RELATIVE COMPACTNESS CONDITIONS FOR TOPOSES

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Introduction.

In this paper a systematic study is made of various notions of "proper map" in the context of toposes.

Modulo some separation conditions, a proper map $Y \to X$ of spaces is generally understood to be a continuous function which preserves compactness of subspaces under inverse image, and which therefore in particular has compact fibers. In this spirit, a first definition of proper map between toposes was put forward by Johnstone in []. There, a map of toposes $f: \mathcal{F} \to \mathcal{E}$ was called proper if $f_*(\Omega_{\mathcal{F}})$ is a compact lattice object in the topos $\mathcal{E}$. This is probably the most direct way of expressing that $\mathcal{F}$ is compact when viewed as a topos over the base $\mathcal{E}$. (In fact, Johnstone used the term "perfect" rather than "proper", and developed the theory mostly in the context of localic maps between toposes, see []).

A related — indeed more restrictive — definition was proposed by M. Tierney and subsequently investigated by T. Lindgren []. They called a map $f: \mathcal{F} \to \mathcal{E}$ of toposes proper if the direct image functor f_* commutes with directed colimits (in a sufficiently strong, "indexed" sense). This notion had earlier been considered for the canonical morphism $\mathcal{E} \to \mathbf{Set}$ associated with a topos $\mathcal{E}$ by K. Edwards in her thesis [], where it had been shown to be equivalent to a finiteness condition (a strong kind of "compactness") for $\mathcal{E}$.

In this paper, both senses of "propriety" will play a fundamental role. To distinguish the two concepts, we shall reserve the term "proper" for the Johnstone version, and refer to Tierney-Lindgren proper maps as "tidy". Our exposition will contain most of the basic results about proper and tidy maps proved by these authors, although our proofs are generally quite different and, we believe, easier. (There is moreover a qualitative difference, in that our proofs are completely constructive and therefore apply over an arbitrary base topos.)

Besides these known results with new proofs, we also present many new results. On the one hand, these new results are partly motivated by our attempt to complete the parallel between proper and tidy maps. For example, parallel to the "classical" Bourbaki characterization of proper maps as stably closed maps, we develop a natural notion of "firmly closed map" and show that the tidy maps are exactly the stably firmly closed ones. On the other hand, we also present new results of a more specific nature. For example, in the context of proper maps we prove a Reeb stability theorem for the compact fiber of a map between toposes, which generalises the classical Reeb stability theorem for foliations. We also characterise the classifying toposes of profinite groups as exactly those hyperconnected pointed toposes with proper diagonal, and prove a similar characterization for classifying toposes of profinite groupoids. In order to deal with descent problems in the context of coherent toposes, we examine a relative notion of tidiness.

Let us describe the contents of the paper.

In Chapter I we study proper maps. We give several examples, and prove the main closure properties of the class of proper maps. Some of these properties are stated in their full generality in [], whereas others were only known for special cases (e.g. under extra separation conditions []). In particular, we show that the pullback of any proper map is again proper, thus providing the full solution of a problem raised by Johnstone in [] and partially answered there. This pullback stability of proper maps is in fact an immediate consequence of an appropriate characterization of such maps in terms of (internal) sites, which will be one of our basic technical tools.

For any pullback square

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] & \mathcal{E} \end{tikzcd} , \tag{1}$$

there is a canonical transformation

$$a^* f_* \longrightarrow g_* b^*.$$

The map f is said to satisfy the (weak) Beck-Chevalley condition if for any morphism a this transformation is an isomorphism (a monomorphism). We shall prove that f is proper precisely when f and any pullback of f satisfies the weak Beck-Chevalley condition.

After introducing a natural notion of closedness for topos morphisms, we obtain the familiar Bourbaki-style characterization of a proper map as one for which all pullbacks are closed. We end Chapter I by developing some of the theory from [] for open maps in the context of proper maps, showing that proper maps are of effective descent, for sheaves as well as for internal locales.

Any notion of propriety is accompanied by a separation condition. In particular, it is natural to define a topos $\mathcal{E}$ to be separated (or Hausdorff) if its diagonal $\mathcal{E} \to \mathcal{E} \times \mathcal{E}$ is proper. Similarly, a map $f\colon \mathcal{F} \to \mathcal{E}$ is said to be separated if the diagonal $\mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is proper. Separated maps are introduced in Chapter II. We establish the elementary closure properties of separated maps, and prove various new results. In particular, we give a characterization of hyperconnected Hausdorff toposes in a surprisingly simple way: they are exactly the classifying toposes of compact groups!

As a more elaborate application, we formulate and prove a topos version of the well known Reeb stability theorem for foliations. It states that, under suitable conditions, a separated map of toposes $f\colon \mathcal{F} \to \mathcal{E}$ has the property that in the neighbourhood of any given compact fiber, all the fibers must be compact. Our proof was to some extent inspired by the treatment of Reeb stability in Haefliger's thesis []. The classical Reeb result for foliations is a consequence of our topos theoretic version, as we shall show explicitly. It also has other applications in foliation theory, as discussed in [].

In Chapter III, we study the basic properties of tidy maps. Two (related) fundamental results were proved in []. Firstly, the class of tidy maps is stable under pullback; and secondly, a map is tidy iff it as well as any of its pullbacks satisfies the Beck-Chevalley condition. The change-of-base formula $a^*f_* \cong g_*b^*$ in (1) above is of course familiar for proper maps between paracompact Hausdorff spaces [], which are special instances of tidy maps.

We shall use a relative form of a criterion due to K. Edwards to derive a description of tidy maps in terms of sites which is appropriately "geometric", hence stable. Lindgren's results follow more or less directly from this description, as does the stability of tidy maps under filtered inverse limits. The final part of Chapter III is devoted to a description of tidy maps as those for which all pullbacks are "firmly" closed. En route, we shall extend to arbitrary tidy maps variuous results obtained in [] for the special case of proper separated maps.

We shall call a topos $\mathcal{E}$ strongly Hausdorff if the diagonal $\mathcal{E} \to \mathcal{E} \times \mathcal{E}$ is a tidy map. Chapter IV contains a discussion of some properties of such strongly Hausdorff toposes. In particular, we present a basepoint-free version of Grothendieck's Galois theory. More precisely, we prove that a coherent topos is strongly Hausdorff iff it is the classifying topos of a profinite groupoid, iff every coherent object in that topos is locally constant. This result of course has as an immediate corollary that a pointed connected coherent topos is strongly Hausdorff iff it is the classifying topos of a profinite group, which is the result underlying Grothendieck's treatment of the fundamental group.

In the final chapter, we introduce *relatively* tidy maps. We shall say a map $f: \mathcal{F} \to \mathcal{E}$ over a base topos $\mathcal{B}$ is tidy relative to $\mathcal{S}$ when its direct image functor commutes with colimits indexed by directed categories in $\mathcal{S}$. Thus, as an extreme case, a map $f: \mathcal{F} \to \mathcal{E}$ is tidy relative to $\mathcal{E}$ iff it is tidy in the ordinary sense. At the other extreme, such a map is tidy relative to the "universal" base topos of sets iff f_* preserves directed colimits in the ordinary naive sense. For example, any coherent map between coherent toposes is tidy relative to **Set**. The main result of Chapter V states that, in a *lax* pullback of toposes over a given base topos $\mathcal{C}$,

$$\begin{tikzcd} (\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{E} . \end{tikzcd}$$

the map d_0 will be tidy even if f is only relatively so; and moreover, in this case the lax pullback square will satisfy the Beck-Chevalley condition

$$g^*f_* \cong {d_0}_*{d_1}^*.$$

This result was conjectured by A. Pitts for coherent toposes and maps. By standard tripleability theory, it will follow that relatively tidy maps are lax descent maps for sheaves. We therefore obtain the so-called lax descent theorem for pretoposes as a special case. The first proof of this theorem, due to Zawadowski [], relied heavily on Makkai's theory of ultracategories and Stone duality [,,].

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CHAPTER I. PROPER MAPS

We begin with an account of the formal properties of proper maps between toposes. Our exposition will proceed along lines which generalise naturally to the treatment of tidy maps in Chapter III, with a pivotal role being played by Beck-Chevalley-type conditions (see section 3).

Our starting point is Johnstone's Louvain-la-Neuve notes [], where a number of the basic examples and results of sections 1 and 2 already appear. Proper maps of locales were extensively studied in []. Although we extend many results of [] to toposes, our approach is rather different in style, and for the most part independent. The exception is the final section, where we shall use standard tools to "lift" the descent properties of localic proper maps obtained in [] to the general case (Theorem 7.2).

The central results are contained in section 5: the stability of proper maps under pullback (Theorem 5.8) and filtered inverse limits (Theorem 5.10). Our proofs of these use a "geometric" site description of compactness (Lemma 5.4), based on a careful analysis of the interplay between finite and directed covers in a site with "enough" finite covers (section 4). We also give a Bourbaki-style characterization of proper maps as those closed maps which remain so upon pullback (section 6).

§1 Definition and examples

Let X be a topological space, and consider the topos $\mathrm{Sh}(X)$ of sheaves on X. Since open subsets of X correspond to subobjects of the terminal object 1 in $\mathrm{Sh}(X)$, *compactness* of X can be expressed as a property of $\mathrm{Sh}(X)$: every cover of 1 by subobjects has a finite subcover, or equivalently, the global sections functor $\Gamma\colon (X) \to \mathbf{Set}$ maps directed covers of 1 to covers of the one-point set. Generalising, one says a topos $\mathcal{E}$ is *compact* if right direct image along the (unique) geometric morphism $\gamma\colon \mathcal{E} \to \mathbf{Set}$, namely the functor $\gamma_*\colon \mathcal{E} \to \mathbf{Set}$ which which assigns to an object of $\mathcal{E}$ its "global" elements, preserves directed suprema of subobjects of 1:

$$\gamma_*(\bigvee U_i) = \bigvee \gamma_*(U_i) \tag{1}$$

for any directed family $\{U_i\}$, $U_i \subseteq 1$.

1.1. Remark. An *object* E of a topos $\mathcal{E}$ is said to be compact if any epimorphic family $\{E_i \to E\}$ contains a finite subfamily (or equivalently, is refined by a finite family) which is still epimorphic [, 7.31]. That is, E is compact precisely when the localization $\mathcal{E}/E$ of $\mathcal{E}$ at E is compact as a topos.

1.2. Examples. (1) For a set I, the topos of I-indexed families of sets is compact iff I is finite.

(2) A topos of G-sets for any group G is compact. More generally, a topos of presheaves $\hat{C}$ for a small category C is compact if C has a finite set of objects F which is *final* in the sense that every object c of C admits an arrow $c \to f$ into some $f \in F$.

(3) For any locale X (like a spatial locale as above), the sheaf topos $\mathrm{Sh}(X)$ is compact iff X is. A general notion of *compact site* for a topos will be introduced in §4.

(4) Any coherent topos is compact.

The fundamental notion of proper map between toposes is that of a morphism which is "relatively" compact. Recall that a topos $\mathcal{E}$ can be viewed as a "universe of sets," and that any topos morphism $f: \mathcal{F} \to \mathcal{E}$ can be regarded as a single topos "inside" this universe $\mathcal{E}$, i.e. as an $\mathcal{E}$-topos.

1.3. Definition []. A map $f: \mathcal{F} \to \mathcal{E}$ is *proper* if it renders $\mathcal{F}$ compact as an $\mathcal{E}$-topos.

Later, in §5, we shall characterise a proper map in the style of Bourbaki [], as a morphism for which all pullbacks are closed maps.

1.4. Examples. Each example in (1.2) can be interpreted in an arbitrary topos $\mathcal{E}$ in place of **Set**, with *finite* meaning "enumerated by $[n] = \{0, 1, 2, \ldots, n-1\}$ for some natural number n," or *Kuratowski-finite* [, 9.11]. Thus, relativised, (1.2.1) states that for an object $I \in \mathcal{E}$, the canonical morphism $\mathcal{E}/I \to \mathcal{E}$ is proper iff I is Kuratowski-finite in $\mathcal{E}$. The relativised form of (1.2.2) says that for a locale X in $\mathcal{E}$, the induced morphism $\mathrm{Sh}_{\mathcal{E}}(X) \to \mathcal{E}$, from the topos of internal sheaves on X, is proper iff X is a compact locale in $\mathcal{E}$.

We shall often use Definition 1.3 as it stands, treating $\mathcal{E}$ as if it were the category of (naive) sets while taking care to argue "constructively" in the sense required for a valid interpretation in any topos. It will nevertheless be useful to give an "external" version (as prescribed by the standard interpretation in a topos of statements made in the language of set theory) of at least one of the equivalent definitions of compactness.

Before doing so, let us remark that it is implicit in the form of a definition like (1.3), and easily provable from the explicit version (1.8) below, that propriety is a *local* property. To state this explictly, consider the morphism $f/E: \mathcal{F}/f^*E \to \mathcal{E}/E$ induced by an object E in $\mathcal{E}$, which is the pullback of f along the canonical morphism $\mathcal{E}/E \to \mathcal{E}$.

1.5. Proposition. *If f is proper, then so is f/E. Conversely, if $E \to 1$ is an epimorphism and f/E is proper, then so is f.*

1.6. Example. For a group-homomorphism $p: H \to G$, the induced morphism $\hat{p}: \hat{H} \to \hat{G}$ between the corresponding toposes of respectively (right) H-sets and G-sets (where $\hat{p}^*$ restricts the action of G along p) is proper iff $G/p(H)$ is finite. Indeed, there is a pullback square

$$\begin{array}{ccc} \hat{C} & \longrightarrow & \hat{H} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\hat{p}} \\ \mathbf{Set} & \longrightarrow & \hat{G}, \end{array}$$

of (presheaf) toposes, where C is the groupoid whose objects are the elements $g \in G$, and whose arrows $g \to g'$ are $h \in H$ with $g \cdot p(h) = g'$. By letting G act on itself from the right by multiplication, one has $\hat{G}/G \cong \mathbf{Set}$, and the bottom map in the pullback square is equivalent to the canonical morphism $\hat{G}/G \to \hat{G}$. It follows from (1.5) that $\hat{p}$ is proper iff γ is. By (1.2.1), the latter is the case precisely when the groupoid C has a finite set of components, i.e. when $G/p(H)$ is finite.

Returning to Definition 1.3, consider an internal category

$$I \equiv (\ I_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} I_0\ )$$

in $\mathcal{E}$. An *I-indexed family* of objects of $\mathcal{E}$ is an object in the topos $\mathcal{E}^I$ of internal diagrams on I [, 2.14], that is, of covariant (or "left") actions of I on objects of $\mathcal{E}$. An I-indexed family of subobjects of 1 in $\mathcal{E}$ then corresponds to a subobject of 1 in $\mathcal{E}^I$, which is to say, a subobject $R \subseteq I_0$ such that $d_0^* R \subseteq d_1^* R$. Of course, via the classifying map of R, we could also view such a family as a functor $I \to \Omega_{\mathcal{E}}$ in $\mathcal{E}$, where $\Omega_{\mathcal{E}}$ is the subobject-classifyer of $\mathcal{E}$ equipped with its usual order. A directed family of subobjects of 1 is one indexed by a directed (or filtered, see [, 2.51]) category I.

The canonical morphism $\pi{=}\pi_I : \mathcal{E}^I \to \mathcal{E}$ has inverse image π^* sending an object E to the corresponding "constant" diagram (E with trivial I-action); π^* has, apart from a right adjoint $\pi_* = \varprojlim_I$, also a left adjoint $\pi_! = \varinjlim_I$. Now, if I is a directed category, the functor $\pi_!$ is exact [, 2.58], and the pair $\pi_! \dashv \pi^*$ defines a canonical section of π, which we denote ∞_I or ∞ (as representing a virtual object of I "at infinity"):

$$\mathcal{E} \xrightarrow{\infty} \mathcal{E}^I, \quad \infty^* = \pi_!,\ \infty_* = \pi^*.$$

In this case the colimit of an I-indexed family of subobjects of 1 coincides with its supremum. It follows that direct image for $f : \mathcal{F} \to \mathcal{E}$ *preserves suprema of I-indexed families of subobjects of* 1 precisely when the square

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\infty"] \arrow[d, "f"'] & \mathcal{F}^{f^* I} \arrow[d, "f^I"] \\
\mathcal{E} \arrow[r, "\infty"] & \mathcal{E}^I
\end{tikzcd}$$

has the property that

$$\infty^*(f^I)_*(V) = f_* \infty^*(V),$$

for any subobject $V \subseteq 1$ in $\mathcal{F}^{f^* I}$. We shall have more to say about the form of this property, a so-called *Beck-Chevalley condition* for the square, in §3.

1.7. Examples. (1) If I is a constant directed category in $\mathcal{E}$, say $I = \gamma^* J$ where $\gamma : \mathcal{E} \to \mathbf{Set}$ is the canonical map and J an ordinary small category, then f_* preserves I-indexed suprema of subobjects of 1 iff it preserves J-indexed suprema of subobjects of 1 in the usual sense.

(2) Let U and V be subobjects of 1 in, respectively, $\mathcal{E}$ and $\mathcal{F}$. Let $I \hookrightarrow 2$ be the ideal of $2 = \{0 < 1\}$ in $\mathcal{E}$ which contains 1 over U (and 0 globally). Then the inclusion

$V + f^*U \subseteq f^*I$ defines an I-indexed family of subobjects of 1 in $\mathcal{F}$, of which the supremum is preserved by f_* iff $f_*V \vee U = f_*(V \vee f^*U)$.

1.8. Definition (of proper map, "indexed" version). A map $f\colon \mathcal{F} \to \mathcal{E}$ is said to be *proper* if, for any object $E \in \mathcal{E}$ and any directed category I in $\mathcal{E}/E$, the associated square

$$\begin{tikzcd}
\mathcal{F}/f^*E \arrow[r, "\infty"] \arrow[d, "f/E"'] & (\mathcal{F}/f^*E)^{f^*I} \arrow[d, "(f/E)^I"] \\
\mathcal{E}/E \arrow[r, "\infty"] & (\mathcal{E}/E)^I
\end{tikzcd}$$

has the property that

$$\infty^*(f/E)^I_*(V) = (f/E)_*\infty^*(V) \tag{2}$$

for any $V \subseteq 1$ in $(\mathcal{F}/f^*E)^{f^*I}$.

1.9. Example. Let $f\colon Y \to X$ be a continuous function between topological spaces. The induced morphism $\mathrm{Sh}(f)\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ between the associated sheaf-toposes is proper when f is a proper map of topological spaces, that is, a closed map with compact fibers; the converse holds if the points of X are locally closed. Indeed, $\mathrm{Sh}(f)$ satisfies (1.8) for constant I as in (1.7.1) when f has compact fibers, and for I as in (1.7.2) when f is closed (here the mild separation is needed for the converse). These two special instances suffice (see []).

1.10. Remarks. (1) Given a surjective family $\{E_i \to E\}$ in $\mathcal{E}$, it is readily seen that (1.8) is satisfied at E as soon as it holds at each E_i. Thus the second (less immediate) part of Proposition 1.5 follows, and also that it is enough to check the definition at those E in any given set of generators for $\mathcal{E}$.

(2) Subobjects of 1 in $(\mathcal{F}/f^*E)^{f^*I}$ correspond to functors $(f/E)^*I \longrightarrow \Omega_{\mathcal{F}/f^*E} \cong f^*E \times \Omega_{\mathcal{F}}$ in $\mathcal{F}/f^*E$, or by the adjunction $(f/E)^* \dashv (f/E)_*$, to functors

$$I \longrightarrow (f/E)_*\Omega_{\mathcal{F}/f^*E} \cong E \times f_*\Omega_{\mathcal{F}} \tag{3}$$

in $\mathcal{E}/E$. Since the supremum of such a family coincides with that of its image, it suffices to take for (3) the *generic directed subobject of* $f_*\Omega_{\mathcal{F}}$, which lives over the object $E \equiv \mathcal{I}f_*\Omega_{\mathcal{F}}$ of directed subobjects of $f_*\Omega_{\mathcal{F}}$. Chasing this family through (2) gives an alternative rendering of Definition 1.3, namely commutativity of the square

$$\begin{tikzcd}
\mathcal{I}f_*\Omega_{\mathcal{F}} \arrow[r, "\mathcal{I}f_*"] \arrow[d, "\bigvee"'] & \mathcal{I}\Omega_{\mathcal{E}} \arrow[d, "\bigvee"] \\
f_*\Omega_{\mathcal{F}} \arrow[r, "f_*"] & \Omega_{\mathcal{E}}
\end{tikzcd} ,$$

which is the obvious equivalent of (1) in $\mathcal{E}$ (here the bottom map f_* classifies

$$1 \cong f_* 1 \xrightarrow{f_*(true)} f_* \Omega_{\mathcal{F}},$$

the top map is "image along f_*" and the side maps internalize the supremum operation).
(3) The "indexed" version in $\mathcal{E}$ of compactness of $\mathcal{F}$ as "every cover of 1 by objects is refined by a finite cover" runs as follows: Any factorization of an epimorphism $S \to f^* E$ of $\mathcal{F}$ in the form

$$S \xrightarrow{\sigma} f^* I \xrightarrow{f^* \rho} f^* E$$

can locally be refined by another such factored epimorphism

$$T \xrightarrow{\tau} f^* K \xrightarrow{f^* \varphi} f^* E'$$

where $K \to E'$ is finite in $\mathcal{E}$. Here "locally refined" means there are maps $\alpha: E' \twoheadrightarrow E$ epi, $\beta: K \to I$ and $\gamma: T \to S$ such that the diagrams

$$\begin{tikzcd} K \arrow[r, "\beta"] \arrow[d, "\varphi"'] & I \arrow[d, "\rho"] \\ E' \arrow[r, "\alpha"] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "\gamma"] \arrow[d, "\tau"'] & S \arrow[d, "\sigma"] \\ f^* K \arrow[r, "f^* \beta"] & f^* I \end{tikzcd}$$

commute. (Intuitively, one should think of the first, given factorization as a cover $\{S_i \to 1 \mid i \in I\}$ in $\mathcal{E}/E$ and the second factorization as a finite refinement $\{T_k \to 1 \mid k \in K\}$ of $\{S_i \to 1 \mid i \in I\}$ in a further localization $\mathcal{E}/E'$.)

1.11. Example (generalising (1.6)). For a functor $p: D \to C$ between small categories, the induced morphism $\hat{p}: \hat{D} \to \hat{C}$ is proper iff for any object c in C and any final (1.2.1) family of the form

$$\{p(d_i) \xrightarrow{\delta_i} c_i \xrightarrow{\gamma_i} c\}$$

in the comma category p/c, there exists a finite final family $\{p(f_k) \xrightarrow{\varphi_k} c\}$ with the property that, for each index k, there is some index i and a commutative diagram of the form

$$\begin{tikzcd} p(f_k) \arrow[r, "p(\beta)"] \arrow[d, "\varphi_k"'] & p(d_i) \arrow[d, "\delta_i"] \\ c \arrow[r, "\alpha"] & c_i \end{tikzcd}$$

in C such that α splits γ_i, i.e. $\gamma_i \circ \alpha = \text{id}$.

§2 First properties

In this section we collect those (closure) properties of the class of proper maps which follow from the definition in a more or less formal fashion. Not treated here are stability under pullback and filtered inverse limits, for which will shall depend on a site description of propriety (§4).

A straightforward calculation with (1.8) yields:

2.1. Proposition. (i) *Any equivalence $\mathcal{F} \cong \mathcal{E}$ of toposes is proper.* (ii) *If $\mathcal{G} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{E}$ are proper, so is their composite $\mathcal{G} \to \mathcal{E}$.* ∎

2.2. Proposition. *In a commutative diagram*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "g"] \arrow[dr, "h"'] & & \mathcal{F} \arrow[dl, "f"] \\ & \mathcal{E}\ , & \end{tikzcd}$$

if g is a surjection and h is proper, then so is f.

Proof. We use (1.8). Consider for an internal category I in $\mathcal{E}$ the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "\infty_{\mathcal{G}}"] \arrow[dr, "g"] \arrow[dd, "h"'] & & \mathcal{G}^I \arrow[dr, "\bar{g}"] \arrow[dd, "\bar{h}"' near start] & \\ & \mathcal{F} \arrow[rr, "\infty_{\mathcal{F}}" near start] \arrow[dl, "f"] & & \mathcal{F}^I \arrow[dl, "\bar{f}"] \\ \mathcal{E} \arrow[rr, "\infty_{\mathcal{E}}"] & & \mathcal{E}^I & \end{tikzcd}$$

(where we write I also for the category f^*I in $\mathcal{F}$ and for $g^*f^*I = h^*I$ in $\mathcal{G}$). For any subobject $V \subseteq 1$ in $\mathcal{F}^I$, propriety of h gives

$$h_*\infty^*\bar{g}^*V = \infty^*\bar{h}_*\bar{g}^*V. \tag{1}$$

Since g is a surjection, so is $\bar{g}$. Hence $\bar{g}_*\bar{g}^*V = V$, and thus $\bar{h}_*\bar{g}^*V = \bar{f}_*\bar{g}_*\bar{g}^*V = \bar{f}_*V$. So (1) yields $h_*\infty^*\bar{g}^*V = \infty^*\bar{f}^*V$. But

$$\begin{aligned} h_*\infty^*\bar{g}^*V &= h_*g^*\infty^*V \\ &= f_*g_*g^*\infty^*V \\ &= f_*\infty^*V, \end{aligned}$$

again because g is surjective. Thus $f_*\infty^*V = \infty^*f_*V$, as desired. The same argument applied to any slice $\mathcal{E}/E$ proves the proposition. ∎

2.3. Proposition. *In a commutative diagram as in (2.2), if h is proper and f is an embedding, then g is proper.*

PROOF. We use the notation as in the previous proof. For a subobject $W \subseteq 1$ in $\mathcal{G}^I$, we want to show

$$g_*\infty^* W = \infty^* \bar{g}_* W. \tag{2}$$

Since f is an embedding, and the inequality $\geq$ in (2) always holds (by adjunction), it suffices to show that $f_* g_* \infty^* W \leq f_* \infty^* \bar{g}_* W$. But

$$\begin{aligned} f_* g_* \infty^* W &= h_* \infty^* W \\ &= \infty^* \bar{h}_* W \qquad (h \text{ proper }) \\ &= \infty^* \bar{f}_* \bar{g}_* W \\ &\leq f_* \infty^* \bar{g}_* W \quad (\text{adjunction}). \end{aligned}$$

The same argument applied to any slice $\mathcal{E}/E$ proves the proposition. ∎

Recall that a map $f: \mathcal{F} \to \mathcal{E}$ is *hyperconnected* if f induces an isomorphism

$$\mathrm{Sub}_{\mathcal{E}}(E) \xrightarrow{\sim} \mathrm{Sub}_{\mathcal{F}}(f^* E)$$

for any E; or equivalently, the canonical map $f_* \Omega_{\mathcal{F}} \to \Omega_{\mathcal{E}}$ is an isomorphism.

2.4. Proposition. *Any hyperconnected map is proper.*

PROOF. Suppose $f: \mathcal{F} \to \mathcal{E}$ is hyperconnected, and consider a diagram of the form

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\infty"] \arrow[d, "f"'] & \mathcal{F}^I \arrow[d, "\bar{f}"] \\ \mathcal{E} \arrow[r, "\infty"] & \mathcal{E}^I \end{tikzcd}$$

for directed I. Since $\bar{f}$ is a pullback of f, it is hyperconnected too, which means any given $V \subseteq 1$ in $\mathcal{F}^I$ is of the form $V = \bar{f}^* U$ for a unique $U \subseteq 1$ in $\mathcal{E}^I$. It follows that

$$\begin{aligned} f_* \infty^* V &= f_* \infty^* \bar{f}^* U \\ &= f_* f^* \infty^* U \\ &= \infty^* U, \end{aligned}$$

the latter since f is (in particular) connected. But $U = \bar{f}_* \bar{f}^* U = \bar{f}_* V$, so $f_* \infty^* V = \infty^* \bar{f}^* V$, as desired.

The same argument applied to any slice proves the proposition. ∎

Next, recall that any map $f: \mathcal{F} \to \mathcal{E}$ can be factored as $f = l \circ h$ where h is hyperconnected (hence surjective) and l is localic; thus l is of the form $\mathrm{Sh}_{\mathcal{E}}(X) \to \mathcal{E}$ for a locale X in $\mathcal{E}$.

2.5. Corollary. *A map $f: \mathcal{F} \to \mathcal{E}$ is proper iff its localic reflection* $\mathrm{Sh}_{\mathcal{E}}(X) \to \mathcal{E}$ *is, that is, iff X is a compact locale in $\mathcal{E}$.*

PROOF. Immediate from (2.1), (2.2), (2.4) (and using (1.4)). ∎

2.6. Remark. By Corollary 2.5, showing properties of proper maps of toposes reduces to adding the "hyperconnected part" to corresponding properties of proper maps of locales [], a strategy which we shall employ in §6 for establishing descent properties. In the meantime however, we continue our independent build-up of the basic properties along lines which are designed to generalise to the treatment of *tidy* maps in Chapter III.

2.7. Proposition. *Consider a pullback square*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "m"] \arrow[d, "q"'] & \mathcal{F} \arrow[d, "p"] \\ \mathcal{G} \arrow[r, "l"] & \mathcal{E} \end{tikzcd} ,$$

where l is open and surjective. If q is proper, then so is p.

PROOF. Suppose that q is proper. Let I be a directed category in $\mathcal{E}$ (and write I also for the induced categories p^*I, etc.). The square in the proposition induces a similar square

$$\begin{tikzcd} \mathcal{H}^I \arrow[r, "\bar{m}"] \arrow[d, "\bar{q}"'] & \mathcal{F}^I \arrow[d, "\bar{p}"] \\ \mathcal{G}^I \arrow[r, "\bar{l}"] & \mathcal{E}^I \end{tikzcd} ,$$

with $\bar{l}$ again an open surjection. Form the cube

$$\begin{tikzcd}[row sep=small, column sep=small] & \mathcal{F} \arrow[rr, "d"] \arrow[dd, "p"' near start] & & \mathcal{F}^I \arrow[dd, "\bar{p}"] \\ \mathcal{H} \arrow[ur, "m"] \arrow[rr, "c" near start] \arrow[dd, "q"'] & & \mathcal{H}^I \arrow[ur, "\bar{m}"] & \\ & \mathcal{E} \arrow[rr, "b" near start] & & \mathcal{E}^I \\ \mathcal{G} \arrow[ur, "l"] \arrow[rr, "a"] & & \mathcal{G}^I \arrow[ur, "\bar{l}"'] \arrow[from=uu, crossing over, "\bar{q}" near start] & \end{tikzcd}$$

in which a, b, c and d denote "points at infinity." Since l is open, $l^*p_*V = q_*m^*V$ for any $V \subseteq 1$ in $\mathcal{F}$; similarly, by openness of $\bar{l}$, the identity $\bar{l}^*\bar{p}_* = \bar{q}_*\bar{m}^*$ holds on subobjects of 1 in the pullback on the right.

Given now any $W \subseteq 1$ in $\mathcal{F}^I$, we claim that $b^*\bar{p}_*W = p_*d^*W$; since l is surjective, it suffices to show $l^*b^*\bar{p}_*W = l^*p_*d^*W$. But

$$\begin{aligned} l^*b^*\bar{p}_*W &= a^*\bar{l}^*\bar{p}_*W \\ &= a^*\bar{q}_*\bar{m}^*W && \text{since } \bar{l} \text{ is open} \\ &= q_*c^*\bar{m}^*W && \text{since } q \text{ is proper} \\ &= q_*m^*d^*W \\ &= l^*p_*d^*W && \text{since } l \text{ is open.} \end{aligned}$$

The same argument applied in an arbitrary slice of $\mathcal{E}$ proves that p is proper. ■

§3 Beck-Chevalley conditions

Consider a pullback square of toposes

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] & \mathcal{E} . \end{tikzcd} \tag{1}$$

Just by commutativity of this square (up to a given natural isomorphism), one obtains a natural transformation

$$a^*f_* \longrightarrow g_*b^*. \tag{2}$$

The square (1) is commonly said to satisfy the *Beck-Chevalley condition* (BCC) if this natural transformation is invertible; that is, for any object F in $\mathcal{F}$, (2) is an isomorphism

$$a^*f_*F \xrightarrow{\sim} g_*b^*F.$$

We shall study this property in Chapter III. For the moment, the following weakening is more relevant.

3.1. Definition. The square (1) is said to satisfy the *weak* Beck-Chevalley condition if, for any object F in $\mathcal{F}$, the canonical map (2) is a mono

$$a^*f_*F \hookrightarrow g_*b^*F.$$

We observe that this condition is stable under localization at an object of $\mathcal{E}$. More precisely, if a square (1) satisfies the weak BCC, then so does the localized square

$$\begin{tikzcd} \mathcal{H}/g^*a^*E \cong \mathcal{H}/b^*f^*E \arrow[r, "\bar{b}"] \arrow[d, "\bar{g}"'] & \mathcal{F}/f^*E \arrow[d, "\bar{f}"] \\ \mathcal{G}/a^*E \arrow[r, "\bar{a}"] & \mathcal{E}/E . \end{tikzcd} \tag{3}$$

for any object E in $\mathcal{E}$. Indeed, this follows easily from the description of the direct image functor of the morphism $\bar{f} = f/E$ in (3): for an object $F \to f^*E$ in $\mathcal{F}/f^*E$, the object $\bar{f}_*(F \to f^*E)$ is the pullback of $f_*F \to f_*f^*E$ along the unit $\eta: E \to f_*f^*E$.

3.2. Proposition. *For a pullback square* (1)*, the following are equivalent:*

(i) *The square satisfies the weak BCC.*

(ii) *For any mono $V \hookrightarrow F$ in $\mathcal{F}$, the square*

$$\begin{tikzcd} a^*f_*V \arrow[r] \arrow[d, hook] & b_*g^*V \arrow[d, hook] \\ a^*f_*F \arrow[r] & b_*g^*F \end{tikzcd} \tag{4}$$

is a pullback.

