A BOUND FOR THE GENERALIZED TJURINA NUMBERS

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Abstract. We define a series of non-negative integer valued invariants $\tau^{(1)}, \dots, \tau^{(n-1)}$ of an analytic n-dimensional isolated complete intersection singularity (the generalized Tjurina numbers). The Tjurina number is equal to $\tau^{(1)}$. We show that all these numbers are bounded by the Milnor number μ by establishing explicit formulas for the differences $\mu - \tau^{(p)}$ involving the mixed Hodge numbers of the cohomology of the link. This generalizes earlier results of Looijenga and Steenbrink on the relation between the Milnor number and the Tjurina number. For singularities with $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$ (among which one finds the weighted homogeneous ones) we obtain some purity results for various mixed Hodge structures associated with the singularity and its resolution.

Introduction

Let (X, x) be a germ of an n-dimensional $\mathbb{C}$-analytic complete intersection with an isolated singularity at x (an *icis* for short). The Milnor number μ and the Tjurina number τ are two well-known analytic invariants of such a singularity; by definition, μ is the middle Betti-number of a Milnor fibre for (X, x) and τ is the $\mathbb{C}$-dimension of the space $T^1_{X,x}$ of infinitesimal deformations of (X, x) (see e.g. [G2, 0.1, 0.2]).

If (X, x) is a hypersurface in $(\mathbb{C}^{n+1}, 0)$ with an isolated singularity defined by an analytic map-germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ then we have the expressions

$$\begin{aligned} &\tau = \dim_{\mathbb{C}} T^1_{X,x} \quad \text{and} \quad T^1_{X,x} \cong \mathcal{O}_{\mathbb{C}^{n+1},0}/(f, \tfrac{\partial f}{\partial z_1}, \dots, \tfrac{\partial f}{\partial z_{n+1}}) \\ &\mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+1},0}/(\tfrac{\partial f}{\partial z_1}, \dots, \tfrac{\partial f}{\partial z_{n+1}}) \end{aligned}$$

with $z_1, \dots, z_{n+1}$ coordinates on $(\mathbb{C}^{n+1}, 0)$. From these formulas it follows that $\mu \geq \tau$. The relation $\mu \geq \tau$ actually holds for any *icis* of dimension $n \geq 1$ but is less immediate in this generality: For curve singularities this is because $\tau = \tau'$ by local duality, and $\tau' \leq \mu$ (see (1.2) for a definition of τ'); for $n \geq 2$ it is a corollary to the following theorem of Looijenga and Steenbrink.

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Theorem 1 (**[LS]**). *Let (X,x) be an* icis *of dimension $n \geq 2$. Then*

$$\mu - \tau = \sum_{i=0}^{n-2} h^{i,0} + a_1 + a_2 + a_3.$$

Here $h^{i,j}$ is the i,j'th Hodge number of the canonical mixed Hodge structure on $H^n_{\{x\}}(X,\mathbb{C})$, and the numbers a_1, a_2 and a_3 are the (non-negative!) dimensions of some modules that will be defined later (cf. Lemma 2.7 below).

The proof of Theorem 1 is based on a study of the mixed Hodge theory of the singularity and a resolution of it, and the point of departure is the equality $\tau = \dim_{\mathbb{C}} H^1_{\{x\}}(\Omega^{n-1}_{X,x})$ [G2, p. 168]. In this perspective τ is merely one of many invariants of (X,x) of the form $\dim_{\mathbb{C}} H^q_{\{x\}}(\Omega^p_{X,x})$. Most of these dimensions, however, are either infinite or trivial, but among the remaining ones we find a distinguished sequence of numbers what we will call the *generalized Tjurina numbers.*

Definition 2. For $n \geq 2$ and $1 \leq p \leq n-1$ we define the p'th generalized Tjurina number as

$$\tau^{(p)} = \dim_{\mathbb{C}} H^p_{\{x\}}(\Omega^{n-p}_{X,x}).$$

In particular, $\tau^{(1)} = \tau$.

See Section 1 for properties and alternative definitions of the generalized Tjurina numbers. We prove the following generalization of Theorem 1.

Theorem 3. *Let (X,x) be an* icis *of dimension $n \geq 3$. The following formula holds for $1 \leq p \leq n-1$.*

$$\mu - \tau^{(p)} = \sum_{i=0}^{n-1-p} \sum_{j=0}^{p-1} h^{i,j} + b_1^{(p)} + b_2^{(p)} + \sum_{j=0}^{p-1} \left(\tilde{e}^{n-j,j+1}_{p+1-j} + e^{n-j,j}_{p+1-j} \right).$$

In the formula the $h^{i,j}$ are defined as above, while the rest of the terms are non-negative numbers (dimensions of particular modules) that will be defined during the proof in (2.3) and (2.5). The proof is really a comparison between three spectral sequences all of which converge to cohomology of the link of (X,x). One of the spectral sequences defines the Hodge filtration, while another contains information about the numbers $\tau^{(p)}$. In Lemma 2.7 we show that $a_1 = \tilde{e}^{n,1}_2 + b_1^{(1)}$ for $n \geq 3$ and $a_2 + a_3 = e^{n,0}_2 + b_2^{(1)}$ for $n \geq 2$. Via these identities Theorem 1 is equivalent to Theorem 3 ($p=1$) when $n \geq 3$. There is an immediate corollary to Theorem 1 and 3

Corollary 4. *Let (X,x) be an* icis *of dimension $n \geq 2$. Then $\mu \geq \tau^{(p)}$ for $1 \leq p \leq n-1$.*

The following theorem is essentially due to Greuel and shows that the inequalities in Corollary 4 become equalities in the weighted homogeneous case.

Theorem 5. *If (X, x) is a weighted homogeneous icis of dimension $n \geq 2$ then $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$.*

Proof. Greuel proves that $\mu = \tau$ under the given assumptions [G2, Satz 0.3], but his proof of this theorem gives our stronger conclusion. □

Our main motivation for studying the generalized Tjurina numbers comes from the following question:

Is the extreme condition $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$ also a sufficient condition for (X, x) to be weighted homogeneous when $n \geq 2$?

The question can be answered affirmatively in two cases, namely for hypersurface singularities and for surface singularities. We will have to leave the question open in general. If (X, x) is a hypersurface then (by Proposition 1.3(e)) $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$ is equivalent to $\mu = \tau$, which again by the famous theorem of K. Saito [Sa] is equivalent to weighted homogeneity. If (X, x) is an *icis* of dimension $n = 2$, then there is only one generalized Tjurina number, namely τ itself, and J. Wahl [W, Thm. 3.2] proved that $\mu = \tau$ is equivalent to weighted homogeneity even for isolated *Gorenstein* singularities.

Notice that there are other promising approaches in the literature to finding sufficient conditions for weighted homogeneity, see e.g. [HM][1].

The following theorem offers some evidence to our question.

Theorem 6. *Let X be a contractible Stein representative of an icis (X, x) of dimension $n \geq 2$. Put $U = X - x$. If $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$, then the canonical mixed Hodge structure on $H^r(U, \mathbb{C})$ is pure of weight r (if $r < n$) or $r + 1$ (if $r \geq n$). Furthermore, the Hodge spectral sequence*

$$E_1^{p,q}(\Omega_U^{\cdot}) = H^q(\Omega_U^p) \implies \mathbb{H}^{p+q}(U, \mathbb{C})$$

degenerates at E_2 and we have a canonical isomorphism $E_2^{p,q}(\Omega_U^{\cdot}) \cong \mathrm{Gr}_F^p H^{p+q}(U, \mathbb{C})$. Hence, the Hodge decomposition reads

$$H^r(U, \mathbb{C}) = \bigoplus_p H^p(H^{r-p}(U, \Omega_U^{\cdot})).$$

Proof. This is an immediate consequence of Theorem 3.3 □

The conclusion of Theorem 6 is already known to hold for weighted homogeneous singularites. Naruki [N] (see also [G2, Satz 0.5]) was the first to establish degeneration of $E(\Omega_U^{\cdot})$ at E_2 and the decomposition statement for the cohomology groups of U for these

[1] Their comment on p. 168 on the Gorenstein property of G is intriguingly close to our question via Proposition 1.3(d) below.

singularities. In [A, Prop. 5.7] Aleksandrov then proved that Naruki's decomposition in fact is the Hodge decomposition of the canonical (pure!) mixed Hodge structure on the cohomology groups. His method of proof is quite different from the present proof of Theorem 6 as a corollary to Theorem 3.3. Actually, 3.3 gives much more: For any good resolution $(Y, D) \longrightarrow (X, x)$ of the singular point x with exceptional fiber D the mixed Hodge structures on $H^{\cdot}(D)$ and $H^{\cdot}_D(Y)$ will be pure if $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$, and we get a rather explicit description of the Hodge decompositions. Purity of $H^{\cdot}(D)$, for instance, implies that the 'dual graph' of D has $\mathbb{C}$-cohomology as a point.

The article is organized as follows. Section 1 gives a broader introduction to the generalized Tjurina numbers, while the proof of Theorem 3 will occupy section 2. Section 3 is devoted to a study of the various mixed Hodge structures associated with a singularity (and its resolution) satisfying the extreme condition $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$. The main result of that section is Theorem 3.3, most of whose conclusions seem to be new even for weighted homogeneous singularities.

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1. Properties of the generalized Tjurina numbers

1.1. Let (X, x) be an *icis* of dimension $n \geq 1$, and recall from Definition 2 that $\tau^{(p)} = \dim_{\mathbb{C}} H^p_{\{x\}}(\Omega^{n-p}_{X,x})$ for $p = 1, \ldots, n-1$. Greuel showed [G2, Prop. 2.3] that when p is in the range $0, \ldots, n$ then $\dim_{\mathbb{C}} H^q_{\{x\}}(\Omega^{n-p}_{X,x}) = \infty$ exactly if $q = n$, and $H^q_{\{x\}}(\Omega^{n-p}_{X,x}) = 0$ unless $q \in \{p, p+1, n\}$. Furthermore, $\dim_{\mathbb{C}} H^p_{\{x\}}(\Omega^{n-p}_{X,x}) = \dim_{\mathbb{C}} H^{p+1}_{\{x\}}(\Omega^{n-p}_{X,x})$ when $p = 0, \ldots, n-2$. This shows that the $\tau^{(p)}$ are finite and are somewhat distinguished among the invariants of the form $\dim_{\mathbb{C}} H^q_{\{x\}}(\Omega^p_{X,x})$.

