Analytic Fourier Integral Operators, Monge-Ampère Equation and Holomorphic Factorization

By

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Abstract. We will show that the factorization condition for the Fourier integral operators $I^\mu_\rho(X,Y;\Lambda)$ leads to a parametrized parabolic Monge-Ampère equation. In case of an analytic operator the fibration by the kernels of the Hessian of phase function is shown to be analytic in a number of cases by considering more general continuation problem for the level sets of a holomorphic mapping. The results are applied to obtain L^p-continuity for translation invariant operators in $\mathbf{R}^n$ with $n \le 4$ and for arbitrary $\mathbf{R}^n$ with $d\pi_{X\times Y}|_\Lambda \le n+2$.

1. Introduction. Let X, Y be smooth paracompact manifolds of dimension n. Let $T \in I^\mu(X,Y;\Lambda)$ be a Fourier integral operator with the Lagrangian distributional kernel of order μ and the wavefront set contained in $\Lambda' = \{(x,\xi,y,\eta) : (x,\xi,y,-\eta) \in \Lambda\}$. We will always assume that Λ is locally a graph of a symplectomorphism between $T^*X\backslash 0$ and $T^*Y\backslash 0$, equipped with the standard symplectic forms $d\sigma_X$ and $d\sigma_Y$. The theory of such operators is discussed in [4], [2], [13], [12]. Let $\pi_{X\times Y}$ be the natural projection from $T^*X \times T^*Y$ to $X \times Y$. It is well know that the operators of order 0 are continuous in L^2-spaces and this result does not depend on the singularities of $\pi_{X\times Y}$. The important result of Seeger, Sogge and Stein [11] is that the Fourier integral operators $T \in I^\mu(X,Y;\Lambda)$ of order $\mu \le -(n-1)|1/p-1/2|$, $1<p<\infty$, are continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$. This conclusion is sharp if $d\pi_{X\times Y}|_\Lambda$ has full rank equal to $2n-1$ somewhere and T is elliptic. However, if $d\pi_{X\times Y}|_\Lambda$ does not attain the rank of $2n-1$, then the estimate for the order μ is not sharp and may depend on the singularities of $d\pi_{X\times Y}|_\Lambda$. Thus, it was shown in [11] that the continuity properties of Fourier integral operators in L^p-spaces with $p \ne 2$ depend on the singularities and the maximal rank of the canonical projection. The important ingredient is the following smooth factorization condition for $\pi_{X\times Y}$ introduced in [11]. Assume that there exists k, $0 \le k \le n-1$, such that for every $\lambda_0 = (x_0,\xi_0,y_0,\eta_0) \in \Lambda$ there is a conic neighborhood $U_{\lambda_0} \subset \Lambda$ of λ_0, and a smooth homogeneous of degree 0 map $\pi_{\lambda_0} : \Lambda \cap U_{\lambda_0} \to \Lambda$ with constant rank $d\pi_{\lambda_0} = n+k$,

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such that $\pi_{X\times Y} = \pi_{X\times Y}\circ\pi_{\lambda_0}$. Under this assumption the operators $T \in I^\mu_\rho(X,Y;\Lambda)$, $1/2 \le \rho \le 1$, are continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$ for $1 < p < \infty$ and the order $\mu \le -(k\rho + n(1-\rho))|1/p - 1/2|$.

We will show that the factorization of $d\pi_{X\times Y}|_\Lambda$ is equivalent to the factorization of the Hessian of the phase function, which leads to the parametrized Monge-Ampère equation (see also [5]). Then we will produce and discuss some examples showing that in general the factorization condition is not trivial, but in comparison with [11] it is not always sufficient to consider phase functions corresponding to the translation invariant operators. In case of the analytic operators we will show that in a number of cases the factorization condition is satisfied. This will be applied to the questions of the L^p-continuity of Fourier integral operators in $\mathbf{R}^n$ with $n \le 4$ and arbitrary $\mathbf{R}^n$ with rank $d\pi_{X\times Y}|_\Lambda \le n+2$.

By the analytic Fourier integral operators we understand the operators acting in real analytic manifolds X and Y for which the canonical relation is analytic. For such operators the phase function is analytic and the factorization condition can be extended to the complex domain for the reduced function after factorizing away the conic variable. We solve this problem partially in a more general setting with the gradient of a function replaced by an arbitrary holomorphic mapping with affine fibres. We will show, that in the case of fibres of dimension larger or equal than $n-2$ and in 3-dimensional space the fibration extends holomorphically to the whole domain. In other cases, by means of examples we will show that in general this conclusion does not hold for lower dimensional fibres.