(iii) *The square satisfies the BCC for subobjects of* 1 *(i.e. $a^*f_*V \xrightarrow{\sim} b_*g^*V$ for $V \subseteq 1$ in $\mathcal{F}$), and the same is true for any localized square* (3)*.*

Proof. (i) $\Rightarrow$ (ii). Consider for the subobject classifier $\Omega_{\mathcal{F}}$ of $\mathcal{F}$ the square

$$\begin{tikzcd} a^*f_*\Omega_{\mathcal{F}} \arrow[r, hook] & b_*g^*\Omega_{\mathcal{F}} \\ a^*f_*1 \arrow[r, "\sim"] \arrow[u, "a^*f_*(true)"] & b_*g^*1 \arrow[u, "g_*b^*(true)"'] \end{tikzcd} \tag{5}$$

Here the upper arrow is mono, by assumption (i), so the square is a pullback. For any mono $V \hookrightarrow F$ in $\mathcal{F}$, its classifying map $c_V: F \to \Omega_{\mathcal{F}}$ fits into a pullback square

$$\begin{tikzcd} F \arrow[r, "c_V"] & \Omega_{\mathcal{F}} \\ V \arrow[r] \arrow[u] & 1 \arrow[u] \end{tikzcd},$$

of which the images under a^*f_* and g_*f^* span a cube with (5). The side of this cube opposite to (5),

$$\begin{tikzcd} a^*f_*F \arrow[r] & b_*g^*F \\ a^*f_*V \arrow[r] \arrow[u, hook] & b_*g^*V \arrow[u, hook] \end{tikzcd},$$

must then also be a pullback, proving (ii).

(ii) $\Rightarrow$ (iii). Again, the explicit description of $\bar{f}_*$ and a carefully drawn cubical diagram will show that property (ii) is stable under localization at an object E of $\mathcal{E}$. Therefore it suffices to prove (iii) for the case $E = 1$. But, if a square of the form (4) with $F = 1$ is a pullback, then $a^*f_*V \to b_*g^*V$ must be an iso.

(iii) $\Rightarrow$ (ii). For an object $V \overset{m}{\hookrightarrow} f^*E$ in $\mathcal{F}/f^*E$, write $\bar{f}_*V \hookrightarrow E$ for $\bar{f}_*m$, so that the definition of $\bar{f}_*$ gives a pullback

$$\begin{tikzcd} \bar{f}_*V \arrow[r] \arrow[d, hook] & f_*V \arrow[d, hook] \\ E \arrow[r, "\eta"] & f_*f^*E\ . \end{tikzcd}$$

Then condition (iii) can be rephrased by stating that the rectangle

$$\begin{tikzcd} a^*\bar{f}_*V \arrow[r] \arrow[d, hook] & a^*f_*V \arrow[r] \arrow[d, hook] & g_*b^*V \arrow[d, hook] \\ a^*E \arrow[r, "a^*\eta"] & a^*f_*f^*E \arrow[r] & g_*b^*f^*E \end{tikzcd} \tag{6}$$

is a pullback (because, modulo the isomorphism $g^*b^*f^*E \cong g_*g^*a^*E$, the lower composite is the unit $a^*E \to g_*g^*a^*E$, the pullback along which is, by definition, $\bar{g}_*\bar{a}^*(V \to f^*E)$).

Now consider any mono $W \hookrightarrow F$ in $\mathcal{F}$, and putting $E = f_*F$, form the pullback

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook] & W \arrow[d, hook] \\ f^*E \arrow[r, "\varepsilon"] & F \end{tikzcd} \tag{7}$$

of W along the counit of the adjuction. Notice that in this special case, $\bar{f}_*V \to E$ is $f_*W \hookrightarrow f_*F$, by the triangular identities for an adjunction. Composing the image of (7) under g_*b^* with the pullback (6) yields that $a^*\bar{f}_*V = a^*f_*W$ as the pullback of W along the composite

$$a^*E \xrightarrow{a^*\eta} a^*f_*f^*E \longrightarrow g_*b^*f^*E \xrightarrow{g_*b^*\epsilon} g_*b^*F.$$

By the naturality of ϵ and the triangular identities, this composite is the canonical map $a^*E = a^*f_*F \to g_*b^*F$. Thus the square

$$\begin{tikzcd} a^*f_*W \arrow[r] \arrow[d, hook] & b_*g^*W \arrow[d, hook] \\ a^*f_*F \arrow[r] & g_*b^*F \end{tikzcd}$$

is a pullback, as required for (ii).

(ii) $\Rightarrow$ (i). Consider for any object F in $\mathcal{F}$ the equalizer

$$F \hookrightarrow F \times F \rightrightarrows F$$

formed by the projections and the diagonal. This gives a diagram with similar equalizer rows,

$$\begin{tikzcd} a^* f_* F \arrow[r, hook] \arrow[d] & a^* f_*(F \times F) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & a^* f_* F \arrow[d] \\ g_* b^* F \arrow[r, hook] & g_* b^*(F \times F) \arrow[r, shift left] \arrow[r, shift right] & g_* b^* F\ . \end{tikzcd}$$

By assumption (ii), the lefthand square is a pullback. The exactness properties of this diagram together now imply that the right-hand map must be mono. ∎

Next, we shall say that a *map* $f: \mathcal{F} \to \mathcal{E}$ satisfies the weak BCC if, for any morphism $a: \mathcal{G} \to \mathcal{E}$, the pullback square (1) satisfies the weak BCC. Say f satisfies the *stable* weak BCC if any pullback of f satisfies the weak BCC.

3.3. Proposition. *If $f: \mathcal{F} \to \mathcal{E}$ satisfies the stable weak BCC, then f is proper.*

(In the next section, we shall show that the converse is also true.)

PROOF. Consider a directed category I in $\mathcal{E}$, and the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\infty"] \arrow[d, "f"'] & \mathcal{F}^{f^* I} \arrow[r] \arrow[d, "f^I"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E} \arrow[r, "\infty"] & \mathcal{E}^I \arrow[r] & \mathcal{E}\ . \end{tikzcd}$$

Since the (total) rectangle and the right-hand squares are pullbacks, so is the left-hand square. By assumption, the weak BCC holds for the left-hand square, which, by Proposition 3.2, implies that for any $U \subseteq 1$ in $\mathcal{F}^{f^* I}$,

$$\infty^*(f^I)_*(U) = f_* \infty^*(U).$$

The same argument applies to any slice $\mathcal{F}/f^*E \to \mathcal{E}/E$, since these slices are pullbacks of $f: \mathcal{F} \to \mathcal{E}$. This shows that f is proper in terms of Definition 1.8. ∎

3.4. Remark. The morphism $\mathcal{E} \xrightarrow{\infty} \mathcal{E}^I$ is a subtopos inclusion. Thus it is sufficient to require the weak BCC stably for pullbacks to *subtoposes* in Proposition 3.3.

§4 Pretopos sites

In this section we introduce a special kind of site which will turn out to be useful when dealing with compactness properties of toposes.

Since we shall work with internal sites in toposes, we need to be precise about the basic definitions. For many purposes, a convenient notion of site is that of a pair $(\mathbb{C}, J)$ consisting of a small category $\mathbb{C}$, together with a Grothendieck topology J on $\mathbb{C}$ [] (called a "pre-topology" in []), an operation assigning to each object $C \in \mathbb{C}$ a family $J(C)$ of "covers" $\{C_i \to C\}$ of C, such that the following three conditions are satisfied:

(i) (identities) The singleton family $\{C \xrightarrow{\mathrm{id}} C\}$ is a cover of C.

(ii) (stability) If $\{C_i \to C\}$ is a cover of C and $D \xrightarrow{f} C$ is any arrow in $\mathbb{C}$, then there exists a cover $\{D_j \to D\}$ such that each composite $D_j \to D \to C$ factors through some $C_i \to C$ (the family $\{D_j \to C\}$ "refines" $\{C_i \to C\}$).

(iii) (transitivity) If $\{C_i \to C\}$ is a cover, and for each index i the family $\{D_{ij} \to C_i\}$ is a cover, then the family of composites $\{D_{ij} \to C\}$ is a cover.

4.1. Remark. If $\mathbb{C}$ has pullbacks, it is sometimes convenient to ask for "strict" stability in (ii):

(ii') If $\{C_i \to C\}$ is a cover of C and $D \xrightarrow{f} C$ is any arrow in $\mathbb{C}$, then $\{C_i \times_C D \to D\}$ is a cover of D.

For every topos $\mathcal{E}$ there exists a site $(\mathbb{C}, J)$ such that $\mathcal{E} \cong \mathrm{Sh}(\mathbb{C}, J)$, the topos of sheaves on $(\mathbb{C}, J)$ [, 0.45]. More generally, for any morphism $p: \mathcal{E} \to \mathcal{S}$, there exists a site $(\mathbb{C}, J)$ in the base $\mathcal{S}$ for p (or, by abuse of language, for $\mathcal{E}$ as $\mathcal{S}$-topos), giving an equivalence of toposes over $\mathcal{S}$:

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\sim"] \arrow[dr, "p"'] & & \mathrm{Sh}_{\mathcal{S}}(\mathbb{C}, J) \arrow[dl, "\gamma"] \\ & \mathcal{S}, & \end{tikzcd}$$

where $\mathrm{Sh}_{\mathcal{S}}(\mathbb{C}, J)$ is the topos of $\mathcal{S}$-internal sheaves on $(\mathbb{C}, J)$ and γ the canonical map [, 4.46].

The notion of site as above unfortunately becomes awkward to work with in situations where a change of base topos is involved: if $(\mathbb{C}, J)$ is a site in a topos $\mathcal{S}$, its inverse image $\varphi^*(\mathbb{C}, J) \equiv (\varphi^*\mathbb{C}, \varphi^* J)$ along a morphism $\varphi: \mathcal{S}' \to \mathcal{S}$ generally fails to satisfy (iii) (unless all covers in J are finite), hence is not a site in $\mathcal{S}'$. It then becomes necessary to deal with a Grothendieck topology in terms of a "basis" for it, a system of covering families which is only required to satisfy the stability condition (ii): such a system generates a Grothendieck topology J under (i) and (iii) and defines the same sheaves on $\mathbb{C}$ as J.

4.2. Convention. A *site* (in an arbitrary topos) is a pair $(\mathbb{C}, J)$ where $\mathbb{C}$ is a small category and J is a system of covers satisfying the stability condition (ii). We refer to the covers in J as the *basic covers* of the site, and to the covers in the full Grothendieck topology obtained by adding the singleton covers (i) and closing under composition of covers (iii), as the *generated covers* of the site. The term "cover" — unqualified — will refer to any family refined by a generated cover. We say an arrow $D \to C$ in the site "covers" if the singleton family $\{D \to C\}$ is a cover. We shall mostly abuse notation, and just write $\mathbb{C}$ for the site $(\mathbb{C}, J)$.

In this terminology now, if $\mathbb{C} = (\mathbb{C}, J)$ is a site in $\mathcal{S}$, then its inverse image $\varphi^*\mathbb{C} = (\varphi^*\mathbb{C}, \varphi^* J)$ along any morphism $\varphi: \mathcal{S}' \to \mathcal{S}$ remains a site in $\mathcal{S}'$. Moreover, given a pullback square of toposes

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\psi"] \arrow[d, "p'"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{S}' \arrow[r, "\varphi"] & \mathcal{S} \end{tikzcd} , \tag{1}$$

if $\mathbb{C}$ is a site for $\mathcal{E}$ in $\mathcal{S}$, so is $\varphi^*\mathbb{C}$ for $\mathcal{E}'$ in $\mathcal{S}'$.

If $\mathbb{C}$ is a site for $\mathcal{E}$, the canonical map $h: \mathbb{C} \to \mathcal{E}$ is flat (see []), hence preserves all finite limits which exist in $\mathbb{C}$. The covers of $\mathbb{C}$ are exactly those families mapped to epimorphic families in $\mathcal{E}$ under h. Moreover, any epimorphic family of the form $\{E_i \to h(C)\}$ in $\mathcal{E}$ is refined by the image of some (generated) cover of $\mathbb{C}$. These facts forn part of the statement that h is *universal* (in the obvious appropriate sense) amongst flat, cover-preserving functors from $\mathbb{C}$ into a topos.

By a *morphism* of sites $F: \mathbb{C} \to \mathbb{D}$ we mean a functor which is flat (expressed in terms of the covers of $\mathbb{D}$) and which maps basic covers to covers. A morphism of sites induces a map of toposes $\mathrm{Sh}(\mathbb{D}) \to \mathrm{Sh}(\mathbb{C})$, as the unique such (up to ismomorphism) which makes the square

$$\begin{tikzcd} \mathbb{D} \arrow[r, "h"] & \mathcal{F} \\ \mathbb{C} \arrow[u, "F"] \arrow[r, "h"] & \mathcal{E} \arrow[u, "f^*"'] \end{tikzcd}$$

commute. Any map of toposes is equivalent to a morphism induced by sites (more generally, any small diagram of toposes is induced by a corresponding diagram of sites, see below).

4.3. Definition. A *pretopos site* $(\mathbb{C}, J)$ is a site for which the underlying category $\mathbb{C}$ is a pretopos (see e.g. [,]) and the system J of basic covers is the union of two subsystems P and S, where P is the topology of finite epimorphic families in $\mathbb{C}$ and S is a system of directed families of monomorphisms in $\mathbb{C}$ which is stable (ii) and moreover satisfies

(iv) (compatibility) If $\{S_i \rightarrowtail C\}$ is a basic S-cover, then so is the family of sums $\{S_i + D \rightarrowtail C + D\}$ for any D, and the family of images $\{f(S_i) \rightarrowtail f(C)\}$ for any arrow $f: C \to D$ in $\mathbb{C}$.

We refer to the covers in P and S as, respectively, the *P-covers* and *basic S-covers* of the pretopos site; the latter covers give rise to a sub-topology of that generated by J, the topology of generated S-covers.

The canonical functor $h: \mathbb{C} \to \mathcal{E}$ into a topos $\mathcal{E}$ from a pretopos site $\mathbb{C}$ for it, is characterized by being universal amongst pretopos morphisms from $\mathbb{C}$ into a topos which transform S-covers into epimorphic families.

We can construct a "subcanonical" pretopos site $(\mathbb{C}, J)$ for any given topos $\mathcal{E}$ as follows. Take any full subcategory of $\mathcal{E}$ spanned by a set of generators for $\mathcal{E}$, and close this category under "canonical" finite limits, finite sums and coequalizers of equivalence relations (if necessary). The result is a sub-pretopos $\mathbb{C}$ of $\mathcal{E}$, which will become a site for $\mathcal{E}$ provided its covers are exactly the epimorphic families in $\mathbb{C}$. But any such cover is clearly decomposable into a family of finite covers followed by a directed cover of monomorphisms. Thus, if we let S consist of the latter covers (and let the basic covers J be the union of these with the finite covers), we obtain a pretopos site for $\mathcal{E}$.

The additional data associated with a pretopos site can be interpreted in any topos, and, being evidently "geometric" [, 6.5], is preserved under change of base. Thus, given a map $p: \mathcal{E} \to \mathcal{S}$, there exists a pretopos site $\mathbb{C}$ for $\mathcal{E}$ in $\mathcal{S}$, and this situation is stable under pullback as in (1).

The decomposition property of covers in the subcanonical site described above has a generalization to arbitrary pretopos sites, as we now go on to show. First, we need:

4.4. Lemma. *Let $\mathbb{C}$ be a pretopos site.*

(i) *All generated S-covers of $\mathbb{C}$ consist of monomorphisms and satisfy the compatibility condition* (iv).

(ii) *If $\{S_i \rightarrowtail C\}$ and $\{T_j \rightarrowtail D\}$ are generated S-covers of $\mathbb{C}$, then so is their sum $\{S_i + T_j \rightarrowtail C + D\}$.*

Proof. (i) Both properties are clearly possessed by trivial covers and preserved under composition of covers, hence are properties of generated S-covers by induction.

(ii) The sum can be written as the composition of the families $\{S_i + D \rightarrowtail C + D\}$ and, for each i, $\{S_i + T_j \rightarrowtail S_i + D\}$; each of these is a generated S-cover by (i). ∎

4.5. Lemma. *Any cover in a pretopos site $\mathbb{C}$ is refined by the composition of a P-cover followed by a generated S-cover.*

Proof. It is enough to show that each composition of a generated S-cover followed by a P-cover is refined by the composition of a P-cover followed by a generated S-cover. For then the property of covers as stated is (trivially) satisfied by all basic covers, and preserved at each generating step, hence (by induction) inherited by all generated covers. Since any cover is refined by a generated cover, the lemma will follow.

Consider such a composition of a P-cover $\{f_i: D_i \to C \mid 1 = 1, \ldots, n\}$ and a family of generated S-covers $\{E_{i\lambda} \rightarrowtail D_i \mid \lambda \in \Lambda_i\}$. Write $D = D_1 + \cdots + D_n$, and $E_\lambda = E_{1\lambda_1} + \cdots E_{n\lambda_n}$ for each $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Lambda = \Lambda_1 \times \cdots \times \Lambda_n$. Then by Lemma 4.4(ii), each individual sum $\{E_\lambda \rightarrowtail D \mid \lambda \in \Lambda\}$ of generated S-covers is again a generated S-cover, as is, by Lemma 4.4(i), the image $\{f(E_\lambda) \rightarrowtail\mid \lambda \in \Lambda\}$ along the induced epimorphism $f: D \to C$. But for each $\lambda \in \Lambda$, the family $\{E_{i\lambda_i} \to f(E_\lambda) \mid i = 1, \ldots, n\}$

is a P-cover, and the composition $\{E_{i\lambda_i} \to f(E_\lambda) \rightarrowtail C \mid \lambda \in \Lambda,\ i = 1, \ldots, n\}$ clearly refines the given composition $\{E_{i\lambda} \rightarrowtail D_i \to C \mid \lambda \in \Lambda_i,\ i = 1, \ldots, n\}$ we started out with. ∎

4.6. Corollary. *In a pretopos site $\mathbb{C}$, a directed cover of monomorphisms is an S-cover.* ∎

4.7. Remark. Let $\mathbb{C}$ be a category with pullbacks and universal (that is, stable under pullback) coproducts and coequalizers of equivalence relations. Call a finite epimorphic family $\{C_i \to C\}$ in $\mathbb{C}$ *regular* if the induced map $\coprod_i C_i \to C$ is a coequalizer. Then the last three results remain true for any site of the form $(\mathbb{C}, P \cup S)$, where P is the topology of finite regular epimorphic families and S is a stable (ii) system of directed covers which is compatible in the sense of (4.3) (iv) with sums and (regular) images. For want of a descriptive name, we shall refer to this situation as a "site with stable compatible system of directed covers."

A morphism between pretopos sites $F: \mathbb{C} \to \mathbb{D}$ preserves the pretopos structure (equivalently, is flat and preserves P-covers) and maps basic S-covers to (directed) covers. Any morphism between toposes is induced by a morphism between pretopos sites. Indeed, suppose, more generally, that we are given a diagram $\{\mathcal{E}_i\}$ of toposes indexed by a small category I (here we tacitly assume that this entails giving explicitly, for each indexing object i, a set $\mathcal{G}_i$ of generators for $\mathcal{E}_i$). For each i, let $\mathbb{C}_i$ be the subcanonical pretopos site for $\mathcal{E}_i$ constructed, in the way described before, from the set of generators $\bigcup_\alpha {t_\alpha}^*[\mathcal{G}_j]$ where $\alpha: i \to j$ varies over all arrows out of i. Then for each arrow $\alpha: i \to j$ in I, inverse image for the corresponding transition map $t_\alpha: \mathcal{E}_i \to \mathcal{E}_j$ has a restriction to a map of pretopos sites $T_\alpha: \mathbb{C}_j \to \mathbb{C}_i$, and T_α induces t_α.

We end this section with a description of filtered inverse limits of toposes in terms of pretopos sites. Consider any (pseudo-)limit

$$\begin{tikzcd} & \mathcal{E}_i \arrow[dd, "t_\alpha"] & i \arrow[dd, "\alpha"] \\ \mathcal{E} \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & & \\ & \mathcal{E}_j & j \end{tikzcd} \tag{2}$$

of a diagram $\{\mathcal{E}_i\}$ of toposes, indexed by an inversely filtered small category I, and induced by a corresponding diagram $\{\mathbb{C}_i\}$ of pretopos sites (e.g. as constructed above).

4.8. Lemma. *The filtered inverse limit of toposes* (2) *is induced by a diagram of pretopos sites*

$$\begin{tikzcd} & \mathbb{C}_i \arrow[dl, "P_i"'] & i \arrow[dd, "\alpha"] \\ \mathbb{C} & & \\ & \mathbb{C}_j \arrow[ul, "P_j"] \arrow[uu, "T_\alpha"'] & j \end{tikzcd} \tag{3}$$

such that each finite commutative diagram (resp. basic S-cover) in $\mathbb{C}$ lifts up to isomorphism through some P_i, to a finite commutative diagram (resp. S-cover) in $\mathbb{C}_i$.

Proof. We construct the site $\mathbb{C}$ for the limit $\mathcal{E}$ in the standard way (see []). Let $\mathbb{C}$ be the category with as objects the disjoint union of those of the categories $\mathbb{C}_i$, and with arrows between $C \in \mathbb{C}_i$ and $D \in \mathbb{C}_j$ given by the filtered colimit of sets

$$\mathbb{C}(C, D) = \varinjlim_{(\alpha,\beta)} \mathbb{C}_k(T_\alpha(C), T_\beta(D)),$$

where (α, β) varies over pairs of maps $i \xleftarrow{\alpha} k \xrightarrow{\beta} j$ with common domain k. For each $i \in I$, let $P_i: \mathbb{C}_i \to \mathbb{C}$ be the functor which takes an object to itself (or more accurately, its representative) in the disjoint union and maps arrows between $C, D \in \mathbb{C}_i$ by the colimit function

$$\mathbb{C}_i(C, D) \to \mathbb{C}(C, D)$$

at $(\mathrm{id}, \mathrm{id})$.

It is then a straightforward matter to check that each arrow $\alpha: i \to j$ in I gives a (pseudo-)commutative diagram as in (3), and that $\mathbb{C}$ is the (pseudo-)colimit of the diagram of categories $\{\mathbb{C}_i\}$. It is also clear from the construction that finite commutative diagrams in $\mathbb{C}$ can be lifted as stated. This, together with the filteredness of I and the fact that the transition functors in (3) are pretopos morphisms, in turn implies that $\mathbb{C}$ inherits the (finitary) pretopos structure from its components so as to make the functors $P_i: \mathbb{C}_i \to \mathbb{C}$ pretopos morphisms. Thus, (3) is in fact a (pseudo-)colimit in the category of pretoposes.

Now let the S-covers of $\mathbb{C}$ be those families which (up to isomorphism) lift through some P_i to an S-cover in $\mathbb{C}_i$. Then the P-covers and S covers are compatible, since the data involved in the compatibility condition (4.3) (iv) can always be lifted to some single $\mathbb{C}_i$, where compatibility is assured by Lemma 4.4. Thus, $\mathbb{C}$ becomes a pretopos site and the P_i morphisms of pretopos sites having the stated lifting property of basic S-covers by construction. Finally, $\mathbb{C}$ is indeed a pretopos site for $\mathcal{E}$, since the pretopos morphism $h: \mathbb{C} \to \mathcal{E}$ induced by the canonical functors $h_i: \mathbb{C}_i \to \mathcal{E}_i$ is easily seen to be universal in mapping S-covers in $\mathbb{C}$ to epimorphic families. ∎

4.9. Remark. It is clear from the proof of Lemma 4.8 that it can be extended to a corresponding result for a limit $f: \mathcal{F} \to \mathcal{E}$ of a diagram of maps $\{f_i: \mathcal{F}_i \to \mathcal{E}_i\}$. That is, if $\{F_i: \mathbb{C}_i \to \mathbb{D}_i\}$ is a diagram of pretopos site morphisms inducing $\{f_i: \mathcal{F}_i \to \mathcal{E}_i\}$, there is a morphism $F: \mathbb{C} \to \mathbb{D}$ between the pretopos sites for $\mathcal{E}$ and $\mathcal{F}$ as in Lemma 4.8, which induces f.

§5 Preservation under pullback and filtered inverse limits

Our main purpose in this section is to show that proper maps are stable under pullback. We shall do this by encoding propriety of a map $f: \mathcal{F} \to \mathcal{E}$ in terms of an inductive property of a pretopos site for $\mathcal{F}$ in $\mathcal{E}$ (see Lemma 5.4 below) which is preserved under change of base. The same property will be involved in the proofs of various other facts, like the converse of Proposition 3.3, and stability of propriety under filtered inverse limits.

First we need a site version of compactness.

5.1. Definition. A pretopos site $\mathbb{C}$ (4.3) is *compact* if any directed cover of 1 by monomorphisms in $\mathbb{C}$ has a single member which already covers.

5.2. Remark. Definition 5.1 makes sense for any "site with stable compatible system of directed covers" (4.7). We make the blanket observation that all results in this section remain true (and most proofs unaltered) upon substitution of this notion for "pretopos site."

We have:

5.3. Proposition. *A pretopos site $\mathbb{C}$ for a topos $\mathcal{E}$ is compact iff $\mathcal{E}$ is compact.*

PROOF. Immediate from the fact that the canonical functor $h\colon \mathbb{C} \to \mathcal{E}$ preserves 1, preserves and reflects covers, and that any directed cover of 1 in $\mathcal{E}$ is refined by the image under h of a directed cover of 1 in $\mathbb{C}$. ∎

The stability properties of compactness that concern us in this secton, are unlocked by the following lemma.

5.4. Lemma. *Let $\mathbb{C}$ be a pretopos site equipped with a system M of distinguished covering monomorphisms $U \rightarrowtail 1$ such that*

(i) *The trivial cover $1 \rightarrowtail 1 \in M$.*
(ii) *If $V \rightarrowtail U \rightarrowtail 1$, then $U \rightarrowtail 1 \in M$ whenever $V \rightarrowtail 1 \in M$.*
(iii) *For any basic S-cover $\{U_i \rightarrowtail U\}$, if $U \rightarrowtail 1 \in M$ then $U_i \rightarrowtail 1 \in M$ for some i.*

Then M contains all monomorphic covers $U \rightarrowtail 1$ of $\mathbb{C}$, and $\mathbb{C}$ is compact.

PROOF. It will be enough to prove that any generated S-cover of 1 contains a member of M. For then any directed cover of 1 contains a member of M by (4.6) and (ii). To this end, consider the property of families $\{U_i \to U\}$ stating that, if $U \to 1 \in M$, then there is some i for which $U_i \to 1 \in M$. Since this property is given to hold for basic S-covers (iii), trivially holds for the family $\{1 \to 1\}$ and is preserved by composition, it must hold for generated S-covers by induction. But then any generated S-cover contains a member of M, since $1 \to 1 \in M$. ∎

5.5. Example. Suppose $\mathbb{C}$ is a pretopos site in which the basic S-covers of 1 are trivial (i.e. contain an isomorphism). Then, by letting the isomorphisms $U \cong 1$ be the distinguished covers, it follows that $\mathbb{C}$ is a compact site in which all directed covers of 1 are trivial (as in the case when $\mathbb{C}$ is compact and subcanonical).

5.6. Corollary. *A pretopos site $\mathbb{C}$ is compact iff the system of all covering subobjects of 1 in $\mathbb{C}$ satisfies the conditions of (5.4).*

PROOF. Basic S-covers consist of monomorphisms and are directed, a property inherited under (post-)composition with a monomorphism. The statement is therefore an immediate consequence of Lemma 5.4. ■

5.7. Lemma. *Let $\varphi: \mathcal{E}' \to \mathcal{E}$ be a morphism of toposes and suppose $\mathbb{C}$ is a compact pretopos site in $\mathcal{E}$. Then the pretopos site $\varphi^*\mathbb{C}$ is compact in $\mathcal{E}'$. Moreover, if M denotes the object of subobjects of 1 which cover in $\mathbb{C}$, then φ^*M is the corresponding object for $\varphi^*\mathbb{C}$.*

PROOF. By Corollary 5.6, M is a system of covers of 1 satisfying the conditions of Lemma 5.4 internally in $\mathcal{E}$. But these conditions are "geometric" and hence preserved under change of base. This means that φ^*M is a system of "distinguished" monomorphic covers of 1 for $\varphi^*\mathbb{C}$ in $\mathcal{E}'$. Since (the proof of) Lemma 5.4 is constructive, it can be interpreted in $\mathcal{E}'$ to yield the result. ■

5.8. Theorem. *In a pullback square*

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\psi"] \arrow[d, "f'"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}' \arrow[r, "\varphi"] & \mathcal{E} \end{tikzcd},$$

suppose that f is proper. Then f' is proper and the weak BCC is satisfied.

PROOF. We write as if $\mathcal{E} \equiv \mathbf{Set}$ and argue constructively.

Let $\mathbb{C}$ be a pretopos site for $\mathcal{F}$. Then $\mathbb{C}$ is compact by Proposition 5.3, and it follows that $\varphi^*\mathbb{C}$ is a compact site for $\mathcal{F}'$ in $\mathcal{E}'$ by Lemma 5.7. Thus, f' is proper, by applying Proposition 5.3 in $\mathcal{E}'$.

To deduce the weak BCC, consider any subobject $V \subseteq 1$ of $\mathcal{F}$, represented by a closed sieve R on $1 \in \mathbb{C}$. It will be enough to deduce, in the internal language of $\mathcal{E}'$, that $1 \to 1$ is in φ^*R whenever φ^*R contains a cover of 1 in $\varphi^*\mathbb{C}$. But — arguing in $\mathcal{E}'$ — φ^*R remains closed under all finite covers of $\varphi^*\mathbb{C}$ in the image of ϕ^*, in particular under P-covers and the singleton covers φ^*M of Lemma 5.7. Therefore, if ϕ^*R contains a cover of 1 at all, it must also contain a directed cover of 1 and consequently, by Lemma 5.7, an element of φ^*M. Since the latter is only possible if $1 \to 1 \in \varphi^*R$ already, we are done. ■

5.9. Corollary. *A map $f: \mathcal{F} \to \mathcal{E}$ is proper iff it satisfies the stable weak BCC.*

PROOF. One direction is Proposition 3.3, the other is immediate from Theorem 5.8. ■

5.10. Theorem. *Suppose $f: \mathcal{F} \to \mathcal{E}$ is the limit of a diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "p_i"] \arrow[dr, "f"'] & & \mathcal{F}_i \arrow[dl, "f_i"] \\ & \mathcal{E} & \end{tikzcd} \tag{1}$$

of proper maps $\{f_i: \mathcal{F}_i \to \mathcal{E}\}$ indexed by a filtered category I. Then f is proper. Moreover, for any $i \in I$ and $V \subseteq 1$ in $\mathcal{F}_i$, the natural inclusion

$$\bigvee\{{f_j}_* {t_\alpha}^* V \mid \alpha: j \to i\} \subseteq f_* {p_i}^* V, \tag{2}$$

where $t_\alpha: \mathcal{F}_j \to \mathcal{F}_i$ denotes the transition map induced by α, is an isomorphism.

PROOF. We can regard I as an internal category in $\mathcal{E}$, so it will be enough to treat the case $\mathcal{E} \equiv$ **Set** constructively.

Let $\{\mathbb{C}_i\}$ be a diagram of pretopos sites inducing $\{\mathcal{F}_i\}$, and let $\mathbb{C}$ be a pretopos site for the limit $\mathcal{F}$ as given by Lemma 4.8. For each i, let M_i be the set of covering subobjects of 1 in $\mathbb{C}_i$, and let M be the set of covering subobjects of 1 in $\mathbb{C}$ which are (up to isomorphism) in the joint image of the M_i under the morphisms $P_i: \mathbb{C}_i \to \mathbb{C}$ which induce the projections $p_i: \mathcal{F} \to \mathcal{F}_i$.

By Proposition 5.3, each $\mathbb{C}_i$ is compact, so that M_i satisfies the conditions of Lemma 5.4. Using the directedness of I and the lifting property of commutative diagrams and basic S-covers in $\mathbb{C}$, the system M is readily seen to inherit these conditions from the M_i. It follows that any covering subobject $U \rightarrowtail 1$ in $\mathbb{C}$ lifts to some $\mathbb{C}_i$, and that $\mathbb{C}$ is compact. Thus $\mathcal{F}$ is compact.

For the second part, consider $i \in I$ and $V \subseteq 1$ in $\mathcal{F}_i$. Represent V by a closed sieve R of $1 \in \mathbb{C}_i$. It will be enough to show that the family $P_i(R)$ in $\mathbb{C}$ covers 1 only if there is some $\alpha: j \to i$ and $U \rightarrowtail 1 \in R$ such that $T_\alpha U \rightarrowtail T_\alpha(1) \cong 1$ covers in $\mathbb{C}_i$ (where $T_\alpha: \mathbb{C}_j \to \mathbb{C}_i$ induces the transition map $t_\alpha: \mathcal{F}_i \to \mathcal{F}_j$). But R is closed under P-covers, hence is generated as a sieve by a directed family of subobjects of 1. Thus, if $P_i(R)$ covers 1, there exists, by compactness of $\mathbb{C}$, some $U \rightarrowtail 1$ in R such that $P_i(U) \rightarrowtail P_i(1) \cong 1$ covers. Thus, by the lifting property of subobject covers of 1 in $\mathbb{C}$ and the directedness of I, we can find an arrow $\alpha: j \to i \in I$ such that $T_\alpha(U) \rightarrowtail T_\alpha(1) \cong 1$ already covers in $\mathbb{C}_j$, as required. ∎

5.11. Corollary. *Suppose in (1) that for each $\alpha: j \to i$ in I and $V \subseteq 1$ in $\mathcal{F}_i$, the natural inclusion ${f_i}_* V \subseteq {f_j}_* {t_\alpha}^* V$ induced by the transition morphism $t_\alpha: \mathcal{F}_j \to \mathcal{F}_i$ is an isomorphism. Then the natural inclusion ${f_i}_* V \subseteq f_* {p_i}^* V$ is an isomorphism for each $i \in I$.* ∎

5.12. Remark. The natural inclusion (2) is the component at $V \subseteq 1 \in \mathcal{F}_i$ of a canonical natural transormation

$$\varinjlim_\alpha {f_j}_* {t_\alpha}^* \to f_* {p_i}^*, \tag{3}$$

where $\alpha: j \to i$ varies over the category I/i. Since the data in (1) localizes, we could have stated equivalently in Theorem 5.10 that the transormation (3), and similarly in Corollary 5.11 the canonical natural transormation ${f_i}_* \to f_* {p_i}^*$, are monomorphisms (see the proof of Proposition 3.2).

§6 Propriety and closed maps

This section is devoted to proving the following result:

6.1. Theorem. *A map $f: \mathcal{F} \to \mathcal{E}$ between toposes is proper iff f is stably closed.*

Here f is said to be stably (or "universally") closed if the pullback of f along an arbitrary map is closed. Before proving the theorem, we define the notion of closed map between toposes and deduce some elementary properties.

6.2. Definition. A map $f: \mathcal{F} \to \mathcal{E}$ is said to be *closed* if, for any $E \in \mathcal{E}$ and any closed subtopos $\mathcal{C} \subseteq \mathcal{F}/E$, the image of $\mathcal{C}$ along $f/E: \mathcal{F}/f^*E \to \mathcal{E}/E$ is a closed subtopos of $\mathcal{E}/E$.