1.2. One may define $\tau^{(0)} = \dim_{\mathbb{C}} H^0_{\{x\}}(\Omega^n_{X,x})$. Then $\tau^{(0)}$ coincides with the invariant τ' of [G2, p. 158]. The relation $\mu \geq \tau'$ with equality in the weighted homogeneous case was already shown in [G1, 5.7 and 5.8], but it has not been possible to find a formula for the difference $\mu - \tau'$ in the style of Theorem 3. It is still not known if the relation $\tau' \geq \tau$ holds in general, as examples suggest, but if $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$ turns out to be a sufficient condition for weighted homogeneity then this must at least imply $\tau' = \mu = \tau$. This, for instance, happens when $n = 2$.

Proposition 1.3. *The generalized Tjurina numbers have the following properties*

(a) $\tau^{(p)} = \tau^{(n-p)}$ *for* $1 \leq p \leq n-1$.
(b) $\tau^{(p)} = \dim H^{p+1}_{\{x\}}(\Omega^{n-p}_X)$ *for* $1 \leq p \leq n-2$.
(c) $\tau^{(p)} = \dim \operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\Omega^p_{X,x}, \mathcal{O}_{X,x})$ *for* $1 \leq p \leq n-1$.
(d) $\tau^{(p)} = \dim S_p T^1_{X,x}$ *for* $1 \leq p \leq n-1$

(S_p denotes the p'th symmetric power over $\mathcal{O}_{X,x}$).

(e) *If (X,x) is a hypersurface singularity then $\tau = \tau^{(1)} = \cdots = \tau^{(n-1)}$.*

Proof. Property (b) follows from the discussion above. Let $'$ denote the $\mathbb{C}$-dual, then we have

$$H^q_{\{x\}}(\Omega^{n-p}_{X,x}) \cong H^{n+1-q}_{\{x\}}(\Omega^p_{X,x})' \quad \text{for } 0 \le p \le n,\ 2 \le q \le n-1,$$
$$\operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\Omega^p_{X,x}, \mathcal{O}_{X,x}) \cong H^{n-q}_{\{x\}}(\Omega^p_{X,x})' \quad \text{for all } p \text{ and } 0 \le q \le n-1.$$

The first isomorphism comes from Serre-duality in the version of Naruki [N, 1.1.1] (see also [G2, 1.3]), the second from local duality (see [G2, 1.2]). (a) and (c) now follow immediately. (e) follows from (d): if (X,x) is a hypersurface then the description of $T^1_{X,x}$ in the introduction shows that $S_pT^1_{X,x} \cong T^1_{X,x}$ for all p. It now only remains to prove (d).

We will sketch how to establish canonical isomorphisms

$$\operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\Omega^p_{X,x}, \mathcal{O}_{X,x}) \cong S_p \operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x}), \qquad \text{for } p = 1, \dots, n.$$

Then (d) follows from (c) via Tjurina's [T] isomorphism $T^1_{X,x} \cong \operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x})$.

Let $f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$ define (X,x) (Then f is actually flat [L, Prop. 6.10]). Define $\Theta_{f,0} = (\Omega^1_{\mathbb{C}^k,0} \otimes_{\mathcal{O}_{\mathbb{C}^k,0}} \mathcal{O}_{\mathbb{C}^{n+k},0})^*$ and $\Theta_{\mathbb{C}^{n+k},0} = (\Omega^1_{\mathbb{C}^{n+k},0})^*$, where $()^*$ denotes the $\mathcal{O}_{\mathbb{C}^{n+k},0}$-dual, and let $tf : \Theta_{\mathbb{C}^{n+k},0} \to \Theta_{f,0}$ be the $\mathcal{O}_{\mathbb{C}^{n+k},0}$-linear homomorphism induced from the tangent map.

Then $\Theta_{\mathbb{C}^{n+k},0}$ and $\Theta_{f,0}$ are $\mathcal{O}_{\mathbb{C}^{n+k},0}$-free and the depth of the ideal of maximal minors in a matrix representation of tf is the maximal possible, namely $n+1$ [L, Prop. 4.4]. By [E, Cor. A2.13] the dual of the Buchsbaum-Eisenbud complex for $p = 1, \dots, n$ gives rise to a minimal free $\mathcal{O}_{X,x}$-resolution of $\Omega^p_{X,x}$ of the form

$$\begin{aligned} 0 &\to (S_p\Theta_{f,0} \otimes \bigwedge^0 \Theta_{\mathbb{C}^{n+k},0})^* \otimes \mathcal{O}_{X,x} \to (S_{p-1}\Theta_{f,0} \otimes \bigwedge^1 \Theta_{\mathbb{C}^{n+k},0})^* \otimes \mathcal{O}_{X,x} \to \dots \\ &\to (S_1\Theta_{f,0} \otimes \bigwedge^{p-1} \Theta_{\mathbb{C}^{n+k},0})^* \otimes \mathcal{O}_{X,x} \to (S_0\Theta_{f,0} \otimes \bigwedge^{p} \Theta_{\mathbb{C}^{n+k},0})^* \otimes \mathcal{O}_{X,x}. \end{aligned}$$

Dualizing this complex we see that $\operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\Omega^p_{X,x}, \mathcal{O}_{X,x})$ is isomorphic to the cokernel of the map

$$S_{p-1}\Theta_{f,0} \otimes \Theta_{\mathbb{C}^{n+k},0} \otimes \mathcal{O}_{X,x} \xrightarrow{\phi_p} S_p\Theta_{f,0} \otimes \mathcal{O}_{X,x}.$$

induced from the composite $S_{p-1}\Theta_{f,0} \otimes \Theta_{\mathbb{C}^{n+k},0} \xrightarrow{1\otimes tf} S_{p-1}\Theta_{f,0} \otimes \Theta_{f,0} \to S_p\Theta_{f,0}$. Thus,

$$\operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\Omega^p_{X,x}, \mathcal{O}_{X,x}) \cong \operatorname{Coker} \phi_p \cong S_p \operatorname{Coker} \phi_1 \cong S_p \operatorname{Ext}^1_{\mathcal{O}_{X,x}}(\Omega^1_{X,x}, \mathcal{O}_{X,x}),$$

as claimed. A more careful study shows that this isomphism is canonical in the sense that it does not depend on the choice of f. $\square$

2. Proof of Theorem 3

2.1 Notation. We start by recalling the setup from [LS]. Let X be a contractible Stein representative for (X,x), which is non-singular away from x. Choose a resolution $Y \xrightarrow{\tilde{p}} X$ of the singularity $\{x\}$ such that $\tilde{p}$ is an isomorphism over $U = X - \{x\}$, and $D = p^{-1}(x)$ is a divisor on Y with normal crossings. In the following we will always suppress the map $\tilde{p}$ and identify U with $Y - D$.

In [St1] a filtered complex $A^{\cdot+1}_{\{x\}}$ on Y is defined. Hypercohomology $\mathbb{H}^{\cdot}(Y, A^{\cdot+1}_{\{x\}})$ of this complex calculates the cohomology groups $H^{\cdot}(U,\mathbb{C})$, and the induced filtration defines the Hodge-filtration of their canonical mixed Hodge structures (in the sense of Deligne [D]). An argument from [LS] shows that the complex $A^{\cdot+1}_{\{x\}}$ is quasi-isomorphic to the complex $\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D$ and that the spectral sequence $E(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D)$ with E_1-term

$$E_1^{p,q}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D) = H^q(Y, \Omega^p_Y(\log D) \otimes \mathcal{O}_D)$$

converging to $\mathbb{H}^{\cdot}(Y, \Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D) \cong \mathbb{H}^{\cdot}(Y, A^{\cdot+1}_{\{x\}}) \cong \mathbb{H}^{\cdot}(U,\mathbb{C})$ degenerates at E_1. In this way we get isomorphisms $\operatorname{Gr}^p_F H^{p+q}(U,\mathbb{C}) \cong H^q(\Omega^p_Y(\log D) \otimes \mathcal{O}_D)$. For $q \geq 1$ the mixed Hodge structure on $H^{q+1}_{\{x\}}(X,\mathbb{C})$ is defined via the isomorphism $H^{q+1}_{\{x\}}(X,\mathbb{C}) \cong H^q(U,\mathbb{C})$, so in particular we get

$$\operatorname{Gr}^p_F H^{p+q+1}_{\{x\}}(X,\mathbb{C}) \cong H^q(\Omega^p_Y(\log D) \otimes \mathcal{O}_D). \tag{2.1.1}$$

2.2 Two spectral sequences converging to 0. There is a triangle

$$\mathbb{H}_D(\Omega^{\cdot}_Y(\log D)) \longrightarrow \mathbb{H}(\Omega^{\cdot}_Y(\log D)) \xrightarrow{\beta} \mathbb{H}(\Omega^{\cdot}_U) \xrightarrow{+1} \mathbb{H}_D(\Omega^{\cdot}_Y(\log D)), \tag{2.2.1}$$

where the middle morphism is known to be an isomorphism. Hence $\mathbb{H}_D(\Omega^{\cdot}_Y(\log D))$ is isomorphic to 0, and we get a spectral sequence

$$\tilde{E}_1^{p,q}(\Omega^{\cdot}_Y(\log D)) = H^q_D(\Omega^p_Y(\log D)) \implies \mathbb{H}^{p+q}_D(\Omega^{\cdot}_Y(\log D)) = 0.$$

Let $\mathcal{J}$ be the reduced sheaf of ideals defining the divisor D on Y. Then we have a short exact sequence of complexes on Y,

$$0 \to \mathcal{J}\Omega^{\cdot}_Y(\log D) \longrightarrow \Omega^{\cdot}_Y(\log D) \xrightarrow{\gamma} \Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D \to 0.$$

This gives another triangle

$$\mathbb{H}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) \to \mathbb{H}(\Omega^{\cdot}_Y(\log D)) \xrightarrow{\mathbb{H}(\gamma)} \mathbb{H}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D) \xrightarrow{+1} \mathbb{H}(\mathcal{J}\Omega^{\cdot}_Y(\log D)). \tag{2.2.2}$$

Since $\mathbb{H}(\Omega^{\cdot}_Y(\log D))$ and $\mathbb{H}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D)$ are both isomorphic to $\mathbb{H}(U,\mathbb{C})$, then $\mathbb{H}(\gamma)$ is an isomorphism, and $\mathbb{H}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = 0$. From this we get one more spectral sequence converging to 0,

$$E_1^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = H^q(\mathcal{J}\Omega^p_Y(\log D)) \implies \mathbb{H}^{p+q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = 0.$$

2.3 Notational conventions. In the following chapter we will introduce several other spectral sequences, all of which compute either $\mathbb{H}(C^\cdot)$ or $\mathbb{H}_D(C^\cdot)$ for some complex $C^\cdot$ of sheaves on Y. In the former case the spectral sequence will be denoted $E(C^\cdot)$ and in the latter by $\tilde{E}(C^\cdot)$, so the tilde indicates local cohomology supported at D.