2. Parametrized fibrations. We assume now that X and Y are open subsets of $\mathbf{R}^n$. This is not restrictive for the local analysis of Fourier integral operators, as it is demonstrated in [2], [4], [12], [13]. The Lagrangian distribution of Fourier integral operator in $T^*\mathbf{R}^n \times T^*\mathbf{R}^n$ can be viewed as a smooth family of Lagrangian submanifolds of $T^*\mathbf{R}^n$. First we will show that in a suitable choice of the coordinate system the ranks of their projections to the base space differ by n. By the equivalence-of-phase-function theorem [4, Th.3.1.3], [2, Th.2.3.4], we can assume that the phase function of an operator $T \in I^\mu_\rho(X,Y;\Lambda)$ is equal to $\Phi(x,y,\xi) = \langle x,\xi\rangle - \phi(y,\xi)$ and $\Lambda = \Lambda_\Phi$ is locally given by $\{(\nabla_\xi\phi(y,\xi),\xi,y,\nabla_y\phi(y,\xi))\}$. The local graph condition is equivalent to

$$\det \phi''_{y\xi}(y,\xi) \ne 0 \tag{1}$$

on the support of the symbol of T. We start with a simple observation.

Lemma 1 *The mapping $\gamma : Y\times\Xi \to T^*X\times T^*Y$ defined by*

$$\gamma(y,\xi) = (\nabla_\xi\phi(y,\xi),\xi,y,\nabla_y\phi(y,\xi))$$

is a diffeomorphism between $Y\times\Xi$ and Λ. For $y\in Y$ the restriction $\xi \to \gamma(y,\xi)$ is a diffeomorphism from Ξ to $\Lambda\cap(\mathbf{R}^n\times\mathbf{R}^n\times\{y\}\times\mathbf{R}^n)$ with the inverse given by the projection $(x,\xi,y,\eta)\to\xi$.

This implies that $d\pi_{X\times Y}|_\Lambda$ is isomorphic to $d\pi_{X\times Y}\circ d\gamma|_{Y\times\Xi}$ and, in particular, they have isomorphic kernels. But

$$\ker d\pi_{X\times Y}\circ d\gamma|_{Y\times\Xi}(y,\xi)=\{(\delta y,\delta\xi):\frac{\partial^2\phi}{\partial\xi^2}(y,\xi)\delta\xi+\frac{\partial^2\phi}{\partial y\partial\xi}(y,\xi)\delta y=0,\delta y=0\}$$

and in view of (1), we get

$$\ker d\pi_{X\times Y}\circ d\gamma|_{Y\times\Xi}(y,\xi)=(0,\ker\frac{\partial^2\phi}{\partial\xi^2}(y,\xi)). \tag{2}$$

Thus, we obtain a characterization of the projection in terms of the phase function, which follows from (2) and the second part of Lemma 1.

Theorem 1 *Let $\Phi(x,y,\xi)=\langle x,\xi\rangle-\phi(y,\xi)$ with Λ_Φ a canonical graph. Then for $0\le k\le n-1$ the following conditions are equivalent*

(i) rank $d\pi_{X\times Y}|_{\Lambda_\Phi}\le n+k$.

(ii) rank $d\pi_{X\times Y}\circ d\gamma|_{\{y\}\times\Xi}\le k$ *for all $y\in Y$ with γ as in Lemma 1.*

(iii) rank $\frac{\partial^2\phi}{\partial\xi^2}(y,\xi)\le k$ *for all $y\in Y$ and $\xi\in\Xi$.*

Note, that the condition $k\le n-1$ leads to the parametrized Monge-Ampère equation for the phase function:

$$\det\frac{\partial^2\phi}{\partial\xi^2}(y,\xi)=0$$

for all $(y,\xi)\in Y\times\Xi$. The following example shows that the factorization condition is not in general satisfied. In case of rank $d\pi_{X\times Y}|_{\Lambda_\Phi}\le n+k$, it follows from Theorem 1, (iii) that the function $\phi(y,\xi)=\langle y,\xi\rangle+\frac{1}{\xi_n}\sum_{i=2}^{k+1}(y_1\xi_1+y_i\xi_i)^2$ satisfies the required rank conditions in a neighborhood of a point $\xi_n=1$ and the fibration is defined by the quotients y_i/y_1, $2\le i\le k+1$, so that we have

Example 1 *Let $1\le k\le n-2$ and $x,y,\xi\in\mathbf{R}^n$. The function*

$$\Phi(x,y,\xi)=\langle x-y,\xi\rangle-\frac{1}{\xi_n}\sum_{i=2}^{k+1}(y_1\xi_1+y_i\xi_i)^2$$

satisfies the condition

$$\text{rank }d\pi_{X\times Y}|_{\Lambda_\Phi}\le n+k$$

and defines a canonical graph Λ_Φ, for which the fibration of $\pi_{X\times Y}$ is not continuously extendible over $y=0$.

Note, that in the case of $k = 0$ we have conormal operators, which can be transformed to the pseudo-differential operators by a coordinate change and for which the factorization condition is trivially satisfied (see also [9]). The case $k = n-1$ corresponds to the condition $\operatorname{rank} d\pi_{X\times Y}|_\Lambda \le 2n-1$, for which the factorization condition is satisfied in view of the homogeneity of Λ with π_{λ_0} being the projection in the conic direction.

3. Holomorphic factorization. Now we will consider a more general factorization problem. Let Γ be a holomorphic mapping from a connected open subset Ω of $\mathbf{C}^m \times \mathbf{C}^n$ to $\mathbf{C}^p$, let $k < n$, and assume that

(i) $\operatorname{rank} \partial\Gamma(y,\xi)/\partial\xi \le k$ for all $(y,\xi) \in \Omega$.