In order to deal with the definition in more detail, we need to recall some notation related to subtoposes of a given topos.

Subtoposes of $\mathcal{F}$ correspond to closure operators on $\mathcal{F}$ (see [,]). Any subobject $U \subseteq 1$ in $\mathcal{F}$ uniquely determines an *open subtopos* denoted $U \subseteq \mathcal{F}$, with closure operator (for any object F of $\mathcal{F}$)

$$\mathrm{Sub}(F) \to \mathrm{Sub}(F), \quad S \mapsto ((U \times F) \Rightarrow S).$$

U has a complement $\mathcal{F} - U$ in the lattice of subtoposes of $\mathcal{F}$, given by the closure operator

$$\mathrm{Sub}(F) \to \mathrm{Sub}(F), \quad S \mapsto ((U \times F) \cup S).$$

By definition, a *closed* subtopos is such a complement of an open subtopos.

Any subtopos $\mathcal{D} \subseteq \mathcal{F}$ is contained in a smallest closed subtopos, its *closure* $\mathrm{Cl}(\mathcal{D}) \subseteq \mathcal{F}$. It is explicitly described as

$$\mathrm{Cl}(\mathcal{D}) = \mathcal{F} - \bar{0}, \tag{1}$$

(where $\bar{0} \in \mathrm{Sub}(1)$ is obtained by applying the closure operator $\mathrm{Sub}(1) \to \mathrm{Sub}(1)$ associated with $\mathcal{D}$ to the initial (sub-)object 0).

For a map $f: \mathcal{F} \to \mathcal{E}$ and a subtopos $\mathcal{D} \subseteq \mathcal{F}$, the closure of a subobject $S \subseteq E$ in $\mathcal{E}$, for the closure operator corresponding to the image $f(\mathcal{D}) \subseteq \mathcal{E}$, is given by the pullback

$$\begin{tikzcd} \overline{S} \arrow[r] \arrow[d, hook] & f_*(\overline{f^*S}) \arrow[d, hook] \\ E \arrow[r, "\eta"] & f_*f^*E, \end{tikzcd} \tag{2}$$

where $\overline{f^*S}$ is the closure of $f^*S \subseteq f^*E$ for the closure operator corresponding to $\mathcal{D}$ and η is the unit of the adjunction (see []).

6.3. Lemma. *A map $f: \mathcal{F} \to \mathcal{E}$ is closed iff the identity*

$$(f/E)_*((f/E)^*U \cup W) = U \cup (f/E)_*W \tag{3}$$

holds for $E \in \mathcal{E}$ and subobjects $U \subseteq E$, $W \subseteq f^*E$.

PROOF. For a subobject $V \subseteq 1$ in $\mathcal{F}$, consider the closed subtopos $\mathcal{F} - V$ and its image $f(\mathcal{F} - V)$. By (1), the closure of the latter subtopos is $\mathcal{F} - f_*(V)$.

If f is closed, comparing the closure operators for $f(\mathcal{F} - V)$ and $\mathcal{F} - f_*V$ on Sub(1) shows that

$$f_*(f^*U \cup V) = U \cup f_*V$$

for any $U \subseteq 1$. Applied to slices of $\mathcal{E}$, this argument shows that (3) follows if f is closed.

Conversely, the closure operators corresponding to the image $f(\mathcal{F} - V)$ and its closure $\mathcal{F} - f_*V$ are at $E \in \mathcal{E}$ and for $U \subseteq E$ given by

$$U \mapsto (f/U)_*(f/U)^*(U) \cup (V \times E))$$

and

$$U \mapsto U \cup (f/E)_*(V \times E)$$

respectively. If (3) holds, then these are equal by substituting $V \times E \subseteq E$ for W. Thus f is closed. ∎

The next lemma is essentially a reformulation of (6.3). Note already that, by (5.8), it furnishes the forward implication in Theorem 6.1.

6.4. Lemma. *A map $f: \mathcal{F} \to \mathcal{E}$ is closed iff for any $E \in \mathcal{E}$ and closed subtopos $\mathcal{C} \subseteq \mathcal{E}/E$, the pullback square*

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[d, "g"'] & \mathcal{F}/f^*E \arrow[d, "f/E"] \\ \mathcal{C} \arrow[r, hook] & \mathcal{E}/E \end{tikzcd}$$

satisfies the weak BCC.

PROOF. Suppose f is closed. It is enough, for each E, to show the weak BCC restricted to subobjects of 1 in $\mathcal{F}/E$; replacing f by f/E, it then suffices to consider the case $E = 1$. Write $\mathcal{C} = \mathcal{E} - U$ and let $c: \mathcal{C} \hookrightarrow \mathcal{E}$ be the inclusion. Then the pullback is $\mathcal{D} = \mathcal{F} - f^*U$, say with inclusion $d: \mathcal{D} \hookrightarrow \mathcal{F}$:

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook, "d"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, hook, "c"] & \mathcal{E} . \end{tikzcd}$$

For $W \subseteq 1$ in $\mathcal{F}$, the Beck-Chevalley identity $c^* f_* W = g_* d^* W$ holds iff $c_* c^* f_* W = c_* g_* d^* W$, since c is an embedding. But $c_* c^* f_* W = f_* W \cup U$, whereas $c_* g_* d^* W = f_* d_* d^* W = f_*(W \cup f^* U)$, and since f is closed these are identical by Lemma 6.3.

The converse is proved by an obvious inversion of the argument, applying the weak BCC "globally" in an arbitrary slice. ■

For the reverse implication in Theorem 6.1, we shall make use of the idea of a "splitting topos". To explain this notion and its basic properties, fix a topos $\mathcal{E}$, and let Σ be any family of subtoposes of $\mathcal{E}$. A morphism $f: \mathcal{F} \to \mathcal{E}$ is said to *split* Σ if for any subtopos $\mathcal{A} \subseteq \mathcal{E}$ in the family Σ, the pullback $f^{-1}(\mathcal{A})$ is a closed subtopos of $\mathcal{F}$. A *splitting topos* for Σ is an $\mathcal{E}$-topos $s: \mathcal{E}_\Sigma \to \mathcal{E}$ which universally splits Σ. This means that if $f: \mathcal{F} \to \mathcal{E}$ as above also splits Σ, then f will factor through s by an essentially unique map $\mathcal{F} \to \mathcal{E}_\Sigma$ over $\mathcal{E}$.

Observe that, by the universal property, if $s: \mathcal{E}_\Sigma \to \mathcal{E}$ is a splitting topos for Σ then for any morphism $g: \mathcal{G} \to \mathcal{E}$ the pullback $\mathcal{E}_\Sigma \times_\mathcal{E} \mathcal{G} \to \mathcal{G}$ is a splitting topos for the family $g^{-1}(\Sigma) = \{g^{-1}(\mathcal{A}) \mid \mathcal{A} \in \Sigma\}$ of subtoposes of $\mathcal{G}$. Thus, the notion of splitting topos is stable under change of base.

We shall also have to apply this notion to the slightly more involved case of an "internal" family of subtoposes of $\mathcal{E}$. Such a family Σ is generated by a collection $\Sigma(E)$ of subtoposes of the slice $\mathcal{E}/E$, where E ranges over the objects of $\mathcal{E}$. The terminology extends in the obvious way: $f: \mathcal{F} \to \mathcal{E}$ splits such a Σ if for each object E the map $f/E: \mathcal{F}/f^*E \to \mathcal{E}/E$ splits $\Sigma(E)$ in the sense above. The universal such $\mathcal{F}$ is called the splitting topos for Σ and again denoted $\mathcal{E}_\Sigma$.

If Σ is the internal family of *all* subtoposes (of $\mathcal{E}/E$ for all E) then the splitting topos for Σ will be called the *full* splitting topos of $\mathcal{E}$, and will be denoted $s: \mathrm{Spl}(\mathcal{E}) \to \mathcal{E}$.

6.5. Proposition. *Let $\mathcal{E}$ be a topos. For any (internal) family Σ of subtoposes of $\mathcal{E}$, the splitting topos $s: \mathcal{E}_\Sigma \to \mathcal{E}$ for $\mathcal{E}$ exists. Moreover, it has the following properties:*

(i) *$s: \mathcal{E}_\Sigma \to \mathcal{E}$ is a localic (stable) surjection.*
(ii) *Any closed subtopos $\mathcal{D} \subseteq \mathcal{E}_\Sigma$ is of the form $s^{-1}(\mathcal{A})$ for a unique subtopos $\mathcal{A}$ of $\mathcal{E}$ (which must then be the image $f(\mathcal{D})$, by* (i)*).*

PROOF. We give a sketch. Since subtoposes of $\mathcal{E}$ correspond to internal sublocales of the terminal locale 1 in $\mathcal{E}$, it suffices to prove the properties for locales instead of toposes while working constructively (in fact, we only need splitting locales of the terminal locale 1).

If X is any locale, the lattices of closed sublocales and all sublocales of X are both dual to frames, and the inclusion of the first into the second preserves meets and finite joins (see [] or [] for the details). This gives a map of locales $s: X' \to X$, where the frame of opens of X' is isomorphic to the dual of the lattice of sublocales of X and $s^{-1}(U)$ for $U \subseteq X$ open given by (the dual to) the closed complement of U. The map s clearly splits all open sublocales of X and satisfies properties (i) and (ii) by definition. But pulling back sublocales preserves meets and finite joins, and any sublocale of X is the intersection of sublocales of the form $U \cup (X - V)$ for $U, V \subseteq X$ open. These facts imply that splitting all open sublocales is equivalent to splitting all sublocales, and that $s: X' \to X$ does so universally.

It is now immediate that for a general family Σ of sublocales of X, the localic splitting X_Σ is the quotient of X' for which the frame of opens is generated by (the duals of) members of Σ, and that the induced map $t: X_\Sigma \to X$ inherits properties (i) and (ii) from s. ∎

6.6. Remark. For a family Σ of open sublocales of the locale X in the last proof, the splitting locale $t: X_\Sigma \to X$ has has as basis opens of the form $t^{-1}(U) - t^{-1}(V)$ for U, V open sublocales of X, with V a finite (possibly empty) join of members of Σ. Thus, a splitting topos $s: \mathcal{E}' \to \mathcal{E}$ for an internal family of *open* subtoposes of $\mathcal{E}$ is a localic $\mathcal{E}$-topos with (internal) basis of open subtoposes of the form $s^{-1}(\mathcal{A})$ for which the inclusion $\mathcal{A} \subseteq \mathcal{E}$ is *locally closed.*

6.7. Lemma. *Let $f: \mathcal{F} \to \mathcal{E}$ be any map. Let $\mathcal{D} = \mathcal{F} - U$ be a closed subtopos of $\mathcal{F}$, and let $s: \mathcal{E}' \to \mathcal{E}$ be any $\mathcal{E}$-topos which splits the image $f(\mathcal{D}) \subseteq \mathcal{E}$. Then $f(\mathcal{D})$ is closed iff for the pullback*

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "t"] \arrow[d, "f'"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}' \arrow[r, "s"] & \mathcal{E} \end{tikzcd} ,$$

the identity $s^* f_* U = f'_* t^* U$ holds.

PROOF. Let us write $\mathcal{C} = f(\mathcal{D})$. Since $\mathcal{E}'$ splits $\mathcal{C}$, the subtopos $\mathcal{C}' = s^{-1}\mathcal{C}$ of $\mathcal{E}'$ is closed. We observe first that $\mathcal{C}'$ is in fact the closure of $f'(t^{-1}\mathcal{D})$, in other words

$$\mathcal{C}' = \mathcal{E}' - f'_* t^* U \tag{4}$$

Indeed, by Proposition 6.5 (ii), this closure $\mathrm{Cl}(f' t^{-1}\mathcal{D})$ is of the form $s^{-1}(\mathcal{A})$ for a uniquely determined subtopos $\mathcal{A} \subseteq \mathcal{E}$. This $\mathcal{A}$ is the image of $\mathcal{E}$ of the composite

$$t^{-1}\mathcal{D} \hookrightarrow \mathcal{F}' \xrightarrow{f'} \mathcal{E}' \xrightarrow{s} \mathcal{E}$$

or equivalently, since $t: t^{-1}\mathcal{D} \to \mathcal{D}$ is surjective, the image of

$$\mathcal{D} \hookrightarrow \mathcal{F} \xrightarrow{f} \mathcal{E}.$$

It follows that $\mathcal{A}$ coincides with $f(\mathcal{D}) = \mathcal{C}$.

Now $\mathcal{C} = f(\mathcal{F} - U)$ is closed iff $\mathcal{C} = \mathcal{E} - f_* U$, and by the surjectivity of s this holds iff $s^{-1}\mathcal{C} = s^{-1}(\mathcal{E} - f_* U)$, that is, iff $\mathcal{C}' = \mathcal{E}' - s^* f_* U$. By (4), this is equivalent to the identity

$$s^* f_* U = f'_* t^* U,$$

which is what we needed to show. ∎

6.8. Proposition. *A map $f: \mathcal{F} \to \mathcal{E}$ is closed iff the weak BCC holds for the pullback of f with the full splitting topos $\mathrm{Spl}(\mathcal{E}) \to \mathcal{E}$ of $\mathcal{E}$.*

PROOF. Clear from Proposition 3.2 and Lemma 6.7. ■

Proof of Theorem 6.1. As already remarked, the forward implication follows since any proper map f satisfies the weak BCC for a pullback square as in Lemma 6.4, by Theorem 5.8. For the converse, it will by Remark 3.4 be enough to show that the weak BCC holds for a pullback

$$\begin{tikzcd} f^{-1}\mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{A} \arrow[r, hook] & \mathcal{E}\ . \end{tikzcd}$$

of f along an arbitrary embedding $\mathcal{A} \hookrightarrow \mathcal{E}$, given that f is stably closed. Let $s: \mathcal{E}' \to \mathcal{E}$ be a splitting topos for $\mathcal{A}$, so that $\mathcal{A}' = s^{-1}\mathcal{A}$ is closed in $\mathcal{E}'$. Then in the pullback diagram

$$\begin{tikzcd} f'^{-1}\mathcal{A}' \arrow[r, hook] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d, "f'"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{A}' \arrow[r, hook] & \mathcal{E}' \arrow[r] & \mathcal{E} \end{tikzcd}$$

f' is closed, so the left-hand square satisfies the weak BCC by Lemma 6.4. Furthermore, the right-hand square satisfies the weak BCC by Lemma 6.7. Thus, the composed rectangle satisfies the weak BCC. Now write this rectangle as another composite of pullbacks

$$\begin{tikzcd} f'^{-1}\mathcal{A}' \arrow[r, two heads] \arrow[d] & f^{-1}\mathcal{A} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{A}' \arrow[r, two heads] & \mathcal{A} \arrow[r] & \mathcal{E}\ . \end{tikzcd}$$

As indicated, the left horizontal maps are surjections, being pullbacks of the splitting cover $\mathcal{E}' \to \mathcal{E}$. Using the surjectivity of $\mathcal{A}' \to \mathcal{A}$, one sees that the required weak BCC for the right-hand square follows from that for the composite rectangle (already established) and left-hand square (which holds by Lemma 6.7). This completes the proof. ■

Having established Theorem 6.1, we can reformulate the weak BCC for proper maps (Theorem 5.8) as follows.

6.9. Corollary. *In a pullback square*

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\psi"] \arrow[d, "f'"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}' \arrow[r, "\varphi"] & \mathcal{E}\ , \end{tikzcd} \tag{5}$$

with f (and hence f') proper, the identity $\varphi^{-1}f(\mathcal{C}) = f'(\psi^{-1}\mathcal{C})$ holds for closed subtoposes $\mathcal{C} \subseteq \mathcal{F}$. ■

6.10. Corollary. *In the pullback square (5), suppose f is a proper surjection. Then*
(i) *The proper map f' is also surjective.*
(ii) *If ψ is proper, then so is ϕ.*

Proof. (i) follows immediately from (6.9), and (ii) then follows using Proposition 2.1 and Proposition 2.2. ■

6.11. Corollary. *Suppose a proper map $f: \mathcal{F} \to \mathcal{E}$ is given as the limit of a diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "p_i"] \arrow[dr, "f"'] & & \mathcal{F}_i \arrow[dl, "f_i"] \\ & \mathcal{E} & \end{tikzcd}$$

of proper surjections $\{f_i: \mathcal{F}_i \to \mathcal{E}\}$, indexed by a filtered category I. Then f is surjective.

Proof. Let $i_0 \in I$. Then by (5.10) the identity

$$f(p_{i_0}^{-1}\mathcal{C}) = \bigwedge\{f_i(t_\alpha^{-1}\mathcal{C}) \mid \alpha: i \to i_0\}$$

(where $t_\alpha: \mathcal{F}_i \to \mathcal{F}_{i_0}$ denotes the transition map induced by α) holds for closed subtoposes $\mathcal{C} \subseteq \mathcal{F}_{i_0}$. The statement follows by taking $\mathcal{C} = \mathcal{F}_{i_0}$. ■

§7 Descent along proper maps

In this section, we shall use the descent theorem for proper maps between locales [] to deduce some of the properties of descent along proper maps between toposes. We begin by recalling the basic definitions.

Consider for each topos $\mathcal{E}$ the (2)-category ($\mathcal{E}$-locales) of internal locales in $\mathcal{E}$. This category is equivalent to that of localic toposes over $\mathcal{E}$. A map $f: \mathcal{F} \to \mathcal{E}$ of toposes induces a functor

$$f^{\#}: (\mathcal{E}\text{-locales}) \to (\mathcal{F}\text{-locales}) \tag{1}$$

by pullback. The map $f: \mathcal{F} \to \mathcal{E}$ gives a diagram of pullbacks

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "\pi_{12}"] \arrow[r, "\pi_{13}"] \arrow[r, shift right=2, "\pi_{23}"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathcal{F} \arrow[r, "f"] & \mathcal{E}\,. \\ & \mathcal{F} \arrow[u, "\delta"] & & \end{tikzcd} \tag{2}$$

Descent data (*relative to* f) on a locale X in $\mathcal{F}$ consists of a map $\theta: \pi_1^\# X \to \pi_2^\# X$ such that the following two identities hold:

$$\delta^\#(\theta) = 1$$
$$\pi_{23}^\#(\theta) \circ \pi_{12}^\# = \pi_{13}^\#(\theta) \quad \text{cocycle condition}$$

(these identities should of course be expressed more carefully by taking the 2-isomorphisms $\delta^\# \pi_1^\# \cong \mathrm{id}$, $\pi_{12}^\# \pi_2^\# \cong \pi_{23}^\# \pi_1^\#$, etc. into account). If (X, θ) and (T, τ) are locales in $\mathcal{F}$ equipped with descent data, a morphism $(X, \theta) \to (T, \tau)$ is a map of locales $\alpha: X \to Y$ in $\mathcal{F}$ which is compatible with the descent data, i.e. $\pi_2^\#(\alpha) \circ \theta = \tau \circ \pi_1^\#(\alpha)$. In this way, one obtains a category

$$\mathbf{Des}(f)$$

of locales in $\mathcal{F}$ equipped with descent data.

If Z is a locale in $\mathcal{E}$, the natural isomorphism of functors in (2), $\pi_1^\# \circ f^\# \cong \pi_2^\# \circ f$, provides the pullback $f^\# Z$ with canonical descent data. This construction defines a functor

$$f^\#: (\mathcal{E}\text{-locales}) \to \mathbf{Des}(f). \tag{3}$$

7.1. Definition ([]). The map $f: \mathcal{F} \to \mathcal{E}$ is said to be of *effective descent* for locales if the functor (3) is an equivalence of categories.

(One should really speak of equivalence of 2-categories, but we shall not mention straightforward 2-categorical details explicitly.)

One also expresses Definition 7.1 informally by saying *"locales descend along f."* The definition applies of course to any (2-categorical) fibration of toposes. In particular, it applies to subcategories of locales which are stable under pullback along topos morphisms, such as compact locales, discrete locales (i.e. sheaves), etc. Thus, if $f: \mathcal{F} \to \mathcal{E}$ is of effective descent for sheaves, we say that *"sheaves descend along f."*

We shall prove the following:

7.2. Theorem. *Let $f: \mathcal{F} \to \mathcal{E}$ be a proper surjection of toposes. Then locales and sheaves descend along f.*

We prove Theorem 7.2 by reduction to localic descent, by means of the next two lemmas.

7.3. Lemma. *Consider a commutative diagram of toposes*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "g"] \arrow[dr, "h"'] & & \mathcal{F} \arrow[dl, "f"] \\ & \mathcal{E} . & \end{tikzcd}$$

If f is of effective descent for locales and g is hyperconnected, then h is of effective descent for locales.

PROOF. The proof is by elementary category theory, using the fact that for a hyperconnected map $g: \mathcal{G} \to \mathcal{F}$, the functor $g^{\#}: \mathcal{F}$-locales $\to \mathcal{G}$-locales is fully faithful. The argument is otherwise analogous to [, p. 67], and we only give a sketch.

Suppose (X, θ) is a locale in $\mathcal{G}$ with descent data θ for h. To see that X descends to $\mathcal{F}$, first observe that by pullback along the map

$$\mathcal{G} \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G} \times_{\mathcal{E}} \mathcal{G},$$

X also has descent data for g. Since g is open, hence of effective descent [], we have $X \cong g^{\#}(Y)$ for a locale Y in $\mathcal{F}$. Now, the map

$$g \times_{\mathcal{E}} g: \mathcal{G} \times_{\mathcal{E}} \mathcal{G} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$$

is hyperconnected, inherited from g. Thus $(g \times_{\mathcal{E}} g)^{\#}$ is fully faithful, and the descent data θ for h must therefore be of the form $(g \times_{\mathcal{E}} g)^{\#}(\tau)$ where τ is descent data for f on Y. Since f is assumed to be of effective descent, we conclude that $Y \cong f^{\#}(Z)$ for a locale Z in $\mathcal{E}$; that is, X descends to $\mathcal{E}$.

We leave the remaining details to the reader. ∎

7.4. Lemma. *Let $f: \mathcal{F} \to \mathcal{E}$ be a proper surjection between toposes. Suppose $\alpha: X \to Y$ is a map between locales in $\mathcal{E}$. If $f^{\#}\alpha$ is open, then so is α.*

PROOF. Factor f as $h \circ l$ where $\mathcal{F} \xrightarrow{\langle} \mathcal{L}$ is hyperconnected and $l: \mathcal{L} \to \mathcal{E}$ is localic. Thus, $\mathcal{L}$ is equivalent (as an $\mathcal{E}$-topos) to the topos $\mathrm{Sh}_{\mathcal{E}}(L)$ of sheaves on an internal locale L in $\mathcal{E}$. By Corollary 2.5, l is proper, or equivalently, L is a compact locale in $\mathcal{E}$. Consider now first the pullback squares of toposes

$$\begin{tikzcd}
\mathrm{Sh}_{\mathcal{F}}(f^{\#}X) \arrow[r, "f^{\#}\alpha"] \arrow[d, "h''"] & \mathrm{Sh}_{\mathcal{F}}(f^{\#}Y) \arrow[r] \arrow[d, "h'"] & \mathcal{F} \arrow[d, "h"] \\
\mathrm{Sh}_{\mathcal{L}}(l^{\#}X) \arrow[r, "l^{\#}\alpha"] & \mathrm{Sh}_{\mathcal{L}}(l^{\#}Y) \arrow[r] & \mathcal{L} .
\end{tikzcd}$$

Since h is hyperconnected, so are h' and h''. Since any hyperconnected map is an open surjection, our assumption that $f^{\#}\alpha$ is open implies (see [,]) that $l^{\#}\alpha$ is open.

Next, consider the similar diagram

$$\begin{tikzcd}
\mathrm{Sh}_{\mathcal{L}}(l^{\#}X) \arrow[r, "l^{\#}\alpha"] \arrow[d, "l''"] & \mathrm{Sh}_{\mathcal{L}}(l^{\#}Y) \arrow[r] \arrow[d, "l'"] & \mathcal{L} \arrow[d, "l"] \\
\mathrm{Sh}_{\mathcal{E}}(X) \arrow[r, "\alpha"] & \mathrm{Sh}_{\mathcal{E}}(Y) \arrow[r] & \mathcal{E} .
\end{tikzcd} \tag{4}$$

As a diagram of localic toposes over $\mathcal{E}$, (4) corresponds to the diagram of locales in $\mathcal{E}$

$$\begin{tikzcd}
L \times X \arrow[r, "l^{\#}\alpha"] \arrow[d, "l''"] & L \times Y \arrow[r] \arrow[d, "l'"] & L \arrow[d, "l"] \\
X \arrow[r, "\alpha"] & Y \arrow[r] & 1 ,
\end{tikzcd}$$

where the projections l' and l'' are proper surjections by pullback-stability. Since $l^{\#}\alpha$ is open, it follows from [, 5.10] that α is open. ∎

Proof of Theorem 7.2. As in the proof of the last lemma, we factor the proper surjection $f\colon \mathcal{F} \to \mathcal{E}$ as a hyperconnected map $h\colon \mathcal{F} \to \mathcal{L}$ followed by a localic proper surjection $l\colon \mathcal{L} \to \mathcal{E}$, where $\mathcal{L} = \mathrm{Sh}_{\mathcal{E}}(L)$ for an internal compact locale in $\mathcal{E}$. Then locales in $\mathcal{L}$ correspond to locales over L in $\mathcal{E}$. By applying [, 5.6] to the proper surjection $L \to 1$ of locales in $\mathcal{E}$, it follows that locales descend along l. Also, since h is an open surjection, locales descend along h [, Ch VIII, Thm 1]. By Lemma 7.3, we conclude that $f = l \circ h$ is of effective descent for locales.

To show that sheaves descend, one first identifies a sheaf S with a discrete locale, i.e. a locale S with the property that $S \to 1$ and the diagonal $S \hookrightarrow S \times S$ are open maps. Since open maps are preserved by pullback, and descend down proper surjections by Lemma 7.4, descent of sheaves now follows formally from that of locales.

This proves Theorem 7.2. ∎

A useful application of Theorem 7.2 concerns the representation of toposes by localic groupoids. For a groupoid G in the category of locales, we write G_0, G_1 for the locales of objects and arrows respectively, and denote the structure maps by

$$\begin{tikzcd}
G_1 \times_{G_0} G_1 \arrow[r, "\circ"] & G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0 \arrow[r, "u"] & G_1 \arrow[r, "i"] & G_1
\end{tikzcd}$$

(u for units, i for inverse, s and t for source and target). The associated topos of (right) G-sheaves is denoted $\mathcal{B}G$. It is extensively discussed in [,]. We recall in particular the following invariance property from []. A homomorphism $\varphi\colon G \to H$ between localic groups is called a *weak* (or *essential*) *equivalence* if

(i) The map $G_0 \times_{H_0} H_1 \xrightarrow{t\pi_2} H_0$ is an open surjection.

(ii) The square

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d] & H_1 \arrow[d] \\
G_0 \times G_0 \arrow[r] & H_0 \times H_0
\end{tikzcd}$$

is a pullback.

Here $G_0 \times_{H_0} H_1$ in (i) is the pullback along $s: H_1 \to H_0$. It is shown that such a weak equivalence induces an equivalence of toposes

$$\varphi: \mathcal{B}G \simeq \mathcal{B}H.$$

We shall also refer to a weak equivalence φ of this kind as *open*, and contrast it with the notion of *proper* weak equivalence, defined by replacing "open" by "proper" in (i).

7.5. Proposition. *Any proper weak equivalence $\varphi: G \to H$ induces an equivalence of toposes $\varphi: \mathcal{B}G \simeq \mathcal{B}H$.*

PROOF. As in [, 5.15], with the use of descent of sheaves along an open map replaced by an application of Theorem 7.2. ∎

We shall see some particular applications of this result in the next chapter.

CHAPTER II. SEPARATED MAPS

In this chapter we consider the separation property which accompanies propriety of a map, namely that of having a proper diagonal. As a typical illustration of the rôle played by this property, we show that the classifying toposes of compact localic groups are precisely the hyperconnected pointed toposes which are separated or "Hausdorff" in this sense (section 3). We also use it to formulate and prove a topos-version of the so-called Reeb stability theorem for foliations (sections 5 and 6).

The definition and elementary formal properties of separated maps are dealt with in the first two sections. For Reeb stability, we shall also need to recall various properties of locally connected and locally compact internal locales in a topos (section 4).

§1 Definition and examples

Recall that a topological space X is Hausdorff precisely when the diagonal embedding $\Delta\colon X \hookrightarrow X \times X$ is closed, that is, a proper map of topological spaces. Based on this idea, we say a topos $\mathcal{E}$ is *Hausdorff* if the diagonal map $\Delta\colon \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ is a proper map of toposes.

1.1. Examples. (1) Let X be a locale. Since the construction of $\mathrm{Sh}(X)$ from X preserves finite limits, $\mathrm{Sh}(X)$ is a Hausdorff topos iff X is a (strongly) Hausdorff locale [,]. A Hausdorff topological space need not be Hausdorff as locale (since the localic product is in general bigger than the topological one); those which are include the locally compact Hausdorff spaces.

(2) Let G be a discrete group. Then the topos $\hat{G}$ of G-sets is Hausdorff iff G is finite. Indeed, let $p\colon \mathbf{Set} \to \hat{G}$ be the unique point. Then p is an open surjection (in fact, a slice). So (I 4.7) applied to the pullback

$$\begin{array}{ccc} \mathbf{Set}/G & \longrightarrow & \hat{G} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow{p \times p} & \hat{G} \times \hat{G} \end{array}$$

implies that $\hat{G}$ is Hausdorff iff $\mathbf{Set}/G \to \mathbf{Set}$ is proper, that is, iff G is finite (I 1.2).

The Hausdorff property extends to maps of toposes in the obvious way, to give the following general notion of separated map:

1.2. Definition. A map $f\colon \mathcal{F} \to \mathcal{E}$ between toposes is said to be *separated* if $\mathcal{F}$ is Hausdorff as an $\mathcal{E}$-topos, that is, if its diagonal $\Delta_f\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is a proper map.

1.3. Examples. (1) Let E be any object in a topos $\mathcal{E}$. The canonical morphism $\mathcal{E}/E \to \mathcal{E}$ is separated iff the map $\mathcal{E}/E \to \mathcal{E}/E\times E$ induced by the diagonal $E \to E\times E$ is proper. Since this map is an embedding, (I 5.1) tells us this is the case iff $E \hookrightarrow E \times E$ defines a closed subtopos of $\mathcal{E}/E \times E$. This means that the diagonal is a complemented subobject of $E \times E$. Thus, $\mathcal{E}/E \to \mathcal{E}$ is separated iff E is decidable.

Recall from (I 1.4) that $\mathcal{E}/E \to \mathcal{E}$ is proper iff E is Kuratowski-finite. It follows that $\mathcal{E}/E$ is proper and separated iff E is a finite locally constant object in $\mathcal{E}$, i.e. iff $\mathcal{E}/E \to \mathcal{E}$ is a finite covering projection of toposes.

(2) Let $Y \to X$ be a map of locales. Then the associated map $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is separated iff $Y \to Y\times_X Y$ is closed.

Example 1.1 (2) of a Hausdorff topos is "typical" in a sense which we now explain. Recall that for a localic groupoid G, its topos of G-equivariant sheaves is denoted $\mathcal{B}G$. We say the localic groupoid G is *open* (resp. *proper*) if its source and target maps

$$G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0 \tag{1}$$

are open (resp. proper). Following [], we say a (not necessarily open) groupoid G is *étale complete* if the diagram

$$\begin{array}{ccc} \mathrm{Sh}(G_1) & \xrightarrow{t} & \mathrm{Sh}(G_0) \\ {\scriptstyle s}\downarrow & & \downarrow \\ \mathrm{Sh}(G_0) & \longrightarrow & \mathcal{B}G \end{array} \tag{2}$$

is a pullback. We recall that any topos can be represented as $\mathcal{B}G$ for some open étale complete G []. Moreover, the notion of étale completeness is invariant both under open weak equivalence (see [, 3.2]) and (by a similar formal argument) proper weak equivalence.

1.4. Proposition. *For an open or proper étale complete groupoid G, its classifying topos $\mathcal{B}G$ is separated iff $(s,t)\colon G_1 \to G_0 \times G_0$ is proper.*

PROOF. The pullback (2) can be rewritten as the pullback

$$\begin{array}{ccc} \mathrm{Sh}(G_1) & \longrightarrow & \mathcal{B}G \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathrm{Sh}(G_0)\times\mathrm{Sh}(G_0) & \longrightarrow & \mathcal{B}G\times\mathcal{B}G\,. \end{array}$$

Since the bottom map is an open or a proper surjection, the diagonal Δ of $\mathcal{B}G$ is proper iff $(s,t)\colon \mathrm{Sh}(G_1) \to \mathrm{Sh}(G_0)\times\mathrm{Sh}(G_0) \cong \mathrm{Sh}(G_0\times G_0)$ is by (I 2.7), (I 5.8) and (I 6.9), iff $(s,t)\colon G_1 \to G_0\times G_0$ is proper by (I 1.4). ∎

1.5. Example. Let G be a discrete group acting on a space X. The topos $\mathrm{Sh}_G(X)$ of G-equivariant sheaves on X is separated iff the action by G on X is proper. The canonical map $\mathrm{Sh}(X) \to \mathrm{Sh}_G(X)$ (with the forgetful functor as its inverse image) is proper iff G is finite.

1.6. Example. Recall that a localic groupoid G is called étale if its source and target maps (1) are local homeomorphisms. Any étale groupoid is étale complete []. The toposes of the form $\mathcal{B}G$ for étale G are exactly the (localic) étendues ([, VIII 3]). Separated étendues are closely related to *orbifolds*; see [] for details.

§2 Formal properties

Separated maps have the following elementary closure properties:

2.1. Proposition. (i) *Any embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ is separated.*

(ii) *In a commutative triangle*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "g"] \arrow[dr, "h"'] & & \mathcal{F} \arrow[dl, "f"] \\ & \mathcal{E}, & \end{tikzcd} \tag{1}$$

if f and g are separated, then so is h;

(iii) *if g is a proper surjection and h is separated, then so is f; and*

(iv) *if h is proper and f is separated then g is proper.*

Proof. These all follow from properties of proper maps by elementary diagram arguments of a well-known kind:

(i) The diagonal of an embedding is an equivalence, hence proper (I 2.1(i)).