Within this section we will for convenience use the following shorthand notation

$$E_r^{p,q} = E_r^{p,q}(\mathcal{J}\Omega_Y^\cdot(\log D)) \quad \text{and} \quad \tilde{E}_r^{p,q} = \tilde{E}_r^{p,q}(\Omega_Y^\cdot(\log D)).$$

We let $e_r^{p,q} = \dim E_r^{p,q}$ and $\tilde{e}_r^{p,q} = \dim \tilde{E}_r^{p,q}$. These are the numbers that appear in the formula of Theorem 3.

By Serre-duality the terms $\tilde{E}_1^{p,q}$ are finite-dimensional and dual to $E_1^{n-p,n-q}$ when $q < n$.

Proposition 2.4. *Let p, q be integers. Then*

$(a)_{p,q}$	$\tilde{E}_1^{p,q} = 0$	*if* $p+q \neq n, n+1$ *and* $q < n$, *or if* $q \geq n+1$.
$(b)_{p,q}$	$E_1^{p,q} = 0$	*if* $p+q \neq n-1, n$ *and* $q > 0$, *or if* $q \geq n$.
$(c)_{p,q}$	$H^q(\Omega_Y^p(\log D)) = 0$	*if* $p+q \neq n-1, n, 2n-1$ *and* $q > 0$, *or if* $q \geq n$.
$(d)_{p,q}$	$H^q(\Omega_Y^p(\log D) \otimes \mathcal{O}_D) = 0$	*if* $p+q \neq 0, n-1, n, 2n-1$, *or if* $q \geq n$.
$(e)_{p,q}$	$H^q(\Omega_U^p) = 0$	*if* $p+q \neq n-1, n$ *and* $0 < q < n-1$, *or if* $q \geq n$.

Furthermore, all of the above modules vanish when $p > n$.

Proof. The vanishing for $p > n$ is clear since both Y and U are non-singular of dimension n. From Serre-duality and the two exact sequences

$$H^q(\mathcal{J}\Omega_Y^p(\log D)) \longrightarrow H^q(\Omega_Y^p(\log D)) \longrightarrow H^q(\Omega_Y^p(\log D) \otimes \mathcal{O}_D)$$
$$H^{q-1}(\Omega_U^p) \longrightarrow H_D^q(\Omega_Y^p(\log D)) \longrightarrow H^q(\Omega_Y^p(\log D))$$

we derive the following logical implications

$$\begin{array}{rll} (a)_{p,q} \Longleftrightarrow & (b)_{n-p,n-q} & \text{for all } p \text{ and } q \leq n \\ (b)_{p,q} \wedge (d)_{p,q} \Longrightarrow & (c)_{p,q} & \text{for all } p, q \\ (e)_{p,q-1} \wedge (c)_{p,q} \Longrightarrow & (a)_{p,q} & \text{for } n < p+q \neq 2n-1 \text{ and } q > 1. \end{array}$$

It is easy to check that we are done by establishing the validity of $(e)_{p,q}$ and $(d)_{p,q}$ for all p, q, and of $(b)_{p,q}$ when $p+q > n$ or $(p,q) = (0,1)$. But $(e)_{p,q}$ follows from [G2, Prop. 2.3], and since U is homotopy equivalent to the link of (X, x) then $(d)_{p,q}$ follows from [G2, 0.3] (which is an immediate consequence of Hamm's theorem [H, Kor. 1.3] on the $(n-2)$-connectedness of the link) and our discussion in (2.1). The vanishing theorem of Steenbrink in [St2] proves $(b)_{p,q}$ for $p+q > n$. Hence, it only remains to verify $(b)_{0,1}$. Since $\Omega_Y^0(\log D) = \mathcal{O}_Y$, this amounts to proving $H^1(Y, \mathcal{J}) = 0$ for $n \geq 3$. The following argument was shown to me by J. Steenbrink.

Assume $n \geq 3$. The Serre-dual of $(b)_{n,n-1}$ states that $H_D^1(\mathcal{O}_Y) = 0$ and together with the vanishing of $H^1(\mathcal{O}_U)$ (by $(e)_{0,1}$ since $n \geq 3$) we obtain $H^1(\mathcal{O}_Y) = 0$. This implies that $H^1(\mathcal{J})$ is isomorphic to the cokernel of $H^0(\mathcal{O}_Y) \longrightarrow H^0(\mathcal{O}_D)$; but this map is surjective [St1, Lemma 2.14], so $H^1(\mathcal{J})$ is trivial as asserted. $\square$

2.5 Formulas for $\tau^{(p)}$. From now on we assume that $n \geq 3$. Then $n+1 < 2n-1$, and the modules $H^{p+1}(\mathcal{J}\Omega_Y^{n-p}(\log D))$, $H^{p+1}(\Omega_Y^{n-p}(\log D))$, $H_D^{p-1}(\Omega_Y^{n-p}(\log D))$ and $H^{p-2}(\Omega_Y^{n-p}(\log D) \otimes \mathcal{O}_D)$ vanish for all p by Proposition 2.4. From (2.2.1) and (2.2.2) we then get the following 6-term exact sequences

$$\begin{aligned} 0 \to H^{p-1}(\Omega_Y^{n-p}(\log D)) &\longrightarrow H^{p-1}(\Omega_U^{n-p}) \longrightarrow H_D^p(\Omega_Y^{n-p}(\log D)) \\ \to H^p(\Omega_Y^{n-p}(\log D)) &\longrightarrow H^p(\Omega_U^{n-p}) \longrightarrow H_D^{p+1}(\Omega_Y^{n-p}(\log D)) \to 0 \end{aligned} \tag{2.5.1}$$

and

$$\begin{aligned} 0 \to &H^{p-1}(\mathcal{J}\Omega_Y^{n-p}(\log D)) \to H^{p-1}(\Omega_Y^{n-p}(\log D)) \to H^{p-1}(\Omega_Y^{n-p}(\log D) \otimes \mathcal{O}_D) \\ \to &H^p(\mathcal{J}\Omega_Y^{n-p}(\log D)) \to H^p(\Omega_Y^{n-p}(\log D)) \to H^p(\Omega_Y^{n-p}(\log D) \otimes \mathcal{O}_D) \to 0 \end{aligned} \tag{2.5.2}$$

Let $\beta^{n-p,p}$ denote the homomorphism $H^p(\Omega_Y^{n-p}(\log D)) \to H^p(\Omega_U^{n-p})$, and let $\rho^{n-p,p}$ be the homomorphism $H^p(\mathcal{J}\Omega_Y^{n-p}(\log D)) \to H^p(\Omega_Y^{n-p}(\log D))$. We define $B_1^{(p)} = \operatorname{Ker} \beta^{n-p,p}$ and $B_2^{(p)} = \operatorname{Ker} \rho^{n-p,p}$ (also when $n = 2$). Then the sequences (2.5.1) and (2.5.2) split into the following four-term exact sequences

$$0 \to H^{p-1}(\Omega_Y^{n-p}(\log D)) \to H^{p-1}(\Omega_U^{n-p}) \to \tilde{E}_1^{n-p,p} \to B_1^{(p)} \to 0 \tag{2.5.3}$$

$$0 \to B_1^{(p)} \to H^p(\Omega_Y^{n-p}(\log D)) \to H^p(\Omega_U^{n-p}) \to \tilde{E}_1^{n-p,p+1} \to 0 \tag{2.5.4}$$

$$0 \to E_1^{n-p,p-1} \to H^{p-1}(\Omega_Y^{n-p}(\log D)) \to \operatorname{Gr}_F^{n-p} H_{\{x\}}^n(X, \mathbb{C}) \to B_2^{(p)} \to 0 \tag{2.5.5}$$

$$0 \to B_2^{(p)} \to E_1^{n-p,p} \to H^p(\Omega_Y^{n-p}(\log D)) \to \operatorname{Gr}_F^{n-p} H_{\{x\}}^{n+1}(X, \mathbb{C}) \to 0. \tag{2.5.6}$$

Since $\dim E_1^{i,j}$ is finite for $j > 0$ then from (2.5.5) and (2.5.6) we get the following equalities of finite sums

$$e_1^{n-p,p-1} + \dim \operatorname{Gr}_F^{n-p} H_{\{x\}}^n(X, \mathbb{C}) = b_2^{(p)} + \dim H^{p-1}(\Omega_Y^{n-p}(\log D)) \tag{2.5.7}$$

$$e_1^{n-p,p} + \dim \operatorname{Gr}_F^{n-p} H_{\{x\}}^{n+1}(X, \mathbb{C}) = b_2^{(p)} + \dim H^p(\Omega_Y^{n-p}(\log D)), \tag{2.5.8}$$

valid for $2 \leq p$ and $1 \leq p$ respectively (with $b_2^{(p)} = \dim B_2^{(p)}$ and $e_k^{i,j} = \dim E_k^{i,j}$).

For $1 \leq p \leq n-2$ we have $H^p(\Omega_U^{n-p}) \cong H_{\{x\}}^{p+1}(\Omega_X^{n-p})$, whose dimension is $\tau^{(n-p)} = \tau^{(p)}$. For $2 \leq p \leq n-1$ we similarly have $H^{p-1}(\Omega_U^{n-p}) \cong H_{\{x\}}^p(\Omega_X^{n-p})$, also of dimension

$\tau^{(n-p)} = \tau^{(p)}$. From (2.5.3) and (2.5.4) we thus obtain the following equalities of finite sums

$$\tau^{(p)} + b_1^{(p)} = \tilde{e}_1^{n-p,p} + \dim H^{p-1}(\Omega_Y^{n-p}(\log D)) \tag{2.5.9}$$
$$\tau^{(p)} + b_1^{(p)} = \tilde{e}_1^{n-p,p+1} + \dim H^{p}(\Omega_Y^{n-p}(\log D)) \tag{2.5.10}$$

valid for $2 \leq p \leq n-1$ and $1 \leq p \leq n-2$ respectively (with $b_1^{(p)} = \dim B_1^{(p)}$ and $\tilde{e}_k^{i,j} = \dim \tilde{E}_k^{i,j}$).