(ii) $\exists (y,\xi) \in \Omega$ so that $\operatorname{rank} \partial\Gamma(y,\xi)/\partial\xi = k$.

The set Ω can be decomposed into a union of $\Omega^{(i)}$ of the points $(y,\xi) \in \Omega$ with $\operatorname{rank} \partial\Gamma(y,\xi)/\partial\xi = i$, $i = 0,\dots,k$. Then the set $\Omega' = \Omega\backslash\Omega^{(k)}$ where $\operatorname{rank} \partial\Gamma(y,\xi)/\partial\xi < k$ is an analytic subset of Ω without interior points and in the open dense subset $\Omega^{(k)}$ of Ω the mapping

$$\kappa : (y,\xi) \mapsto \ker \frac{\partial\Gamma(y,\xi)}{\partial\xi}$$

is holomorphic from $\Omega^{(k)}$ to the Grassmann manifold $\mathbf{G}_{n-k}(\mathbf{C}^n)$ of all $(n-k)$-dimensional subspaces of $\mathbf{C}^n$. Let us denote by Ω^{sing} the subset of $\omega = (y,\xi) \in \Omega'$ such that κ can not be extended to a holomorphic mapping $U \to \mathbf{G}_{n-k}(\mathbf{C}^n)$ on a open neighborhood U of ω in Ω.

Lemma 2 *If $\omega \in \Omega^{\mathrm{sing}}$, then for every k-dimensional linear subspace C of $\mathbf{C}^n$ there exists a sequence $\omega_j \in \Omega^{(k)}$, such that ω_j converges to ω as $j \to \infty$, $\kappa(\omega_j)$ converges to $\kappa \in \mathbf{G}_{n-k}(\mathbf{C}^n)$ as $j \to \infty$, and $\kappa \cap C \ne \{0\}$.*

Proof. The set $G(C) = \{L \in \mathbf{G}_{n-k}(\mathbf{C}^n) : L \cap C = \{0\}\}$ is holomorphically diffeomorphic to $\mathbf{C}^{k(n-k)}$ (see, for example, [6, B.6.6] and [6, Prop.,p.367]). It follows that if there exists a neighborhood U of ω in Ω such that $\kappa(U \cap \Omega^{(k)})$ is contained in a compact subset of $G(C)$, then $\omega \notin \Omega^{\mathrm{sing}}$. This implies the statement of Lemma 2.

Lemma 3 *Assume that in addition to* (i), (ii) *holds*

(iii) *If $(y,\xi) \in \Omega^{(k)}$, then Γ is constant on the set of $(y,\xi+z)$, for all $z \in \kappa(y,\xi) = \ker \partial\Gamma(y,\xi)/\partial\xi$, such that $(y,\xi+z) \in \Omega$.*

Then for any $(y,\xi) \in \Omega^{\mathrm{sing}}$ and for any k-dimensional linear subspace C of $\mathbf{C}^n$ there exists a linear subspace L of C with $\dim L \ge 1$, such that for each $l \in L$ we have $\Gamma(y,\xi+l) = \Gamma(y,\xi)$.

Proof. By Lemma 2 there exists $\lim_j \kappa(\omega_j) = \kappa$ as $\omega_j \to (y,\xi)$ with $\omega_j \in \Omega^{(k)}$ and $\kappa \cap C \ne \{0\}$. We take $L = \kappa \cap C$. By (iii) for each $\omega_j = (y_j,\xi_j)$ and $z \in \kappa(\omega_j)$ we have $\Gamma(y_j,\xi_j+z) = \Gamma(y_j,\xi_j)$. By continuity of Γ we obtain the statement of Lemma 3.

As a consequence, in the case $k = 1$ we have

Theorem 2 *Let Γ satisfy* (i),(ii) *and* (iii) *with $k = 1$ and let $(y, \xi) \in \Omega^{\mathrm{sing}}$. Then the mapping $\eta \mapsto \Gamma(y, \eta)$ is constant.*

4. Translation invariant case. Now we will concentrate on the case when $\Gamma(y, \xi) = \Gamma(\xi)$ for all $y \in Y$, or rather on the mapping $\Gamma(y, \xi)$ with a fixed value of y. This reduces to $m = 0$, so that the sets $\{y\} \times \Omega^{(i)}$ we simply denote by $\Omega^{(i)}$ via the identification $\{y\} \times \mathbf{C}^n \cong \mathbf{C}^n$ and $\Gamma : \Omega \subset \mathbf{C}^n \to \mathbf{C}^p$. From now on we will always assume the conditions (i), (ii) and (iii) of the previous section satisfied. Thus, we have the mapping $\kappa : \Omega^{(k)} \to \mathbf{G}_{n-k}(\mathbf{C}^n)$ defined on an open dense subset $\Omega^{(k)}$ of Ω. The graph of the mapping κ is

$$G = \{(\xi, L) \in \Omega \times \mathbf{G}_{n-k}(\mathbf{C}^n) : \xi \in \Omega^{(k)}, L = \kappa(\xi)\}$$