(ii) Consider the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[d, "\Delta_g"'] \arrow[dr, "g"] & \\ \mathcal{G} \times_{\mathcal{F}} \mathcal{G} \arrow[r] \arrow[d, "p"'] & \mathcal{F} \arrow[d, "\Delta_f"] \\ \mathcal{G} \times_{\mathcal{E}} \mathcal{G} \arrow[r, "g \times g"] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \end{tikzcd} \tag{2}$$

where the (bottom) square is a pullback. Then p is proper since Δ_f is (I 4.7). Hence $\Delta_h\colon \mathcal{G} \to \mathcal{G} \times_{\mathcal{E}} \mathcal{G}$, as the composite of Δ_g and p, is proper by (I 2.1(ii)).

(iii) The diagram (2) contains a triangle

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "g"] \arrow[dr, "\Delta_h \circ (g \times g)"'] & & \mathcal{F} \arrow[dl, "\Delta_f"] \\ & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} & \end{tikzcd} .$$

If h is separated and g is a proper surjection, then $(g \times g) \circ \Delta_h$ is proper, and hence by (I 2.2) so is Δ_f.

(iv) The map g is the composition $g = \pi_2 \circ (\mathrm{id}, g)$ in the following diagram where both squares are pullbacks:

$$\begin{tikzcd} & \mathcal{G} \arrow[r, "g"] \arrow[d, "(\mathrm{id},g)"'] & \mathcal{F} \arrow[d, "\Delta_f"] \\ \mathcal{G} \arrow[d, "h"'] & \mathcal{G} \times_{\mathcal{E}} \mathcal{F} \arrow[l] \arrow[r, "g \times \mathrm{id}"] \arrow[d, "\pi_2"] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \\ \mathcal{E} & \mathcal{F} \arrow[l] & \end{tikzcd}$$

Since h and Δ_f are proper by assumption, so are π_2 and (id, g), and hence g. ■

2.2. Proposition. *In a pullback square*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{E} \end{tikzcd} ,$$

(i) *if f is separated, then so is $\bar{f}$;*
(ii) *the converse holds if g is a proper (or open) surjection.*

PROOF. By the equivalence θ in the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "(\bar{f},\Delta_f)"] \arrow[dr, "\Delta_{\bar{f}}"'] & \mathcal{G} \times_{\mathcal{E}} (\mathcal{F} \times_{\mathcal{E}} \mathcal{F}) \arrow[d, "\theta"] \\ & \mathcal{H} \times_{\mathcal{G}} \mathcal{H} \end{tikzcd} ,$$

the diagonal $\Delta_{\bar{f}}$ is a pullback of the diagonal Δ_f. Thus the proposition follows from (I 4.7, 4.9). ■

2.3. Corollary. *In the triangle* (1), *if h is separated and the diagonal $\Delta_f: \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is separated (for example, if f is localic), then g is separated.*

PROOF. Form the diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\Delta_g"] \arrow[dr, "\Delta_h"'] & \mathcal{G} \times_{\mathcal{F}} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_f"] \\
& \mathcal{G} \times_{\mathcal{E}} \mathcal{G} \arrow[r, "g \times g"] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} .
\end{tikzcd}$$

If Δ_f is separated, then so is its pullback $\mathcal{G} \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G} \times_{\mathcal{E}} \mathcal{G}$ by (2.2). But then Δ_g is proper, by Proposition 2.1 (iv). ■

2.4. Proposition. *Suppose $f: \mathcal{F} \to \mathcal{E}$ is the limit*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "p_i"] \arrow[dr, "f"'] & & \mathcal{F}_i \arrow[dl, "f_i"] \\
& \mathcal{E} &
\end{tikzcd} \tag{3}$$

of a diagram of separated maps $\{f_i: \mathcal{F}_i \to \mathcal{E}\}$ indexed by a filtered category I. Then f is separated.

PROOF. The diagonal $\Delta_f: \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is the limit of the diagram $\{g_i: \mathcal{G}_i \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}\}$ obtained by pulling back each diagonal $\Delta_{f_i}: \mathcal{F}_i \to \mathcal{F}_i \times_{\mathcal{E}} \mathcal{F}_i$ along $\mathcal{F} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}_i \times_{\mathcal{E}} \mathcal{F}_i$ (and the obvious induced transition maps). The statement therefore follows from the stability of proper maps under pullback and filtered inverse limits (I 5.8, 5.10). ■

2.5. Proposition. *A map $f: \mathcal{F} \to \mathcal{E}$ is separated iff both parts of its hyperconnected-localic factorization are.*

PROOF. Let X be the localic reflection of f in $\mathcal{E}$. Then, writing $\mathcal{L} = \mathrm{Sh}_{\mathcal{E}}(X)$ for the $\mathcal{E}$-topos of internal sheaves on the locale X, the map f factors as a hyperconnected map $h: \mathcal{F} \to \mathcal{L}$ followed by a localic map $l: \mathcal{L} \to \mathcal{E}$. If h and l are separated, so is $f = l \circ h$ by 2.1(i). Conversely, suppose f is separated. Then, first of all, since h is a proper surjection (I 2.4), l must be separated by 2.1(iii). To prove that h is also separated, we use a diagram as in the proof of 2.1(iv), in this case

$$\begin{tikzcd}
 & \mathcal{F} \arrow[r, "h"] \arrow[d, "(\mathrm{id},h)"'] & \mathcal{L} \arrow[d, "\Delta_l"] \\
\mathcal{F} \arrow[d, "f"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{L} \arrow[l] \arrow[r, "h\times\mathrm{id}"] \arrow[d, "\pi_2"] & \mathcal{L} \times_{\mathcal{E}} \mathcal{L} \\
\mathcal{E} & \mathcal{L} . \arrow[l] &
\end{tikzcd}$$

Note that since l is localic, Δ_l, and therefore its pullback (id, h), is an embedding, hence separated. Furthermore, by 2.2(i), the pullback π_2 of f is separated. By 2.1(ii), the composition $h = \pi_2 \circ (\mathrm{id}, h)$ must be separated. ∎

§3 Hyperconnected Hausdorff toposes

In this section we characterize hyperconnected Hausdorff toposes with a base point. Fix an arbitrary base topos $\mathcal{S}$. For a localic group G in $\mathcal{S}$, there is a topos $\mathcal{B}G = \mathcal{B}_{\mathcal{S}}G$ of internal G-sets in $\mathcal{S}$ with a canonical point $q: \mathcal{S} \to \mathcal{B}G$. Clearly $\mathcal{B}G$ is hyperconnected, and separated if G is compact and étale complete, by Proposition 1.4 (applied to the case where G_0 is the one-point space). Our first theorem states that every pointed hyperconnected Hausdorff topos is of this form.

3.1. Theorem. *Let $f: \mathcal{E} \to \mathcal{S}$ be a topos over $\mathcal{S}$ with a base point (section) $s: \mathcal{S} \to \mathcal{E}$. Then $\mathcal{E}$ is hyperconnected and Hausdorff over $\mathcal{S}$ iff there exists a compact étale complete localic group G such that $\mathcal{E} \cong \mathcal{B}G$ (as pointed $\mathcal{S}$-toposes).*

Proof. ($\Leftarrow$) This implication is proved before the statement of the theorem.

($\Rightarrow$) Let

$$H = \; H_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} H_0$$

be an open étale complete localic groupoid in $\mathcal{S}$ so that $\mathcal{E} \cong \mathcal{B}H$ as $\mathcal{S}$-toposes. We can choose H so large that p lifts to a point (again denoted) $p: 1 \to H_0$. Since $\mathcal{E}$ is separated and hyperconnected, the map $(d_0, d_1): H_1 \to H_0 \times H_0$ is proper while

$$H_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} H_0 \longrightarrow 1$$

is a coequaliser of locales in $\mathcal{S}$. Let R be the image of (d_0, d_1) in $H_0 \times H_0$. Then R is a closed sublocale of $H_0 \times H_0$ while the projections $R \rightrightarrows H_0$ are open. Thus by [], R is the kernel pair of its coequaliser, that is, $R = H_0 \times H_0$ and $H_1 \to H_0 \times H_0$ is a proper surjection. But this means that for the vertex group $G = H_p$ at the point $p \in H_0$, the full inclusion $G \hookrightarrow H$ is a proper weak equivalence of localic groupoids. By (I 7.5), it induces an equivalence of $\mathcal{S}$-toposes

$$\mathcal{B}G \xrightarrow{\sim} \mathcal{B}H \cong \mathcal{E},$$

and it is clear that under this equivalence the point $p: \mathcal{S} \to \mathcal{B}H$ corresponds to the canonical point of $\mathcal{E}$.

This proves the theorem. ■

In the case where the base topos is **Set** (or any other Boolean topos) this can be sharpened:

3.2. Theorem. *Let $\mathcal{E}$ be a pointed topos over* **Set**. *Then $\mathcal{E}$ is hyperconnected and Hausdorff iff $\mathcal{E}$ is the topos $\mathcal{B}G$ of continuous G-sets for some profinite group G.*

Note in particular that this implies that every pointed hyperconnected Hausdorff topos is coherent.

For the proof of this second theorem, we recall the construction of the étale completion of a localic group from []. Let G be a localic group, and consider the topos $\mathcal{B}G$ of continuous G-sets with its canonical point q: **Set** $\to \mathcal{B}G$. The monoid of endomorphisms of q can be explicitly described in terms of G, as

$$\mathrm{End}\,(q) = \varprojlim_U G/U = M(G).$$

Here U ranges over all open subgroups of G (ordered by inclusion), and G/U is the discrete space of right cosets. So a point in $M(G)$ can be denoted

$$t = \{U \cdot t_U\}_U,$$

and multiplication is then described as

$$U \cdot (t \cdot s)_U = U \cdot t_U \cdot s_{t_U^{-1} U t_U} \in G/U.$$

Let $A(G) \subseteq \varprojlim_U G/U$ denote the localic group of invertible elements of this monoid. There are canonical maps

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[drr, "\mathrm{proj}"'] & A(G) \arrow[r, hook] & M(G) \arrow[d, "\pi_U"] \\ & & G/U\ . \end{tikzcd}$$

Here $\varphi: G \to M(G)$ is defined by $\varphi(g) = \{U \cdot g\}_U$. This is a homomorphism of localic monoids. Note that it follows from this diagram that each projection $A(G) \to G/U$ is open.

Proof of Theorem 3.2. We only need to establish the forward implication. By Theorem 3.1, there is a compact localic group G so that $\mathcal{E} \cong \mathcal{B}G$. If G is étale complete, then the map $\varphi: G \to A(G)$ is an isomorphism. If G is compact, then each G/U is a finite set, so $M(G)$ is a compact Hausdorff monoid. Since $G \cong A(G)$ is compact, it is closed in $M(G)$. Since $A(G)$ maps surjectively onto each G/U, it is also dense in $M(G)$. Thus $G = A(G) \cong M(G)$. In particular, the compact group G is Hausdorff and totally disconnected, hence profinite [, 8.41]. ∎

§4 Locally connected and locally compact maps of locales

In this section, we review some definitions and facts involving locally connected and locally compact locales in a topos. These locales will play a rôle in our treatment of the Reeb stability theorem. Most of the material presented here is well-known, although our approach to the stability of local compactness in the spirit of (I §5) is to some extent novel. Our arguments, presented in the language of set theory, will be constructive throughout to ensure a valid interpetation in an arbitrary base topos $\mathcal{S}$ (fixed for the duration of the present section).

Our review of local connectedness is primarily based on the Appendix of []. Let X be a locale. Recall that X has "global support" (the map $X \to 1$ is surjective) if and only if any covering family of opens of X has an element (in other words, is non-empty in a strong sense). An open $U \subseteq X$ which (considered as locale) has global support is said to be *positive*. A cover $\{U_i\}$ of X by positive opens U_i is said to be *connected* if, for any U_i and $U_{i'}$, there is a chain.

$$U_i = U_{i_0},\, U_{i_1},\, \ldots,\, U_{i_n} = U_{i'}$$

with $U_{i_k} \cap U_{i_{k+1}}$ positive for each $k = 0, \ldots, n-1$. The locale X is *connected* if it has global support, and every cover of X by positive opens is connected. X is *locally connected* if it has a basis consisting of connected opens; if X is also connected (clc), this basis can of course be chosen to contain X itself. A locally connected locale is in particular open, by [, V 3.2].

Like compactness, any constructively defined property of locales can be made to apply to a map between locales by "relativising" to a sheaf topos, that is, by using the well-known equivalence (see e.g. [] or []) between localic maps $f\colon Y \to X$ and internal locales in $\mathrm{Sh}(X)$. Thus, we say a map $f\colon Y \to X$ is *(locally) connected* if f is (locally) connected when viewed as locale in $\mathrm{Sh}(X)$. Interpreting the definitions given above in the topos $\mathrm{Sh}(X)$ yields:

4.1. Lemma. *A map $f\colon Y \to X$ is locally connected iff f is an open map, and Y has a basis $\mathcal{B}$ with the following property: If $B = \bigvee_i B_i$ in Y where B, B_i belong to $\mathcal{B}$, then for any pair of indices i and i', $f(B_i) \cap f(B_{i'})$ is covered by those open $U \subseteq X$ for which there is a chain $B_i = B_{i_0}, B_{i_1}, \ldots, B_{i_n} = B_{i'}$ with $U \subseteq f(B_{i_k} \cap B_{i_{k+1}})$ for $k = 0, \ldots, n-1$. The map f is in addition connected iff f is surjective and $\mathcal{B}$ can be chosen to contain Y.* ∎

As can be shown directly from this description:

4.2. Lemma. *The class of (connected and) locally connected maps is closed under composition and pullback.* ∎

Our review of local compactness draws upon [] (but see also []). Recall that for two opens U, $V \subseteq X$, one says that U is "way below" V, denoted $U \ll V$, if every cover of V contains a finite cover of U. Thus, X is compact precisely when $X \ll X$. The locale X is said to be *locally compact* if, for every open $V \subseteq X$, one has $V = \bigvee\{U \mid U \ll V\}$. In a locally compact locale, the way below relation interpolates: $U \ll V$ only if there exists $W \subseteq X$ open such that $U \ll W \ll V$.

We first extend the notions of compactness and local compactness to a suitable *presentation* for a locale, namely a site $(\mathbb{P}, C)$ as defined in (I 4.2) where the underlying category $\mathbb{P}$ is a preordered set. Thus, for $x \in \mathbb{P}$, the members of $C(x)$ (the basic covers of x) are families $\{x_i\}$ of elements of $\downarrow(x)$. The stability condition states that for any basic cover $\{x_i\}$ of x and $y \leq x$ in $\mathbb{P}$, there is a basic cover $\{y_j\}$ of y with members in $\downarrow\{x_i\}$ (thus, C is a *covering system* in the sense of []).

The data $(\mathbb{P}, C)$ presents the locale X if the frame of opens of X can be reconstructed as the downsets D of $\mathbb{P}$ which are *closed* in the sense that $C(x) \subseteq D \Rightarrow x \in D$. This can also be formulated by saying that there is an association $x \mapsto B_x \subseteq X$ of elements of $\mathbb{P}$ with opens of X such that

(i) The family $\{B_x\}$ constitutes a basis for X, in the strong sense that each $B_x \cap B_y$ is covered by $\{B_z \mid z \leq x \text{ and } z \leq y\}$;

(ii) For a family $\{x_i\} \subseteq \mathbb{P}$, the corresponding family of opens $\{B_{x_i}\}$ covers B_x in X iff $\downarrow\{x_i\}$ contains a generated cover of x.

If $\varphi: \mathcal{S}' \to \mathcal{S}$ is a topos over (our chosen base topos) $\mathcal{S}$, then $\varphi^*(\mathbb{P}, C)$ remains a presentation for the locale $\varphi^{\#}X$ in $\mathcal{S}'$.

4.3. Definition. Let $(\mathbb{P}, C)$ be a presentation with a "stable compatible system of directed covers" (I 4.7), or *directed* presentation for short. Explicitly, $\mathbb{P}$ has finite meets and joins satisfying the distributive law, and $C = P \cup S$ where P is the topology given by finite joins and S is a system of stable directed covers which are compatible with binary joins: if $\{x_i\}$ is a basic S-cover, then so is $\{x_i \vee y\}$. Say y is "way below" x in $\mathbb{P} = (\mathbb{P}, C)$, and write $y \ll x$, if any directed cover x_i of x has an element x_i such that $y \wedge x_i$ is a cover of y. $\mathbb{P}$ is *compact* if the terminal element $1 \in \mathbb{P}$ satisfies $1 \ll 1$ (as anticipated in (I 5.2)) and *locally compact* if the (directed) family $\{y \mid y \leq x \text{ and } y \ll x\}$ is a cover for each $x \in \mathbb{P}$.

Note that any locale X has a directed presentation, namely its own frame of opens, which is compact (resp. locally compact) in the sense just defined precisly when X is compact (resp. locally compact). More generally, we have:

4.4. Proposition. *A directed presentation $\mathbb{P}$ for a locale X is compact (resp. locally compact) iff X is compact (resp. locally compact).*

PROOF. The assignment $x \mapsto B_x$ preserves and reflects the waybelow relation (since it preserves binary meets, preserves and reflects covers, and any directed cover of a basis element B_x in the frame of opens of X is refined by a directed cover of basis elements).

Since $B_1 = X$, the equivalence for the case of compactness is now clear (by (I 5.2), we could also have referred to (I 5.3)).

If $\mathbb{P}$ is locally compact, then for each $x \in \mathbb{P}$, the basis element B_x is covered by the family $\{B_y \mid y \ll x\}$. Thus, for any open U of X, $U = \bigvee\{B_x \mid B_x \leq U\} = \bigvee\{B_y \mid B_y \ll U\}$, which shows that X is locally compact. Conversely, suppose X is locally compact. Then for any $x \in \mathbb{P}$,

$$\begin{aligned} B_x &= \bigvee\{U \mid U \ll B_x\} \\ &= \bigvee\{B_y \mid B_y \leq U \ll B_x \text{ and } y \leq x\} \\ &= \bigvee\{B_y \mid y \leq x \text{ and } y \ll x\}, \end{aligned}$$

which says that $\{y \mid y \leq x \text{ and } y \ll x\}$ is a cover of x. ■

4.5. Lemma. *Let $\mathbb{P}$ be a directed presentation equipped with a binary relation $\prec$ — or "strong inclusion" — with the following properties:*

(i) *If $z \prec y \leq x$, then $z \prec x$.*
(ii) *If $y \prec x$ and $\{x_i\}$ is a basic S-cover of x, then $y \prec x_i$ for some i.*
(iii) *The family $\{y \mid y \prec x\}$ is a cover of x.*

Then $y < x \Rightarrow y \ll x$ and $\mathbb{P}$ is locally compact.

Proof. The system of families $\{x_i\} \subseteq \downarrow(x)$ for $x \in \mathbb{P}$ with the property that $y \prec x \Rightarrow y \prec x_i$ for some i is easily seen to be a full (i.e. upclosed under refinement, using (i)) topology on $\mathbb{P}$. Since it contains the basic S-covers by (ii), it contains all S-covers, and in particular the directed ones (here we may apply (I 4.6) in view of (I 4.7)). This shows that $y \prec x \Rightarrow y \ll x$. But then local compactness follows by (iii). ■

4.6. Lemma. *Let $\varphi: \mathcal{S}' \to \mathcal{S}$ be a $\mathcal{S}$-topos and let $\mathbb{P}$ be the presentation of a locale X by its frame of opens.*

(i) *If X is compact, then the (directed) presentation $\varphi^*\mathbb{P}$ is compact in $\mathcal{S}'$, with $\{1\}$ the only directed cover of $1 \in \varphi^*\mathbb{P}$.*
(ii) *If X is locally compact, then $\varphi^*\mathbb{P}$ is locally compact and $\varphi^*(\ll_{\mathbb{P}})$ is contained in $\ll_{\varphi^*\mathbb{P}}$.*

Proof. (i) If X is compact, then $\mathbb{P}$ is compact by Proposition 4.4. Since a compact directed presentation is a special instance of a compact site with stable compatible directed covers (I 5.2), the may apply (I 5.7) (after substituting "directed presentation" for "pretopos site") to obtain the result.

(ii) If $\mathbb{P}$ is locally compact, then the way below relation $\ll$ on $\mathbb{P}$ is a strong inclusion (4.5). Since the defining properties of a strong inclusion are "geometric" it follows that $\varphi^* \ll$ is a strong inclusion on $\varphi^*\mathbb{P}$. So the result follows by an application of Lemma 4.5. ■

Using (4.4) we conclude:

4.7. Corollary. *Local compactness of a locale is preserved under change of base.* ■

A locale with a basis of compact neighbourhoods is evidently locally compact. The converse is not true in general, but does hold for (strongly) Hausdorff locales, as we shall now show. Recall that a locale X is said to be *regular* if every open $U \subseteq X$ can be written as $U = \bigvee\{V \mid \overline{V} \subseteq U\}$, where $\overline{V}$ denotes the closure of V.

4.8. Proposition. *A compact or locally compact Hausdorff locale is regular.*

PROOF. Let X be a Hausdorff locale, with $\mathbb{P}$ the presentation of X by its lattice of opens. Since X is Hausdorff, we have for any open $U \subseteq X$:

$$\begin{aligned} X \times U &\subseteq (U \times X) \cup (X \times X - \Delta) \\ &= \bigvee\{P \times Q \mid P \subseteq U \text{ or } P \cap Q = 0\}, \\ &\subseteq \bigvee\{P \times Q \mid P \subseteq \neg Q \cup U\} \end{aligned} \tag{1}$$

where $\neg Q$ denotes $X - \overline{Q}$, the largest open of X disjoint from Q.

Suppose first that X is compact. Given an open sublocale $i\colon U \hookrightarrow X$, let $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ be the set of pairs $\{(P,Q) \mid P \subseteq \neg Q \cup U\}$. Consider the topos $\mathrm{Sh}(U)$, with $\phi\colon \mathrm{Sh}(U) \to \mathbf{Set}$ the canonical map. Identify the projection $\pi_2\colon X \times U \to U$ with the locale $\varphi^{\#}X$ in $\mathrm{Sh}(U)$, so that the embedding $(\mathrm{id}, i)\colon U \hookrightarrow X \times U$ becomes a point p of $\varphi^{\#}X$. Any $P \in \varphi^*\mathbb{P}$ gives internally an open B_P of $\varphi^{\#}X$ in $\mathrm{Sh}(U)$. In particular, any $P \in \mathbb{P}$ gives for the corresponding "constant" element $\hat{P} \in \varphi^*(\mathbb{P})$ an internal open $B_{\hat{P}}$ of $\varphi^{\#}X$, corresponding to the external open $P \times U \subseteq X \times U$. Now (1) is easily seen to imply the internal truth of the statement

$$B_{\hat{X}} \subseteq \bigvee\{B_P \mid \text{ For some } Q \in \varphi^*\mathbb{P},\ (P,Q) \in \varphi^*\mathcal{R} \text{ and } p \in B_Q\}$$

in $\mathrm{Sh}(U)$. But this then says that the (internally) directed family $\{P \mid$ For some $Q \in \varphi^*\mathbb{P}$, $(P,Q) \in \varphi^*\mathcal{R}$ and $p \in B_Q\}$ is a cover of the terminal element $\hat{X}$ of φ^*P. By Lemma 4.6 (i), it therefore contains $\hat{X}$: it is true in $\mathrm{Sh}(U)$ that there is some $Q \in \varphi^*\mathbb{P}$ such that $p \in Q$ while $(\hat{X}, Q) \in \varphi^*\mathcal{R}$. Externally,

$$U = \bigvee\{V \subseteq U \mid X \subseteq \neg Q \cup U \text{ and } V \subseteq Q\} = \bigvee\{V \mid \overline{V} \subseteq U\}.$$

Since U was arbitrary, this shows that X is regular.

Next, suppose that X is locally compact. Consider any $U, V \subseteq X$ open such that $V \ll U$. We show that $\overline{V} \subseteq U$, which will prove regularity of X. To this end, we regard the projection $\pi_1\colon X \times X \to X$ as the locale $\varphi^{\#}X$ in the topos $\mathrm{Sh}(X)$, where $\phi\colon \mathrm{Sh}(X) \to \mathbf{Set}$ again denotes the canonical map. Let q be the "generic" point of φ^*X, defined by the diagonal $\Delta\colon X \hookrightarrow X \times X$. With notation as before (but now applying to $\mathrm{Sh}(X)$), (1) implies the internal statement

$$B_{\hat{X}} \subseteq \bigvee\{B_Q \mid Q \in \varphi^*\mathbb{P},\ q \notin B_Q \text{ or } q \in B_{\hat{U}}\}$$

in $\mathrm{Sh}(X)$, in other words, the internal ideal $\{Q \mid q \notin B_Q$ or $q \in B_{\hat{U}}\}$ is a cover of $\hat{U}$. But by Lemma 4.6 (ii), $\hat{V} \ll \hat{U}$ in the site $\varphi^*\mathbb{P}$. We conclude that it is true in $\mathrm{Sh}(X)$ that $q \notin B_{\hat{V}}$ or $q \in B_{\hat{U}}$. Externally, $X \subseteq \neg V \cup U$, or $\overline{V} \subseteq U$, as required. This completes the proof. ∎

4.9. Corollary. *A compact Hausdorff locale has a basis of compact neighbourhoods, hence is locally compact.*

PROOF. Immediate from Proposition 4.8 and the fact that a compact regular locale has a basis of closed, hence compact neighbourhoods. ∎

4.10. Lemma. *For any open U of a locally compact Hausdorff locale X, its closure $\overline{U}$ is compact whenever $U \ll X$.*

PROOF. First note that for $U, V \subseteq X$ open in any locale X, $\overline{U} \subseteq V$ and $U \ll X$ together imply $U \ll V$.

Suppose X is locally compact Hausdorff, hence regular by Proposition 4.8. Let $U \ll X$, $U \subseteq X$ open. If $\overline{U}$ is covered by a directed family $\{U_i\}$ of opens, then $U \ll \bigvee\{U_i\}$ by our starting comment, and we can choose $W \subseteq X$ open such that $U \ll W \ll \bigvee\{U_i\}$; it follows that $U \ll U_i$ for some i, and then that $\overline{U} \leq U_i$ by regularity of X. ∎

4.11. Proposition. *The following conditions are equivalent to local compactness for a Hausdorff locale X:*

(i) *X is covered by the interiors of a family of compact sublocales.*
(ii) *X has a basis of compact neighbourhoods.*

PROOF. If X is locally compact, then $X = \bigvee\{U \mid U \ll X\}$, and (i) follows from (4.10). In turn, (i) implies (ii) by (4.9). That X is locally compact if (ii) holds is clear. ∎

As usual, a map $f\colon Y \to X$ of locales is said to be *locally compact* if Y is locally compact as a locale in $\mathrm{Sh}(X)$. By Corollary 4.7:

4.12. Proposition. *Locally compact maps are stable under pullback.* ∎

Like local connectedness, local compactness for a map $f\colon Y \to X$ can be translated into a property directly expressed in terms of f. In particular, if f is separated, i.e. Y is Hausdorff as a locale in $\mathrm{Sh}(X)$, we have the following.

4.13. Lemma. *Let $f\colon Y \to X$ be a separated map. Then f is locally compact iff Y has an open cover $\mathcal{V}$ such that for each $V \in \mathcal{V}$, f can be restricted to a proper map $\overline{V} \to U$ into some open $U \subseteq X$.*

PROOF. If Y has a cover $\mathcal{V}$ as described, then the family of maps

$$\{g{\restriction}V\colon V \to U \mid V \in \mathcal{V} \text{ and } \overline{V} \xrightarrow{g} U \text{ a proper restriction of } f\}$$

defines an internal cover of $f\colon Y \to X$ as a locale in $\mathrm{Sh}(X)$, by open sublocales each contained in a compact sublocale. So f is locally compact by Proposition 4.11 (i).

Conversely, if f is locally compact as locale in $\mathrm{Sh}(X)$, then Proposition 4.11 (i) gives a cover $\mathcal{V}$ of Y such that for each $V \in \mathcal{V}$, there exists a sublocale C of Y with $V \subseteq C \subseteq Y$ and some open $U \subseteq X$ such that f restricts to a proper map $C \to U$. But then such C is closed by Proposition 2.1 (iv). Hence, $\overline{V} \subseteq C$ and the further restriction of f to $\overline{V}$ remains proper. Thus, the cover $\mathcal{V}$ has the property required by the lemma. ∎

§5 A topos version of the Reeb stability theorem

In this section we shall present a topos-theoretic generalisation of the "Reeb stability theorem." Its relation to the classical Ehresman-Reeb stability theorem for foliations will be explained in §6.

Before stating the result, we recall from [] that for a point $x: \mathbf{Set} \to \mathcal{E}$ of a topos $\mathcal{E}$, an (*"étale"*) *neighbourhood* of x is a pair $(U, \tilde{x})$, where U is an object in $\mathcal{E}$ and $\tilde{x} \in x^*(U)$. This element $\tilde{x}$ may be identified with a lifting of x to a point of $\mathcal{E}/U$:

$$\begin{tikzcd} & \mathcal{E}/U \arrow[d] \\ \mathbf{Set} \arrow[ur, dashed, "\tilde{x}"] \arrow[r, "x"'] & \mathcal{E} \end{tikzcd}$$

For a map $f: \mathcal{F} \to \mathcal{E}$ between toposes, we denote by $\mathcal{F}_x$ the fiber over x, and by $\mathcal{F}_U$ the pullback over $\mathcal{E}/U$, as in the diagram

$$\begin{tikzcd} \mathcal{F}_x \arrow[r] \arrow[d] & \mathcal{F}_U \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ \mathbf{Set} \arrow[r, "\tilde{x}"] & \mathcal{E}/U \arrow[r] & \mathcal{E}\,, \end{tikzcd} \tag{1}$$

where both squares are pullbacks. Thus $\mathcal{F}_U = \mathcal{F}/f^*U$.

If L is a locale in the topos $\mathcal{E}$, we write L_x for the locale in $\mathbf{Set}$ obtained by pullback along $x: \mathbf{Set} \to \mathcal{E}$, and call it the *fiber* of L over x. We shall also write L_U for the pullback of L along $\mathcal{E}/U \to \mathcal{E}$. Thus, taking toposes of internal sheaves, in the diagram (1) for $\mathcal{F} = \mathrm{Sh}_{\mathcal{E}}(L)$, the topos $\mathcal{F}_x = \mathrm{Sh}_{\mathcal{E}}(L)_x$ is $\mathrm{Sh}(L_x)$, while $\mathcal{F}_U = \mathrm{Sh}_{\mathcal{E}}(L)_U$ is $\mathrm{Sh}_{\mathcal{E}/U}(L_U)$.

5.1. Theorem. *Let $\mathcal{E}$ be a topos. Let L be a connected, locally connected, locally compact Hausdorff locale in $\mathcal{E}$, and let x be a point of $\mathcal{E}$. If L_x is compact, then there is an étale neighbourhood $(V, \tilde{x})$ of x such that L_V is a compact locale in $\mathcal{E}/V$.*

We shall reduce the proof of Theorem 5.1 to the following lemma for locales.

5.2. Lemma. *Let $\varphi: Y \to X$ be a map of locales. Assume φ is connected, locally connected, locally compact and separated. Let x be a point of X for which the fiber $\varphi^{-1}(x) \subseteq Y$ is compact. Then there exists an open neighbourhood $U \subseteq X$ of x such that the restriction $\varphi^{-1}(U) \to U$ of φ is proper.*

Proof. Using Lemma 4.13 and Proposition 2.1 (iv), the set

$$\{V \subseteq X \mid V \text{ open and } \overline{V} \xrightarrow{\varphi} U \text{ proper for some open neighbourhood } U \text{ of } x\}$$

is easily seen to be a directed cover of the compact fiber $\varphi^{-1}(x)$. Thus, we can find opens U, V of X with $x \in U \subseteq X$ and $\varphi^{-1}(x) \subseteq V \subseteq Y$, and such that the restriction $\overline{V} \to U$

is proper. It follows that $\varphi(\overline{V}-V)$ is closed in U. Since $\varphi^{-1}(x)\subseteq V$, there is an open neighbourhood W of x such that $\varphi(\overline{V}-V)\cap W=0$, giving $(\overline{V}-V)\cap\varphi^{-1}(W)=0$; since ϕ is open, we can assume that $W\subseteq\varphi(V)$. It folows that $V\cap\varphi^{-1}(W)\subseteq V$ and $(Y-\overline{V})\cap\varphi^{-1}(W)$ form an open disjoint cover of $\varphi^{-1}(W)$. But $\varphi\colon Y\to X$ is stably connected, which means its restriction $\varphi^{-1}(W)\to W$ remains connected; since $W\subseteq\varphi(V)$ this implies $\varphi^{-1}(W)\subseteq V$. Thus, the square

$$\begin{tikzcd}
\varphi^{-1}(W) \arrow[r, hook] \arrow[d] & \overline{V} \arrow[d] \\
W \arrow[r, hook] & X
\end{tikzcd}$$

is a pullback. Since $\varphi\colon\overline{V}\to U$ is proper, so is $\varphi^{-1}(W)\to W$. ∎

Proof of Theorem 5.1. Let X be a locale for which there exists an open surjection $\pi\colon\mathrm{Sh}(X)\to\mathcal{E}$ (see []). We may choose X so large that that the point x can be lifted to a point $\bar{x}$ of X with $\pi(\bar{x})=x$. The fibered product $\mathrm{Sh}(X)\times_{\mathcal{E}}\mathrm{Sh}_{\mathcal{E}}(L)=\mathrm{Sh}_{\mathrm{Sh}(X)}(\pi^{\#}L)$ is the topos of sheaves on a locale L_X with the projection $\mathrm{Sh}(X)\times_{\mathcal{E}}\mathrm{Sh}_{\mathcal{E}}(L)\to\mathrm{Sh}(X)$ corresponding to a map $\varphi\colon L_X\to X$. The fiber $\varphi^{-1}(\bar{x})$ is the locale L_x, which we assumed to be compact. Moreover, since $\mathrm{Sh}_{\mathcal{E}}(L)\to\mathcal{E}$ is clc, locally compact and separated, so is the map $\varphi\colon L_X\to X$, by (4.2), (4.7), and (2.2). Thus, Lemma 5.2 applies, to give an open neighbourhood $U\subseteq X$ of $\bar{x}$ for which φ restricts to a proper map $\varphi^{-1}(U)\to U$. Now let $\mathcal{E}/V=\pi(U)$ be the corresponding open subtopos of $\mathcal{E}$, and consider the diagram (where we write X for the topos $\mathrm{Sh}(X)$, and similarly for L_X, etc.)

$$\begin{tikzcd}
 & \mathrm{Sh}_{\mathcal{E}/V}(L_V) \arrow[rr] \arrow[dd] & & \mathrm{Sh}_{\mathcal{E}}(L) \arrow[dd] \\
\varphi^{-1}(U) \arrow[ur] \arrow[rr] \arrow[dd] & & L_X \arrow[ur] \arrow[dd] & \\
 & \mathcal{E}/V \arrow[rr, hook] & & \mathcal{E} \\
U \arrow[ur] \arrow[rr, hook] & & X \arrow[ur] &
\end{tikzcd}$$

In this diagram, the front, back, left and right squares are pullbacks. Since $U\to\mathcal{E}/V$ is an open surjection, the propriety of $\varphi^{-1}(U)\to U$ implies that of $\mathrm{Sh}_{\mathcal{E}/V}(L_V)\to\mathcal{E}/V$ (I 2.7). Thus, L_V is a compact locale in $\mathcal{E}/V$ ∎

There is a version of Theorem 5.1, purely in terms of toposes. We say a map $f\colon\mathcal{F}\to\mathcal{E}$ between toposes is *locally compact* if the localic reflection $\mathrm{Sh}_{\mathcal{E}}(L)\to\mathcal{E}$ of f is given by a locally compact locale L in $\mathcal{E}$.