By combining (2.5.9) with (2.5.7) and (2.5.10) with (2.5.8) we obtain the following two formulas for $\tau^{(p)}$,

$$\begin{aligned}\tau^{(p)} &= \tilde{e}_1^{n-p,p} + \dim H^{p-1}(\Omega_Y^{n-p}(\log D)) - b_1^{(p)} \\ &= \tilde{e}_1^{n-p,p} + e_1^{n-p,p-1} + \dim \operatorname{Gr}_F^{n-p} H^n_{\{x\}}(X,\mathbb{C}) - (b_1^{(p)} + b_2^{(p)})\end{aligned} \tag{2.5.11}$$

and

$$\begin{aligned}\tau^{(p)} &= \tilde{e}_1^{n-p,p+1} + \dim H^{p}(\Omega_Y^{n-p}(\log D)) - b_1^{(p)} \\ &= \tilde{e}_1^{n-p,p+1} + e_1^{n-p,p} + \dim \operatorname{Gr}_F^{n-p} H^{n+1}_{\{x\}}(X,\mathbb{C}) - (b_1^{(p)} + b_2^{(p)})\end{aligned} \tag{2.5.12}$$

valid for $2 \leq p \leq n-1$ and $1 \leq p \leq n-2$ respectively.

2.6 The difference $\mu - \tau^{(p)}$. Let $n \geq 3$ and $1 \leq p \leq n-1$. We can rewrite $\tau - \tau^{(p)} = \tau^{(1)} - \tau^{(p)}$ as the sum $\sum_{i=2}^{p}(\tau^{(i-1)} - \tau^{(i)})$ and in each term of this sum substitute for $\tau^{(i-1)}$ the expression given in (2.5.12) and for $\tau^{(i)}$ the expression from (2.5.11). This gives

$$\begin{aligned}\tau - \tau^{(p)} &= \sum_{i=2}^{p}\Big(\dim \operatorname{Gr}_F^{n-i+1} H^{n+1}_{\{x\}}(X,\mathbb{C}) - \dim \operatorname{Gr}_F^{n-i} H^{n}_{\{x\}}(X,\mathbb{C})\Big) \\ &\qquad + \sum_{i=2}^{p}\Big((\tilde{e}_1^{n-i+1,i} + e_1^{n-i+1,i-1}) - (\tilde{e}_1^{n-i,i} + e_1^{n-i,i-1})\Big) \\ &\qquad - \sum_{i=2}^{p}\Big((b_1^{(i-1)} + b_2^{(i-1)}) - (b_1^{(i)} + b_2^{(i)})\Big) \\ &= \sum_{i=1}^{p-1}\Big(\dim \operatorname{Gr}_F^{n-i} H^{n+1}_{\{x\}}(X,\mathbb{C}) - \dim \operatorname{Gr}_F^{n-i-1} H^{n}_{\{x\}}(X,\mathbb{C})\Big) \\ &\qquad \sum_{i=2}^{p}\Big(\tilde{e}_1^{n-i+1,i} - \tilde{e}_1^{n-i,i}\Big) + \sum_{i=1}^{p-1}\Big(e_1^{n-i,i} - e_1^{n-i-1,i}\Big) \\ &\qquad + b_1^{(p)} + b_2^{(p)} - b_1^{(1)} - b_2^{(1)}.\end{aligned} \tag{2.6.1}$$

We want to combine this with the following formula for $\mu-\tau$ ([LS, eqn. (8)])

$$\mu - \tau = \dim \operatorname{Gr}_F^n H^{n+1}_{\{x\}}(X,\mathbb{C}) - \dim \operatorname{Gr}_F^{n-1} H^n_{\{x\}}(X,\mathbb{C}) + a_1 + a_2 + a_3, \tag{2.6.2}$$

and for that purpose we need to find another expression for $a_1+a_2+a_3$ (the numbers a_1, a_2, a_3 are the dimensions of the modules A_1, A_2, A_3 to be defined in the proof of the following lemma).

Lemma 2.7. *The identities*

$$a_1 = \tilde{e}_2^{n,1} + b_1^{(1)}, \qquad a_2 + a_3 = e_2^{n,0} + b_2^{(1)}.$$

hold for $n \geq 3$ and $n \geq 2$ respectively.

Proof. For the first equation, assume $n \geq 3$. Since $H^0_D(\Omega^n_Y(\log D)) = \tilde{E}_1^{n,0} = \tilde{E}_\infty^{n,0} = 0$ and $\tilde{E}_1^{n,1} = 0$ (here we use $n \geq 3$), we have the following commutative diagram with exact rows (compare with sequence (2.5.3) and (2.5.4)).

$$\begin{array}{ccccccccc}
0 \to & H^0(\Omega^{n-1}_Y(\log D)) & \longrightarrow & H^0(\Omega^{n-1}_U) & \xrightarrow{\pi} & H^1_D(\Omega^{n-1}_Y(\log D)) & \longrightarrow & B_1^{(1)} & \to 0 \\
& \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_2} & & & \\
0 \to & H^0(\Omega^n_Y(\log D)) & \longrightarrow & H^0(\Omega^n_U) & \longrightarrow & H^1_D(\Omega^n_Y(\log D)) & \longrightarrow & 0 &
\end{array}$$

with δ_1, δ and δ_2 induced from the exterior derivative. By definition ([LS]),

$$A_1 = \operatorname{Coker}\big(H^0(\Omega^n_Y(\log D)) \oplus H^0(\Omega^{n-1}_U) \to H^0(\Omega^n_U)\big)$$

which is clearly isomorphic $\operatorname{Coker}(\delta_2 \circ \pi)$, so it fits into an exact sequence

$$0 \to \frac{\operatorname{Im}\delta_2}{\operatorname{Im}(\delta_2 \circ \pi)} \to A_1 \to \operatorname{Coker}\delta_2 \to 0.$$

By Proposition 2.4, $\operatorname{Ker}\delta_2 = \tilde{E}_2^{n-1,1} = \tilde{E}_\infty^{n-1,1} = 0$, so δ_2 is injective. Hence,

$$\frac{\operatorname{Im}\delta_2}{\operatorname{Im}(\delta_2 \circ \pi)} \cong \operatorname{Coker}\pi = B_1^{(1)}$$

Since $\operatorname{Coker}\delta_2 = \tilde{E}_2^{n,1}$ we now see that $a_1 = \tilde{e}_2^{n,1} + b_1^{(1)}$.

For the other equation we will assume $n \geq 2$ and consider another commutative diagram with exact rows (compare with (2.5.5) and (2.5.6))

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathcal{J}\Omega^{n-1}_Y(\log D)) & \to & H^0(\Omega^{n-1}_Y(\log D)) & \xrightarrow{\gamma} & H^0(\Omega^{n-1}_Y(\log D) \otimes \mathcal{O}_D) & \to & B_2^{(1)} & \to 0 \\
& \downarrow{\scriptstyle \delta_3} & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_4} & & & \\
0 \to & H^0(\mathcal{J}\Omega^n_Y(\log D)) & \to & H^0(\Omega^n_Y(\log D)) & \longrightarrow & H^0(\Omega^n_Y(\log D) \otimes \mathcal{O}_D) & \to & 0, &
\end{array}$$

where again the vertical maps are induced from exterior derivation. From the degeneration at E_1 of the spectral sequence $E(\Omega^\cdot_Y(\log D) \otimes \mathcal{O}_D)$ introduced in (2.1) we see that δ_4 is trivial, so δ_1 factors through a map $\tilde{\delta}_1 : H^0(\Omega^{n-1}_Y(\log D)) \longrightarrow H^0(\mathcal{J}\Omega^n_Y(\log D))$ with the same kernel as δ_1.

By definition, $A_3 = \operatorname{Coker} \gamma|_{\operatorname{Ker} \delta_1}$, which fits into the exact sequence

$$0 \to \frac{\operatorname{Im} \gamma}{\gamma(\operatorname{Ker} \delta_1)} \longrightarrow A_3 \longrightarrow B_2^{(1)} \to 0.$$

There is another exact sequence

$$0 \to \frac{\operatorname{Im} \tilde{\delta}_1}{\operatorname{Im} \delta_3} \longrightarrow \operatorname{Coker} \delta_3 \longrightarrow \operatorname{Coker} \tilde{\delta}_1 \to 0,$$

in which we find $E_2^{n,0} = \operatorname{Coker} \delta_3$ and, by definition, $A_2 = \operatorname{Coker} \tilde{\delta}_1$. From the following isomorphisms

$$\frac{\operatorname{Im} \gamma}{\gamma(\operatorname{Ker} \delta_1)} \underset{\cong}{\xleftarrow{\gamma}} \frac{H^0(\Omega^{n-1}_Y(\log D))}{\operatorname{Ker} \gamma + \operatorname{Ker} \delta_1} = \frac{H^0(\Omega^{n-1}_Y(\log D))}{\operatorname{Ker} \gamma + \operatorname{Ker} \tilde{\delta}_1} \underset{\cong}{\xrightarrow{\tilde{\delta}_1}} \frac{\operatorname{Im} \tilde{\delta}_1}{\operatorname{Im} \delta_3}.$$

it now follows that

$$a_3 = b_2^{(1)} + \dim \frac{\operatorname{Im} \gamma}{\gamma(\operatorname{Ker} \delta_1)} = b_2^{(1)} + e_2^{n,0} - a_2,$$

as desired. $\square$

2.8. Now combine (2.6.1), (2.6.2) and the equations from the lemma (still assuming $n \geq 3$) to give

$$\begin{aligned} \mu - \tau^{(p)} = &\sum_{i=0}^{p-1} \Big(\dim \operatorname{Gr}_F^{n-i} H^{n+1}_{\{x\}}(X, \mathbb{C}) - \dim \operatorname{Gr}_F^{n-i-1} H^n_{\{x\}}(X, \mathbb{C}) \Big) \\ &+ \left(\tilde{e}_2^{n,1} + \sum_{i=2}^{p} \left(\tilde{e}_1^{n-i+1,i} - \tilde{e}_1^{n-i,i} \right) \right) + \left(e_2^{n,0} + \sum_{i=1}^{p-1} \left(e_1^{n-i,i} - e_1^{n-i-1,i} \right) \right) \\ &+ \left(b_1^{(p)} + b_2^{(p)} \right) \end{aligned} \tag{2.8.1}$$

To finish the proof of Theorem 3 we will find other expressions for those terms in (2.8.1) that contain negative signs.

