X, such that $U_n \cap \Delta_E$ is a basis of open sets in the relative topology on Δ_E, each $\overline{U}_n$ is compact and for each $x \in U_n$, there is an m such that $x \in U_m \subset \overline{U}_m \subset U_n$. It follows from the regularity of A that, given any pair U_m, U_n such that $\overline{U}_m \subset U_n$, there exists a function $\tau_{m,n} \in A$ such that $\tau_{m,n} = 1$ on U_m and $\tau_{m,n} = 0$ outside U_n. The collection of such $\tau_{m,n}$ is countable and we relable it as τ_k, $k \in \mathbb{N}$, in such a way that each $\tau_{m,n}$ occurs infinitely often. The sequence τ_k now has the required properties. $\square$

Theorem 3.3. *Suppose that the Gelfand space X of the Banach function algebra A is metrizable. Let E be a compact non-synthesis subset of X. Let functions $f_j \in I_E \setminus J_E$ $(j \geq 1)$ be given. Then there exist $\alpha_j \in \mathbb{C}$ $(j \geq 1)$ such that $g := \sum_{j=1}^{\infty} \alpha_j f_j$ exists and*

$$\Delta_g = \overline{\bigcup_{j=1}^{\infty} \Delta_{f_j}}\,.$$

Proof. As in the proof of Lemma 3.1, one inclusion is obvious. As the set Δ_g is closed, it therefore suffices to show that for every $n \geq 1$ one has $\Delta_{f_n} \subset \Delta_g$. We can further assume, without loss of generality, that $||f_j|| = 1$ for all $j \geq 1$.

Take a sequence $(\tau_j)_{j\geq 1}$ as in Lemma 3.2. We shall choose the numbers α_k $(k \geq 1)$ inductively, and we shall denote the partial sums $\sum_{j=1}^{k} \alpha_j f_j$ by g_k. Take $\alpha_1 = 1$. Suppose that for some $k \geq 2$ the numbers $\alpha_1, \ldots, \alpha_{k-1}$ have been chosen. Take $\alpha_k \in \mathbb{C} \setminus \{0\}$ such that

(i) $\Delta_{g_k} = \bigcup_{j=1}^{k} \Delta_{f_j}$,

(ii) $|\alpha_k| \leq |\alpha_{k-1}|/2$,

(iii) if $g_{k-1}\tau_k \notin J_E$, then $|\alpha_k| < \dfrac{\mathrm{dist}(g_{k-1}\tau_k, J_E)}{2||\tau_k||}$,

where, for $f \in A$ and $\Phi \subset A$, we define $\mathrm{dist}(f, \Phi) = \inf\{||f - g|| \mid g \in \Phi\}$. It follows from Lemma 3.1 that (i) can be satisfied for arbitrarily small values of α. Therefore a choice as above is always possible.

Now take any $x \in \Delta_{f_n}$ and suppose that $x \notin \Delta_g$. From (i) it follows that $x \in \Delta_{g_n}$. Write $U = X \setminus \Delta_g$. By Lemma 3.2, there is a $k > n$ such that $\tau_k =_x 1$ and $\tau_k =_y 0$ for every $y \notin U$. If $y \in U$ then $y \notin \Delta_g$, hence $g \in_y J_E$, hence also $g\tau_k \in_y J_E$. If $y \notin U$ then $g\tau_k =_y 0 \in J_E$. It follows that $\Delta_{g\tau_k} = \emptyset$, hence $g\tau_k \in J_E$.

On the other hand, $\Delta_{f_n} \subset \Delta_{g_{k-1}}$ (by (i)), so $g_{k-1}\tau_k =_x g_{k-1} \notin_x J_E$, and hence $g_{k-1}\tau_k \notin J_E$. We have

$$||g\tau_k - g_{k-1}\tau_k|| = \left\| \sum_{j=k}^{\infty} \alpha_j\, f_j\, \tau_k \right\| \leq \sum_{j=k}^{\infty} |\alpha_j|\, ||\tau_k|| \leq 2\, |\alpha_k|\, ||\tau_k|| < \operatorname{dist}(g_{k-1}\tau_k, J_E)\,,$$

from (ii) and (iii) above. It follows that $g\tau_k \notin J_E$, a contradiction. □

We now give an example showing that the metrizability of the space X is crucial.