5.3. Corollary. *Let $\mathcal{E}$ be a Hausdorff topos. Let $f\colon\mathcal{F}\to\mathcal{E}$ be a connected, locally connected, locally compact and separated map of toposes, and let x be a point of $\mathcal{E}$. If* *the fiber $\mathcal{F}_x$ is compact, then there is a neighbourhood $(U, \tilde{x})$ of x so that f restricts to a proper map $\mathcal{F}_U \to \mathcal{E}/U$.*

PROOF. By (I 2.5), it suffices to prove the conclusion for the localic reflection of f in $\mathcal{E}$. The result then follows from (5.1). ∎

For some applications, it is useful to state explicitly a version of Theorem 5.1 where the map $f\colon \mathcal{F} \to \mathcal{E}$ is not necessarily connected. Let $f\colon \mathcal{F} \to \mathcal{E}$ be a locally connected map, and let $\mathcal{C}$ be a connected component of the fiber $\mathcal{F}_x$. Let U be an étale neighbourhood of x, so that $\mathcal{F}$ restricts to a map $\mathcal{F}_U \to \mathcal{E}/U$. An étale neighbourhood of $\mathcal{C}$ over U is an object V of $\mathcal{F}_U$ together with a lifting of $\mathcal{C} \hookrightarrow \mathcal{F}_U$ to $\mathcal{F}_U/V$:

$$\begin{tikzcd}
 & \mathcal{F}_x/V_x \arrow[r] \arrow[d] & \mathcal{F}_U/V \arrow[d] & \\
\mathcal{C} \arrow[r, hook] \arrow[ur, dashed] & \mathcal{F}_x \arrow[r] \arrow[d] & \mathcal{F}_U \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\
 & \mathbf{Set} \arrow[r] & \mathcal{E}/U \arrow[r] & \mathcal{E} .
\end{tikzcd}$$

We say that V has *compact* (*connected*) *fibers* if the map $\mathcal{F}_U/V \to \mathcal{E}/U$ is proper (connected). Now (5.1) has the following generalization.

5.4. Corollary. *Let $f\colon \mathcal{F} \to \mathcal{E}$ be a locally connected, locally compact and separated map. Let $x\colon \mathbf{Set} \to \mathcal{E}$ be a point of $\mathcal{E}$, and let $\mathcal{C}$ be a compact connected component of the fiber $\mathcal{F}_x$. Then there exists (étale) neighbourhoods U of x in $\mathcal{E}$ and V of x in $\mathcal{F}_U$ so that V has connected and compact fibers.*

PROOF. This follows formally from (5.1). Let $\pi_0(f)$ be the object in $\mathcal{E}$ of connected components of f, so that f factors as

$$\mathcal{F} \xrightarrow{\bar{f}} \mathcal{E}/\pi_0(f) \to \mathcal{E},$$

where $\bar{f}$ is connected (and otherwise retains all the $\mathcal{E}$-local properties of f). The pair x, $\mathcal{C}$ together define a point $\bar{x} = (x, \mathcal{C})$ of $\mathcal{E}/\pi_0(f)$, with fiber $\mathcal{F}_{\bar{x}} = \mathcal{C}$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathbf{Set} \arrow[r, "\bar{x}"] & \mathcal{E}/\pi_0(f) .
\end{tikzcd}$$

By (5.1) there is a map $U \to \pi_0(f)$ with a lifting y of $\bar{x}$ so that the pullback $\mathcal{V}$ over $\mathcal{E}/\pi_0(f)$ maps properly into $\mathcal{E}/U$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\bar{f}"] \\
\mathbf{Set} \arrow[r, "y"] \arrow[rr, bend right, "\bar{x}"'] & \mathcal{E}/U \arrow[r] & \mathcal{E}/\pi_0(f) .
\end{tikzcd}$$

The map $\mathcal{V} \to \mathcal{E}/U$ is also connected, as a pullback of $\bar{f}: \mathcal{F} \to \mathcal{E}/\pi_0(f)$. We claim that $\mathcal{V}$ is the required étale neighbourhood of $\mathcal{C}$. Indeed, it only remains to be verified that there is an object V of $\mathcal{F}$ so that $\mathcal{V} = \mathcal{F}/V$, and this is indeed the case, for $V = f^*(U \to \pi_0(f))$, by the righthand pullback above. ■

§6 The classical Reeb stability theorem

In this section we shall explain the relation between Theorem 5.1 and the well-known Reeb stability theorem for foliations (see e.g. []). We first recall various notions from foliation theory (holonomy, leaves, etc.) in topos-theoretic terms.

Let G be a localic (or topological) groupoid, and assume that G is étale, i.e. the source and target maps $s, r: G_1 \rightrightarrows G_0$ are local homeomorphisms. Let $\mathcal{B}G$ be the classifying topos of G. Recall [] that for any locale X, topos morphisms $\mathrm{Sh}(X) \to \mathcal{B}G$ can be described in terms of groupoid homomorphisms

$$\varphi: U^X \to G \tag{1}$$

where U^X is the obvious groupoid $U \times_X U \rightrightarrows U$ defined from an open cover $X = \bigcup U_i$ with associated étale surjection $U = \coprod U_i \to X$. Note that $\mathcal{B}U^X = \mathrm{Sh}(X)$ (because there is an open weak equivalence $U^X \to X$ (I §7), if we view the locale X as a groupoid with identity arrows only).

Such a groupoid homomorphism (1) can equivalently be described by maps $g_i: U_i \to G_0$ and $c_{ij}: U_{ij} = U_i \cap U_j \to G_1$ satisfying the evident conditions ($s \circ c_{ij} = g_j$, $t \circ c_{ij} = g_i$, $c_{ij} \circ c_{jk} = c_{ik}$ on U_{ijk}). The system (g_i, c_{ij}) is called a *cocycle* on X with values in G. If $\tau: \varphi \to \varphi'$ is a continuous natural transformation between two homomorphisms as in (1), the two corresponding cocycles are conjugate (via mappings $\tau_i: U_i \to G_1$.

We remark that a topos map $\mathrm{Sh}(X) \to \mathcal{B}G$ is locally connected iff it can be represented by a cocycle for which the maps $g_i: U_i \to G_0$ are all locally connected. A Haefliger G-structure on X, or a *G-foliation* on X, is by definition an isomorphism class of locally connected topos morphims $X \to \mathcal{B}G$. It is represented by a "locally connected" cocycle, unique up to conjugacy and up to refinement of the cover $U \twoheadrightarrow X$.

6.1. Remark. Later, in Theorem 6.6, we shall require the map $\mathrm{Sh}(X) \to \mathcal{B}G$ to be separated. We note that this is the case if X is Hausdorff while the locale G_0 of objects is *locally* Hausdorff (that is, has an open cover of Hausdorff locales).

6.2. Example. Let $G = \Gamma^q$ be the "Haefliger groupoid," with $\mathbb{R}^q$ as space of objects and germs of diffeomorphisms as arrows. For a C^∞-manifold, a C^∞-foliation of codimension q is by definition a topos morphism $\mathrm{Sh}(X) \to \mathcal{B}\Gamma^q$, which is represented by a cocycle for which all the $g_i: U_i \to \mathbb{R}^q$ are C^∞-submersions (hence are locally connected maps).

If $y_0 \in G_0$ is a point in the space of objects of G, we write G_{y_0} for the vertex group at y_0. It is a discrete group because G is assumed to be étale. There are obvious topos maps

$$\mathbf{Set} \xrightarrow{\tilde{y}_0} \mathcal{B}G_{y_0} \xrightarrow{i} \mathcal{B}G,$$

where $\tilde{y}_0$ is the canonical point of the topos $\mathcal{B}G_0$ of G_0-sets, and i is induced by the inclusion $G_{y_0} \hookrightarrow G$ (but i need not be an embedding of toposes). If X is any locale, and $\varphi: \mathrm{Sh}(X) \to \mathcal{B}G$ is any topos morphism, we obtain by pullback a diagram

$$\begin{tikzcd}
\mathrm{Sh}(\tilde{L}) \arrow[r, "\pi"] \arrow[d] & \mathrm{Sh}(L) \arrow[r] \arrow[d] & \mathrm{Sh}(X) \arrow[d, "\varphi"] \\
\mathbf{Set} \arrow[r, "\tilde{y}_0"] & \mathcal{B}G_{y_0} \arrow[r, "i"] & \mathcal{B}G\ .
\end{tikzcd} \tag{2}$$

Here π is automatically a covering projection of locales with group G_{y_0}, because $\mathbf{Set} \to \mathcal{B}G_{y_0}$ is one of toposes.

Now suppose $\varphi: \mathrm{Sh}(X) \to \mathcal{B}G$ is locally connected. One can then factor φ as a connected and locally connected morphism followed by a local homeomorphism (i.e. a slice), say

$$\mathrm{Sh}(X) \xrightarrow{\psi} (\mathcal{B}G)/E \xrightarrow{\lambda} \mathcal{B}G.$$

There is an étale groupoid H, up to weak equivalence uniquely determined, for which $(\mathcal{B}G)/E \cong \mathcal{B}H$ and λ is induced by an étale groupoid homomorphism $H \to G$.

6.3. Definition. For a G-foliation $\varphi: \mathrm{Sh}(X) \to \mathcal{B}G$ on a locale X, its *holonomy groupoid* is an étale localic groupoid H for which φ can be factored as a connected, locally connected morphism $\psi: \mathrm{Sh}(X) \to \mathcal{B}H$ followed by a slice $\lambda: \mathcal{B}H \to \mathcal{B}G$. (This groupoid H is uniquely determined up to weak equivalence.)

6.4. Example. If $\varphi: \mathrm{Sh}(X) \to \mathcal{B}\Gamma^q$ is an ordinary foliation on a smooth manifold X (see Example 6.2), this defines (an étale groupoid weakly equivalent to) the usual holonomy groupoid (cf. []).

Now let x_0 be a point of X. Its image $\psi(x_0)$ is a point of the topos $\mathcal{B}H$. Since the canonical morphism $\mathrm{Sh}(H_0) \to \mathcal{B}H$ is an étale surjection, we can choose a point $y_0 \in H_0$ such that the corresponding point of the topos $\mathcal{B}H$ is isomorphic to $\psi(x_0)$. We shall abuse notation and also write $\psi(x_0)$ for such a chosen point y_0 of H_0. Let us form a pullback diagram analogous to (2):

$$\begin{tikzcd}
\mathrm{Sh}(\tilde{L}_{x_0}) \arrow[r, "\pi"] \arrow[d] & \mathrm{Sh}(L_{x_0}) \arrow[r] \arrow[d] & \mathrm{Sh}(X) \arrow[d, "\psi"] \arrow[dr, "\varphi"] & \\
\mathbf{Set} \arrow[r, "\psi(x_0)"] & \mathcal{B}H_{\psi(x_0)} \arrow[r, "i"] & \mathcal{B}H \arrow[r, "\lambda"] & \mathcal{B}G\,.
\end{tikzcd} \tag{3}$$

We can now define the following notions, which specialize to the usual ones in the case of an ordinary foliation $\varphi\colon \mathrm{Sh}(X) \to \mathcal{B}\Gamma^q$ on a manifold X.

6.5. Definition. (cf. diagram (3)). The vertex group $H_{\psi(x_0)}$ is called the *holonomy group* at x_0 of the G-foliation on X given by φ. (In view of the implicit choice of $y_0 = \psi(x_0)$, it is uniquely defined up to conjugation.) The locale L_{x_0} is called the *leaf* of x_0, and the map $\pi\colon \tilde{L}_{x_0} \to L_{x_0}$ the *holonomy covering* of this leaf.

The following theorem for a G-foliation on a locale X is now an immediate consequence of Theorem 5.1. For ordinary foliations, it is exactly the Reeb stability theorem.

6.6. Theorem. *Let G be an étale localic groupoid. Let $\varphi\colon \mathrm{Sh}(X) \to \mathcal{B}G$ be a G-foliation on a locale X, and let x_0 be a point of X. Suppose the map φ is separated (see* (6.1)*), while the leaf L_{x_0} is compact and the holonomy group at x_0 is finite. Then the same is true for all points in an open neighbourhood of x_0.*

Proof. Since $\tilde{L}_{x_0} \to L_{x_0}$ is a covering with group $\mathrm{Hol}(x_0)$, which is assumed to be finite, the locale $\tilde{L}_{x_0}$ is compact since L_{x_0} is. Thus, the fiber of $\psi\colon \mathrm{Sh}(X) \to \mathcal{B}H$ at $\psi(x_0)$ is compact (cf. diagram (3)). Since $\lambda\colon \mathcal{B}H \to \mathcal{B}G$ is a slice, H_0 is again locally Hausdorff, so we can apply Theorem 5.1 to find an étale neighbourhood V of $\psi(x_0)$ such that ψ restricts to a proper map over V. This neighbourhood V is an object of $\mathcal{B}H$, i.e. V is an étale H-space. We may assume V is of the form

$$t^{-1}(V_0) \xrightarrow{s} H_0$$

for an open neighbourhood V_0 of $\psi(x_0)$ in H_0, since such étale H-spaces generate the topos $\mathcal{B}H$ (see []). Thus, if $x \in X$ is any point in X with $\psi(x) \in_0$, the point $\psi(x)\colon \mathbf{Set} \to \mathcal{B}H$ factors through $(\mathcal{B}H)/V$. Therefore $\mathrm{Sh}(\tilde{L}_x)$ is compact, because $\mathrm{Sh}(L_x) \to \mathbf{Set}$ is the pullback of the proper map ψ/V, hence is itself proper:

$$\begin{tikzcd}
\mathrm{Sh}(\tilde{L}_x) \arrow[r, "\pi"] \arrow[d] & \bullet \arrow[r] \arrow[d] & \mathrm{Sh}(X) \arrow[d, "\psi"] \\
\mathbf{Set} \arrow[r, "\psi(x)"] & (\mathcal{B}H)/V \arrow[r, "i"] & \mathcal{B}H\,.
\end{tikzcd} \tag{4}$$

Since $\tilde{L}_x \to L_x$ is a covering projection with groups $\mathrm{Hol}(x)$, it follows that L_x is compact and $\mathrm{Hol}(x)$ is finite. ∎

CHAPTER III. TIDY MAPS

In Chapter III we study the fundamental properties of tidy maps between toposes, maps which are proper in the strong sense considered by K.E. Edwards [] and T. Lindgren []. We shall build upon the methods and results of Chapter I: as before, our strategy for showing the non-trivial closure properties of the class of tidy maps will rest on a good site-description of tidiness.

After giving the definition and basic examples (section 1), we deduce various elementary formal properties of tidy maps (sectons 2). We also show that tidiness is implied by the stable BCC (section 3). In section 4, which is the most technical, we first introduce, and establish needed properties of, a covenient type of "strongly compact" site. We then show tidy maps are stable under pullback with BCC (Theorem 4.8) and filtered inverse limits (Theorem 4.11).

The profinite reflection of a map between toposes was considered by P.T. Johnstone in [], where it was called the "pure-entire" factorization. After compiling a number of relevant properties of this factorization (section 5), we show that tidy maps are exactly those for which the pure part is connected, and stably so in an appropriate sense. The result is a Bourbaki-style characterization of tidiness (section 6).

§1 Definition and examples

Let $\mathcal{E}$ be a topos, and let $\gamma\colon \mathcal{E} \to \mathbf{Set}$ be its canonical morphism into the "terminal" topos of sets. We shall call $\mathcal{E}$ strongly compact if the global sections functor commutes with all directed colimits; i.e. if

$$\varinjlim_I \gamma_*(E_i) \longrightarrow \gamma_*(\varinjlim_I E_i)$$

is an isomorphism for every diagram $\{E_i\}$ of objects of $\mathcal{E}$ indexed by a small directed ($\equiv$ filtered) category I.

Comparing this definition with the one at the beginning of §1 of Chapter I, one sees that "strongly compact" is indeed a strengthening of "compact." Following are various elementary examples, which also serve to illustrate the difference between the ordinary and strong versions of compactness for toposes.

1.1. Examples. (1) For a group G, the topos $\mathcal{B}G$ of G-sets has for its global sections functor $\gamma_*\colon \mathcal{B}G \to \mathbf{Set}$ the fixed point functor, $\gamma_* S = S^G = \{s \mid s \cdot g = s \text{ for all } g \in G\}$. So, clearly, $\mathcal{B}G$ is strongly compact if G is finite. (In fact it is not difficult to show that $\mathcal{B}G$ is strongly compact iff G is finitely generated.)

(2) Any coherent topos is strongly compact. To see this, recall ([, 7.31]) that a topos $\mathcal{E}$ is coherent if it has a site $(\mathbb{C}, J)$ with finite limits, all of whose covers are finite. For such a site, the inclusion $\mathrm{Sh}(\mathbb{C}, J) \hookrightarrow \hat{\mathbb{C}}$, of sheaves into presheaves, preserves filtered colimits. From this property it follows that the global sections functor $\gamma_*\colon \mathcal{E} \to \mathbf{Set}$, given by evaluation at the terminal object of $\mathbb{C}$, commutes with filtered colimits.

(3) For any compact Hausdorff space X, the sheaf topos $\mathrm{Sh}(X)$ is strongly compact. To see this, consider for any diagram $\{S_i\}$ of sheaves indexed by a directed category I its colimit $S = \varinjlim S_i$. Just by compactness of X, the canonical mapping

$$c\colon \varinjlim \Gamma S_i \to \Gamma(\varinjlim S_i) = \Gamma S$$

is injective (I 1.8, 3.2). To see it is also surjective, write $c_i\colon S_i \to S$ for the evident map, and take any $s \in \Gamma S$. Then there is an open cover $X = U_1 \cup \ldots \cup U_n$ such that $s{\restriction}U_k = c_{i_k}(s_k)$ for some $s_k \in S_{i_k}(U_k)$. And by directedness of I, we may assume that $S_{i_k} = S_i$ does not depend on k. Let $X = V_1 \cup \ldots \cup V_n$ be a refinement with $\bar{V}_k \subseteq U_k$, and write $\bar{V}_{kl}$ for $\bar{V}_k \cap \bar{V}_l$. Then $s_k{\restriction}\bar{V}_{kl}$ and $s_l{\restriction}\bar{V}_{kl}$ are both mapped to $s{\restriction}\bar{V}_{kl}$. So, by directedness of I and compactness of $\bar{V}_{kl}$, we can find a transition $S_i \to S_j$ in the colimit such that the images of $s_1, \ldots s_k$ in S_j form a compatible family for the closed cover $\{\bar{V}_1, \ldots \bar{V}_n\}$. Thus, they glue to an element $s' \in \Gamma S_j$, mapped to the given $s \in \Gamma S$. This shows that c is surjective.

The definition of strong compactness can be relativised in the evident way, to give the notion of a *tidy map* between toposes. Thus $f\colon \mathcal{F} \to \mathcal{E}$ is tidy if, internally in $\mathcal{E}$, $\mathcal{F}$ is strongly compact as an $\mathcal{E}$-topos. An "external" form of this definition, in the style of (and using the notation from) the earlier definition (I 1.8) is as follows.

1.2. Definition []. A map $f\colon \mathcal{F} \to \mathcal{E}$ is said to be *tidy* if, for any object $E \in \mathcal{E}$ and any directed category I in $\mathcal{E}/E$, the associated square

$$\begin{tikzcd}
\mathcal{F}/f^*E \arrow[r, "\infty"] \arrow[d, "f/E"'] & (\mathcal{F}/f^*E)^{f^*I} \arrow[d, "(f/E)^I"] \\
\mathcal{E}/E \arrow[r, "\infty"] & (\mathcal{E}/E)^I
\end{tikzcd}$$

has the property that the canonical map

$$\infty^*(f/E)^I_*(V) \to (f/E)_*\infty^*(V)$$

is an isomorphism for any object V in $(\mathcal{F}/f^*E)^{f^*I}$.

Like propriety, tidiness is of "local nature":

1.3. Proposition. *If f is tidy, then so is f/E for any $E \in \mathcal{E}$. Conversely, if $E \to 1$ is an epimorphism in $\mathcal{E}$ and f/E is tidy, then so is f.*

1.4. Examples. (1) Consider, for an object E in a topos $\mathcal{E}$, the slice $\mathcal{E}/E$ and the canonical map $\mathcal{E}/E \to \mathcal{E}$. Internally in $\mathcal{E}$, the object E can be written (i) as a directed colimit of its Kuratowski-finite subobjects, or, using the natural numbers object of $\mathcal{E}$, (ii) as a directed colimit of finite cardinals

$$E \cong \varinjlim_{\substack{n \in \mathbb{N} \\ \alpha : \boldsymbol{n} \to E}} \boldsymbol{n}$$

where $\boldsymbol{n} = \{0, 1, \ldots, n-1\}$. We have already seen that $\mathcal{E}/E \to \mathcal{E}$ is proper iff E is Kuratowski-finite, and this can be proved using (i). From (ii), it follows that that $\mathcal{E}/E \to \mathcal{E}$ is tidy iff E is Kuratowski-finite and decidable, i.e. E is a locally constant finite object of $\mathcal{E}$, or in other words, $\mathcal{E}/E \to \mathcal{E}$ is a finite covering projection.

We can easily "relativise" the examples and the proof of (1.1), to obtain:

(2) For a Kuratowski-finite group G in a topos $\mathcal{E}$, the map $\mathcal{B}_{\mathcal{E}}G \to \mathcal{E}$ is tidy.

(3) For any coherent site $(\mathbb{C}, J)$ in $\mathcal{E}$, the map $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, J) \to \mathcal{E}$ is tidy.

(4) If X is any compact Hausdorff (hence compact regular, (II 4.9)) locale in a topos $\mathcal{E}$, the map $\mathrm{Sh}_{\mathcal{E}}(X) \to \mathcal{E}$ is tidy.

As a final example, we mention the following generalization of (2) above.

1.5. Proposition. *For any compact localic group G in a topos $\mathcal{E}$, the map $\mathcal{B}_{\mathcal{E}}(G) \to \mathcal{E}$ is tidy. ($\mathcal{B}_{\mathcal{E}}(G)$ is the topos of continuous G-objects in $\mathcal{E}$.)*

PROOF. We reduce to $\mathcal{E} = \mathbf{Set}$ by arguing constructively. For a G-set X, write X^G for the subset of fixed points for the action. We have to show, for any filtered system $\{X_i\}$ of G-sets, that the canonical map

$$\varinjlim X_i^G \to (\varinjlim X_i)^G \tag{1}$$

is a bijection.

Since $\mathcal{B}G$ has a surjective point $\mathbf{Set} \to \mathcal{B}G$, $\mathcal{B}G$ is compact, which means (1) is injective. To show that (1) is surjective, consider any element of the right-hand side, say $\eta(x)$ where $\eta\colon X_{i_0} \to \varinjlim X_i$ is a colimit map and $x \in X_{i_0}$. For each $\alpha\colon i_0 \to j$ in I, let $U_\alpha \subseteq G$ be the stabiliser of $\alpha \cdot x \in X_j$, that is (applying set-theoretic notation to locales)

$$U_\alpha = \{g \in G \mid g \cdot (\alpha \cdot x) = \alpha \cdot x\}.$$

Since $\eta(x)$ is fixed, these U_α form an open cover of G. Explicitly, consider the two maps of locales $\phi, \psi\colon G \to X_{i_0}$, $\phi(g) = g \cdot x$ and $\psi(g) = x$. Then $\eta\phi = \eta\psi$, so that $((\phi, \psi)$ factors through the kernel pair R of η,

$$R = \{(y, z) \in X_{i_0} \times X_{i_0} \mid \exists \alpha\colon i_0 \to j \text{ such that } \alpha \cdot y = \alpha \cdot z\}.$$

But for $(y, z) \in R$,

$$\begin{aligned}(\phi, \psi)^{-1}(y, z) =& \{g \mid g \cdot x = y \text{ and } x = z\} \\ &\subseteq \{g \mid \exists \alpha\colon i_0 \to j \text{ such that } g \cdot x = y \text{ and } \alpha \cdot y = \alpha \cdot x\} \\ &= \textstyle\bigvee_\alpha U_\alpha.\end{aligned}$$

Now, since G is compact, there are $\alpha_k\colon i_0 \to j_k$ $(k = 1, \ldots, n)$ so that G is covered by $U_{\alpha_1}, \ldots, U_{\alpha_n}$. Since I is filtered, there is a map $\beta\colon i_0 \to l$ dominating all the α_k, as in the commutative diagram

$$\begin{tikzcd} & j_k \arrow[dr] & \\ i_0 \arrow[ur, "\alpha_k"] \arrow[rr, "\beta"] & & l \, . \end{tikzcd} \qquad (k = 1, \ldots, n)$$

Thus $\beta \cdot x \in X_l$ is fixed by G, so that $\eta(x)$ is in the image of $X_l^G \to (\varinjlim X_i)^G$. This proves that (1) is surjective. ■

§2 First properties

In this section we catalogue some of the immediate properties of the class of tidy maps. The fact that tidy maps are stable under pullback will be established in §4.

2.1. Proposition. *Any tidy map is proper.* ■

2.2. Proposition. (i) *Any equivalence $\mathcal{F} \cong \mathcal{E}$ of toposes is tidy.* (ii) *If $\mathcal{G} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{E}$ are tidy, so is their composite $\mathcal{G} \to \mathcal{E}$.*

Proof. Obvious from Definition 1.2. ■

2.3. Proposition. *In a commutative diagram*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "g"] \arrow[dr, "h"'] & & \mathcal{F} \arrow[dl, "f"] \\ & \mathcal{E} \, , & \end{tikzcd}$$

if g is connected and h is tidy, then so is f.

Proof. The proof is almost verbally the same as that for (I 2.2), now using the fact that for a connected map $g: \mathcal{G} \to \mathcal{F}$, the induced map $\bar{g}: \mathcal{G}^I \to \mathcal{F}^I$ is again connected, so that the unit $V \to \bar{g}_* \bar{g}^* V$ is an isomorphism for each object V of $\mathcal{F}^I$. ■

2.4. Corollary. *If $f: \mathcal{F} \to \mathcal{E}$ is tidy, then so is its localic reflection.* ■

2.5. Remark. There is no reason for the hyperconnected part of a tidy map to be tidy in general, but as we shall see later, a tidy map does indeed factor as a connected map followed by a localic map in such a way that both factors are tidy, in analogy with the image factorization of a proper map, see (6.6) below.

2.6. Proposition. *In a commutative diagram as in (2.3), if h is tidy and f is an embedding, then g is tidy.*

PROOF. The proof is analogous to that of (I 2.3), and we use the same notation. Let W be any object in $\mathcal{G}^I$. We have to show that the canonical map

$$\infty^* \bar{g}_* W \to g_* \infty^* W \tag{1}$$

is an isomorphism. Since g is proper by (2.1) and (I 2.3), we already know (by (I 3.2)) that this map (1) is mono. To show that it is also epi, it suffices to prove that its image under f_* is, because f is assumed to be an embedding. Consider for this the commutative diagram

$$\begin{tikzcd}
f_* \infty^* \bar{g}_* W \arrow[r] & f_* g_* \infty^* W \arrow[dd, equal] \\
\infty^* \bar{f}_* \bar{g}_* W \arrow[u] \arrow[d, equal] & \\
\infty^* \bar{h}_* W \arrow[r, "\sim"] & h_* \infty^* W \, .
\end{tikzcd}$$

Here the lower arrow is an isomorphism because h is assumed proper. Hence the upper horizontal arrow must be epi. ∎

2.7. Remark. The analogue of (I 2.4) for tidy maps is false, as is clear from Example 1.1.1.

The following proposition generalises Example 1.1.3.

2.8. Proposition. *Any proper and separated map of toposes is tidy.*

PROOF. Our argument needs the fact that tidiness descents down open surjections, the proof of which is postponed until §4 (Proposition 4.10).

Since every map $f: \mathcal{F} \to \mathcal{E}$ factors as a hyperconnected map followed by a localic map, and since these two maps are both proper and separated whenever f is, it suffices to prove the theorem for the two special cases where f is either localic or hyperconnected. The first case is taken care of by Proposition 1.5.

For the second case, suppose $f: \mathcal{F} \to \mathcal{E}$ is hyperconnected (hence proper) and separated. Consider the pullback

$$\begin{tikzcd}
\mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r] \arrow[d, "\pi_1"'] & \mathcal{F} \arrow[d] \\
\mathcal{F} \arrow[r, "f"] & \mathcal{E} \, .
\end{tikzcd}$$

The map π_1 is again hyperconnected and separated and has the diagonal Δ_f as a section. Thus, up to equivalence, this map is of the form $\mathrm{Sh}_{\mathcal{F}}(G) \to \mathcal{F}$ for some compact localic group in $\mathcal{F}$, by (II Theorem 3.1). By (1.5), π_1 is tidy. Since f is an open surjection, we conclude by (4.10) that f is itself tidy. ∎

§3 The Beck-Chevalley condition

Recall from (I §3) that a commutative square

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] & \mathcal{E}\,. \end{tikzcd} \tag{1}$$

is said to satisfy the *Beck-Chevalley condition* (BCC) if the canonical natural transformation

$$a^* f_* \xrightarrow{\sim} g_* b^*.$$

is an isomorphism. The *map* f is said to satisfy the BCC if for any map $a: \mathcal{G} \to \mathcal{E}$, the pullback square of f along a satisfies the BCC. If any pullback of f satisfies the BCC, we say f satisfies the *stable* BCC. This terminology is analogous to that introduced for the weak Beck-Chevalley condition in (I §3).

3.1. Proposition. *If $f: \mathcal{F} \to \mathcal{E}$ satisfies the stable BCC then f is tidy.*

Proof. Consider a directed category I in $\mathcal{E}$, and the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\infty"] \arrow[d, "f"'] & \mathcal{F}^{f^* I} \arrow[r] \arrow[d, "f^I"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E} \arrow[r, "\infty"] & \mathcal{E}^I \arrow[r] & \mathcal{E}\,. \end{tikzcd}$$

Since the (total) rectangle and the right-hand squares are pullbacks, so is the left-hand square. By assumption, the BCC holds for the left-hand square, which says that for any object U of $\mathcal{F}^{f^* I}$, the canonical map

$$\infty^* (f^I)_* U \to f_* \infty^* U.$$

is an isomorphism. The same argument applies to any slice $\mathcal{F}/f^* E \to \mathcal{E}/E$, since these slices are pullbacks of $f: \mathcal{F} \to \mathcal{E}$. But this is tidiness of f, according to Definition 1.2. ∎

3.2. Remark. As in (I 3.4) we observe that the morphism $\mathcal{E} \xrightarrow{\infty} \mathcal{E}^I$ is a subtopos inclusion. Thus it is enough to require the BCC stably for pullbacks to *subtoposes* in Proposition 3.1.

One of the main results of this chapter is the converse of Proposition 3.1, to be proved in the next section.

§4 Stability under change of base

In this section we give a description of tidy maps in terms of sites, based on a result of K. Edwards []. As a first application, we obtain new proofs of two theorems of T. Lindgren [], namely a characterisation of strongly proper maps in terms of the Beck-Chevalley condition (the converse of (3.1)), as well as the preservation of strong propriety under pullback. Our proofs are simpler than those of [], and also constructive (avoiding the transfinite iteration involved in the original arguments), hence are valid over an arbitrary base topos.

We begin with a formulation of the Edwards criterion for a topos $\mathcal{E}$ to be strongly compact. Although we state it in the informal language of sets, it applies over an arbitrary base topos.

4.1. Proposition []. *A topos $\mathcal{E}$ is strongly compact iff $\mathcal{E}$ is compact and, moreover, for any object E in $\mathcal{E}$ with global support (i.e. $E \to 1$ epi) the following condition holds: for any directed epimorphic family $\{R_i \subseteq E \times E\}$ of equivalence relations on E there exists a subobject $U \subseteq E$ with global support such that $U \times U \subseteq R_i$ for some i.*

Proof. ($\Rightarrow$) Suppose $\mathcal{E}$ is strongly compact, and let $\{R_i \subseteq E \times E\}$ be as in the statement of the proposition. Then the directed diagram of quotients E/R_i has colimit $\varinjlim E/R_i = E/\bigvee R_i = 1$. Since $\mathcal{E}$ is assumed tidy, it follows that $\varinjlim \Gamma(E/R_i) = 1$. In particular, we find for some i a global section $s\colon 1 \to E/R_i$. The pullback of s along $E \to E/R_i$ is a subobject $U \subseteq E$ with the required properties.

($\Leftarrow$) To show that $\mathcal{E}$ is strongly compact, consider any directed diagram $\{D_i\}$ of objects of $\mathcal{E}$, and write $D = \varinjlim D_i$. Since $\mathcal{E}$ is compact, the canonical map

$$\varinjlim \Gamma D_i \to \Gamma(\varinjlim D_i) = \Gamma D$$

is injective (I 3.2). To see that it is also surjective, take $x \in \Gamma(D)$, and write $E_i \subseteq D_i$ for the pullback of $D_i \to D$ along $x\colon 1 \to D$. Then $\varinjlim E_i = 1$. So by compactness of $\mathcal{E}$, there exists an index i_0 such that $E_{i_0} \to 1$ is epi. Each transition map $D_{i_0} \to D_i$ in the diagram restricts to a map $E_{i_0} \to E_i$, with kernel pair $R_i \subseteq E_{i_0} \times E_{i_0}$, say. Since $\varinjlim E_i = 1$, the family $\{R_i\}$ covers $E_{i_0} \times E_{i_0}$. By the assumption, there exists a $U \subseteq E_{i_0}$ such that $U \twoheadrightarrow 1$ and $U \times U \subseteq R_i$ for some i. Then the composite map $U \to E_{i_0} \to E_i$ factors through $U \twoheadrightarrow 1$, providing the required section $1 \to E_i \to D_i$ mapping to x. ■

Before proceeding to the next definition, we need to introduce some notation concerning equivalence relations in a pretopos site (I 4.3). Given a subobject $U \rightarrowtail C \times C$ in a pretopos site $\mathbb{C}$, one can define a sequence of subobjects $U^{(n)} \rightarrowtail C \times C$ which jointly form the equivalence relation generated by U in the usual way: let $U^{(0)} = \Delta_C$ (the diagonal), $U^{(1)} = U^{(0)} \cup U \cup U^{\mathrm{op}}$, and let $U^{(n+1)}$ be the image of $(\pi_1, \pi_3)\colon U^{(n)} \times_C U^{(1)} \rightarrowtail C \times C \times C \to C \times C$. We shall call a family of monomorphisms of the form $\{U_i \rightarrowtail C \times C \mid i \in I\}$ *effective* if there exists a subobject $D \rightarrowtail C$ such that both $D \to 1$ and the induced family $\{(D \times D) \cap U^{(n)} \rightarrowtail D \times D \mid i \in I, n \in \mathbb{N}\}$ are covers of $\mathbb{C}$. A subobject $U \rightarrowtail C \times C$ is effective if it is so as a singleton family.