and we also define $G_0 = \{(\xi, L) \in \Omega \times \mathbf{G}_{n-k}(\mathbf{C}^n) : L \subset \ker \partial\Gamma/\partial\xi(\xi)\}$. Clearly G_0 is a closed analytic subset of $\Omega \times \mathbf{G}_{n-k}(\mathbf{C}^n)$ and

$$G = (\Omega^{(k)} \times \mathbf{G}_{n-k}(\mathbf{C}^n)) \cap G_0 = G_0 \backslash \{(\xi, L) \in \Omega \times \mathbf{G}_{n-k}(\mathbf{C}^n) : \xi \in \Omega \backslash \Omega^{(k)}\}, \qquad (3)$$

which is the complement in G_0 of the closed analytic subset $G_0 \cap ((\Omega \backslash \Omega^{(k)}) \times \mathbf{G}_{n-k}(\mathbf{C}^n))$. Let $\bar{\kappa}(\xi) \subset \mathbf{G}_{n-k}(\mathbf{C}^n)$ be the set of all limits of $\kappa(\xi_j)$ as $\xi_j \to \xi$, $\xi_j \in \Omega^{(k)}$. (See also [8]). With $V \subset \mathbf{G}_{n-k}(\mathbf{C}^n)$ we associate the cone

$$\tilde{V} = \{\xi \in \mathbf{C}^n : \exists \lambda \in V, \xi \in \lambda\}.$$

It is easy to see that if V is analytic in $\mathbf{G}_{n-k}(\mathbf{C}^n)$, then $\tilde{V}$ is analytic in $\mathbf{C}^n$. Further, we will often identify V and $\tilde{V}$ for the kernels $\kappa(\xi)$ if it is clear in the context.

Proposition 1 *The closure $\bar{G}$ of G is analytic. The set $\bar{\kappa}(\xi)$ is analytic and connected for every $\xi \in \Omega$. The point $\xi \in \Omega^{\mathrm{sing}}$ if and only if $\dim \bar{\kappa}(\xi) \geq 1$, or $\dim \tilde{\bar{\kappa}}(\xi) \geq n - k + 1$. Moreover, if C is an irreducible component of $\tilde{\bar{\kappa}}(\xi)$, then $\dim C \geq n - k + 1$.*

Proof. It follows from (3) that G is a difference of two analytic sets. Hence its closure $\bar{G}$ is analytic in view of [6, IV.2.10]. It follows that $\bar{\kappa}(\xi) = \{L \in \mathbf{G}_{n-k}(\mathbf{C}^n) : (\xi, L) \in \bar{G}\}$ is an analytic subset of $\mathbf{G}_{n-k}(\mathbf{C}^n)$. Let U, V be open disjoint subsets of $\mathbf{G}_{n-k}(\mathbf{C}^n)$ and $\bar{\kappa}(\xi) \subset (U \cup V)$. Let $A = \{\eta \in \Omega^{(k)} : \kappa(\eta) \in U\}$ and $B = \{\eta \in \Omega^{(k)} : \kappa(\eta) \in V\}$. Then A and B are disjoint open subsets of $\Omega^{(k)}$. There is a connected open neighborhood W of ξ, such that $W \cap \Omega^{(k)} \subset A \cup B$ and $W \cap \Omega^{(k)}$ is connected. It follows that the intersection of $W \cap \Omega^{(k)}$ with either A or B is empty and hence $\bar{\kappa}(\xi) \cap U$ or $\bar{\kappa}(\xi) \cap V$ is empty. Therefore, $\bar{\kappa}(\xi)$ is connected. If $\xi \notin \Omega^{\mathrm{sing}}$, then $\bar{\kappa}(\xi)$ consists of one point, so that $\dim \bar{\kappa}(\xi) = 0$. Conversely, if $\xi \in \Omega^{\mathrm{sing}}$ we have that for every $C \in \mathbf{G}_k(\mathbf{C}^n)$ the intersection of the hypersurface $\{L \in \mathbf{G}_{n-k}(\mathbf{C}^n) : L \cap C \neq \{0\}\}$ with $\bar{\kappa}(\xi)$ is not empty by Lemma 2. This implies that $\bar{\kappa}(\xi)$ is infinite, hence $\dim \bar{\kappa}(\xi)$ can not be equal to zero. The analyticity of $\tilde{\bar{\kappa}}(\xi)$ implies $\dim \tilde{\bar{\kappa}}(\xi) \geq n - k + 1$.

Finally, let C be an irreducible component with $\dim C \leq n - k$. Then C is an element of $\bar{\kappa}(\xi) \subset \mathbf{G}_{n-k}(\mathbf{C}^n)$. The set $\bar{\kappa}(\xi)$ is connected and $\dim \bar{\kappa}(\xi) \geq 1$, implying that C is contained in the closure of a smooth part of $\bar{\kappa}(\xi)$ of positive dimension and that C is contained in an irreducible component of $\tilde{\bar{\kappa}}(\xi)$ of dimension strictly larger than $n-k$, a contradiction with definition of C.

We have the following general property of Ω^{sing} as the indeterminacy set of a meromorphic mapping (see [8]).

Theorem 3 *The set Ω^{sing} is an analytic subset of Ω with $\dim \Omega^{\text{sing}} \leq n-2$.*

Finally we want to mention that in general the fibration need not be holomorphically extendible.