Lemma 2.9. *Let $h^{i,j}$ be the (i,j)'th Hodge number of $H^n_{\{x\}}(X,\mathbb{C})$. Then*

$$\sum_{i=0}^{p-1}\Big(\dim \operatorname{Gr}_F^{n-i} H^{n+1}_{\{x\}}(X,\mathbb{C}) - \dim \operatorname{Gr}_F^{n-i-1} H^{n}_{\{x\}}(X,\mathbb{C})\Big) = \sum_{i=0}^{p-1}\sum_{j=0}^{n-p-1} h^{i,j}.$$

Proof. The mixed Hodge structure on $H^n_{\{x\}}(X,\mathbb{C})$ is of weight $\leq n-1$ [St1, Cor. 1.12], so $\dim \operatorname{Gr}_F^{n-i-1} H^n_{\{x\}}(X,\mathbb{C}) = \sum_{j=0}^{i} h^{n-i-1,j} = \sum_{j=0}^{i} h^{j,n-i-1}$. Hence,

$$\sum_{i=0}^{p-1} \dim \operatorname{Gr}_F^{n-i-1} H^n_{\{x\}}(X,\mathbb{C}) = \sum_{i=0}^{p-1}\sum_{j=0}^{i} h^{j,n-i-1} = \sum_{j=0}^{p-1}\sum_{i=j}^{p-1} h^{j,n-i-1} = \sum_{j=0}^{p-1}\sum_{i=n-p}^{n-j-1} h^{j,i}.$$

On the other hand, Poincaré duality [St1, Cor. 1.15] gives $\dim \operatorname{Gr}_F^{n-i} H^{n+1}_{\{x\}}(X,\mathbb{C}) = \dim \operatorname{Gr}_F^{i} H^{n}_{\{x\}}(X,\mathbb{C}) = \sum_{j=0}^{n-i-1} h^{i,j}$, so

$$\sum_{i=0}^{p-1} \dim \operatorname{Gr}_F^{n-i} H^{n+1}_{\{x\}}(X,\mathbb{C}) = \sum_{i=0}^{p-1}\sum_{j=0}^{n-i-1} h^{i,j}.$$

Now the result is obtained by combining the two displayed equations. $\square$

Lemma 2.10. *Assume $n \geq 2$ and $1 \leq p \leq n-1$. Then*

$$\tilde{e}_2^{n,1} + \sum_{i=2}^{p}\Big(\tilde{e}_1^{n-i+1,i} - \tilde{e}_1^{n-i,i}\Big) = \tilde{e}_2^{n-p+1,p} + \tilde{e}_3^{n-p+2,p-1} + \tilde{e}_4^{n-p+3,p-2} + \cdots + \tilde{e}_{p+1}^{n,1}.$$

Proof. For $p=1$ the formula is trivially true. We will proceed by induction in p. It follows from Proposition 2.4 that $\tilde{E}_1^{i,j} = 0$ when $i+j = n-1, n+2$ and $j \leq n-1$. This gives us the exact sequences

$$\begin{array}{ccccccccc}
0\to \tilde{E}_2^{n-p,p} & \longrightarrow & \tilde{E}_1^{n-p,p} & \longrightarrow & \tilde{E}_1^{n-p+1,p} & \longrightarrow & \tilde{E}_2^{n-p+1,p} & \to 0 \\
0\to \tilde{E}_3^{n-p,p} & \longrightarrow & \tilde{E}_2^{n-p,p} & \longrightarrow & \tilde{E}_2^{n-p+2,p-1} & \longrightarrow & \tilde{E}_3^{n-p+1,p-1} & \to 0 \\
\vdots & & \vdots & & \vdots & & \vdots & \\
0\to \tilde{E}_{p+1}^{n-p,p} & \longrightarrow & \tilde{E}_p^{n-p,p} & \longrightarrow & \tilde{E}_p^{n,1} & \longrightarrow & \tilde{E}_{p+1}^{n,1} & \to 0,
\end{array}$$

where $\tilde{E}_{p+1}^{n-p,p} = \tilde{E}_{\infty}^{n-p,p} = 0$, since $\tilde{E}_{p+1}^{n+1,0} = 0$. From this we get

$$\begin{aligned}
&\tilde{e}_2^{n-p+1,p} + \tilde{e}_3^{n-p+2,p-1} + \tilde{e}_4^{n-p+3,p-2} + \cdots + \tilde{e}_{p+1}^{n,1} \\
&\qquad = (\tilde{e}_1^{n-p+1,p} - \tilde{e}_1^{n-p,p}) + (\tilde{e}_2^{n-p+2,p-1} + \tilde{e}_3^{n-p+3,p-2} + \cdots + \tilde{e}_p^{n,1})
\end{aligned}$$

and, by the induction hypothesis,

$$\begin{aligned}
&= (\tilde{e}_1^{n-p+1,p} - \tilde{e}_1^{n-p,p}) + \tilde{e}_2^{n,1} + \sum_{i=2}^{p-1} \left(\tilde{e}_1^{n-i+1,i} - \tilde{e}_1^{n-i,i}\right) \\
&= \tilde{e}_2^{n,1} + \sum_{i=2}^{p} \left(\tilde{e}_1^{n-i+1,i} - \tilde{e}_1^{n-i,i}\right)
\end{aligned}$$

as claimed. □

Lemma 2.11. *Assume $n \geq 2$ and $1 \leq p \leq n-1$. Then*

$$e_2^{n,0} + \sum_{i=1}^{p-1} \left(e_1^{n-i,i} - e_1^{n-i-1,i}\right) = e_2^{n-p+1,p-1} + e_3^{n-p+2,p-2} + e_4^{n-p+3,p-3} + \cdots + e_{p+1}^{n,0}$$

Proof. The case $p = 1$ is trivially true. It follows from Proposition 2.4 that $E_1^{i,j} = 0$ when $i + j = n-2, n+1$ and $j \geq 1$. The induction step for $p > 1$ can now be performed as in the proof of Lemma 2.10 by replacing all occurences of $\tilde{E}_k^{i,j}$ and $\tilde{e}_k^{i,j}$ by $E_k^{i,j-1}$ and $e_k^{i,j-1}$ respectively. □

Now enter the expressions found in Lemmas 2.9, 2.10 and 2.11 in the right-hand side of (2.8.1) to establish the formula of Theorem 3 and thereby completing the proof.

3. Some mixed Hodge structures

In this section we study some of the mixed Hodge structures associated with singularities satisfying $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$. Notice that this includes the weighted homogeneous singularities.

3.1 Quite a few more spectral sequences. Recall the notation introduced in (2.1). Since $\mathbb{H}(\Omega_U^{\cdot}) \cong \mathbb{H}(\mathbb{C}_U)$, $\mathbb{H}(\Omega_Y^{\cdot}(\log D)) \cong \mathbb{H}(\mathbb{C}_U)$ and $\mathbb{H}(\Omega_Y^{\cdot}(\log D)) \cong \mathbb{H}(\mathbb{C}_U)^2$, we get

[2] When convenient we regard the constant sheaf $\mathbb{C}_U$ on U as a complex of sheaves concentrated in degree 0. Similarly with $\mathbb{C}_Y$.

the following 3 spectral sequences all converging to cohomology, $\mathbb{H}(\mathbb{C}_U)$, of U

$$\begin{aligned} E_1^{p,q}(\Omega_U^{\cdot}) &\implies H^{p+q}(U,\mathbb{C}) \\ E_1^{p,q}(\Omega_Y^{\cdot}(\log D)) &\implies H^{p+q}(U,\mathbb{C}) \\ E_1^{p,q}(\Omega_Y^{\cdot}(\log D)\otimes\mathcal{O}_D) &\implies H^{p+q}(U,\mathbb{C}). \end{aligned}$$

In all three cases, E_∞ is the quotient space of the filtration on $\mathbb{H}(\mathbb{C}_U)$ induced by the *filtration bête* of the underlying complex (i.e. of $\Omega_U^{\cdot}$, $\Omega_Y^{\cdot}(\log D)$ and $\Omega_Y^{\cdot}(\log D)\otimes\mathcal{O}_D$, respectively.) In (2.1) it was recalled that the sequence $E(\Omega_Y^{\cdot}(\log D)\otimes\mathcal{O}_D)$ degenerates at E_1 and that the induced filtration on $\mathbb{H}(\mathbb{C}_U)$ is the Hodge filtration of the canonical mixed Hodge structure.

The chain homomorphisms $\Omega_Y^{\cdot}(\log D)\to\Omega_U^{\cdot}$ and $\Omega_Y^{\cdot}(\log D)\to\Omega_Y^{\cdot}(\log D)\otimes\mathcal{O}_D$, induce homomorphisms

$$\begin{aligned} &E_r^{p,q}(\Omega_Y^{\cdot}(\log D)) \xrightarrow{\beta_r^{p,q}} E_r^{p,q}(\Omega_U^{\cdot}) \quad\text{and} \\ &E_r^{p,q}(\Omega_Y^{\cdot}(\log D)) \xrightarrow{\gamma_r^{p,q}} E_r^{p,q}(\Omega_Y^{\cdot}(\log D)\otimes\mathcal{O}_D) \end{aligned} \tag{3.1.1}$$

for all integers p,q and $r=1,2,\ldots,\infty$. In particular we see that the filtration on $\mathbb{H}(\mathbb{C}_U)$ induced by $\Omega_Y^{\cdot}(\log D)$ is contained in each of the filtrations induced by $\Omega_U^{\cdot}$ and $\Omega_Y^{\cdot}(\log D)\otimes\mathcal{O}_D$, but there are no a priori relations between the latter two filtrations. However, we will see below that the three filtrations coincide provided $\mu=\tau^{(1)}=\cdots=\tau^{(n-1)}$.