The definition of strong compactness for a pretopos site is an appropriate reformulation of the Edwards criterion.

4.2. Definition. A pretopos site $\mathbb{C}$ is *strongly compact* if $\mathbb{C}$ is compact (I 5.1) and, moreover, for any $C \to 1$ which covers, any directed cover of $C \times C$ by monomorphisms in $\mathbb{C}$ has an effective member.

4.3. Remark. Like the definition of compactness (I 5.1), Definition 4.2 makes sense for any "site with stable compatible system of directed covers" (I 5.2). The results of this section remain true (and most proofs unaltered) if we work with such a site $\mathbb{C}$ instead of a pretopos site, provided we add the requirement that the coproducts in $\mathbb{C}$ are *disjoint* (see []), to ensure that "preservation of covers" entails "preservation of sums."

We have:

4.4. Proposition. *A pretopos site $\mathbb{C}$ for a topos $\mathcal{E}$ is strongly compact iff $\mathcal{E}$ is strongly compact.*

Proof. We need to add to the proof of (I 5.3) the verification that a compact $\mathbb{C}$ satisfies the additional condition for strong compactness precisely when the (compact) topos $\mathcal{E}$ satisfies the Edwards criterion.

Suppose $\mathbb{C}$ is indeed strongly compact, and let $E \twoheadrightarrow 1$ and $\{R_i \hookrightarrow E \times E\}$ be as assumed in (4.1). By compactness and the existence of finite sums and images in $\mathbb{C}$, preserved by the canonical functor $h\colon \mathbb{C} \to \mathcal{E}$, we find an object C of $\mathbb{C}$ such that $C \to 1$ covers and $h(C) \twoheadrightarrow h(1) \cong 1$ refines $E \twoheadrightarrow 1$, say by a map $e\colon h(C) \to E$. Let $\{S_j \rightarrowtail C \times C\}$ be a cover by monomorphisms in $\mathbb{C}$ such that the family $\{h(S_j) \hookrightarrow h(C \times C) \cong h(C) \times h(C)\}$ refines $\{(e \times e)^{-1}(R_i) \hookrightarrow h(C) \times h(C)\}$. Since the family $\{R_i \hookrightarrow E \times E\}$ is directed, we can assume that $\{S_j \rightarrowtail C \times C\}$ is directed too. By strong compactness of $\mathbb{C}$, some $S_j \rightarrowtail C \times C$ is effective. Thus, we can find a subobject $D \rightarrowtail C$ with $D \to 1$ a cover such that the family $\{D \times D \cap S_j^{(n)} \rightarrowtail D \times D\}$ is a cover. By construction $h(S_j) \hookrightarrow h(C \times C) \to E \times E$ factors through some equivalence relation $R_i \hookrightarrow E \times E$. But then $h(S_j^{(n)}) \hookrightarrow h(C \times C) \to E \times E$ factors through R_i for all n, and this is easily seen to imply that the map $h(D) \times h(D) \to E \times E$ factors through R_i. Let $V \subseteq E$ be the image of $h(D) \to E$. Then $V \to 1$ is epi and $V \times V \subseteq R_i$, as required in (4.1).

Conversely, suppose $\mathcal{E}$ satisfies the Edwards criterion. For a suboject $V \hookrightarrow E \times E$ in $\mathcal{E}$, let $\widehat{V} = \bigvee_n V^{(n)} \hookrightarrow E \times E$ denote the equivalence relation on E generated by V. Consider any directed cover $\{S_j \rightarrowtail C \times C\}$ in $\mathbb{C}$, where $C \to 1$ covers. Since the family $\{\widehat{h(S)}_i \hookrightarrow h(C) \times h(C)\}$ of equivalence relations in $\mathcal{E}$ is directed, (4.1) gives some $U \subseteq h(C)$ with global support in $\mathcal{E}$ and some j such that $U \times U \subseteq \widehat{h(S_i)}$, that is, such that the family $\{U \times U \cap h(S_i^{(n)}) \hookrightarrow U \times U \mid n \in \mathbb{N}\}$ is epimorphic. But by compactness and the existence of finite sums and images in $\mathbb{C}$, we can assume that the inclusion $U \subseteq h(C)$ lies in the image of h, say $U \cong h(D)$ where $D \rightarrowtail C$ is a subobject in $\mathbb{C}$ such that $D \to 1$ covers. But this says that $S_i \rightarrowtail C \times C$ is an effective subobject of $C \times C$. Thus, we have shown that $\mathbb{C}$ is strongly compact. ∎

The next "induction" lemma is the counterpart of (I 5.3) for dealing with strong compactness.

4.5. Lemma. *Let $\mathbb{C}$ be a compact pretopos site equipped with a system N of distinguished effective subobjects (of products $C \times C$) such that*

(i) *The trivial effective subobject $C \times C \xrightarrow{\text{id}} C \times C \in N$ whenever $C \to 1$ covers.*
(ii) *In $V \rightarrowtail U \rightarrowtail D \times D \rightarrowtail C \times C$ (where $D \rightarrowtail C$), if $V \rightarrowtail D \times D \in N$ then $U \rightarrowtail C \times C \in N$.*
(iii) *$U^{(n)} \rightarrowtail C \times C \in N$ only if $U \rightarrowtail C \times C \in N$.*
(iv) *For any basic S-cover $\{U_i \rightarrowtail U\}$, if $U \rightarrowtail C \times C \in N$ then $U_i \rightarrowtail C \times C \in N$ for some i.*

Then N contains all effective subobjects $U \rightarrowtail C \times C$ of $\mathbb{C}$, and $\mathbb{C}$ is strongly compact.

PROOF. Consider the following property of families $\{U_i \to U\}$: for any $U \rightarrowtail C \times C \in N$, there is some i for which $U_i \to U$ is a monomorphism and $U_i \rightarrowtail C \times C \in N$. Since this property is given to hold for basic S-covers (iv), trivially holds for the family $\{1 \to 1\}$ and is preserved by composition, it must hold for generated S-covers by induction. But then, if $C \to 1$ is a cover, any generated S-cover $\{S_i \rightarrowtail C \times C\}$ contains a member of N, since the identity $C \times C \to C \times C \in N$. By (I 4.3) and condition (ii), the same is true for any directed cover of $C \times C$. This shows that $\mathbb{C}$ is strongly compact.

To prove that N contains all effective subobjects, consider any such, say $U \to C \times C$, and let $D \rightarrowtail C$ be a monomorphism such that $D \to 1$ and the family $\{(D \times D) \cap U^{(n)} \rightarrowtail D \times D \mid n \in \mathbb{N}\}$ are covers. Then, by what we have just shown, some $(D \times D) \cap U^{(n)} \rightarrowtail D \times D$ is in N, whence $U \rightarrowtail C \times C \in N$ by conditions (ii) and (iii). ■

4.6. Corollary. *A compact pretopos site $\mathbb{C}$ is strongly compact iff the system of all effective subobjects satisfies the conditions of* (4.5).

4.7. Lemma. *Let $\varphi: \mathcal{E}' \to \mathcal{E}$ be a morphism of toposes and suppose $\mathbb{C}$ is a strongly compact pretopos site in $\mathcal{E}$. Then the pretopos site $\varphi^*\mathbb{C}$ is strongly compact in $\mathcal{E}'$. Moreover, if L denotes the object of objects which cover 1, and N the object of effective subobjects in $\mathbb{C}$, then φ^*L and φ^*N are the corresponding objects, respectively, for $\varphi^*\mathbb{C}$.*

PROOF. By (I 5.4), φ^* preserves both the compactness and the object of covering subobjects of 1 of $\mathbb{C}$. It therefore also preserves the object of objects with covering support. It follows that the conditions of Lemma 4.5, which are satisfied by N (Corollary 4.6), are "geometric" and hence inherited by φ^*N. Thus, the lemma follows by an application of (4.5) in $\mathcal{E}'$. ■

4.8. Theorem []. *In a pullback square*

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\psi"] \arrow[d, "f'"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}' \arrow[r, "\varphi"] & \mathcal{E} \end{tikzcd} \qquad (1)$$

suppose that f is tidy. Then f' is tidy and the BCC is satisfied.

PROOF. We reduce to the case $\mathcal{E} \equiv \mathbf{Set}$ and argue constructively.

Let $\mathbb{C}$ be a pretopos site for $\mathcal{F}$. Then $\mathbb{C}$ is strongly compact by Proposition 4.4 and it follows that $\varphi^*\mathbb{C}$ is a strongly compact site for $\mathcal{F}'$ in $\mathcal{E}'$ by Lemma 4.7. Thus, f' is tidy, by applying Proposition 4.4 in $\mathcal{E}'$.

To deduce the BCC, consider any object F of $\mathcal{F}$, represented by a sheaf P on $\mathbb{C}$. The corresponding sheaf for ψ^*F made in the topos $\mathcal{E}'$, is given by the sheafification $Q \equiv (\varphi^*P)^{++}$ in $\mathcal{E}'$ of the presheaf φ^*P, and the map $\varphi^*f_*F \to f'_*\psi^*F$ by the component at the terminal object $1 \in \mathbb{C}$ of the canonical natural transformation $\eta\colon \varphi^*P \to Q$. We need to show that this map is epi (we already know it is mono by the weak BCC which holds since f is proper (I 5.8)).

The sheaf P has the following property: for any effective subobject $U \rightarrowtail C \times C$ and element $p \in P(C)$ such that the restrictions of p along the projections $U \rightrightarrows C$ agree, there is a subobject $D \rightarrowtail C$ where $D \to 1$ covers, and a unique "global" element $s \in P(1)$ such that $s{\restriction}D = p{\restriction}D$ in $P(D)$. For, choose this $D \rightarrowtail C$ to be any subobject for which $D \to 1$ covers and for which the family $\{D \times D \cap U^{(n)} \rightarrowtail D \times D\}$ covers ($U \rightarrowtail C \times C$ is effective). Then the restrictions of p along each pair of projections $U^{(n)} \rightrightarrows C$ agree, which implies that $p{\restriction}D$ is *locally compatible*, that is to say, compatible over a cover of $D \times D$. By the sheaf condition, it has a global element s of the form claimed. Since the notions of "object with covering support" and "effective subobject" for a strongly compact $\mathbb{C}$ are preserved under change of base by Lemma 4.7, this property is inherited by φ^*P (despite not being a sheaf in general) in $\mathcal{E}'$.

We now argue internally in $\mathcal{E}'$. An element $q \in Q(1)$ is given by a cover $\{C_j \to 1\}$ in $\varphi^*\mathbb{C}$ and a family of elements $p_j \in \varphi^*P(C_j)$ which are locally compatible in the sense that p_j and $p_{j'}$ agree on a cover of $C_j \times C_{j'}$. By compactness and the existence of sums in $\varphi^*\mathbb{C}$, we can take this cover of 1 to consist of a single arrow $C \to 1$, with q given by some $p \in \varphi^*P(C)$. Local compatibility of p means that there is a cover $\{S_i \to C \times C\}$ such that for each i, the restrictions of p along the induced maps $S_i \rightrightarrows C$ agree; since φ^*P still satisfies the sheaf property for P-covers, we can assume that each $S_i \to C \times C$ is a monomorphism and that the family $\{S_i \rightarrowtail C \times C\}$ is directed. By strong compactness, some $S_i \rightarrowtail C \times C$ is effective. As shown above, we can find some $s \in \varphi^*P(1)$ and a subobject $D \rightarrowtail C$ such that $D \to 1$ covers and such that $s{\restriction}D = p{\restriction}D$. Since such s is mapped to (the equivalence class of) p by $\eta_1\colon \varphi^*P(1) \to Q(1)$, this shows that the map $\varphi f_*F \to f'_*\psi^*F$ is epi.

We have therefore established the BCC. ∎

4.9. Corollary []. *A map $f\colon \mathcal{F} \to \mathcal{E}$ is tidy iff it satisfies the stable BCC.*

PROOF. One direction is Proposition 3.1, the other is immediate from Theorem 4.8. ∎

4.10. Proposition. *In the pullback square* (1), *suppose φ is an open or a proper surjection. If f' is tidy, then so is f.*

PROOF. We write as if $\mathcal{E} \equiv \mathbf{Set}$ and argue constructively. Suppose f' is tidy and let $\mathbb{C}$ be a pretopos site for $\mathcal{F}$. Tidiness of f' implies that the pretopos site $\varphi^*\mathbb{C}$ is strongly compact. We want to conclude that $\mathcal{F}$ is strongly compact by showing that $\mathbb{C}$ is strongly compact.

Since $\mathcal{F}$ is compact by (I 2.7) and (I 6.9), we already know that $\mathbb{C}$ is compact. Consider any $C \in \mathbb{C}$ such that $C \to 1$ covers and any directed cover $\{S_i \rightarrowtail C \times C \mid i \in I\}$ of monomorphisms in $\mathbb{C}$. Let

$$K = \{(i, D) \mid i \in I,\ D \rightarrowtail C \in \mathbb{C} \text{ such that } D \to 1 \text{ and } \{S_i^{(n)} \cap (D \times D) \rightarrowtail D \times D \mid n \in \mathbb{N}\} \text{ are covers in } \mathbb{C}\}.$$

We need to show that K has an element. Since $\varphi^*\mathbb{C}$ is strongly compact, the corresponding object in $\mathcal{E}'$,

$$K' = \{(i, D) \mid i \in \varphi^*I,\ D \rightarrowtail \varphi^*C \in \varphi^*\mathbb{C} \text{ such that } D \to 1 \text{ and } \{\varphi^*(S_i)^{(n)} \cap (D \times D) \rightarrowtail D \times D \mid n \in \mathbb{N}\} \text{ are covers in } \varphi^*\mathbb{C}\}$$

is inhabited, in other words, $K' \to 1$ is epi.

Suppose first that φ is open. Then φ^* preserves first-order logic (see []), hence in particular the notion of "closed sieve." This implies that, for a fixed family $\{C_k \to C\}$ in $\mathbb{C}$, the set $\{D \to C \mid \{D \times_C C_k \to D\} \text{ covers}\}$ is mapped by φ^* to the object defined by the expression $\{D \to \varphi^*C \mid \{D \times_{\varphi^*C} \varphi^*C_k \to D\} \text{ covers}\}$ in $\mathcal{E}'$. It follows that the definition of K is preserved, that is, $K' = \varphi^*K$. Since φ^* is faithful, we conclude that K has an element.

On the other hand, suppose φ is proper, that is, $\mathcal{E}'$ is compact. Then ψ is a surjection by (I 6.9). Let $k_{U'}$ for $U' \subseteq 1$ in $\mathcal{E}'$ denote the functor $\mathbb{C} \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'/U'$. We can express the image of $K' \to 1$ as a directed join

$$\operatorname{supp}(K') = \bigvee\{U' \subseteq 1 \mid \text{For some } i \in I \text{ and } D \rightarrowtail C \in \mathbb{C},\ k_{U'} \text{ maps } D \to 1 \text{ and } \{S_i^{(n)} \cap (D \times D) \rightarrowtail D \times D \mid n \in \mathbb{N}\} \text{ to covers in } \mathcal{F}/U'\}.$$

Since $\operatorname{supp}(K') \cong 1$, compactness of $\mathcal{E}'$ gives some $i \in I$ and $D \rightarrowtail C$ in $\mathbb{C}$ such that $D \to 1$ is mapped to an epimorphism, and $\{S_i^{(n)} \cap (D \times D) \rightarrowtail D \times D \mid n \in \mathbb{N}\}$ to an epimorphic family in $\mathcal{F}$' by the functor $\mathbb{C} \to \mathcal{F} \to \mathcal{F}'$. But ψ^* is faithful, which implies that $D \to 1$ and $\{S_i^{(n)} \cap (D \times D) \rightarrowtail D \times D \mid n \in \mathbb{N}\}$ are covers in $\mathbb{C}$; that is, (i, D) is an element of K.

This completes the proof. ∎

4.11. Theorem. *Suppose $f\colon \mathcal{F} \to \mathcal{E}$ is the limit*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "p_i"] \arrow[dr, "f"'] & & \mathcal{F}_i \arrow[dl, "f_i"] \\ & \mathcal{E} & \end{tikzcd} \tag{2}$$

of a diagram of tidy maps $\{f_i\colon \mathcal{F}_i \to \mathcal{E}\}$ indexed by a filtered category I. Then f is tidy. Moreover, for any $i \in I$, the canonical natural transformation

$$\varinjlim_\alpha {f_j}_* {t_\alpha}^* \to f_* {p_i}^*,$$

where $\alpha: j \to i$ varies over the category I/i and $t_\alpha: \mathcal{F}_j \to \mathcal{F}_i$ denotes the transition map induced by α, is an isomorphism.

PROOF. By regarding I as an internal category in $\mathcal{E}$, it will suffice to treat the case $\mathcal{E} \equiv \mathbf{Set}$ constructively.

Let $\{\mathbb{C}_i\}$ be a diagram of pretopos sites inducing $\{\mathcal{F}_i\}$, and let $\mathbb{C}$ be a pretopos site for the limit $\mathcal{F}$ as given by (I 4.8). Denote the canonical functors associated with an arrow $\alpha: j \to i \in I$ as indicated in the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[d, "h"'] & \mathbb{C}_j \arrow[l, "P_j"'] \arrow[d, "h_j"] & \mathbb{C}_i \arrow[l, "T_\alpha"'] \arrow[d, "h_i"] \arrow[ll, bend right, "P_i"'] \\
\mathcal{F} & \mathcal{F}_j \arrow[l, "{p_j}^*"'] & \mathcal{F}_i \arrow[l, "{t_\alpha}^*"'] \arrow[ll, bend left, "{p_i}^*"]
\end{tikzcd}$$

For each i, let N_i be the set of effective subobjects subobjects in $\mathbb{C}_i$, and let N be the set of effective subobjects in $\mathbb{C}$ which are (up to isomorphism) in the joint image of the N_i under the morphisms $P_i: \mathbb{C}_i \to \mathbb{C}$ which induce the projections $p_i: \mathcal{F} \to \mathcal{F}_i$.

Since each $\mathbb{C}_i$ is strongly compact, we can show, as in the proof of (I 5.10), that $\mathbb{C}$ is compact, and also that each covering subobjects of 1 in $\mathbb{C}$ is (up to isomorphism) in the image of some P_i. Since epimorphisms of $\mathbb{C}$ lift similarly, it follows that the same can be said for any cover $C \to 1$ in $\mathbb{C}$. Using this fact, the lifting property of commutative diagrams and basic S-covers in $\mathbb{C}$, and the directedness of I, it is not hard to check that the system N inherits the conditions of Lemma 4.5 from the N_i. It follows that any effective suobject $U \rightarrowtail C \times C$ in $\mathbb{C}$ lifts to some $\mathbb{C}_i$, and that $\mathbb{C}$ is strongly compact. Thus $\mathcal{F}$ is strongly compact.

To show the second part, fix $i \in I$ and consider any $F \in \mathcal{F}_i$. We need to show that any global element $s: 1 \to {p_i}^*F$ in $\mathcal{F}$ is of the form ${p_j}^*x: 1 \cong {p_j}^*1 \to {p_i}^*F$ for some $\alpha: j \to i \in I$ and a global element $x: 1 \to {t_\alpha}^*F$ in $\mathcal{F}_j$, and further that if $x': 1 \to {t_{\alpha'}}^*F$ with $\alpha': j' \to i$ is another such lifting of s, then there is a commutative diagram

$$\begin{tikzcd}
k \arrow[r, "\beta'"] \arrow[d, "\beta"'] & j' \arrow[d, "\alpha'"] \\
j \arrow[r, "\alpha"] & i
\end{tikzcd} \tag{3}$$

in I such that ${t_\beta}^*x = {t_{\beta'}}^*x'$ in $\mathcal{F}_k$.

Now, if each $\mathcal{F}_j$ were the presheaf topos on $\mathbb{C}_j$, then $\mathcal{F}$ would be the presheaf topos on $\mathbb{C}$ and the global element s would indeed have this lifting property. For then $f_*{p_i}^*$ would be calculated explicitly as "left Kan-extension" along P_i at $1 \in \mathbb{C}$, which can be expressed as the (filtered) colimit of extensions along T_α for $\alpha: j \to i$, evaluated at $1 \in \mathbb{C}_j$, as is readily verified using the directedness of I and the lifting property of finite commutative diagrams in $\mathbb{C}$.

This implies, firstly, that we can "locally lift s, in a locally compatible way." More precisely, using strong compactness and the existence of sums and images in $\mathbb{C}$, we can find $\alpha: j \to i$, C and $U \rightarrowtail C \times C$ in $\mathbb{C}_j$, and $y: h_j(C) \to t_\alpha{}^*F$ in $\mathcal{F}_j$ such that $P_j(C) \to P_j(1) \cong 1$ covers and $P_j(U) \rightarrowtail P_j(C) \times P_j(C)$ is effective in $\mathbb{C}$, while s restricts to $p_j{}^*y$, and the restrictions of y along the projections $h_j(U) \rightrightarrows h_j(C)$ are equal in $\mathcal{F}_j$. But since singleton covers of 1 and effective subobjects in $\mathbb{C}$ lift, we can (using directedness of I) further arrange that $C \to 1$ covers and that $U \rightarrowtail C \times C$ is effective in $\mathbb{C}_j$. It follows that y is the restriction to $h_j(C)$ of a unique global element $x: 1 \to t_\alpha{}^*F$, and then that $p_j{}^*x = s$. This proves the existence of a lifting for s.

Secondly, if $x': 1 \to t^*_{\alpha'}F \in \mathcal{F}_{j'}$ also satisfies $p_{j'}{}^*x' = s$, then x and x' can "locally" be forced to become equal in the way required. Explicitly, incorporating compactness and the existence of finite images in $\mathbb{C}$, we can find a commutative diagram (3) in I and $V \rightarrowtail 1$ in $\mathbb{C}_k$ such that $P_k(V) \rightarrowtail P_k(1) \cong 1$ covers in $\mathbb{C}$ and the restrictions of $t_\beta{}^*x$ and $t_{\beta'}{}^*x'$ to $h_k(V) \rightarrowtail h_k(1) \cong 1$ agree. Again, since singleton covers of 1 in $\mathbb{C}$ lift, we can (using directedness of I) arrange that $V \rightarrowtail 1$ covers already in $\mathbb{C}_k$. Thus, $t_\beta{}^*x = t_{\beta'}{}^*x'$ in $\mathcal{F}_k$, which proves the "uniqueness" part of the lifting for s. ■

4.12. Corollary. *Suppose in (2) that for each $\alpha: j \to i$ in I, the canonical natural transformation ${f_i}_* \to {f_j}_* t_\alpha{}^*$ induced by the transition morphism $t_\alpha: \mathcal{F}_j \to \mathcal{F}_i$ is an isomorphism. Then the canonical natural transformation ${f_i}_* \to f_* p_i{}^*$ is an isomorphism for each $i \in I$.* ■

§5 Entire maps

In this section we catalogue various results involving profinite localic maps, needed for an alternative description of tidy maps to be given in §6. In particular, we recall the *pure-entire* factorization of a map $f: \mathcal{F} \to \mathcal{E}$ introduced by P.T. Johnstone [].

We start with some well-known facts concerning profinite sets and locales (see [, Chapters II and VI]), interpreted in an arbitrary topos $\mathcal{S}$ (fixed for the moment as base). As usual we write as if $\mathcal{S}$ is the category of sets.

Let $\mathbb{F} = \mathbb{F}_\mathcal{S}$ denote the full (internal) subcategory of $\mathcal{S}$ of finite cardinals as in Example 1.4 (1). We recall that the category of formal inverse limits of finite objects, or *profinite objects* in $\mathcal{S}$ is defined to be the dual of the category of filtered (i.e. finite limit-preserving or "left exact") internal diagrams on $\mathbb{F}$. Any profinite object P can be turned into a locale $|P|$ by constructing the (obvious) limit

$$|P| = \varprojlim_{\substack{n \in \mathbb{F} \\ x \in P(n)}} \boldsymbol{n} \tag{1}$$

The functor $|-|$ has a left adjoint, which assigns to a locale X the diagram $\mathcal{P}(X)$ of partitions of X, or equivalently, the diagram "under" X spanned by $\mathbb{F}$ in the category of locales, whose value at $\boldsymbol{n}$ is the set of maps from X into the discrete locale $\boldsymbol{n}$.

The next proposition says that profinite objects of $\mathcal{S}$ are the "same" as profinite internal locales, that is, inverse limits of finite discrete locales.

5.1. Proposition. *The realization (1) of a profinite object as a locale embeds the category of profinite objects of $\mathcal{S}$ as the (full) reflective subcategory of the category of locales in $\mathcal{S}$, consisting of those locales X (called "Stone locales" in []) satisfying any of the following equivalent conditions:*

(i) *X is coherent and regular.*
(ii) *X is coherent with complemented compact opens (that is, the basis of compact opens of X form a boolean algebra).*
(iii) *X is compact and has a clopen basis.*

PROOF. The usual lattice-theoretic proofs of these facts (see []) are essentially constructive, hence interpretable in the topos $\mathcal{S}$. ∎

5.2. Remarks. (1) The functor

$$X \mapsto |\mathcal{P}(X)| = \varprojlim_{\substack{n \in \mathbb{F} \\ \delta: X \to n}} \boldsymbol{n}$$

assigns to a locale X its profinite reflection, and the reflection map $X \to |\mathcal{P}(X)|$ is an isomorphism iff X satisfies any of the equivalent conditions above. In terms of condition (ii), the compact opens of $|\mathcal{P}(X)|$ correspond to the complemented opens of X, which in turn correspond to the maps $X \to 1 + 1 \cong \mathbf{2}$.

(2) By [, III 1.3] and (II 4.8)), condition (i) equivalently states that X is coherent and Hausdorff.

(3) Any map between coherent locales satisfying condition (ii) is clearly coherent, hence it follows that the category of profinite objects in $\mathcal{S}$ is dual to the category of boolean algebras in $\mathcal{S}$. (More directly, the category of profinite objects is a "pro-completion" of $\mathbb{F}$, and the category of boolean algebras an "ind-completion" of the category of finite ($\equiv$ finitely presentable) boolean algebras, which is dual to $\mathbb{F}$.)

Combining the localic reflection of a topos with the profinite reflection of a locale gives:

5.3. Corollary. *Any $\mathcal{S}$-topos $p: \mathcal{E} \to \mathcal{S}$ has a profinite localic reflection*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "g"] \arrow[dr, "p"'] & & \mathrm{Sh}_{\mathcal{S}}(\pi_0^{\mathrm{pf}}(\mathcal{E})) \arrow[dl] \\ & \mathcal{S}, & \end{tikzcd}$$

a universal map into an $\mathcal{S}$-topos of sheaves on a profinite locale. More precisely, $\pi_0^{\mathrm{pf}}(\mathcal{E})$ is the (internal) limit of locales

$$\pi_0^{\mathrm{pf}}(\mathcal{E}) \cong \varprojlim_{(n,\delta)} \boldsymbol{n}$$

where n varies over $\mathbb{N}$ and δ over n-fold partitions $\mathcal{E} = \mathcal{E}_0 + \cdots \mathcal{E}_{n-1}$ of $\mathcal{E}$ into open subtoposes (with an arrow $(n, \delta) \to (n', \delta')$ being a function $\alpha: \boldsymbol{n} \to \boldsymbol{n}'$ such that $\mathcal{E}_k \subseteq$

$\mathcal{E}_{\alpha(k)}$ for $k < n$). The locale $\pi_0^{\mathrm{pf}}(\mathcal{E})$ has for its basis of compact opens the boolean algebra $p_* \mathbf{2}_{\mathcal{E}}$, where $\mathbf{2}_{\mathcal{E}} = 1 + 1 \in \mathcal{E}$.

PROOF. Clear from the existence of the localic reflection, (5.1) and (5.2). ■

5.4. Remark. The notation $\pi_0^{\mathrm{pf}}(\mathcal{E})$ is meant to convey the idea of the profinite reflection as "profinite object of connected components" in analogy with the profinite fundamental group $\pi_1^{\mathrm{pf}}(\mathcal{E})$, see []. If $\mathcal{E}$ is connected, then clearly $\pi_0^{\mathrm{pf}}(\mathcal{E}) \cong 1$, but (unless $\mathcal{S}$ is the category of classical sets) the converse need not be true. More generally, the reflection map $\mathcal{E} \to \mathrm{Sh}_{\mathcal{S}}(\pi_0^{\mathrm{pf}}(\mathcal{E}))$ need not be connected.

We are now ready to pursue various formal aspects of profinite localic maps:

5.5. Definition []. A morphism $f \colon \mathcal{F} \to \mathcal{E}$ is said to be *entire* if it is localic for a profinite or "Stone" locale. Thus, in terms of Corollary 5.3, f is entire iff it coincides with its profinite reflection,

$$\mathcal{F} \simeq \mathrm{Sh}_{\mathcal{E}}(\pi_0^{\mathrm{pf}}(\mathcal{F})) \simeq \mathrm{Sh}_{\mathcal{E}}(f_* \mathbf{2}_{\mathcal{F}})$$

as $\mathcal{E}$-toposes (where the boolean algebra $f_* 2_{\mathcal{F}}$ is regarded as a coherent internal site). Entirety is clearly a "local" property.

5.6. Examples. (1) In any topos $\mathcal{E}$, amongst the discrete locales, the profinite ones are precisely the finite cardinals. Thus, for an object $E \in \mathcal{E}$, the map $\mathcal{E}/E \to \mathcal{E}$ is entire iff it is a finite covering projection.

(2) An inclusion of toposes is entire iff it is closed. For, the terminal locale is profinite, and clearly by condition (iii) in Proposition 5.1, profiniteness is inherited by closed sublocales. Conversely, an entire embedding is proper, hence closed. More generally:

5.7. Proposition. *Any entire map is tidy, in fact proper and separated.*

PROOF. Clear from (1.4) (3) and Remark 5.2 (2). ■

Applied to the second example above, this gives:

5.8. Corollary. *Any closed inclusion of toposes is tidy.* ■

5.9. Proposition. *In a commutative diagram*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "g"] \arrow[dr, "h"'] & & \mathcal{F} \arrow[dl, "f"] \\ & \mathcal{E}\ , & \end{tikzcd}$$

(i) *if f and g are entire, so is h;*
(ii) *if f is separated and h is entire, then g is entire.*

PROOF. We use the third characterization of profinite locales in Proposition 5.1. Then by (I 2.1) and (II 2.1(iv)), it will be enough to show for (i) that the property of a map of being localic with clopen basis is preserved under composition and for (ii) that this property is always inherited by g from h (for any f).

We argue in $\mathcal{E}$ as if it were the category of sets. Since localic maps are preserved under composition, we can assume that $\mathcal{G} \simeq \mathrm{Sh}(Z)$ for a locale Z, with opens given by the subobjects of 1 in $\mathcal{G}$. Let $\mathbb{C}$ be any site for $\mathcal{F}$ with a terminal object, and $e: \mathbb{C} \to \mathcal{F}$ the canonical functor. The map g is induced by a locale Y in $\mathcal{F}$, with frame of opens represented by the sheaf A on $\mathbb{C}$ which has subobjects of $g^*e(C)$ as sections at $C \in \mathbb{C}$, and with restriction along $D \to C$ defined by pullback along $g^*e(D) \to g^*e(C)$.

A basis for Y is a subsheaf $B \subseteq A$ such that for any $C \in \mathbb{C}$, the images of maps of the form

$$W \hookrightarrow g^*e(D) \xrightarrow{g^*e(\alpha)} g^*e(C)$$

for $\alpha: D \to C \in \mathbb{C}$ and $W \in B(D)$ together form a basis for the frame $A(C)$ in the ordinary sense. In particular the subsheaf generated by elements of $A(1)$ is a basis, and if Z has a clopen basis, the complemented elements of $A(1)$ will suffice, resulting in an internal basis of clopens $B \subseteq A$ for Y. This establishes (ii).

On the other hand, if $\mathcal{F} \to \mathcal{E}$ is localic with clopen basis, we can choose $e: \mathbb{C} \to \mathcal{F}$ to be the inclusion of complemented subobjects of 1 in $\mathcal{F}$. Then, given any internal base $B \subseteq A$, the generating elements of $A(1)$ (which is the frame of opens of Z) coming from $B(V)$ for complemented $V \subseteq 1$ are complemented, and it follows that Z has a clopen basis. This proves (i). ∎

5.10. Proposition. *In a pullback square*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"'] & \mathcal{E}, \end{tikzcd} \tag{2}$$

suppose f is a localic map. Then

(i) *If f is entire, so is g.*

(ii) *If g is entire and a is a proper or open surjection, then f is entire.*

PROOF. (i) If f is entire, then $\mathcal{H}$ is the category of sheaves on the boolean algebra (site) $a^*f_*\mathbf{2}_{\mathcal{F}}$ as $\mathcal{G}$-topos, whence g is entire.

(ii) If g is entire, then g is tidy (5.7). Thus, by Proposition 4.10, f is tidy and the BCC is satisfied in (2). It follows that

$$g_*\mathbf{2}_{\mathcal{H}} \cong g_*b^*\mathbf{2}_{\mathcal{F}} \cong a^*f_*\mathbf{2}_{\mathcal{F}}.$$

But this means that pullback along $a: \mathcal{G} \to \mathcal{E}$ preserves the entire reflection of f, in other words, forces the reflection unit to be an isomorphism. Since a is of effective descent for locales, it follows that f is entire. ∎

5.11. Definition []. A morphism $f\colon \mathcal{F} \to \mathcal{E}$ is said to be *pure* if its entire reflection is trivial. Thus, by Corollary 5.3, f is pure precisely when the canonical map $\mathbf{2}_{\mathcal{E}} \to f_*\mathbf{2}_{\mathcal{F}}$ is an isomorphism. Any connected map is clearly pure. Note again that purity is a "local" property of a map.