Theorem 4 *For every $3 \leq k \leq n-1$ and $2 \leq d \leq \min\{k-1, n-k+1\}$ there exist holomorphic mapping $\Gamma : \mathbf{C}^n \to \mathbf{C}^n$ with affine fibres, satisfying* $\operatorname{rank} D\Gamma \leq k$ *and* $\dim \Omega^{\text{sing}} = n-d$. *Moreover, Γ can be chosen such that $\Omega \backslash \Omega^{(k)} = \Omega^{\text{sing}}$.*

Note that the bounds for k and d are essential and, in fact, necessary, but we will not pursue it here because of the different purpose of this paper. See [10] for more details.

5. The case $n = 3$. In this section we consider holomorphic mappings $\Gamma : \Omega \subset \mathbf{C}^3 \to \mathbf{C}^r$ satisfying conditions (i)-(iii) of Section 3. In view of Theorem 3 the set Ω^{sing} in $\mathbf{C}^3$ is at most one dimensional. For $k = 1$ Theorem 1 shows that the singular set Ω^{sing} is empty. The same holds for $k = 2$:

Theorem 5 *If $n = 3$ and $k = 2$, then Ω^{sing} is empty.*

Proof. Assume first that $\Omega^{\text{sing}} \neq \emptyset$ and that $\dim \Omega^{\text{sing}} = 1$. For $\xi \in \Omega^{\text{sing}}$ the set $\tilde{\bar{\kappa}}(\xi)$ is contained in $\Omega \backslash \Omega^{(2)}$, which is an analytic subset of dimension less or equal to 2, so any smooth part of $\tilde{\bar{\kappa}}(\xi)$ is an open subset of $\Omega \backslash \Omega^{(2)}$, and therefore each 2-dimensional irreducible component of $\tilde{\bar{\kappa}}(\xi)$ is equal to an irreducible component of $\Omega \backslash \Omega^{(2)}$. Because the latter set has only finitely many irreducible components we get that the 2-dimensional irreducible components of the infinitely many $\tilde{\bar{\kappa}}(\xi)$, $\xi \in \Omega^{\text{sing}}$, can not all be distinct from each other. Suppose $\xi, \eta \in \Omega^{\text{sing}}$, $\xi \neq \eta$ and C is a 2-dimensional irreducible component of both $\tilde{\bar{\kappa}}(\xi)$ and $\tilde{\bar{\kappa}}(\eta)$. Then, for each $\zeta \in C$ not on the line between ξ and η, the line from ξ to ζ and the line from η to ζ both are limits of regular fibres, which implies that $\zeta \in \Omega^{\text{sing}}$. But this would imply that $C \subset \Omega^{\text{sing}}$, in contradiction with $\dim \Omega^{\text{sing}} = 1$. Now we will show that the condition $\dim \Omega^{\text{sing}} = 1$ is necessary, which will in turn imply the statement of Theorem.

For $\xi \in \Omega$ let $a_{ij}(\xi) = \partial_j \Gamma_i(\xi)$, $\{a_{ij}(\xi)\} = D\Gamma(\xi) \in \mathbf{C}^{r \times 3}$, and let $\Delta_{pi}^{mj}(\xi)$ be the determinant of 2 by 2 matrix obtained by the intersection of rows p, i with columns m, j in $D\Gamma(\xi)$. Let $x \in \ker D\Gamma(\xi)$ and let i and j be such that $a_{ij} \not\equiv 0$. For a function f by Z_f we denote its zero locus. Then the set of ξ for which $a_{ij}(\xi) = 0$ is a hypersurface in $\mathbf{C}^3$ and on its complement $\Omega \backslash Z_{a_{ij}}$ we have $x_j = -\sum_{k \neq j} a_{ik}(\xi)/a_{ij}(\xi) x_k$. Substitution of this into the other $r-1$ equations leads for $p \neq i$ to

$$\sum_{k \neq j} (a_{pk} - a_{pj} a_{ik}/a_{ij})(\xi) x_k = 0. \tag{4}$$

This means that on $\Omega^{(2)}\backslash Z_{a_{ij}}$ the projection of the fibre $\kappa(\xi)$ to the hyperplane $x_j = 0$ is given by the equation (4), which define the same line for any $p \neq i$ in view of the condition on the rank $D\Gamma \leq 2$. Equation (4) is equivalent to