3.2. Since the complex $\Omega_Y^{\cdot}$ is a resolution of the constant sheaf $\mathbb{C}_Y$ on Y, then we have natural isomorphisms $\mathbb{H}(\Omega_Y^{\cdot})\cong\mathbb{H}(\mathbb{C}_Y)$ and $\mathbb{H}_D(\Omega_Y^{\cdot})\cong\mathbb{H}_D(\mathbb{C}_Y)$. Thus we have two more spectral sequences

$$E_1^{p,q}(\Omega_Y^{\cdot})=H^q(\Omega_Y^p) \implies \mathbb{H}^{p+q}(\Omega_Y^{\cdot})$$

and

$$\tilde{E}_1^{p,q}(\Omega_Y^{\cdot})=H_D^q(\Omega_Y^p) \implies \mathbb{H}_D^{p+q}(\Omega_Y^{\cdot})$$

(please recall our notational conventions described in (2.3)). The filtrations on $\mathbb{H}(\Omega_Y^{\cdot})$ and $\mathbb{H}_D(\Omega_Y^{\cdot})$ implicitly defined by $E_\infty(\Omega_Y^{\cdot})$ and $\tilde{E}_\infty(\Omega_Y^{\cdot})$ are induced from the *filtration bête* on $\Omega_Y^{\cdot}$.

By Deligne [D]$\mathbb{H}(\mathbb{C}_Y)$ and $\mathbb{H}_D(\mathbb{C}_Y)$ have canonical mixed Hodge structures compatible with the triangle

$$\mathbb{H}_D(\mathbb{C}_Y)\to\mathbb{H}(\mathbb{C}_Y)\to\mathbb{H}(\mathbb{C}_U)\xrightarrow{+1}\mathbb{H}_D(\mathbb{C}_Y).$$

The Hodge filtrations can be defined by two spectral sequences

$$\begin{aligned} E_1^{p,q}(\Omega_Y^{\cdot}/\mathcal{J}\Omega_Y^{\cdot}(\log D)) &= H^q(\Omega_Y^p/\mathcal{J}\Omega_Y^p(\log D)) \implies H^{p+q}(Y,\mathbb{C}) \\ E_1^{p,q}(\Omega_Y^{\cdot}(\log D)/\Omega_Y^{\cdot}) &= H^q(\Omega_Y^p(\log D)/\Omega_Y^p) \implies H_D^{p+q+1}(Y,\mathbb{C}) \end{aligned}$$

both of which degenerate at E_1.

Theorem 6 is a corollary to the following.

Theorem 3.3. *Assume $n \geq 2$ and $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$ (satisfied e.g. if (X, x) is weighted homogeneous). Then the isomorphisms*

$$\mathbb{H}_D(Y, \mathbb{C}) \cong \mathbb{H}_D(\Omega^{\cdot}_Y), \quad \mathbb{H}(Y, \mathbb{C}) \cong \mathbb{H}(\Omega^{\cdot}_Y), \quad \mathbb{H}(U, \mathbb{C}) \cong \mathbb{H}(\Omega^{\cdot}_U)$$

are isomorphisms of filtered objects, when $\mathbb{H}_D(Y, \mathbb{C})$, $\mathbb{H}(Y, \mathbb{C})$ and $\mathbb{H}(U, \mathbb{C})$ are equipped with their Hodge filtrations. The spectral sequences $\tilde{E}(\Omega^{\cdot}_Y)$, $E(\Omega^{\cdot}_Y)$ and $E(\Omega^{\cdot}_U)$ all degenerate at E_2 giving isomorphisms

$$\begin{aligned} \tilde{E}_2^{p,q}(\Omega^{\cdot}_Y) &\cong \operatorname{Gr}_F^p H_D^{p+q}(Y, \mathbb{C}), \\ E_2^{p,q}(\Omega^{\cdot}_Y) &\cong \operatorname{Gr}_F^p H^{p+q}(Y, \mathbb{C}), \\ E_2^{p,q}(\Omega^{\cdot}_U) &\cong \operatorname{Gr}_F^p H^{p+q}(U, \mathbb{C}). \end{aligned}$$

Furthermore, the mixed Hodge structures on $H_D^r(Y, \mathbb{C})$ and $H^r(Y, \mathbb{C}) = H^r(D, \mathbb{C})$ are pure of weight r for all $r \geq 0$, whereas the mixed Hodge structure on $H^r(U, \mathbb{C})$ is pure of weight r for $r \leq n-1$ and of weight $r+1$ for $r \geq n$.

3.4 Beginning of proof of Theorem 3.3. The proof of the theorem is quite long, and will be pieced together from several lemmas below. We will start by holding our assumptions up against the conclusions of Theorem 1 and Theorem 3.

Assume that $n \geq 2$ and $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$. From Theorem 1 and 3 we then see that the Hodge numbers $h^{p,q}$ vanish for $p + q < n - 1$. On the other hand, Steenbrink proves in [St1, Cor. 1.12] that the mixed Hodge structure on $H^{n-1}(U, \mathbb{C})$ has weights at most $n-1$, so this proves that the Hodge structure on $H^{n-1}(U, \mathbb{C})$ is pure of weight $n-1$. By Serre-duality [St1, Cor. 1.15] we immediately get the purity and weight assertions of our theorem for $H^r(U, \mathbb{C})$. Purity of weight r of the mixed Hodge structures on $H_D^r(Y, \mathbb{C})$ and $H^r(Y, \mathbb{C})$ is then another consequence of Steenbrinks [St1, Cor. 1.12].

There are a few more immediate consequences of our assumption. From Theorem 1 it follows that $a_1 = a_2 = a_3 = 0$, and provided $n \geq 3$ we get from Theorem 3 that $b_1^{(q)} = b_2^{(q)} = 0$ for $q = 1, \ldots, n-1$. For $n = 2$ we will still have $b_2^{(1)} = 0$ because of Lemma 2.7, but $b_1^{(1)}$ may be non-trivial.

Lemma 3.5. *Assume $n \geq 2$ and $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$. Then the spectral sequences from (2.2) degenerate at E_2, that is,*

$$\tilde{E}_2^{p,q}(\Omega^{\cdot}_Y(\log D)) = 0 \quad \text{and} \quad E_2^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = 0 \quad \text{for all } p, q.$$

Proof. The assumptions imply that $e_2^{n,0} = e_2^{n-1,1} = \cdots = e_2^{2,n-2} = 0$; in fact, for $n \geq 3$ this follows immediately from Theorem 3, whereas for $n = 2$ one has to combine Theorem 1 with Lemma 2.7. Together with the vanishing results from Proposition 2.4 this is enough to ensure that $E_2^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = E_\infty^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = 0$ for all p, q. By duality, $\tilde{E}_2^{p,q}(\Omega^{\cdot}_Y(\log D)) = 0$. □

Proposition 3.6. *Assume $n \geq 2$ and $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$. Then $\mathbb{H}(\Omega^{\cdot}_Y(\log D)) \xrightarrow{\cong} \mathbb{H}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D)$ is a filtered isomorphism, and the spectral sequence $E(\Omega^{\cdot}_Y(\log D))$ degenerates at E_2.*

Proof. We claim that there is an exact sequence

$$0 \to H^q(\mathcal{J}\Omega^p_Y(\log D)) \longrightarrow H^q(\Omega^p_Y(\log D)) \longrightarrow H^q(\Omega^p_Y(\log D) \otimes \mathcal{O}_D) \to 0 \qquad (3.6.1)$$

for all p, q. To prove this fact, let K denote the kernel of $H^q(\mathcal{J}\Omega^p_Y(\log D)) \longrightarrow H^q(\Omega^p_Y(\log D))$. We must show that $K = 0$.

For $q = 0$ this is obvious, and for $p + q \neq n-1, n$, $q > 0$ it follows from Proposition 2.4 (b). Since $E_1^{0,n}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = E_\infty^{0,n}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = 0$ then $K = 0$ also when $(p,q) = (0,n)$. For $p + q = n$ and $1 \leq q \leq n-1$, $K = B_2^{(q)}$, which is trivial by our assumptions. It only remains to prove that $K = 0$ when $p + q = n-1$ and $q > 0$. But in that case K is isomorphic to the cokernel of

$$E_1^{p,n-2-p}(\Omega^{\cdot}_Y(\log D)) \xrightarrow{\gamma_1^{p,n-2-p}} E_1^{p,n-2-p}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D) \cong \operatorname{Gr}_F^p H^{n-2}(U),$$

and $\operatorname{Gr}_F^p H^{n-2}(U) = 0$ for $n \geq 3$ and also for $n = 2$ provided $p > 0$. We are thus left with proving that $\gamma_1^{0,0}$ is surjective also in case $n = 2$. For that purpose observe that we always have an injection $\iota : E_\infty^{0,0}(\Omega^{\cdot}_Y(\log D)) \hookrightarrow E_1^{0,0}(\Omega^{\cdot}_Y(\log D))$, and that $\gamma_\infty^{0,0}$ factors as $\gamma_1^{0,0} \circ \iota$. Clearly, $\gamma_\infty^{0,0}$ is an isomorphism, so $\gamma_1^{0,0}$ must be surjective, as desired.

We have now proved that $K = 0$ for all p, q, and exactness of (3.6.1) follows.

We may rewrite (3.6.1) as the exact sequence of complexes

$$0 \to E_1^{\cdot,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) \longrightarrow E_1^{\cdot,q}(\Omega^{\cdot}_Y(\log D)) \xrightarrow{\gamma_1^{\cdot,q}} E_1^{\cdot,q}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D) \to 0.$$

From this we get a long-exact sequence

$$\begin{aligned} \ldots \longrightarrow E_2^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) &\longrightarrow E_2^{p,q}(\Omega^{\cdot}_Y(\log D)) \\ &\xrightarrow{\gamma_2^{p,q}} E_2^{p,q}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D) \longrightarrow E_2^{p+1,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) \longrightarrow \ldots, \end{aligned}$$

and since $E_2^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D))$ vanishes by Proposition 2.4, then $\gamma_2^{p,q}$ is an isomorphism for all p, q.