5.12. Lemma. *Any morphism $f\colon \mathcal{F} \to \mathcal{E}$ factors as a pure morphism followed by an entire morphism.*

PROOF. Factor f as $\mathcal{F} \xrightarrow{p} \mathcal{P} \xrightarrow{g} \mathcal{E}$ where g is the entire reflection of f, and then factor p as $\mathcal{F} \xrightarrow{q} \mathcal{Q} \xrightarrow{h} \mathcal{P}$ where h is the entire reflection of p. It will be enough to show that h is an equivalence, since this will show that p is pure. By Proposition 5.9, the composite $\mathcal{Q} \to \mathcal{P}$ is entire, whence it follows by the universal property of p that there exists a morphism $k\colon \mathcal{P} \to \mathcal{Q}$ over $\mathcal{E}$ such that $h \circ k \cong \mathrm{id}_{\mathcal{P}}$ and $k \circ p \cong q$. But then $k \circ h \cong \mathrm{id}_{\mathcal{Q}}$ by the universal property of q, and we are done. ∎

5.13. Lemma. *The following are equivalent for a morphism $f\colon \mathcal{F} \to \mathcal{E}$:*

(i) *f is pure.*

(ii) *Any commutative square of the form below in which the map g is entire has a unique (up to isomorphism) commuting diagonal fill-in d:*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{H} \arrow[d, "g"] \\
\mathcal{E} \arrow[r, "a"'] \arrow[ur, dashed, "d"] & \mathcal{G}\,.
\end{tikzcd}$$

(iii) *For any $E \in \mathcal{E}$, the morphism $f/E\colon \mathcal{F}/f^*E \to \mathcal{E}/E$ satisfies the property in* (ii) *for the special case where g is the (étale) map $\mathcal{G} + \mathcal{G} \simeq \mathcal{G}/\mathbf{2}_{\mathcal{G}} \to \mathcal{G}$.*

PROOF. First note that by the pullback-stability of entire maps (5.10), condition (ii) is equivalent to its restriction to the case where the map $a\colon \mathcal{E} \to \mathcal{G}$ in the diagram above is the identity. It is then clear that (ii) as a property of f is preserved under pullback along étale maps, since for any $\mathcal{E}$-topos $\mathcal{H}$, any $E \in \mathcal{E}$ and any profinite locale X in $\mathcal{E}/E$, there is a correspondence (or more precisely, an equivalence) between maps

$$\mathcal{H}/f^*E \to \mathrm{Sh}_{\mathcal{E}/E}(X) \quad \rightleftarrows \quad \mathcal{H} \to \mathrm{Sh}(\prod_E X)$$

over $\mathcal{E}/E$ and $\mathcal{E}$ respectively. Here $\prod_E X$ denotes the internal localic product of the "E-indexed family X of profinite locales", which is profinite.

The implication from (i) to (ii) now follows from the universal property of a pure map, as unit for an entire reflection. Further, by its just-mentioned stability under étale change of base, property (ii) implies (iii) as a special case. Finally, to complete the circle of implications, assume f has property (iii) and consider the composite of commutative rectangles and canonical maps

$$\begin{tikzcd}
\mathcal{F}/\mathbf{2}_{\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{F}/f^*f_*\mathbf{2}_{\mathcal{F}} \arrow[r] \arrow[d] & \mathcal{F}/\mathbf{2}_{\mathcal{F}} \arrow[r] & \mathcal{E}/\mathbf{2}_{\mathcal{E}} \arrow[d] \\
\mathcal{E}/\mathbf{2}_{\mathcal{E}} \arrow[r, "\mathcal{E}/c"] & \mathcal{E}/f_*\mathbf{2}_{\mathcal{F}} \arrow[rr] \arrow[urr, dashed, "\mathcal{E}/s"] & & \mathcal{E}\,.
\end{tikzcd}$$

By the existence clause of (iii) applied in the left-hand rectangle, there is a commuting diagonal as indicated, induced by a map $s\colon f_*\mathbf{2}_{\mathcal{F}} \to \mathbf{2}_{\mathcal{E}}$. But then the arrows in the top triangle are the sides of a pullback square, which says that $f^*s\colon f^*f_*\mathbf{2}_{\mathcal{F}} \to \mathbf{2}_{\mathcal{F}}$ coincides with the counit of the adjunction $f^* \dashv f_*$. It follows that s is a right inverse for the unit $c\colon \mathbf{2}_{\mathcal{E}} \to f_*\mathbf{2}_{\mathcal{F}}$. But s is also a left inverse for c, by the uniqueness clause of (iii) applied in the total rectangle. Thus, f is pure. ∎

5.14. Remark. Lemma 5.12, together with the orthogonality relationship (ii) between pure and entire maps in Lemma 5.13, say that the classes of pure and entire maps form a so-called *factorization system* in the category of (Grothendieck) toposes, a result first obtained in [].

5.15. Corollary. *The pure-entire factorization of a map $f\colon \mathcal{F} \to \mathcal{E}$ is essentially unique.* ∎

Recall that a morphism $f\colon \mathcal{F} \to \mathcal{E}$ is connected precisely when it satisfies property (iii) of the last lemma with $\mathbf{2}_{\mathcal{G}}$ replaced by an arbitrary object $G \in \mathcal{G}$. The next technical lemma will enable us to give a simple description of connected maps as (certain) pure maps.

5.16. Lemma. *Any object E in a topos $\mathcal{E}$ can be presented as an equalizer*

$$\begin{tikzcd}
E \arrow[r, hook] & B_0 \arrow[r, shift left] \arrow[r, shift right] & B_1
\end{tikzcd}$$

where B_0 and B_1 are the underlying objects of boolean algebras.

Proof. The functor which assigns to E the free boolean algebra object $F(E)$ on E reflects isomorphisms. This follows from a few facts which may be read off from any standard construction of $F(E)$ valid in a topos. Writing as if $\mathcal{E}$ is the category of sets, any element $x \in F(E)$ can be written as a finite join

$$x_0 \vee x_1 \vee \cdots \vee x_{n-1}, \tag{3}$$

where each x_i is a finite meet of elements or complements of elements, in the image of the canonical map $i_E\colon E \to F(E)$. Also, the map i_E is an inclusion, and for x as in in (3), $x = i_E(e)$ for $e \in E$ only if $i_E(e) = x_k$ for some k, while each x_k is either of this form or zero. Using this, one checks that a bijection of the form $F(s)\colon F(E) \to F(E')$ restricts to a bijection $s\colon E \to E'$.

Now, being left adjoint to the forgetful functor, F maps the canonical diagram

$$\begin{tikzcd} E \arrow[r, hook, "i_E"] & F(E) \arrow[r, shift left, "i_{F(E)}"] \arrow[r, shift right, "F(i_E)"'] & F(F(E)) \end{tikzcd} \tag{4}$$

to a split equalizer of boolean algebras. It follows by a standard argument that (4) is an equalizer in $\mathcal{E}$. ∎

5.17. Remark. The free boolean algebra functor is in fact comonadic (in any topos), since it also preserves equalizers of reflexive pairs. However, a bare-hands constructive prove of this fact (which will not be needed) is just a little too technical to be replicated here.

5.18. Proposition. *A morphism $f\colon \mathcal{F} \to \mathcal{E}$ is connected iff the pullback of f along any entire map is pure.*

Proof. Consider a pullback square

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] & \mathcal{E} \end{tikzcd}$$

in which the horizontal maps are entire, say $\mathcal{G} \simeq \mathrm{Sh}_{\mathcal{E}}(B)$ and $\mathcal{H} \simeq \mathrm{Sh}_{\mathcal{F}}(f^*B)$ for a boolean algebra (coherent site) B in $\mathcal{E}$. In particular, $f^*B = b_*\mathbf{2}_{\mathcal{H}}$. Saying now that g is pure is equivalent to saying that g followed by a is the pure-entire factorization of the composite $\mathcal{H} \to \mathcal{F} \to \mathcal{E}$, that is, that the canonical map $B \to f_*f^*B$ is an isomorphism. Thus, the lemma states that the unit of the adjunction $f^* \dashv f_*$ is an isomorphism iff it is an isomorphism on underlying objects of boolean algebras. One direction is trivial, and the other is immediate from Lemma 5.16. ∎

5.19. Lemma. *The BCC is satisfied in a pullback square*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] & \mathcal{E} \end{tikzcd}$$

iff the pure-entire factorization of any composite $\mathcal{P} \to \mathcal{F} \to \mathcal{E}$ of f with an entire map is preserved under change of base along $a\colon \mathcal{G} \to \mathcal{E}$.

Proof. Arguing as in the last proof, we see that the lemma claims that the canonical natural transformation $a^*f_* \to g_*b^*$ is an isomorphism iff its components on underlying objects of boolean algebras in $\mathcal{F}$ are isomorphisms. But using Lemma 5.16, this is clear. ∎

5.20. Remark. By definition, a morphism $f\colon \mathcal{F} \to \mathcal{E}$ is connected iff pulling back internal locales along f restricts to a fully faithful functor on discrete, or "ind-finite" locales. By (the proof of) Proposition 5.18, connectedness of f is equivalent to the same condition on profinite locales. Similarly, Lemma 5.19 essentially states that the Beck-Chevalley condition can equivalently be formulated with reference to either discrete or profinite locales.

§6 Tidiness and closed maps

In this section we complete the analogy between proper and tidy maps by introducing firmly closed maps and proving the following counterpart of (I 6.1):

6.1. Theorem. *A map $f\colon \mathcal{F} \to \mathcal{E}$ between toposes is tidy iff all pullbacks of f are firmly closed.*

To define the notion of firmly closed map, we use the pure-entire factorization $\mathcal{F} \to \mathrm{Sh}_{\mathcal{E}}(\pi_0^{\mathrm{pf}}(\mathcal{F})) \to \mathcal{E}$ of a map $f\colon \mathcal{F} \to \mathcal{E}$, discussed in the previous section.

6.2. Definition. A map $f\colon \mathcal{F} \to \mathcal{E}$ is *firmly closed* if, for any $E \in \mathcal{E}$ and any entire map $\mathcal{P} \to \mathcal{F}/E$, the pure part of the composite $\mathcal{P} \to \mathcal{F}/f^*E \to \mathcal{E}/E$ is connected (see also Remark 5.4).

We observe straightaway that "firmly closed" is indeed a strengthening of "closed".

6.3. Proposition. *Any firmly closed map $f\colon \mathcal{F} \to \mathcal{E}$ is closed.*

Proof. Given $E \in \mathcal{E}$, any closed inclusion $\mathcal{C} \subseteq \mathcal{F}/f^*E$ is an entire map, Example 5.6 (2). Since f is firmly closed and a connected map is surjective, the image $f(\mathcal{C})$ of $\mathcal{C} \to \mathcal{E}/E$ is the same as the image of its entire part. But any entire map is proper, hence closed by (I 6.1). ∎

6.4. Lemma. *A map $f\colon \mathcal{F} \to \mathcal{E}$ is firmly closed iff for arbitrary $E \in \mathcal{E}$, the BCC is satisfied in the left of any pair of pullback squares*

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F}/f^*E \arrow[d, "f/E"] \\
\mathcal{K} \arrow[r, "c"] & \mathcal{G} \arrow[r, "a"] & \mathcal{E}/E
\end{tikzcd}$$

in which the bottom maps are entire.

Proof. Suppose f is firmly closed. Given $E \in \mathcal{E}$, consider successive pullbacks of f/E along entire maps as above. For any entire map $\mathcal{P} \to \mathcal{H}$, the pure part $p\colon \mathcal{P} \to$

$\mathrm{Sh}_{\mathcal{G}}(\pi_0^{\mathrm{pf}}(\mathcal{P}))$ of the composite $\mathcal{P} \to \mathcal{G}$ is also the pure part of the composite $\mathcal{P} \to \mathcal{E}/E$, by entirety of the map $a: \mathcal{G} \to \mathcal{E}/\mathcal{E}$ and preservation of entire maps under composition:

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d, "p"'] & \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F}/f^*E \arrow[d, "f/E"] \\
\mathrm{Sh}_{\mathcal{G}}(\pi_0^{\mathrm{pf}}(\mathcal{P})) \arrow[r] & \mathcal{G} \arrow[r, "a"] & \mathcal{E}/E\ .
\end{tikzcd}$$

It follows that p is connected, hence pure, and stably so for pullbacks along entire maps (5.18). By Lemma 5.19, this demonstrates that the BCC holds for the pullback of g along any entire map $c: \mathcal{K} \to \mathcal{G}$.

To show, conversely, that f is firmly closed if it satisfies the stated condition, let $E \in \mathcal{E}$ be given, and consider an arbitrary entire map $h: \mathcal{P} \to \mathcal{F}$. Factor h as $b \circ a$, as indicated in the diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "a"] \arrow[dr, "p"'] & \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F}/f^*E \arrow[d, "f/E"] \\
& \mathrm{Sh}_{\mathcal{E}/E}(\pi_0^{\mathrm{pf}}(\mathcal{P})) \arrow[r] & \mathcal{E}/E
\end{tikzcd}$$

where p is the pure part of $\mathcal{P} \to \mathcal{E}/E$ and g is the pullback of f/E along the entire part of p. Note that the map $\mathcal{P} \to \mathcal{H}$ is entire by Proposition 5.9 and the fact that b (as the pullback of an entire map) is entire. By assumption, the BCC holds in any pullback square

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "d"] \arrow[d, "h"'] & \mathcal{H} \arrow[d, "g"] \\
\mathcal{K} \arrow[r, "c"] & \mathrm{Sh}_{\mathcal{E}/E}(\pi_0^{\mathrm{pf}}(\mathcal{P}))
\end{tikzcd}$$

in which the map c is entire, In terms of Lemma 5.19, this says that the purity of p is preserved under pullback along entire maps, that is, that p is connected (5.18). We may therefore conclude that f is firmly closed. ∎

For the next lemma, we recall the notion of *splitting topos* from (I §6).

6.5. Lemma. *The BCC holds in any pullback square*

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "t"] \arrow[d, "f'"'] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}' \arrow[r, "s"] & \mathcal{E}
\end{tikzcd}$$

in which the map f is proper and firmly closed, and $s: \mathcal{E}' \to \mathcal{E}$ is a splitting topos for a family of open subtoposes in $\mathcal{E}$.

PROOF. The pure (hence connected) part of any composite $\mathcal{P} \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{E}$ of f with an entire map remains firmly closed and proper, the latter by (II 2.1), (II 2.2) and the fact that an entire map is proper and separated (5.7). Using (5.19), it will therefore be enough to show that the map $f': \mathcal{F}' \to \mathcal{E}'$ is pure given that f is also connected. Thus, assuming that f is connected, we shall be be done if we can verify the existence part of property (iii) in Lemma 5.13 (since f' is surjection (I 6.10), the uniqueness part is assured). Since the assumptions on f localize, we can work internally in $\mathcal{E}$.

To this end, consider any decomposition $\mathcal{F}' = \mathcal{F}'_1 + \mathcal{F}'_2$ into two clopen subtoposes. We need to show that, locally in $\mathcal{E}'$, $\mathcal{F}' = \mathcal{F}'_1$ or $\mathcal{F}' = \mathcal{F}'_2$. Observe now that by (I 6.6), any cover of $\mathcal{E}'$ by open subtoposes is refined by a cover which is the inverse image, along the map $s: \mathcal{E}' \to \mathcal{E}$, of a cover of $\mathcal{E}$ by locally closed subtoposes. Since we only need to reach our conclusion locally in $\mathcal{E}'$, while the restriction of f to a locally closed subtopos of $\mathcal{E}$ remains proper (I 5.8) and connected (5.18), we can assume any simplifaction effected by "passing to a cover in $\mathcal{E}'$". Now, if the given partition of $\mathcal{F}'$ is the inverse image along $t: \mathcal{F}' \to \mathcal{F}$ of a partition $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ of $\mathcal{F}$, then $\mathcal{F}' = \mathcal{F}'_0$ or $\mathcal{F}' = \mathcal{F}'_1$ directly by the purity of f. We show that this is indeed the situation on a cover of $\mathcal{E}'$, thus completing the proof.

First, since f' is proper (I 5.8), each of $\mathcal{F}'_0$ and $\mathcal{F}'_1$ is compact as an $\mathcal{E}'$-topos. Thus, by (I 6.6) we can, after passing to a cover of $\mathcal{E}'$ if necessary, write

$$\mathcal{F}'_i = t^{-1}(f^{-1}\mathcal{A}_{i1} \cap \mathcal{V}_{i1}) \cup t^{-1}(f^{-1}\mathcal{A}_{i2} \cap \mathcal{V}_{i2}) \cup \cdots \cup t^{-1}(f^{-1}\mathcal{A}_{in_i} \cap \mathcal{V}_{in_i}) \qquad (1)$$

where each $\mathcal{A}_{ik} \subseteq \mathcal{E}$ is closed and each $\mathcal{V}_{ik} \subseteq \mathcal{F}$ is open. Then the subtoposes of $\mathcal{E}$ of the form

$$\mathcal{P}_{11} \cap \mathcal{P}_{12} \cap \cdots \cap \mathcal{P}_{1n_1} \cap \mathcal{P}_{21} \cap \mathcal{P}_{22} \cap \cdots \cap \mathcal{P}_{2n_2} \qquad (2)$$

where $\mathcal{P}_{ik}$ is either $\mathcal{A}_{ik}$ or its complement, collectively pull back to an open cover of $\mathcal{E}'$ over which the expressions (1) become "constant" in the sense that $\mathcal{A}_{ik}$ is forced to coincide with $\mathcal{E}$ (if $\mathcal{A}_{ik}$ appears in (2)) or $0 \subseteq \mathcal{E}$ (if the complement of $\mathcal{A}_{ik}$ appears in (2)). This shows that, after passing to a cover of $\mathcal{E}'$ twice if necessary, we can reduce to the case where the given partition of $\mathcal{F}'$ comes from $\mathcal{F}$, as required.

This completes the proof of the lemma. ■

Proof of Theorem 6.1. If f is tidy, it satisfies the stable BCC (4.8), hence is firmly closed by Lemma 6.4. For the converse, assume that all pullbacks of f are firmly closed. It will by Remark 3.2 be enough to show that the BCC holds for a pullback

$$\begin{tikzcd} f^{-1}\mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{A} \arrow[r, hook] & \mathcal{E}\,. \end{tikzcd}$$

of f along an arbitrary embedding $\mathcal{A} \hookrightarrow \mathcal{E}$. Let $s: \mathcal{E}' \to \mathcal{E}$ be any splitting topos for open subtoposes of $\mathcal{E}$ which also splits $\mathcal{A}$ (e.g. the full splitting topos). Then the inclusion $\mathcal{A}' \equiv s^{-1}\mathcal{A} \hookrightarrow \mathcal{E}'$ is closed, hence entire. Now, in the pullback diagram

$$\begin{tikzcd} f'^{-1}\mathcal{A}' \arrow[r, hook] \arrow[d] & \mathcal{F}' \arrow[r] \arrow[d, "f'"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{A}' \arrow[r, hook] & \mathcal{E}' \arrow[r] & \mathcal{E} \end{tikzcd}$$

f' is firmly closed, so the left-hand square satisfies the BCC by Lemma 6.4. Since f is stably closed, it is proper by (I 6.1) so that the right-hand square satisfies the BCC by Lemma 6.5. It follows that the composed rectangle satisfies the BCC. Write this rectangle as another composite of pullbacks

$$\begin{tikzcd} f'^{-1}\mathcal{A}' \arrow[r, two heads] \arrow[d] & f^{-1}\mathcal{A} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{A}' \arrow[r, two heads] & \mathcal{A} \arrow[r] & \mathcal{E} . \end{tikzcd}$$

The left horizontal maps are surjections, being pullbacks of the splitting cover $\mathcal{E}' \to \mathcal{E}$. By the surjectivity of $\mathcal{A}' \to \mathcal{A}$, the required BCC for the right-hand square follows from that for the composite rectangle (already established) and left-hand square (which holds again by Lemma 6.5). This completes the proof. ∎

6.6. Corollary. *Any tidy map $f: \mathcal{F} \to \mathcal{E}$ factors via a connected map*

$$\mathcal{F} \to \mathrm{Sh}_{\mathcal{E}}(\pi_0^{\mathrm{fp}}(\mathcal{F}))$$

through the entire map $\mathrm{Sh}_{\mathcal{E}}(\pi_0^{\mathrm{fp}}(\mathcal{F})) \to \mathcal{E}$ induced by the profinite locale of connected components of $\mathcal{F}$ in $\mathcal{E}$. Moreover, for any pullback square

$$\begin{tikzcd} \mathcal{H} \arrow[r, "b"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] & \mathcal{E} , \end{tikzcd}$$

there is an isomorphism

$$\pi_0^{\mathrm{fp}}(\mathcal{H}) \cong a^{\#}\pi_0^{\mathrm{fp}}(\mathcal{F})$$

in $\mathcal{G}$. In particular, tidy connected maps are preserved under change of base. ∎

6.7. Corollary. *Suppose a tidy map $f\colon \mathcal{F} \to \mathcal{E}$ is obtained as the filtered limit of a diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "p_i"] \arrow[dr, "f"'] & & \mathcal{F}_i \arrow[dl, "f_i"] \\ & \mathcal{E} & \end{tikzcd}$$

of tidy maps $\{f_i\colon \mathcal{F}_i \to \mathcal{E}\}$, each of which is connected. Then f is connected.

PROOF. Let $i_0 \in I$. Then by (4.11), the canonical map

$$\varinjlim_\alpha f_{i*}\mathbf{2}_{\mathcal{F}_i} \cong \varinjlim_\alpha f_{i*}{t_\alpha}^*\mathbf{2}_{\mathcal{F}_{i_0}} \to f_*{p_{i_0}}^*\mathbf{2}_{\mathcal{F}_{i_0}} \cong f_*\mathbf{2}_{\mathcal{F}}$$

(where $\alpha\colon i \to i_0$ varies over the category I/i_0 and $t_\alpha\colon \mathcal{F}_i \to \mathcal{F}_{i_0}$ denotes the transition map induced by α) is an isomorphism. But, since each f_i is pure, $\varinjlim_\alpha f_{i*}{t_\alpha}^*\mathbf{2}_{\mathcal{F}_{i_0}} \cong \mathbf{2}_{\mathcal{E}}$, which implies that f is pure. As f is tidy, hence firmly closed, it follows that f is connected. ∎

Following [], let us call a map $f\colon \mathcal{F} \to \mathcal{E}$ "light" if it is orthogonal to connected morphisms, in other words, if in any commutative square of the form below in which the map g is connected, there exists an essentially unique commuting diagonal d as indicated

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "a"] \arrow[ur, dashed, "d"] & \mathcal{E}\,. \end{tikzcd}$$

The standard formal arguments show that lightness is a local property, preserved under composition, pullback and taking limits indexed in the base. Since the hyperconnected part of any map is in particular connected, light maps are always localic: call the corresponding locales "totally disconnected". Any discrete locale is totally disconnected by definition, and consequently so is any iterated limit of discrete locales. By Corollary 6.6, we have:

6.8. Corollary. *A morphism $f\colon \mathcal{F} \to \mathcal{E}$ is entire iff f is both tidy and light.*

CHAPTER IV. STRONGLY SEPARATED MAPS

In this chapter we study the properties of strongly separated (or "strongly Hausdorff") toposes, i.e. those toposes with tidy diagonal. After dealing with the definition and general facts (sections 1 and 2), we specialize to coherent strongly Hausdorff toposes, showing that these are coherent toposes in which the coherent objects coincide with the locally finite ones (section 3). We then go on to characterize these as profinite toposes, which entails a basepoint-free version of Grothendieck's Galois theory (section 4).

§1 Definition of strong separation

Naturally associated to the notion of tidy map is the following separation condition.

1.1. Definition. A map $f\colon \mathcal{F} \to \mathcal{E}$ between toposes is said to be *strongly separated* if its diagonal $\Delta_f\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is a tidy map. Recall (III 1.2) that this means that the direct image functor $(\Delta_f)_*$ commutes with directed colimits indexed in $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$. If $f\colon \mathcal{F} \to \mathcal{E}$ is strongly separated, we also say that $\mathcal{F}$ is *strongly Hausdorff* as $\mathcal{E}$-topos.

1.2. Examples. (1) For a group G, the topos $\mathcal{B}G$ of G-sets is strongly separated iff G is finite (II 1.1(4)). More generally, using (III 1.4), one finds that for a group G in a topos $\mathcal{E}$, the associated map $\mathcal{B}_{\mathcal{E}}G \to \mathcal{E}$ is strongly separated iff G is Kuratowski-finite and decidable; or equivalently, G is a locally constant finite group in $\mathcal{E}$.

(2) Consider a locale X in a topos $\mathcal{E}$, and the associated topos $\mathrm{Sh}_{\mathcal{E}}(X)$ of internal sheaves on X. The map $\mathrm{Sh}_{\mathcal{E}}(X) \to \mathcal{E}$ is strongly separated iff X is (strongly) Hausdorff, by (II 1.3(2)) and the following proposition.

1.3. Proposition. *Any localic and separated map is strongly separated. In particular, any embedding is strongly separated.*

Proof. A map is localic iff its diagonal is an embedding. But an embedding is proper iff it is closed, and any closed embedding is tidy (III 5.8). The second statement follows by (II 2.1(i)). ∎

Recall now that any diagonal map $\Delta_f\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is already of a very special nature, namely localic and orthogonal to connected maps or "light", see (III 6.7). More explicitly, there exists a locale Y in $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ which is a limit of prodiscrete locales — and hence "totally disconnected" — such that there is an equivalence

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\sim"] \arrow[dr, "\Delta_f"'] & & \mathrm{Sh}_{\mathcal{F}\times_{\mathcal{E}}\mathcal{F}}(Y) \arrow[dl] \\ & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} & \end{tikzcd}$$

of toposes over $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$. To see this, write $f: \mathcal{F} \to \mathcal{E}$ as the classifying topos of a geometric theory T in $\mathcal{E}$, with universal model M. Let $\pi_1, \pi_2: \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}$ be the projections, so that there are models $M_1 = \pi_1^* M$ and $M_2 = \pi_2^* M$ of T in $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$. Let Y be the locale of isomorphisms of T-models from M_1 to M_2. Then Y can be written as a limit involving the locales of T-model homomorphisms $\mathrm{Hom}(M_1, M_1)$, $\mathrm{Hom}(M_1, M_2)$, etc. which in turn are limits of discrete locales. The topos $\mathrm{Sh}_{\mathcal{F} \times_{\mathcal{E}} \mathcal{F}}(Y)$ classifies (as an $\mathcal{E}$-topos) the theory of a pair of T-models with an isomorphism between them. This theory is evidently equivalent to T, so that $\mathrm{Sh}_{\mathcal{F} \times_{\mathcal{E}} \mathcal{F}}(Y)$ is equivalent to $\mathcal{F}$, by an equivalence over $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$.

1.4. Proposition. *A map $f: \mathcal{F} \to \mathcal{E}$ is strongly separated iff its diagonal $\mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is entire.*

PROOF. By the above remarks, the statement follows directly from the fact that a morphism is entire iff it is tidy and "light" (III 6.7). ∎

1.5. Example. Let G be an open or proper étale complete localic groupoid with classifying topos $\mathcal{B}G$. Recall (II 1.4) that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Sh}(G_1) \arrow[r] \arrow[d, "{(s,t)}"'] & \mathcal{B}G \arrow[d, "\Delta"] \\
\mathrm{Sh}(G_0) \times \mathrm{Sh}(G_0) \arrow[r] & \mathcal{B}G \times \mathcal{B}G ,
\end{tikzcd}$$

where the lower arrow is an open or proper surjection. Thus $\mathcal{B}G$ is strongly separated iff (s,t) is, iff (s,t) is an entire map of locales. This holds, for example, if $(s,t): G_1 \to G_0 \times G_0$ is proper while G_1 is a Hausdorff locale (II 2.2).

§2 Elementary properties

In this section we record some elementary closure properties of the class of strongly proper maps. We omit proofs which are analogous to those in (II §2).

2.1. Proposition. (i) *Any embedding $\mathcal{F} \hookrightarrow \mathcal{E}$ is strongly separated.*

(ii) *In a commutative triangle*

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "g"] \arrow[dr, "h"'] & & \mathcal{F} \arrow[dl, "f"] \\
& \mathcal{E} , &
\end{tikzcd} \tag{1}$$

if f and g are strongly separated, then so is h;

(iii) *if g is a tidy surjection and h is strongly separated, then so is f; and*

(iv) *if h is tidy and f is strongly separated then g is tidy.* ∎

2.2. Proposition. *In a pullback square*

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{E}, \end{tikzcd}$$

(i) *if f is strongly separated, then so is $\bar{f}$;*
(ii) *the converse holds if g is a proper (or open) surjection.* ∎

2.3. Proposition. *A map $f\colon \mathcal{F} \to \mathcal{E}$ is strongly separated iff both parts of its hyperconnected-localic factorization are.*

Proof. The proof uses Proposition 1.3 and is otherwise analogous to that of (II 2.4). ∎

2.4. Proposition. *Suppose $f\colon \mathcal{F} \to \mathcal{E}$ is the limit*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "p_i"] \arrow[dr, "f"'] & & \mathcal{F}_i \arrow[dl, "f_i"] \\ & \mathcal{E} & \end{tikzcd} \tag{2}$$

of a diagram of separated maps $\{f_i\colon \mathcal{F}_i \to \mathcal{E}\}$ indexed by a category I. Then f is strongly separated.

Proof. The diagonal $\Delta_f\colon \mathcal{F} \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is the limit of the diagram $\{g_i\colon \mathcal{G}_i \to \mathcal{F} \times_{\mathcal{E}} \mathcal{F}\}$ obtained by pulling back each diagonal $\Delta_{f_i}\colon \mathcal{F}_i \to \mathcal{F}_i \times_{\mathcal{E}} \mathcal{F}_i$ along $\mathcal{F} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}_i \times_{\mathcal{E}} \mathcal{F}_i$ (and the obvious induced transition maps). The statement therefore follows from (1.4) and the stability of entire maps under pullback (III 5.10) and inverse limits (by III 5.3). ∎

§3 Strongly separated coherent toposes

In this section, we wish to examine coherent toposes which are strongly Hausdorff. We show that these are precisely the coherent toposes in which the coherent objects coincide with the locally finite objects. This will lead to a characterization of the class of strongly Hausdorff coherent toposes as "profinite" toposes in Section 4. But first, we recall some terminology.

Recall [] that an object C in a topos $\mathcal{E}$ is said to be *compact* (or "quasi-coherent" []) when the topos $\mathcal{E}/C$ is compact, that is, when every epimorphic family $\{E_i \to C\}$ in $\mathcal{E}$ has a finite epimorphic subfamily. A compact object is said to be *coherent* if for any diagram $D \to C \leftarrow E$ in $\mathcal{E}$ with D and E compact, the pullback $D \times_C E$ is again compact. A topos $\mathcal{E}$ is said to be coherent if its full subcategory $\mathrm{Coh}(\mathcal{E})$ of coherent objects is closed under finite limits and generates $\mathcal{E}$. This is equivalent to the condition that $\mathcal{E}$ is defined by a site with finite limits and finite covering families. The category $\mathrm{Coh}(\mathcal{E})$ is then (essentially small and) a pretopos. It defines a pretopos site without S-covers for $\mathcal{E}$, in the terminology of (I §4)).

Next recall that an object E in a topos $\mathcal{E}$ is said to be *locally constant* if there exists an epimorphic family $\{C_i \to 1\}$ in $\mathcal{E}$, and for each i a set S_i and an isomorphism $E \times C_i \xrightarrow{\sim} \gamma^*(S_i) \times C_i$ over C_i (where $\gamma\colon \mathcal{E} \to \mathbf{Set}$ is the canonical map). If each set S_i can be chosen to be finite, E is said to be *locally finite*. Extending the terminology for sheaves on a space, one says the étale map $\mathcal{E}/E \to \mathcal{E}$ is a covering projection (resp. a finite covering projection) if E is locally constant (resp. locally finite). We denote the full subcategory of locally finite objects of $\mathcal{E}$ by $\mathrm{LF}(\mathcal{E})$.

3.1. Lemma. *An object E in a compact strongly Hausdorff topos $\mathcal{E}$ is locally finite iff the localization $\mathcal{E}/E$ is a compact Hausdorff topos.*

PROOF. Recall first (see (II 1.3(1)) or (III 1.4(1))) that E is locally finite iff the canonical map $\mathcal{E}/E \to \mathcal{E}$ is proper and separated. Thus, if E is locally finite, $\mathcal{E}/E$ is a compact Hausdorff topos by the preservation of proper separated maps under composition, (I 2.1) and (II 2.1(ii)). Conversely, if $\mathcal{E}/E$ is compact Hausdorff, then $\mathcal{E}/E \to \mathcal{E}$ is proper by (II 2.1(iv)). But also, since the diagonal of $\mathcal{E}$ is entire (1.4), and therefore separated itself, (II 2.3) applies, showing that $\mathcal{E}/E \to \mathcal{E}$ is separated. Thus E is locally finite. ∎

3.2. Lemma. *In a compact topos $\mathcal{E}$, $\mathrm{LF}(\mathcal{E}) \subseteq \mathrm{Coh}(\mathcal{E})$. The reverse inclusion holds if $\mathcal{E}$ is coherent and strongly Hausdorff.*

PROOF. Suppose $E \in \mathcal{E}$ is locally finite. Then, using Lemma 3.1(i), $\mathcal{E}/E$ is compact by the preservation of propriety under composition (I 2.1). Thus E is a compact object. Next, since $\mathcal{E}/E \to \mathcal{E}$ is separated, the diagonal $\mathcal{E}/E \hookrightarrow \mathcal{E}/E \times E$ is closed, i.e. $E \subseteq E \times E$ is complemented. But then $C \times_E D$ is complemented in $C \times D$, from which coherence of E follows.