$$\Delta_{pi}^{kj}(\xi)x_k + \Delta_{pi}^{mj}(\xi)x_m = 0. \tag{5}$$

Let π_j be the natural projection $\mathbf{C}^3 \to \{\xi \in \mathbf{C}^3 : \xi_j = 0\}$. Let $\xi_0 \in \Omega^{\mathrm{sing}}$. Then by Proposition 1 we have $\dim \tilde{\tilde{\kappa}}(\xi_0) \geq 2$ and, therefore, there exists j such that $\dim \pi_j \tilde{\tilde{\kappa}}(\xi_0) = 2$. Suppose first that there exists an open neighborhood $U(\xi_0)$ of ξ_0 in Ω such that $a_{ij}(\xi) = 0$ for all $1 \leq i \leq r$ and $\xi \in U(\xi_0)$. By analyticity of a_{ij} and connectedness of Ω we may assume that $U(\xi_0) = \Omega$. Let k, m denote the other two columns of $D\Gamma$. Because the maximal rank of $D\Gamma$ is 2, there exist numbers p, i such that $\Delta_{pi}^{km} \not\equiv 0$ and one of a_{im}, a_{ik}, say a_{im} is not identically equal to zero. Then, as before, on $\Omega^{(2)}\backslash Z_{a_{im}}$ the projection $\pi_m \kappa(\xi)$ is determined by the equation $\Delta_{pi}^{jm}(\xi)x_j + \Delta_{pi}^{km}(\xi)x_k = 0$ and $\Delta_{pi}^{jm} \equiv 0$ implies $x_k = 0$ on $\Omega^{(2)}\backslash(Z_{a_{im}} \cup Z_{\Delta_{pi}^{km}})$, that is $\pi_m\kappa(\xi)$ are parallel to $x_k = 0$. Now, from $\Delta_{pi}^{km} \not\equiv 0$ we also get that a_{ik} or a_{pk} does not vanish identically. We denote it by a_{qk} with $q = i$ or $q = p$. The same argument as before shows that on $\Omega^{(2)}\backslash(Z_{a_{qk}} \cup Z_{\Delta_{pi}^{km}})$ projections $\pi_k\kappa(\xi)$ are determined by $\Delta_{pi}^{jk}(\xi)x_j + \Delta_{pi}^{mk}(\xi)x_m = 0$ and are parallel to $x_m = 0$. It follows that on the open dense subset $U = \Omega^{(2)}\backslash(Z_{a_{im}} \cup Z_{a_{qk}} \cup Z_{\Delta_{pi}^{km}})$ of Ω the fibres $\kappa(\xi)$ belong to the intersection of two transversal families of parallel planes, so that $\kappa(\xi)$ are parallel to each other on U. Since U is dense in Ω, we have the constant extension of κ to Ω and $\Omega^{\mathrm{sing}} = \emptyset$. This is a contradiction with $\xi_0 \in \Omega^{\mathrm{sing}}$ and, therefore, there exists an index i such that $a_{ij} \not\equiv 0$. As before, on $\Omega^{(2)}\backslash Z_{a_{ij}}$ the projection $\pi_j\kappa(\xi)$ is determined by the equation (5) for some $p \neq i$. If one of $\Delta_{pi}^{kj}, \Delta_{pi}^{mj}$ does not vanish identically, our assumption of $\dim \pi_j \tilde{\tilde{\kappa}}(\xi_0) = 2$ implies that the meromorphic function $\Delta_{pi}^{kj}/\Delta_{pi}^{mj}$ or $\Delta_{pi}^{mj}/\Delta_{pi}^{kj}$ is multivalued at ξ_0 and we get $Z_{\Delta_{pi}^{mj}} \cap Z_{\Delta_{pi}^{kj}} \subset \Omega^{\mathrm{sing}}$. On the other hand $\dim Z_{\Delta_{pi}^{mj}} \cap Z_{\Delta_{pi}^{kj}} \geq 1$, implying $\dim \Omega^{\mathrm{sing}} \geq 1$ and completing the proof of Theorem 5.

Thus, the only case which is left is that $\Delta_{pi}^{mj} \equiv \Delta_{pi}^{kj} \equiv 0$ for all $1 \leq p \leq r$. We will show that this is impossible. First, the condition $\Delta_{pi}^{mj} \equiv \Delta_{li}^{mj} \equiv 0$ for l, p, i all different, implies $\Delta_{pl}^{mj} \equiv 0$. Indeed, for each fixed ξ in $U = \Omega\backslash Z_{a_{ij}}$ we denote by A, B, C the vectors $(a_{lm}(\xi), a_{lj}(\xi)), (a_{pm}(\xi), a_{pj}(\xi)), (a_{im}(\xi), a_{ij}(\xi))$ respectively. The condition $\Delta_{pi}^{mj} \equiv \Delta_{li}^{mj} \equiv 0$ implies the existence of $\alpha, \beta, \gamma, \delta$ with $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$, such that $\alpha B + \beta C = 0$ and $\gamma A + \delta C = 0$. Because of $C \neq 0$ we get that the condition $\alpha = 0$ or $\gamma = 0$ imply $\beta = 0$ or $\delta = 0$ respectively. Hence we can assume that $\alpha \neq 0$ and $\gamma \neq 0$. The condition $\beta = 0$ implies $B = 0$ and hence in this case A and B are linearly dependent. If $\beta \neq 0$, then we get $\gamma A - \delta\alpha\beta^{-1}B = 0$ and A and B are linearly dependent again, implying $\Delta_{pl}^{mj}(\xi) = 0$. Because the argument holds for any ξ in the open dense subset U of Ω, we obtain $\Delta_{pl}^{mj} \equiv 0$. The same argument implies $\Delta_{pl}^{kj} \equiv 0$. The same argument shows that $\Delta_{pi}^{kj} \equiv \Delta_{pi}^{mj} \equiv 0$ imply $\Delta_{pi}^{km} \equiv 0$. Now, let $M(\xi) \in \mathbf{C}^{3\times 3}$ be the submatrix of $D\Gamma(\xi)$ with rows p, l, i. The condition rank $D\Gamma \leq 2$ implies $0 = \det M(\xi) = \pm(a_{ik}\Delta_{pl}^{mj} - a_{im}\Delta_{pl}^{kj} + a_{ij}\Delta_{pl}^{km})$ and, therefore, $a_{ij}\Delta_{pl}^{km} \equiv 0$ and $\Delta_{pl}^{km} \equiv 0$ on U and also on Ω because it is holomorphic. The conclusion is that all two dimensional subdeterminants are identically equal to zero on Ω, a contradiction with $k=2$. This completes the proof of Theorem 5.