Since $E(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D)$ degenerates already at E_1, this will be sufficient to ensure that $E(\Omega^{\cdot}_Y(\log D))$ degenerates at E_2 and that $\mathbb{H}(\Omega^{\cdot}_Y(\log D)) \longrightarrow \mathbb{H}(\Omega^{\cdot}_Y(\log D) \otimes \mathcal{O}_D)$ is an isomorphism of filtered objects, as required. $\square$

3.7 Proof of the assertions in Theorem 3.3 concerning $\mathbb{H}(Y, \mathbb{C})$. We have already shown that the Hodge structure on $\mathbb{H}(Y, \mathbb{C})$ is pure of the specified weight. The Hodge filtration on $\mathbb{H}^{\cdot}(Y,\mathbb{C}) \cong \mathbb{H}^{\cdot}(\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D))$ is induced from the *filtration bête* on the complex $\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D)$. The morphism of complexes $\Omega^{\cdot}_Y \to \Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D)$ induces homomorphisms $E_r^{p,q}(\Omega^{\cdot}_Y) \to E_r^{p,q}(\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D))$ for all p,q and $r = 1,\dots,\infty$. It will suffice to show that

$$E_2^{p,q}(\Omega^{\cdot}_Y) \to E_2^{p,q}(\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D))$$

is an isomorphism for all p,q.

To that end, notice that for all p there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}\Omega^p_Y(\log D) \arrow[r] \arrow[d, equal] & \Omega^p_Y \arrow[r] \arrow[d] & \Omega^p_Y/\mathcal{J}\Omega^p_Y(\log D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}\Omega^p_Y(\log D) \arrow[r] & \Omega^p_Y(\log D) \arrow[r] & \Omega^p_Y(\log D)\otimes\mathcal{O}_D \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. From the cohomological long-exact sequences arising from the rows we get (for all q) a commutative diagram

$$\begin{tikzcd}
H^q(\Omega^p_Y/\mathcal{J}\Omega^p_Y(\log D)) \arrow[r] \arrow[d] & H^{q+1}(\mathcal{J}\Omega^p_Y(\log D)) \arrow[d, equal] \\
H^q(\Omega^p_Y(\log D)\otimes\mathcal{O}_D) \arrow[r] & H^{q+1}(\mathcal{J}\Omega^p_Y(\log D)).
\end{tikzcd}$$

By virtue of (3.6.1) the lower horizontal map is trivial; hence, so is the upper. This way we get for all q a short exact sequence of complexes

$$0 \to E_1^{\cdot,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) \to E_1^{\cdot,q}(\Omega^{\cdot}_Y) \to E_1^{\cdot,q}(\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D)) \to 0,$$

which induces the long-exact

$$\begin{aligned}\cdots \to E_2^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) &\to E_2^{p,q}(\Omega^{\cdot}_Y)\\ &\to E_2^{p,q}(\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D)) \to E_2^{p+1,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) \to \dots.\end{aligned}$$

Since $E_2^{p,q}(\mathcal{J}\Omega^{\cdot}_Y(\log D)) = 0$ for all p,q by Proposition 2.4, we obtain the desired isomorphisms $E_2^{p,q}(\Omega^{\cdot}_Y) \xrightarrow{\cong} E_2^{p,q}(\Omega^{\cdot}_Y/\mathcal{J}\Omega^{\cdot}_Y(\log D))$.

This shows all the assertions in Theorem 3.3 concerning $\mathbb{H}(Y,\mathbb{C})$. $\square$

3.8 Proof of the assertions in Theorem 3.3 concerning $\mathbb{H}(U,\mathbb{C})$. Recall from (3.1.1) that for all p,q,r we have morphisms

$$E_r^{p,q}(\Omega^{\cdot}_Y(\log D)) \xrightarrow{\beta_r^{p,q}} E_r^{p,q}(\Omega^{\cdot}_U).$$

Claim: $\beta_2^{p,q}$ is an isomorphism for all p when $q \leq n-2$.

For $p+q \neq n-1, n$ this is clear from Proposition 2.4, so we can restrict our attention to the cases $p+q=n-1$ and $p+q=n$.

Assume $q=0$ and recall the terminology from the proof of Lemma 2.7. Injectivity of δ_2 implies that $\operatorname{Ker}\delta_1 \cong \operatorname{Ker}\delta$. Since $E_2^{n-1,0}(\Omega_Y^{\cdot}(\log D)) = E_\infty^{n-1,0}(\Omega_Y^{\cdot}(\log D))$ is a quotient of $\operatorname{Ker}\delta_1$ and $E_2^{n-1,0}(\Omega_U^{\cdot})$ is a quotient of $\operatorname{Ker}\delta$, then $\beta_2^{n-1,0}$ must be surjective, hence an isomorphism. From the proof of Lemma 2.7 we see that $\beta_2^{n,0}$ coincides with the map $\operatorname{Coker}\delta_1 \longrightarrow \operatorname{Coker}\delta$, whose cokernel is A_1. By our assumptions $A_1 = 0$, so $\beta_2^{n,0}$ is indeed an isomorphism.

Now assume $1 \leq q \leq n-2$. This can only happen if $n \geq 3$, and it follows from our assumptions and Theorem 3 that $B_1^{(q)} = 0$ and $B_1^{(q+1)} = 0$. Hence the rows in the diagram

$$\begin{array}{ccccccccc}
0 \to & H^q(\Omega_Y^{n-q-1}(\log D)) & \longrightarrow & H^q(\Omega_U^{n-q-1}) & \longrightarrow & \tilde{E}_1^{n-q-1,q+1}(\Omega_Y^{\cdot}(\log D)) & \to 0 \\
& \Big\downarrow{\scriptstyle \delta_1^q} & & \Big\downarrow{\scriptstyle \delta^q} & & \Big\downarrow{\scriptstyle \cong} & \\
0 \to & H^q(\Omega_Y^{n-q}(\log D)) & \longrightarrow & H^q(\Omega_U^{n-q}) & \longrightarrow & \tilde{E}_1^{n-q,q+1}(\Omega_Y^{\cdot}(\log D)) & \to 0
\end{array}$$

are exact. The map $\tilde{E}_1^{n-q-1,q+1}(\Omega_Y^{\cdot}(\log D)) \longrightarrow \tilde{E}_1^{n-q,q+1}(\Omega_Y^{\cdot}(\log D))$ is an isomorphism by Lemma 3.5, so we get isomorphisms $\operatorname{Ker}\delta_1^q \longrightarrow \operatorname{Ker}\delta^1$ and $\operatorname{Coker}\delta_1^q \longrightarrow \operatorname{Coker}\delta^q$. But these two isomorphisms coincide with $\beta_2^{n-q-1,q}$ and $\beta_2^{n-q,q}$ respectively.

This proves the claim.

Now notice that since $E(\Omega_Y^{\cdot}(\log D))$ degenerates at E_2 by Lemma 3.5 then surjectivity of $\beta_2^{p,q}$ for $q < n-1$ implies that also $\beta_\infty^{p,q}$ is a surjection. But since the two spectral sequences $E(\Omega_Y^{\cdot}(\log D))$ and $E(\Omega_Y^{\cdot})$ define filtrations on the same space with the first filtration contained in the other, then $\beta_\infty^{p,q}$ can only be surjective for $q < n-1$ if it is an isomorphism. This shows that $E_2^{p,q}(\Omega_U^{\cdot}) = E_\infty^{p,q}(\Omega_U^{\cdot})$ for $q < n-1$. Since $E_1^{p,q}(\Omega_U^{\cdot}) = 0$ and $E_1^{p,q}(\Omega_Y^{\cdot}(\log D)) = 0$ for $q > n-1$ it follows that $E(\Omega_U^{\cdot})$ degenerates at E_2 and that $\beta_2^{p,q}$ is an isomorphism for all p, q. This shows that the Hodge filtration on $\mathbb{H}(\mathbb{C}_U)$ is determined by $E_2(\Omega_U^{\cdot})$.

We have thus proved all the assertions in Theorem 3.3 concerning $\mathbb{H}(U, \mathbb{C})$. □

3.9 End of proof of Theorem 3.3. Under the assumptions of the theorem we have already proved the statements regarding purity and weights of all the Hodge structures, and that the filtrations $F^{\cdot}\mathbb{H}(\mathbb{C}_Y)$ on $\mathbb{H}(\mathbb{C}_Y)$ and $F^{\cdot}\mathbb{H}(\mathbb{C}_U)$ on $\mathbb{H}(\mathbb{C}_U)$ induced from the *filtration bête* on $\Omega_Y^{\cdot}$ and $\Omega_U^{\cdot}$ respectively coincide with the Hodge filtrations. We have also seen that the two associated spectral sequences degenerate at E_2.

Denote by $\tilde{F}^{\cdot}\mathbb{H}_D(\mathbb{C}_Y)$ the filtation on $\mathbb{H}_D(\mathbb{C}_Y)$ whose p'th group is the image of $\mathbb{H}_D(\tau_{\geq p}\Omega_Y^{\cdot}) \longrightarrow \mathbb{H}_D(\Omega_Y^{\cdot}) \cong \mathbb{H}_D(\mathbb{C}_Y)$. Then clearly the long-exact sequence

$$\cdots \to H_D^r(Y) \to H^r(Y) \to H^r(U) \to H_D^{r+1}(Y) \to \dots \tag{3.9.1}$$

(which is an exact sequence of mixed Hodge structures) induces a complex

$$\cdots \to \tilde{F}^p H^r_D(Y) \to F^p H^r(Y) \to F^p H^r(U) \to \tilde{F}^p H^{r+1}_D(Y) \to \dots \tag{3.9.2}$$

for all p, but this complex is not a priori exact.