Proposition 3.4. *There exists a compact non-metrizable abelian group G (i.e., the dual group Γ is discrete and uncountable), a closed set $E \subset G$ of non-synthesis, and functions $f, g \in I_E \setminus J_E$ such that $\Delta_{f+\alpha g} \neq \Delta_f \cup \Delta_g$ for all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$.*

Proof. Denote the set $\{\alpha \in \mathbb{C} \mid |\alpha| \leq 1\}$ by D. We need the following three ingredients:

(i) A function $\varphi \in A(\mathbb{T}^2)$ with $\operatorname{range}(\varphi) = D$: for example $\varphi(x, y) = \mathrm{e}^{2\pi \mathrm{i} x} \cos 2\pi y$.

(ii) A set of non-synthesis $F \subset \mathbb{T}$ and a function $\psi \in I_F \setminus J_F$; we may assume that $\Delta_\psi^F = \Delta_F = F$.

(iii) A non-metrizable compact abelian group H. We take

$$H = \prod_{(x,y)\in\mathbb{T}^2} \{0,1\} = \left\{ (\varepsilon_{x,y}) \mid \varepsilon_{x,y} \in \{0,1\} \quad ((x,y) \in \mathbb{T}^2) \right\}\,.$$

(the corners of an uncountable-dimensional cube). The zero element is $\underline{0} = (0)_{x,y}$. The elements $p_{x,y} = (\varepsilon_{x',y'})$ with $\varepsilon_{x,y} = 1$, $\varepsilon_{x',y'} = 0$ otherwise, constitute the set C_1 of all those corners of the cube which are *adjacent* to $\underline{0}$.

Take $G = \mathbb{T}^2 \times \mathbb{T} \times H$, and consider the subset $\widetilde{F} = \{(0,0)\} \times F \times \{\underline{0}\}$ of G. By the injection theorem for sets of synthesis ([1, p.149]) $\widetilde{F}$ is a subset of non-synthesis of G.

For each $(x, y) \in \mathbb{T}^2$ we now consider the set $E_{x,y} = \{(x, y)\} \times F \times \{p_{x,y}\} \subset G$. All $E_{x,y}$ are translates of $\widetilde{F}$, hence they are sets of non-synthesis in G as well. Let E be the closure of the union of these sets. Then it is not hard to see that

$$E = \left(\bigcup_{(x,y)\in\mathbb{T}^2} \{(x,y)\} \times F \times \{p_{x,y}\} \right) \cup \left(\mathbb{T}^2 \times F \times \{\underline{0}\} \right)\,.$$

Now define $f, g \in I_E$ by $f(x, y, z, t) = -\varphi(x, y)\psi(z)$ and $g(x, y, z, t) = \psi(z)$, where $(x, y, z) \in \mathbb{T}^3$ and $t \in H$. It is easily seen that $\Delta_g = E$ and that $\Delta_f \subset \Delta_g$, thus $\Delta_f \cup \Delta_g = \Delta_g$; here $\Delta = \Delta^E$. Moreover, we have $(f + \alpha g)(x, y, z, t) = (\alpha - \varphi(x, y))\psi(z)$. For (x, y) such that $\varphi(x, y) = \alpha$ the function $f + \alpha g$ vanishes on $\{(x, y)\} \times \mathbb{T} \times \{p_{x,y}\}$, a set of synthesis. Therefore, $\Delta_{f+ag} \cap E_{x,y} = \emptyset$ (when $\varphi(x, y) = \alpha$), and hence $\Delta_{f+ag} \neq \Delta_f \cup \Delta_g$. □

4. Lifting the spectrum of non-synthesis

Let G be a compact abelian group and let H be a closed subgroup. We write π for the canonical mapping $\pi : G \to G/H$. Let E be a closed subset of G/H. In [1, p.145] it is shown that E is a set of non-synthesis for G/H if and only if $\pi^{-1}E$ is a set of non-synthesis for $A(G)$. Does this result extend to the non-synthesis spectrum?