6. The case $k=2$. In this section we consider holomorphic mappings $\Gamma : \Omega \subset \mathbf{C}^n \to \mathbf{C}^p$ satisfying conditions (i)-(iii) of Section 3 with $k=2$.

Theorem 6 *If $k=2$, then the singular set Ω^{sing} is empty.*

PROOF. Let $\xi \in \Omega^{\mathrm{sing}}$. According to Proposition 1, the set $\tilde{\bar{\kappa}}(\xi)$ is at least $(n-1)$-dimensional, which implies $\tilde{\bar{\kappa}}(\xi) \not\subset \Omega^{\mathrm{sing}}$ by Theorem 3. The set $\bar{\kappa}(\xi)$ is connected, therefore, there exist different $\kappa_1, \kappa_2 \in \bar{\kappa}(\xi) \subset \mathbf{G}_{n-2}(\mathbf{C}^n)$, not contained in Ω^{sing}. For $i=1,2$ the sets

$$\mathcal{K}_i = \{H \in \mathbf{G}_3(\mathbf{C}^n) : \dim(H \cap \kappa_i) = 1\}, \mathcal{K}_0 = \{H \in \mathbf{G}_3(\mathbf{C}^n) : H \cap \kappa_1 \neq H \cap \kappa_2\}$$

and $\mathcal{K}$ of all $H \in \mathbf{G}_3(\mathbf{C}^n)$ for which $H \cap \kappa_i \not\subset \Omega^{\mathrm{sing}}$, are open and dense in $\mathbf{G}_3(\mathbf{C}^n)$, their intersection is open and dense in $\mathbf{G}_3(\mathbf{C}^n)$ and we take $H \in \mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_0 \cap \mathcal{K}$. Let $\eta \in \Omega^{(2)}$ be close to ξ with $\kappa(\eta)$ close to one of κ_i. Then, by transversality, $\dim(\eta + \kappa(\eta)) \cap (\xi + H) = 1$. The set $\Omega^{(2)} \cap (\eta + \kappa(\eta))$ is open and dense in $\Omega \cap (\eta + \kappa(\eta))$. Therefore, there exists $\zeta \in \Omega^{(2)} \cap (\eta + \kappa(\eta))$, ζ close to $(\eta + \kappa(\eta)) \cap (\xi + H)$, such that there exists $H_0 \in \mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_0 \cap \mathcal{K}$ with $\zeta \in H_0$. Thus, without loss of generality we may take $H = H_0$ with $\zeta \in \Omega^{(2)} \cap (\eta + \kappa(\eta)) \cap H$. Now, $\kappa(\eta) = \kappa(\zeta)$, implying that the mapping

$$\gamma = \Gamma|_{(\xi+H)\cap\Omega}$$

satisfies $\ker D\gamma(\zeta) = \kappa(\zeta) \cap H$, which is one dimensional, and, therefore, $\operatorname{rank} D\gamma(\zeta) = \dim H - 1 = 2$. Moreover, if $\theta \in \ker D\gamma(\zeta)$, then $\gamma(\zeta+\theta) = \Gamma(\zeta+\theta) = \Gamma(\zeta) = \gamma(\zeta)$ because $\theta \in \kappa(\zeta)$ and property (iii) of Section 3. This means that the conditions (i), (ii) and (iii) are satisfied for γ. Because the set

$$\Omega^{(2)}(H) = \{\zeta \in (\eta + H) \cap \Omega : \operatorname{rank} D\gamma(\zeta) = 2\}$$

is open and dense in $(\eta + H) \cap \Omega$, we can find $\zeta_j \in \Omega^{(2)}(H)$ which are arbitrary close to $(\eta + H) \cap (\eta_j + \kappa(\eta_j))$, from which it follows that $\kappa(\zeta_j)$ is arbitrary close to $\kappa(\eta_j)$. Note, that $\Omega^{(2)}(H) \subset \Omega^{(2)}$ and κ is constant on an open dense subset of $\eta + \kappa(\eta)$, $\eta \in \Omega^{(2)}$.

This proves that at the limit point ξ in $(\xi + H) \cap (\xi + \kappa)$ we get all $H \cap \bar{\kappa}(\xi)$ as limits of $H \cap \kappa(\zeta_j)$, $\zeta_j \in \Omega^{(2)}(H)$, $\zeta_j \to \xi$, in particular two different lines $H \cap \kappa_i$ by $H \in \mathcal{K}_0$, or ξ is in the singular set for the mapping γ. But this is in contradiction with Theorem 5, which says that for $n=3, k=2$ the singular set is empty.