Let us now identify the Hodge filtration on $\mathbb{H}_D(\mathbb{C}_Y)$. Since the complex $\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y$ is supported on D then we get an isomorphism of filtered objects

$$\mathbb{H}_D(\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y) \xrightarrow{\cong} \mathbb{H}(\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y)$$

with the filtations induced from the *filtration bête* on $\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y$. From the exact sequence of complexes

$$0 \to \Omega^\cdot_Y \to \Omega^\cdot_Y(\log D) \to \Omega^\cdot_Y(\log D)/\Omega^\cdot_Y \to 0$$

we then get a morphism of filtered objects

$$\mathbb{H}_D(\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y) \xrightarrow{\iota} \mathbb{H}_D(\Omega^\cdot_Y).$$

Since $\mathbb{H}_D(\Omega^\cdot_Y(\log D)) = 0$ by Proposition 2.4 then ι is an isomorphism if we forget about the filtrations. The Hodge filtration $F^\cdot\mathbb{H}_D(\mathbb{C}_Y)$ on $\mathbb{H}_D(\mathbb{C}_Y)$ is thus the filtration induced from the filtration on $\mathbb{H}_D(\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y)$ via the isomorphism

$$\mathbb{H}_D(\Omega^\cdot_Y(\log D)/\Omega^\cdot_Y) \xrightarrow{\iota} \mathbb{H}_D(\Omega^\cdot_Y) \cong \mathbb{H}_D(\mathbb{C}_Y).$$

Since ι is a morphism of filtered objects it follows that $F^p\mathbb{H}_D(\mathbb{C}_Y) \subset \tilde{F}^p\mathbb{H}_D(\mathbb{C}_Y)$ for all p so we get (for all r) the following diagram

$$\begin{tikzcd}
F^pH^{r-1}(U) \arrow[r] \arrow[d, equal] & F^pH^r_D(Y) \arrow[r] \arrow[d, "\iota"'] & F^pH^r(Y) \arrow[r] \arrow[d, equal] & F^pH^r(U) \arrow[d, equal] \\
F^pH^{r-1}(U) \arrow[r] & \tilde{F}^pH^r_D(Y) \arrow[r] & F^pH^r(Y) \arrow[r] & F^pH^r(U)
\end{tikzcd}$$

whose rows are complexes. From exactness of the top row it now follows immediately that ι is surjective, so in fact the two filtrations $F^\cdot$ and $\tilde{F}^\cdot$ on $H^r_D(Y)$ coincide.

To prove degeneracy of $\tilde{E}(\Omega^\cdot_Y)$ at E_2, we shall use the fact that $\mathrm{Gr}^p_F H^r_D(Y)$ is Poincaré dual to $\mathrm{Gr}^{n-p}_F H^{2n-r}(Y)$ [F] (see also [St1, 1.6]), that is, $\tilde{E}^{p,q}_\infty(\Omega^\cdot_Y)$ is dual to $E^{n-p,n-q}_\infty(\Omega^\cdot_Y)$ for all p,q. But $\tilde{E}^{p,q}_1(\Omega^\cdot_Y) = H^q_D(\Omega^p_Y)$ is the Serre-dual of $E^{n-p,n-q}_1(\Omega^\cdot_Y) = H^{n-q}(\Omega^{n-p}_Y)$, at least for $q < n$, and since the d-operator is self adjoint up to a sign with respect to the Serre-pairing, then we see that $\tilde{E}^{p,q}_2(\Omega^\cdot_Y)$ is dual to $E^{n-p,n-q}_2(\Omega^\cdot_Y)$ for $q < n$. Now degeneration of $E(\Omega^\cdot_Y)$ at E_2 implies that $\tilde{E}^{p,q}_2(\Omega^\cdot_Y)$ and $\tilde{E}^{p,q}_\infty(\Omega^\cdot_Y)$ have the same dimensions, hence are equal, for $q < n$. Since $\tilde{E}^{p,q}_1(\Omega^\cdot_Y) = 0$ for $q > n$, then all the differential maps of $\tilde{E}_2$ must be trivial, as required.

This ends the proof of Theorem 3.3. $\square$

3.10 Description of the Hodge filtration on $H^{\cdot}_{\{x\}}(X,\mathbb{C})$. Under the assumptions of Theorem 6 we can give a rather explicit description of the Hodge filtration on $H^{\cdot}_{\{x\}}(X,\mathbb{C})$ using sheaves of exact forms.

Define $d\Omega_U^{-1} = \mathbb{C}_U$ and let $d\Omega_U^p$ be the image of the sheaf homomorphism $\Omega_U^p \xrightarrow{d} \Omega_U^{p+1}$ for $p > 0$. Similarly we define $d\Omega_X^{-1} = \mathbb{C}_X$ and $d\Omega_X^p = \mathrm{Im}(\Omega_X^p \xrightarrow{d} \Omega_X^{p+1})$ for $p > 0$.

Since the complex $\Omega_U^{\cdot}$ is a resolution of $\mathbb{C}_U$ then $\tau_{\geq p}\Omega_U^{\cdot}$ shifted p positions to the left is a resolution of $d\Omega_U^{p-1}$ for all p. The de Rham evaluation map $dR_{p,q}$ is by definition the following composition of maps

$$H^q(d\Omega_U^{p-1}) = \mathbb{H}^{p+q}(\tau_{\geq p}\Omega_U^{\cdot}) \longrightarrow \mathbb{H}^{p+q}(\Omega_U^{\cdot}) = H^{p+q}(U,\mathbb{C}).$$

Provided the conclusion of Theorem 6 we see that the image of $dR_{p,q}$ coincides with $F^pH^{p+q}(U,\mathbb{C})$, and degeneration of $E(\Omega_U^{\cdot})$ at E_2 implies that

$$H^q(\Omega_U^{p-1}) \xrightarrow{d} \mathbb{H}^{p+q}(\tau_{\geq p}\Omega_U^{\cdot})) \longrightarrow \mathbb{H}^{p+q}(\Omega_U^{\cdot}))$$

is exact. This gives us an isomorphism

$$F^pH^r(U,\mathbb{C}) \cong \mathrm{Coker}\left(H^{r-p}(\Omega_U^{p-1}) \xrightarrow{d} H^{r-p}(d\Omega_U^{p-1})\right) \tag{3.10.1}$$

for all $r, p \geq 0$, where the map from right to left is induced from $dR_{p,r-p}$.

If we translate cohomology on U to local cohomology with support in x, then we get the following simple expressions for the Hodge filtration on the middle cohomology groups

Corollary 3.11. *Let (X,x) be an icis of dimension $n \geq 2$, and assume $\mu = \tau^{(1)} = \cdots = \tau^{(n-1)}$. Then we have isomorphisms*

$$F^pH^n_{\{x\}}(X,\mathbb{C}) \cong H^{n-p}_{\{x\}}(d\Omega_X^{p-1})$$
$$F^pH^{n+1}_{\{x\}}(X,\mathbb{C}) \cong \mathrm{Coker}\left(H^{n-p+1}_{\{x\}}(\Omega_X^{p-1}) \xrightarrow{d} H^{n-p+1}_{\{x\}}(d\Omega_X^{p-1})\right)$$

for $0 \leq p \leq n$, induced (from right to left) from the de Rham evaluation map.

Proof. If X is a contractible Stein representative of (X,x), then Ω_X^p, being coherent, is acyclic for $H^0(X,-)$ for all p. For $0 \leq p \leq n-1$ we have a resolution $0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^p \to d\Omega_X^p \to 0$ of $\mathbb{C}_X$ [G1, Satz 4.4], so by acyclicity of the modules $\Omega_X^{\cdot}$ we get $H^q(d\Omega_X^p) \cong H^{p+q+1}(\mathbb{C}_X) = 0$ for $q \geq 1$, so in this case also $d\Omega_X^p$ is acyclic for $H^0(X,-)$. This gives

$$H^q(\Omega_U^p) \cong H^{q+1}_{\{x\}}(\Omega_X^p) \quad \text{for } q > 0 \text{ and all } p,$$
$$H^q(d\Omega_U^p) \cong H^{q+1}_{\{x\}}(d\Omega_X^p) \quad \text{for } q > 0 \text{ and } 0 \leq p \leq n-1$$

and the formulas for $F^pH^n_{\{x\}}(\mathbb{C})$ for $p \leq n-2$ and $F^pH^{n+1}_{\{x\}}(\mathbb{C})$ for $p \leq n-1$ follow immediately from (3.10.1) and the vanishing statements of Proposition 2.4 (e).

Now $H^0(\Omega^{n-2}_X) \to H^0(\Omega^{n-2}_U)$ is an isomorphism since $H^0_{\{x\}}(\Omega^{n-2}_X) = 0$ and $H^1_{\{x\}}(\Omega^{n-2}_X) = 0$ [G2, 3.3], and $H^0(\Omega^{n-2}_X) \xrightarrow{d} H^0(d\Omega^{n-2}_X)$ is surjective since $H^1(d\Omega^{n-3}_X) = 0$. Hence,

$$\begin{aligned} H^1_{\{x\}}(d\Omega^{n-2}_X) &= \mathrm{Coker}(H^0(d\Omega^{n-2}_X) \to H^0(d\Omega^{n-2}_U)) \\ &= \mathrm{Coker}(H^0(\Omega^{n-2}_U) \xrightarrow{d} H^0(d\Omega^{n-2}_U)) \\ &\cong F^{n-1}H^{n-1}(U, \mathbb{C}) \\ &= F^{n-1}H^n_{\{x\}}(\mathbb{C}). \end{aligned}$$

By [G1, Lemma 4.5] we have $H^0_{\{x\}}(d\Omega^{n-1}_X) = 0$, which is indeed equal to $F^nH^{n-1}_{\{x\}}(\mathbb{C})$.

Consider the diagram

$$\begin{array}{ccccc} H^0(\Omega^{n-1}_X) & \longrightarrow & H^0(\Omega^{n-1}_U) & \longrightarrow & H^1_{\{x\}}(\Omega^{n-1}_X) \to 0 \\ \downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\ H^0(d\Omega^{n-1}_X) & \longrightarrow & H^0(d\Omega^{n-1}_U) & \longrightarrow & H^1_{\{x\}}(d\Omega^{n-1}_X) \to 0 \end{array}$$

with exact rows. The first vertical map is surjective since $H^1(d\Omega^{n-2}_X) = 0$, so we get an isomorphism between the cokernels of the two other vertical maps. Together with (3.10.1) this gives the desired formula for $F^nH^{n+1}_{\{x\}}(X, \mathbb{C})$. □

It would be desirable to have the formula for $F^pH^{n+1}_{\{x\}}(\mathbb{C})$ in Corollary 3.11 valid also for $p = n+1$. A short calculation will show that the dimension of $\mathrm{Coker}(d : H^0_{\{x\}}(\Omega^n_X) \to H^0_{\{x\}}(d\Omega^n_X)) = \mathrm{Coker}(d : T\Omega^n_{X,x} \to d\Omega^n_{X,x})$ is bounded by $\mu - \tau'$ (see 1.2), whereas $F^{n+1}H^{n+1}_{\{x\}}(X, \mathbb{C}) = 0$. Hence, if (X, x) is weighted homogeneous, then the formulas in 3.11 hold for all p.

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