It is not difficult to adapt Reiter's method to show that $\Delta_{\pi^{-1}E} \supset \pi^{-1}\Delta_E$ [2]. But trying to prove the opposite inclusion meets with difficulties, which we now describe. (It was an attempt to circumvent those difficulties which led to the results of the previous section.)

First we discuss a natural way to write any function $f \in A(G)$ as a sum of functions in $A(G)$, each of which is the product of a character by a function that is constant on cosets of H.

Lemma 4.1. *For every $f \in A(G)$ there is a countable (possibly finite) set $R \subset \Gamma$ such that each coset of $H^\perp$ contains at most one element of R and such that for each $\chi \in R$ there is a function $f_\chi \in A(G/H)$ such that*

$$f = \sum_{\chi \in R} \chi \cdot f_\chi \circ \pi .$$

Proof. For $f \in A(G)$, we have $f(x) = \sum_{\gamma \in \Gamma} \widehat{f}(\gamma)\langle x, \gamma\rangle$ with $\sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)| < \infty$. Consider the cosets of $H^\perp$ in Γ. At most countably many of them contain at least one element γ for which $\widehat{f}(\gamma) \neq 0$. Take a set $R \subset \Gamma$ such that each of these cosets contains exactly one element of R (such a set R is called a *set of representatives* for these cosets). Then

$$f(x) = \sum_{\chi\in R} \sum_{\gamma \in H^\perp} \widehat{f}(\gamma+\chi)\langle x, \gamma+\chi\rangle = \sum_{\chi\in R} \langle x, \chi\rangle \sum_{\gamma\in H^\perp} \widehat{f}(\gamma+\chi)\langle \pi(x), \gamma\rangle .$$

Identifying $H^\perp$ with the dual of G/H, and defining, for arbitrary $\chi \in \Gamma$, functions $f_\chi \in A(G/H)$ by $f_\chi = \sum_{\gamma\in H^\perp} \widehat{f}(\gamma+\chi)\,\gamma$, we obtain $f = \sum_{\chi\in R} \chi \cdot f_\chi \circ \pi$, as desired.□

Now take $f \in I_{\pi^{-1}E}$. Consider any $y \in \pi^{-1}E$. We have then, for arbitrary $z \in H$, $y + z \in \pi^{-1}E$ as well. Writing f as in Lemma 4.1 we obtain, for arbitrary $z \in H$,

$$0 = f(y+z) = \sum_{\chi\in R} \left(\langle y, \chi\rangle \cdot f_\chi(\pi(y))\right) \langle z, \chi\rangle .$$

This is the Fourier series for the function $z \mapsto f(y+z)$, which in this case is the zero function. Therefore all its Fourier coefficients are zero. We conclude that the functions f_χ in the above proof satisfy $f_\chi(\pi(y)) = 0$ for all $y \in \pi^{-1}E$: thus $f_\chi \in I_E$.

For $w \notin \Delta_E$ we know that, for all $\chi \in R$, there exists a neighbourhood U_χ of w such that $f_\chi \in_{U_\chi} J_E$. In the case where G/H is metrizable, we can choose U_χ independently of χ and conclude the proof. We do not know how to do this for the general non-metrizable case. On the other hand, if we knew it failed in a specific case, then we would have an example where $\Delta_{\pi^{-1}E} \neq \pi^{-1}\Delta_E$, as the following result shows.

Proposition 4.2. *Let G_0 be a compact abelian group and let E be a closed subset of G_0. Let $w \in E$ be such that there is a sequence $(f_n)_{n\geq 1}$ in I_E such that for all neighbourhoods U of w there is an n such that $f_n \notin_U J_E$. Then there is an element $t \in \mathbb{T}$ such that $(w,t) \in \Delta_{E\times\mathbb{T}}$.*

Proof. Suppose there exists no $t \in \mathbb{T}$ such that $(w,t) \in \Delta_{E\times\mathbb{T}}$. We may assume that $\|f_n\|_{A(G_0)} = 2^{-n}$ $(n \geq 1)$. Let $f \in I_{E\times\mathbb{T}}$ be defined by

$$f(x,t) = \sum_{n=1}^{\infty} f_n(x)e^{int},$$

for $x \in G_0$ and $t \in \mathbb{T} \cong [-\pi, \pi[$.