7. Application. Consider an analytic operator $T \in I^{\mu}_{\rho}(X, Y; \Lambda)$, $1/2 \leq \rho \leq 1$, commuting with translations, X and Y open subsets of $\mathbf{R}^n$. This means that it is equal to the convolution with some distribution p. The theory of such operators as multipliers is well known and they can exhibit quite irregular behavior (e.g. [7], [13]). See also [3] for the case $\rho = 1/2$. This distribution p is a Fourier integral distribution defined by some conic Lagrangian manifold $\Lambda^p \subset T^*(\mathbf{R}^n)$. It follows from the proof of [2, Prop.3.7.3], that locally $\Lambda^p = \Lambda_\phi$ with homogeneous $\phi(z,\xi) = \langle z, \xi\rangle - H(\xi)$ and $\Lambda_\phi = \{(\nabla H(\xi), \xi)\}$. Then p is given by

$$p(z) = \int e^{i\phi(z,\xi)} a(z,\xi) d\xi$$

with some symbol $a \in S^\mu_\rho$. The operator T is then of the form

$$Tu(x) = u * p(x) = \int\int e^{i\Phi(x,y,\xi)} a(x-y,\xi) u(y) d\xi dy$$

with the phase function $\Phi(x,y,\xi) = \langle x-y, \xi\rangle - H(\xi)$. We assume that

$$\text{rank } d\pi_{X\times Y}|_\Lambda \leq n+k \tag{6}$$

for some $0 \leq k \leq n-1$. The condition (iii) of Theorem 1 becomes rank $D^2 H(\xi) \leq k$ for all $\xi \in \Xi$. Now, let $\xi = (\theta, \tau) \in \mathbf{R}^{n-1} \times \mathbf{R}$ be the splitting of ξ with τ being the conic variable. For $\zeta \in \mathbf{R}^{n-1}$ define $F(\zeta) = H(\zeta, 1)$. Then using the homogeneity of H, we obtain $H(\theta,\tau) = \tau F(\theta/\tau)$. Now, we have

$$\begin{array}{lcl}
\nabla_\theta H(\theta,\tau) & = & \nabla F(\theta/\tau), \\
\partial_\tau H(\theta,\tau) & = & -\langle \nabla F(\theta/\tau), \theta/\tau\rangle + F(\theta/\tau), \\
D^2_\theta H(\theta,\tau) & = & D^2 F(\theta/\tau)/\tau, \\
\partial_\tau \nabla_\theta H(\theta,\tau) & = & -\langle D^2 F(\theta/\tau), \theta/\tau\rangle/\tau, \\
\partial^2_\tau H(\theta,\tau) & = & (\theta/\tau)^* D^2 F(\theta/\tau)(\theta/\tau)/\tau,
\end{array}$$

so that for $\zeta = \theta/\tau$ we get

$$D^2 H(\theta,\tau) = \frac{1}{\tau} \begin{pmatrix} D^2 F(\zeta) & -\langle D^2 F(\zeta), \zeta\rangle \\ \langle D^2 F(\zeta), \zeta\rangle^* & \zeta^* D^2 F(\zeta)\zeta \end{pmatrix}.$$

It follows that rank $D^2 H(\theta,\tau) =$ rank $D^2 F(\theta/\tau)$ and, therefore, (6) is equivalent to

$$\text{rank } D^2 F(\zeta) \leq k.$$

The mapping $\nabla F(\zeta)$ is analytic in $\mathbf{R}^{n-1}$, so it allows a holomorphic extension to a mapping $\Gamma : \Omega \to \mathbf{C}^{n-1}$ with some open $\Omega \subset \mathbf{C}^{n-1}$. If $1 \leq k \leq n-2$, then Γ satisfies conditions (i)-(iii) of Section 2, if the maximal rank k is attained somewhere.

Theorem 7 *Let $T \in I^\mu_\rho(X,Y;\Lambda)$ be an analytic translation invariant Fourier integral operator, $1/2 \leq \rho \leq 1$. Let* rank $d\pi_{X\times Y}|_\Lambda \leq n+k$, $0 \leq k \leq 2$. *Then T is bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$, if $\mu \leq -(k+(n-k)(1-\rho))|1/p - 1/2|, 1 < p < \infty$.*

Proof. Follows from Theorems 2, 6 and Theorem 5.1 of [11].

Theorem 8 *Let X, Y be open subsets of $\mathbf{R}^n$, $n \leq 4$ and let $T \in I^\mu_\rho(X,Y;\Lambda)$ be an analytic translation invariant Fourier integral operator, $1/2 \leq \rho \leq 1$. Let $0 \leq k \leq 3$ be such that* rank $d\pi_{X\times Y}|_\Lambda \leq n+k$. *Then T is bounded from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$, if $\mu \leq -(k+(n-k)(1-\rho))|1/p-1/2|, 1 < p < \infty$.*

Proof. For $k = n-1$ and $k = 0$ the statement follows from [11, Th.5.1]. For $k = 1$ it follows from Theorem 7. The last case is $n = 4, k = 2$ and this follows from Theorem 5.

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