Then, for each $t \in \mathbb{T}$ there exist a function $g_t \in J_{E\times\mathbb{T}}$ and open neighbourhoods U_t of w in G_0 and V_t of t in $\mathbb{T}$ such that $f = g_t$ on $U_t \times V_t$. Now choose a finite subcover $V_{t_1}, \ldots V_{t_n}$ of $\mathbb{T}$: we denote the intersection of the corresponding U_{t_j} by U, and we write g_k instead of g_{t_k}.

Let $W_1, \ldots, W_n$ be an open cover of $\mathbb{T}$ such that $W_k \subset \overline{W}_k \subset V_{t_k}$, for $k = 1, \ldots, n$. Let σ_k, for $k = 1, \ldots, n$, be functions in $A(\mathbb{T})$ such that $\sigma_k(t) \geq 0$ for all t, $\sigma_k(t) > 0$ when $t \in \overline{W}_k$ and $\operatorname{supp}\sigma_k \subset V_{t_k}$. Define $\widetilde{\sigma}_k \in A(G_0 \times \mathbb{T})$ by $\widetilde{\sigma}_k(x,t) = \sigma_k(t)$. Set

$$g = (\sum_{k=1}^{n} \widetilde{\sigma}_k g_k)(\sum_{k=1}^{n} \widetilde{\sigma}_k)^{-1}.$$

($\sum \widetilde{\sigma}_k > 0$ on $G_0 \times \mathbb{T}$, so, by Wiener's theorem, $(\sum_{k=1}^{n} \widetilde{\sigma}_k)^{-1} \in A(G_0 \times \mathbb{T})$). Then $g \in J_{E\times\mathbb{T}}$ and $g = f$ on $U \times \mathbb{T}$.

To finish the proof, we must show that this leads to a contradiction. Let $\psi \in A(G_0)$ be such that $\operatorname{supp}\psi \subset U$ and such that there exists an open neighbourhood U' of w with $\psi = 1$ on U'. We may also assume that $0 \leq \psi \leq 1$. Define $\widetilde{\psi} \in A(G_0 \times \mathbb{T})$ by $\widetilde{\psi}(x,t) = \psi(x)$. Then $g\widetilde{\psi} = f\widetilde{\psi} \in J_{E\times\mathbb{T}}$.

This means that, given $\varepsilon > 0$, there exists an open neighbourhood N of $E \times \mathbb{T}$ with a function $h \in A(G_0 \times \mathbb{T})$ which is identically zero on N and such that $\|h - f\widetilde{\psi}\|_A < \epsilon$. We can write $h(x,t) = \sum h_n(x,t)e^{int}$, where $\sum \|h_n\|_{A(G_0)} = \|h\|_{A(G_0\times\mathbb{T})}$. By looking at the Fourier transforms, we see that $\|h_n - f_n\psi\|_{A(G_0)} < \varepsilon$, for $n \geq 1$.

Let M be a neighbourhood of E such that $M \times \mathbb{T} \subset N$. For all $t \in \mathbb{T}$, and each fixed $x \in M$, we have $\sum h_n(x)e^{int} = 0$. Hence, by the uniqueness theorem for Fourier series, each $h_n(x) = 0$ on M.

Since $\varepsilon > 0$ is arbitrary, we conclude that $f_n\psi \in J_E$ and thus $f_n \in_{U'} J_E$, for every $n \geq 1$, contrary to the hypotheses. □

To get an example where Δ_E does not lift, it thus suffices to find a compact non-metrizable abelian group G_0, with E and w as in the proposition but satisfying, in addition, the condition $w \notin \Delta_E$. However, can these two requirements be met simultaneously? As our construction in [2] suggests, a set $E \subset G$ for which Δ_E is not closed may have to be extremely thin (in some sense) at every point $w \in \overline{\Delta_E} \setminus \Delta_E$, which might be an obstruction for the construction of a sequence $(f_n)_{n\geq 1}$ as in Proposition 4.2.

So far we have not been able to settle this problem.

References

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