Suppose next that $\mathcal{E}$ is coherent and strongly Hausdorff. Then if $E \in \mathcal{E}$ is coherent, $\mathcal{E}/E$ is a coherent topos, hence strongly compact (III 1.1(2)). Since $\mathcal{E}$ is strongly Hausdorff, we can apply Proposition 2.1(iv) to conclude that $\mathcal{E}/E \to \mathcal{E}$ is a tidy map, whence E is locally finite (III 1.4(1)). Thus, $\mathrm{Coh}(\mathcal{E}) \subseteq \mathrm{LF}(\mathcal{E})$. ∎

3.3. Corollary. *Let $\mathcal{E}$ be a coherent strongly Hausdorff topos. For any object E of $\mathcal{E}$, the following properties are equivalent.*

(i) *E is locally finite;*
(ii) *The canonical map $\mathcal{E}/E \to \mathcal{E}$ is proper and separated;*
(iii) *The localization $\mathcal{E}/E$ is a compact Hausdorff topos;*
(iv) *E is coherent.*

PROOF. Clear from Lemma 3.1 and Lemma 3.2. ∎

3.4. Proposition. *The following conditions on a topos $\mathcal{E}$ are equivalent:*

(i) $\mathcal{E}$ *is coherent and strongly Hausdorff;*
(ii) $\mathcal{E}$ *is coherent, and* $\mathrm{Coh}(\mathcal{E}) \simeq \mathrm{LF}(\mathcal{E})$*;*
(iii) $\mathcal{E}$ *is strongly compact and generated by* $\mathrm{LF}(\mathcal{E})$.

PROOF. The implication (i)⇒(ii) follows from Lemma 3.2, and since a coherent topos is strongly compact, (ii) clearly implies (iii).

To show that (iii) implies (i), first observe that if $\mathrm{LF}(\mathcal{E})$ generates, so that $\mathcal{E}$ has a site consisting of locally finite objects, then models of the theory T classified by $\mathcal{E}$ are functors with values in the category of finite cardinals. It follows that the diagonal of $\mathcal{E}$, constructed as sheaves on an iterated internal limit of discrete locales in $\mathcal{E} \times \mathcal{E}$ as in section 1, is in fact entire, since the discrete locales involved are finite cardinals. Thus, if $\mathcal{E}$ is generated by its locally finite objects, it is strongly Hausdorff, and then also coherent, by Lemma 3.2.

This completes the proof. ∎

3.5. Examples. (1) The common properties of coherent sheaves on a Stone space hold in any topos. More explicitly, a topos $\mathrm{Sh}(X)$ of sheaves on a profinite locale is coherent and strongly Hausdorff, with $\mathrm{Coh}(\mathrm{Sh}(X)) \simeq \mathrm{LF}(\mathrm{Sh}(X))$ the subcategory of sheaves S for which there is a finite partition $X = U_1 \cup \cdots \cup U_n$ such that $S\restriction U_i$ is the constant sheaf with finite fiber F_i. We shall use this in §4.

(2) The classifying topos $\mathcal{B}(G)$ for a finite discrete groupoid G is coherent and strongly Hausdorff, with $\mathrm{Coh}(\mathcal{B}(G)) \simeq \mathrm{LF}(\mathcal{B}(G)) \sim \mathrm{Gal}(\mathcal{B}(G))$, the full subcategory of (right) actions of G on G_0-indexed families of finite sets.

§4 Galois theory for profinite groupoids

In this section we give a characterization of strongly Hausdorff coherent toposes as "profinite" toposes. More precisely, we shall prove the following.

4.1. Theorem. *A coherent topos $\mathcal{E}$ is strongly Hausdorff iff $\mathcal{E}$ is the classifying topos of a profinite groupoid.*

Recall that a localic (or topological) groupoid is said to be profinite if it can be obtained as an inverse limit

$$\varprojlim_i F^i \tag{1}$$

of finite (discrete) groupoids F^i. By decomposing such an inverse limit into the filtered limit of its finite sublimits, we see that a profinite groupoid can also be written as a *filtered* inverse limit of finite groupoids.

4.2. Remark. If G is a profinite groupoid, then its locales of objects and arrows are profinite, i.e. Stone locales. However, it is not the case that any groupoid in the category of profinite locales is profinite as a groupoid. (For example, let K be any compact Hausdorff locale. Then there exists a profinite locale X and a continuous surjection $p: X \to K$. The locale $R = X \times_K X$ is again profinite, so that the equivalence relation $G = (R \rightrightarrows X)$ is a groupoid in the category of profinite locales. In general, however, G

cannot be a profinite groupoid, because when it is, its classifying topos $\mathcal{B}G \simeq \mathrm{Sh}(K)$ is coherent (Proposition 4.6 below).

4.3. Proposition. *Let $G = \varprojlim_i F^i$ be a profinite groupoid. Then the canonical map*

$$\mathcal{B}G \to \varprojlim_i \mathcal{B}F^i$$

is an equivalence of toposes.

Proof. Since any finite groupoid is evidently étale complete, there is for each index i a pullback of toposes

$$\begin{tikzcd} \mathrm{Sh}(F_1^i) \arrow[r, "t"] \arrow[d, "s"'] & \mathrm{Sh}(F_0^i) \arrow[d] \\ \mathrm{Sh}(F_0^i) \arrow[r] & \mathcal{B}F^i . \end{tikzcd} \tag{2}$$

Since inverse limits commute with pullbacks as well as with the functor sending a locale to its topos of sheaves, we obtain a pullback

$$\begin{tikzcd} \mathrm{Sh}(G_1^i) \arrow[r, "t"] \arrow[d, "s"'] & \mathrm{Sh}(G_0^i) \arrow[d, "p"] \\ \mathrm{Sh}(G_0^i) \arrow[r, "p"] & \varprojlim_i \mathcal{B}F^i , \end{tikzcd} \tag{3}$$

as the inverse limit of the pullbacks (2). It now suffices to show that the map p in (3) is a proper surjection. For then, by (I 7.2), p is of effective descent for sheaves, so that the canonical map $\mathcal{B}G \to \varprojlim_i \mathcal{B}F^i$ is an equivalence, exactly as required.

To prove that p is a proper surjection, consider for each index j the projection $\varprojlim_i \mathcal{B}F^i \to \mathcal{B}F^j$ and form the pullback

$$\begin{tikzcd} \mathcal{P}^j \arrow[r, "t"] \arrow[d] & \mathrm{Sh}(F_0^j) \arrow[d] \\ \varprojlim_i \mathcal{B}F^i \arrow[r] & \mathcal{B}F^j . \end{tikzcd}$$

The map $\mathrm{Sh}(F_0^j) \to \mathcal{B}F^j$ is a finite (surjective) covering projection (i.e. is equivalent to a slice $\mathcal{B}F^j/A \to \mathcal{B}F^j$ for a locally finite object A in $\mathcal{B}F^j$ with global support). Therefore, the same is true for the map $\mathcal{P}^j \to \varprojlim_i \mathcal{B}F^i$. In particular, this map is an entire surjection. By (I 6.11) it follows that that the (relative) inverse limit $\varprojlim_j \mathcal{P}^j \to \varprojlim_i \mathcal{B}F^i$ is again a proper (in fact, entire) surjection. But straightforward manipulation of (2-categorical) limits shows that this map is equivalent to $p\colon \mathrm{Sh}(G_0) \to \varprojlim_i \mathcal{B}F^i$, hence proves that the map p is a proper surjection as well. ∎

4.4. Corollary. *Any profinite groupoid is étale complete.*

PROOF. The statement means that for any profinite groupoid G the canonical diagram of toposes

$$\begin{tikzcd} \mathrm{Sh}(G_1) \arrow[r, "t"] \arrow[d, "s"'] & \mathrm{Sh}(G_0) \arrow[d] \\ \mathrm{Sh}(G_0) \arrow[r] & \mathcal{B}G \,. \end{tikzcd}$$

is a pullback. But by the equivalence of Proposition 4.3, this diagram is equivalent to the pullback (3) above. ■

4.5. Remark. Let $\mathcal{E} = \varprojlim_i \mathcal{E}^i$ be a filtered inverse limit of coherent toposes $\mathcal{E}^i$ and coherent maps $\mathcal{E}^j \to \mathcal{E}^i$ between them. Then the topos $\mathcal{E}$ is again coherent. Indeed, the bonding mappings $\mathcal{E}^j \to \mathcal{E}^i$ induce morphisms

$$\mathrm{Coh}(\mathcal{E}^j) \to \mathrm{Coh}(\mathcal{E}^i)$$

between the pretoposes of coherent objects. The inverse limit $\mathcal{E}$ can be constructed as the topos of sheaves on the (pseudo-)colimit $\mathcal{C} = \varprojlim_i \mathrm{Coh}(\mathcal{E}^i)$ of pretoposes, constructed as the colimit of the underlying categories, see the proof of (I 4.8). This shows that $\mathcal{E}$ is coherent. It also shows (I 4.8) that any coherent object of $\mathcal{E}$ is of the form $\pi_i^*(C)$ for some coherent object C in some $\mathcal{E}^i$, where $\pi_i\colon \mathcal{E} \to \mathcal{E}^i$ is the projection. From this remark and Proposition 3.4 it is evident that:

4.6. Proposition. *The classifying topos of a profinite groupoid is coherent and strongly Hausdorff.* ■

Proposition 4.6 furnishes the reverse implication in Theorem 4.1. To show the forward implication, we use the following (weaker, see Remark 4.2 above) result:

4.7. Lemma. *Any coherent Hausdorff topos $\mathcal{E}$ is of the form $\mathcal{B}(G)$ for a groupoid in the category of profinite locales.*

PROOF. Recall from [] that any coherent topos $\mathcal{E}$ has a (stable) cover of the form $\varphi\colon \mathrm{Sh}(X) \to \mathcal{E}$, where the locale X is profinite (although the formulation and proof of this result in [] refers to Stone topological spaces, each ingredient, in particular the Barr cover construction, is evidently constructive once one sticks to working with profinite locales). Since $\mathcal{E}$ is Hausdorff, the cover φ is entire by (III 5.9(ii)). Let G be the groupoid such that $X = G_0$ and such that the diagram

$$\begin{tikzcd} \mathrm{Sh}(G_1) \arrow[r, "t"] \arrow[d, "s"'] & \mathrm{Sh}(G_0) \arrow[d, "\varphi"] \\ \mathrm{Sh}(G_0) \arrow[r, "\varphi"] & \mathcal{E} \end{tikzcd}$$

is a pullback. Then $\mathcal{E} = \mathcal{B}G$ by the descent theorem for proper maps (I 7.2). Also, the projections s, t are entire by (III 5.10). It follows, using (III 5.9(i)), that G_1 is again a profinite locale, so that G is a groupoid in the category of profinite locales, as required. ∎

Any localic groupoid G has a profinite reflection $P(G)$, which can be constructed as the limit

$$P(G) = \lim_{\leftarrow \mathcal{D}(G)} K$$

where $\mathcal{D}(G)$ is the directed inverse system of functors $G \to K$ into finite groupoids and commuting (transition) functors λ:

$$\begin{tikzcd} & G \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ L \arrow[rr, "\lambda"'] & & K . \end{tikzcd} \tag{4}$$

To complete the proof of Theorem 4.1, we consider a groupoid G as in Lemma 4.7. By Remark 4.5, the "Galois" category $\mathrm{Gal}(P(G)) = \mathrm{LF}(\mathcal{B}(P(G))$ is the (pseudo-)colimit of the categories $\mathrm{Gal}(K)$. We would therefore be done if we can prove that this colimit coincides with $\mathrm{Gal}(G)$. Inspection of the construction of a directed colimit of categories (see the proof of (I 4.8)) shows that we need to verify the following:

(1) Any locally finite object of $\mathcal{B}(G)$ results up to isomorphism as α^*S for some (α, K) in $\mathcal{D}(G)$ and $S \in \mathrm{Gal}(K)$;

(2) Given any $(\alpha, K) \in \mathcal{D}(G)$, $S, T \in \mathrm{Gal}(K)$ and a map $h: \alpha^*S \to \alpha^*T$ in $\mathrm{Gal}(G)$, there is an arrow $\lambda: (\alpha, L) \to (\beta, L)$ in $\mathcal{D}(G)$ (as in the diagram (4)) and a map $g: \lambda^*S \to \lambda^*T$ in $\mathrm{Gal}(L)$ such that $h = \beta^*g$;

(3) Any two choices of g in (2) are "eventually equal" in $\mathcal{D}(G)$.

Now, since G_0 is profinite, any object C of $\mathrm{Gal}(G)$ is of the form $C = C_1 + \cdots + C_n$, where each C_i is a G-sheaf with constant finite fiber F_i. Since the support $U_i \subseteq G_0$ of C_i is G-invariant, we can write $G = G_1 + \cdots + G_n$, where G_i is the restriction of G to U_i. Let K be the groupoid $\mathrm{Aut}(F_1) + \cdots + \mathrm{Aut}(F_n)$. Then the action of G on the C_i gives for each i a functor $G_i \to \mathrm{Aut}(F_i)$, and these combine to give a functor $\alpha: G \to K$. Moreover, $C \cong \alpha^*S$ where S is the disjoint sum $F_1 + \cdots + F_n$ equipped with its canonical K-action. This verifies (1).

To show (2), let $h: \alpha^*S \to \alpha^*T$ be a map in $\mathrm{Gal}(G)$ as stated. By replacing K with an equivalent finite groupoid with more objects, if necessary, we can assume that h decomposes (as a map of étale spaces in $\mathrm{Sh}(G_0)$) into a sum of trivial maps of the form $\mathrm{id} \times g_k: V_k \times S(k) \to V_k \times T(k)$ over $V_k = {\alpha_0}^*\{k\}$, for $k \in K_0$. Let $\lambda: L \to K$ be the subgroupoid of K with the same objects, but with

$$L(k, k') = \{\rho \in K(k, k') \mid g_{k'} \circ S(\rho) = T(\rho) \circ g_k\}.$$

Then $\lambda: G \to K$ factors through L, and the maps $g_k: S(k) \to T(k)$ become the components of a natural transformation $g: \lambda^*S \to \lambda^*T$ which pulls back to h, as required.

Finally, for (3), if there are two choices for g, simply restrict the objects of L to those for which the components of the two choices agree.

This completes the proof of Theorem 4.1.

Grothendieck's classical Galois theorem is of course a special case, and takes the following form:

4.8. Corollary [,]. *Let $\mathcal{E}$ be a pointed topos. The following are equivalent:*

(i) *$\mathcal{E}$ is hyperconnected and (strongly) Hausdorff;*
(ii) *$\mathcal{E}$ is connected, coherent and strongly Hausdorff;*
(iii) *$\mathcal{E} \simeq \mathcal{B}(\mathcal{G})$ for a profinite group;*
(iv) *$\mathcal{E}$ is coherent, with locally finite coherent objects.*

PROOF. Clear from (II 3.2), Proposition 3.4 and Theorem 4.1. ∎

CHAPTER V. RELATIVELY TIDY MAPS AND LAX DESCENT

In this chapter, we consider the following weakening of the notion of tidy map. A morphism $f: \mathcal{F} \to \mathcal{E}$ between toposes is said to be *relatively tidy* if its direct image functor f_* commutes with ordinary (small, external) filtered colimits (the reason for this terminology will become clear below). We shall develop as much of the theory of relatively tidy maps as is needed to prove that in a lax pullback square

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E}
\end{tikzcd},$$

where f is relatively tidy, the map d_0 is tidy and the induced natural transformation $g^* f_* \to d_{0*} d_1{}^*$ is an isomorphism (Theorem 5.1). This result has immediate applications to lax descent of sheaves (section 6). Indeed, it has to a large extent been motivated by the desire to exhibit a proof of Zawadowski's descent theorem for pretoposes [] which is both conceptual and constructive, using standard methods of topos theory. It is thus relevant to point out that the ingredients to the above theorem are mostly well-known (or at least straightforward to prove) when specialised to the special instance of coherent morphisms between coherent toposes (see []).

We start with two preparatory sections, the first on path toposes and localizations and the second on lax pullbacks. After dealing with the formal definition and some elementary facts about relative tidiness (section 3), we introduce relative tidy morphisms between convenient types of sites (section 4) as a vehicle for showing that relatively tidy maps are stable under change of base (Theorem 4.9) and filtered inverse limits (Theorem 4.10). With these properties in place, our main results follow rather straightforwardly in a formal way (section 5).

§1 Path toposes

Most of the material in this preliminary section is based on []. We shall discuss the construction, for any topos $\mathcal{E}$, of a path topos $P(\mathcal{E})$, where the "paths" are parametrised by the Sierpinski space. We work over a fixed base topos $\mathcal{S}$ as if $\mathcal{S}$ is "the" category of sets.

Consider the topos $\mathbb{S}$ of sheaves on the Sierpinski space. It is (equivalent to) the category $\mathbf{Set}^{\mathbf{2}}$, whose objects are given by functions $\alpha: S_0 \to S_1$ and whose arrows are commutative squares. We start with a well-known lemma.

1.1. Lemma. *$\mathbb{S}$ is an exponentiable topos, that is, for any topos $\mathcal{E}$, the exponential topos $\mathcal{E}^{\mathbb{S}}$ exists.*

PROOF. Although exponentiability of $\mathbb{S}$ falls out immediately from the general theory of such toposes developed in [], we describe two easy ways of seeing this directly.

One is based on the formalism of classifying toposes of geometric theories: Let $\mathcal{E}$ be any topos, and let T be a geometric theory classified by $\mathcal{E}$. Let T' be the geometric theory of which the models are homomorphisms of T-models. Then the classifying topos of T' clearly has the universal property required for the exponential $\mathcal{E}^{\mathbb{S}}$.

Alternatively, we use sites: For a given topos $\mathcal{E}$, let $(\mathbb{C}, J)$ be any site for $\mathcal{E}$ with pullbacks. Let $\mathrm{Ar}(\mathbb{C})$ be the arrow category of $\mathbb{C}$, and let J' be the stable system of covers on $\mathrm{Ar}(\mathbb{C})$ which at the object $(C' \to C)$ consists of the following two types of families: first, for each J-cover $\{C_i \to C\}$ of C, the family $\{(C' \times_C C_i \to C_i) \to (C' \to C)\}$, and, secondly, for each J-cover $\{\alpha_j : C'_j \to C'\}$ of C', the family of arrows

$$\begin{tikzcd} C'_j \arrow[r, "\alpha_j"] \arrow[d] & C' \arrow[d] \\ C \arrow[r, "\mathrm{id}"] & C. \end{tikzcd}$$

Then $(\mathrm{Ar}(\mathbb{C}), J')$ is a site for $\mathcal{E}^{\mathbb{S}}$ (see [] for a proof). ■

1.2. Definition. The *path topos* of a topos $\mathcal{E}$ is the exponential $\mathcal{E}^{\mathbb{S}}$, denoted $P(\mathcal{E})$. We shall write

$$\partial_0, \partial_1 : P(\mathcal{E}) \rightrightarrows \mathcal{E}$$

for the evident "evaluation" morphisms, and $\iota : \mathcal{E} \to P(\mathcal{E})$ for the "diagonal" section. The universal transformation $\partial_0^* \to \partial_1^*$ will be denoted by μ. Note that the natural transformation $\iota^*\mu : \iota^*\partial_0^* \to \iota^* d_1^*$ is the identity, modulo the canonical isomorphisms $\partial_0 \iota \cong \mathrm{id}_{\mathcal{E}} \cong \partial_1 \iota$.

The construction of path toposes is related to the theory of local toposes and of localizations developed in [] and []. Recall from the latter source that a morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be *local* if the direct image functor f_* has an $\mathcal{E}$-indexed right adjoint, denoted f^+ (respectively $f^+_E : \mathcal{E}/E \to \mathcal{F}/f^*E$ for its indexed part at E). It follows of course that for a local map f, the direct image preserves all $\mathcal{E}$-internal colimits. A fortiori, we obtain:

1.3. Proposition. *Any local morphism of toposes is tidy.* ■

If $p : \mathcal{S} \to \mathcal{E}$ is a point of an $\mathcal{S}$-topos $\mathcal{E}$, one can "localize" $\mathcal{E}$ at p, that is construct a *local* $\mathcal{S}$-topos

$$\mathrm{Loc}_p(\mathcal{E})$$

with certain universal properties (see []). This construction depends on the base topos $\mathcal{S}$. For our present purposes, it is important to observe that $\mathrm{Loc}_p(\mathcal{E})$ can be obtained as a filtered inverse limit of slices of $\mathcal{E}$,

$$\mathrm{Loc}_p(\mathcal{E}) \cong \varprojlim_U \mathcal{E}/U \tag{1}$$

indexed by the directed category of neighbourhoods of the point p [, Theorem 3.7].

The path topos is $P(\mathcal{E})$ is in some sense the universal localization of $\mathcal{E}$. More explicitly, the $\mathcal{S}$-topos $\mathcal{E}$ pulls back to an $\mathcal{E}$-topos $\pi_1\colon \mathcal{E}\times\mathcal{E}\to\mathcal{E}$, of which the diagonal $\Delta\colon\mathcal{E}\to\mathcal{E}\times\mathcal{E}$ is a "generic point." Now, working over $\mathcal{E}$ as base, one can construct the localization $\mathrm{Loc}_\Delta(\mathcal{E}\times\mathcal{E}\to\mathcal{E})$, which is a local $\mathcal{E}$-topos. This localization is precisely the path topos $P(\mathcal{E})$, viewed as an $\mathcal{E}$-topos via the map $\partial_0\colon P(\mathcal{E})\to\mathcal{E}$. In particular, ∂_0 is a local morphism, hence is tidy by (1.3). The right adjoint to ∂_{0*} is the inverse image ι^* of the diagonal section. In particular, since $\partial_{1*}\iota_*\cong\mathrm{id}$, we find by taking right adjoints that $\partial_{0*}\partial_1{}^*\cong\mathrm{id}$ also (see []).

The next proposition summarizes the above properties of the path topos to be used in this chapter.

1.4. Proposition []. *Let $\mathcal{E}$ be any topos and let $P(\mathcal{E})$ be its path topos, with canonical maps $\partial_0,\ \partial_1\colon P(\mathcal{E})\rightrightarrows\mathcal{E}$.*

(i) *The map $\partial_0\colon P(\mathcal{E})\to\mathcal{E}$ is local, hence strongly proper.*

(ii) *The canonical transformation $\tau\colon\partial_0{}^*\to\partial_1{}^*$ induces an isomorphism $\partial_{0*}\partial_1{}^*\cong\mathrm{id}$.*

(iii) *$P(\mathcal{E})$ is the localization of $\pi_1\colon\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ at the diagonal point $\Delta\colon\mathcal{E}\to\mathcal{E}\times\mathcal{E}$. In particular, there is an equivalence of $\mathcal{E}$-toposes*

$$\begin{tikzcd} P(\mathcal{E}) \arrow[rr, "\sim"] \arrow[dr, "(\partial_0,\partial_1)"'] & & \varprojlim_i (\mathcal{E}\times\mathcal{E})/U_i \arrow[dl] \\ & \mathcal{E}\times\mathcal{E}\ , & \end{tikzcd}$$

where the inverse limit is indexed by by an internal directed category in $\mathcal{E}$ (cf. (1) above). ■

§2 Lax pullbacks of toposes

The sole purpose of this section is to review the definition and construction of lax pullbacks of toposes, also known as "comma-squares." Again, we fix a base topos $\mathcal{S}$, and assume all toposes to be $\mathcal{S}$-toposes.

Given two morphisms $f\colon\mathcal{F}\to\mathcal{E}$ and $g\colon\mathcal{G}\to\mathcal{E}$, the *lax pullback (over $\mathcal{S}$)* of f and g is a square

$$\begin{tikzcd} \mathcal{H} \arrow[r, "a"] \arrow[d, "b"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{E} \end{tikzcd}$$

together with a 2-cell $\tau\colon gb\Rightarrow fa$ (i.e. a natural transformation $\tau^*\colon b^*g^*\to a^*f^*$), and universal with this property. This means roughly that, given any pair of morphisms

$u: \mathcal{K} \to \mathcal{F}$ and $v: \mathcal{K} \to \mathcal{G}$, together with a 2-cell $\sigma: gv \Rightarrow fa$, there is a morphism (unique up to natural isomorphism) $c: \mathcal{K} \to \mathcal{H}$ for which there are isomorphisms $\alpha: ac \Rightarrow u$ and $\beta: bc \Rightarrow v$ such that the square of 2-cells

$$\begin{tikzcd} gbc \arrow[r, "g\beta"] \arrow[d, "\tau c"'] & gv \arrow[d, "\sigma"] \\ fac \arrow[r, "f\alpha"] & fu \end{tikzcd}$$

commutes. The precise formulation of the universal property refers in the usual way to an equivalence of categories between $\mathrm{Hom}(\mathcal{K}, \mathcal{H})$ and the category of such triples (u, v, σ), natural in $\mathcal{K}$. Note that the definition is not symmetric in f and g. We shall denote a square with this universal property by

$$\begin{tikzcd} (\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{G} \arrow[r, "g"] & \mathcal{E} \end{tikzcd} . \qquad (1)$$

The use of the notation is justified by the existence and uniqueness of lax pullbacks expressed in Proposition 2.3 below.

2.1. Remark. Unlike pullbacks, lax pullbacks of toposes depend on the base $\mathcal{S}$. Indeed, the 2-cell $\tau: gb \Rightarrow fa$ is required to be a transformation over $\mathcal{S}$. This means that, if we denote the structure maps to the base by $\gamma_{\mathcal{E}}: \mathcal{E} \to \mathcal{S}$, etc. then $\gamma_{\mathcal{E}}\tau = \mathrm{id}$. Or more precisely, the diagram of 2-cells,

$$\begin{tikzcd} \gamma_{\mathcal{E}}(gb) \arrow[rr, "\gamma_{\mathcal{E}}\tau"] \arrow[dr] & & \gamma_{\mathcal{E}}(fa) \arrow[dl] \\ & \gamma_{\mathcal{H}}, & \end{tikzcd}$$

in which the sloping arrows are (the obvious) isomorphisms, commutes. In particular, in the extreme case where $\mathcal{E}$ is the base topos $\mathcal{S}$, the lax pullback coincides with the ordinary pullback. This is of course not the case in general, as is evident from the following instance of a lax pullback.

2.2. Lemma. *For any topos $\mathcal{E}$, the square*

$$\begin{tikzcd}
P(\mathcal{E}) \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & \mathcal{E} \arrow[d, "\mathrm{id}"] \\
\mathcal{E} \arrow[r, "\mathrm{id}"] & \mathcal{E}
\end{tikzcd}$$

naturally associated to the path topos of $\mathcal{E}$ is a lax pullback over $\mathcal{S}$.

PROOF. Clear from the universal property of the exponential $P(\mathcal{E})$. ∎

2.3. Proposition. *For any pair of morphisms $f: \mathcal{F} \to \mathcal{E}$ and $g: \mathcal{G} \to \mathcal{E}$, the lax pullback* (1) *exists, and is unique up to equivalence.*

PROOF. Constructing the lax pullback (1) amounts to constructing the pullback

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "h"] \arrow[d, "(d_0, d_1)"'] & P(\mathcal{E}) \arrow[d, "(\partial_0, \partial_1)"] \\
\mathcal{G} \times \mathcal{F} \arrow[r, "g \times f"] & \mathcal{E} \times \mathcal{E} ,
\end{tikzcd}$$

with $\tau: gd_0 \Rightarrow fd_1$ obtained from the "universal path" $(\mu: d_0 \Rightarrow d_1): P(\mathcal{E}) \rightrightarrows \mathcal{E}$ in the obvious way. ∎

2.4. Remark. We shall make use below of the following equivalent "stepwise" construction of the lax pullback (1) from the path topos:

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r] \arrow[dd, "d_0"'] \arrow[ddr, phantom, "\overset{\mathrm{pb}}{\cong}"] & (\mathcal{E} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\mathrm{pb}}{\cong}"] & \mathcal{F} \arrow[d, "f"] \\
 & P(\mathcal{E}) \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\overset{\mu}{\Rightarrow}"] & \mathcal{E} \arrow[d, "\mathrm{id}"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E} \arrow[r, "\mathrm{id}"] & \mathcal{E} .
\end{tikzcd}$$

§3 Relatively tidy maps

Recall from chapter III that a morphism $f: \mathcal{F} \to \mathcal{E}$ is said to be tidy if f_* preserves all $\mathcal{E}$-internal directed colimits. In this section we shall be interested in morphisms

$f:\mathcal{F} \to \mathcal{E}$ for which f_* is only required to preserve ordinary directed colimits, that is, such that for any diagram $\{F_i\}$ of objects of $\mathcal{F}$ indexed by a filtered category I in **Set**, the canonical map

$$\theta: \varinjlim_I f_*(F_i) \to f_*(\varinjlim_I F_i) \tag{1}$$

is an isomorphism. As usual, this notion makes sense over any base topos $\mathcal{S}$ in place of **Set**, and leads to the following.

3.1. Definition. A morphism $f:\mathcal{F} \to \mathcal{E}$ of $\mathcal{S}$-toposes is said to be *tidy relative to* $\mathcal{S}$ if f_* commutes with all $\mathcal{S}$-internal filtered colimits.

Note that this definition takes care of "pure" tidiness as the case where $\mathcal{S}$ coincides with $\mathcal{E}$. If $\mathcal{S}$ is, or plays the role of the topos of sets as fixed base topos, we shall for brevity refer to f simply as *relatively tidy*.

3.2. Example. Clearly, any coherent map $f:\mathcal{F} \to \mathcal{E}$ between coherent toposes is relatively tidy [].

As an aid to our exposition, we also introduce the corresponding relative notion of propriety (so that, in particular, a relatively tidy map is a relatively proper map with additional properties).

3.3. Definition. A morphism $f:\mathcal{F} \to \mathcal{E}$ of $\mathcal{S}$-toposes is said to be *proper relative to* $\mathcal{S}$ if for each $E \in \mathcal{E}$, direct image for the induced morphism $f/E:\mathcal{F}/f^*E \to \mathcal{E}/E$ of $\mathcal{S}$-toposes preserves suprema of $\mathcal{S}$-internal directed families of subobjects of 1.

3.4. Remark. Definitions 3.1 and 3.3 can be made more explicit in the usual way. Write $\gamma_{\mathcal{E}}:\mathcal{E} \to \mathcal{S}$ and $\gamma_{\mathcal{F}}:\mathcal{F} \to \mathcal{S}$ for the structure maps into the base topos.

(1) The map $f:\mathcal{F} \to \mathcal{E}$ is tidy relative to $\mathcal{S}$ iff, for any object $S \in \mathcal{S}$ and any directed category I in $\mathcal{S}/S$, the associated square (with notation as in (I 1.8) but writing $\mathcal{E}/S$ for $\mathcal{E}/\gamma_{\mathcal{E}}^*S$, f/S for $f/\gamma_{\mathcal{E}}^*S$, I for $(\gamma_{\mathcal{E}}/S)^*I$, etc.)

$$\begin{tikzcd} \mathcal{F}/S \arrow[r, "\infty"] \arrow[d, "f/S"'] & (\mathcal{F}/S)^I \arrow[d, "(f/S)^I"] \\ \mathcal{E}/S \arrow[r, "\infty"] & (\mathcal{E}/S)^I \end{tikzcd} \tag{2}$$

has the property that the canonical natural transformation

$$\infty^*(f/S)^I_* \to (f/S)_*\infty^* \tag{3}$$

is an isomorphism.

(2) The map $f: \mathcal{F} \to \mathcal{E}$ is proper relative to $\mathcal{S}$ iff, for any object E in $\mathcal{E}$, any object $S \in \mathcal{S}$ and any directed category I in $\mathcal{S}/S$, the square

$$\begin{tikzcd}
(\mathcal{F}/E)/S \arrow[r, "\infty"] \arrow[d, "(f/E)/S"'] & ((\mathcal{F}/E)/S)^I \arrow[d, "((f/E)/S)^I"] \\
(\mathcal{E}/E)/S \arrow[r, "\infty"] & ((\mathcal{E}/E)/S)^I
\end{tikzcd}$$

has the property that the canonical map

$$\infty^*((f/E)/S)^I_*(V) \to ((f/E)/S)_*\infty^*(V)$$

is an isomorphism for any $V \subseteq 1$ in $((\mathcal{F}/E)/S)^I$. By (I 3.2) this is equivalent to the requirement for the square (2) that the natural transformation (3) is *mono*.

3.5. Proposition. *If $f: \mathcal{F} \to \mathcal{E}$ is tidy (resp. proper) relative to $\mathcal{S}$, then so is the induced morphism $f/E: \mathcal{F}/f^*E \to \mathcal{E}/E$ for any object E in $\mathcal{E}$.*

PROOF. Clear from Remark 3.4. ∎

The elementary closure properties of proper and tidy maps have evident relative versions. Thus, propriety and tidiness relative to $\mathcal{S}$ are both "local" properties with respect to $\mathcal{S}$. Also:

3.6. Proposition. (i) *Any equivalence of $\mathcal{S}$-toposes is tidy relative to $\mathcal{S}$.*

(ii) *The composition of two morphisms which are proper (resp. tidy) relative to $\mathcal{S}$ is again proper (resp. tidy).* ∎

Relatively tidy morphisms can be characterised by an "Edwards criterion" analogous to (III 4.1). We note, as before, that the next proposition can be formulated and proved in the internal logic of an arbitrary base topos $\mathcal{S}$, substituting "proper/tidy relative to $\mathcal{S}$" for "relatively proper/tidy".

3.7. Proposition. *A morphism $f: \mathcal{F} \to \mathcal{E}$ is*

(i) *relatively proper iff for any directed epimorphic family $\{F_i \hookrightarrow f^*E\}$ of subobjects in $\mathcal{F}$ there exists an epimorphic family $\{E_j \to E\}$ in $\mathcal{E}$ such that each $f^*E_j \to f^*E$ factors through some $F_i \hookrightarrow f^*E$;*

(ii) *relatively tidy iff f is relatively proper and for any epimorphism $F \twoheadrightarrow f^*E$ and any directed epimorphic family $\{R_i \hookrightarrow F \times_{f^*E} F\}$ of equivalence relations in $\mathcal{F}$, there exists an epimorphic family $\{E_j \to E\}$ in $\mathcal{E}$ and for each index j an epi $A_j \to f^*E_j$ and an index i such that there is a commutative diagram in $\mathcal{F}$:*

$$\begin{tikzcd}
R_i \arrow[r, shift left] \arrow[r, shift right] & F \arrow[r, "\lambda"] & f^*E \\
A_j \times_{f^*E_j} A_j \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & A_j \arrow[u] \arrow[r] & f^*E_j \arrow[u]
\end{tikzcd}$$

PROOF. (i) Given a family $\{F_i \to f^*E\}$ of subobjects in $\mathcal{F}$, its image under the functor $(f/E)_*: \mathcal{F}/f^*E \to \mathcal{E}/E$ is refined by a family $\{E_j \to E\}$ in $\mathcal{E}$ precisely when each $f^*E_j \to f^*E$ factors through some $F_i \hookrightarrow f^*E$. Thus, the statement is just a reformulation of Definition 3.1.

(ii) Suppose $f: \mathcal{F} \to \mathcal{E}$ is relatively proper, so that that for any directed system $\{F_i\}$ in $\mathcal{F}$, the canonical map

$$\theta: \varinjlim_I f_*(F_i) \to f_*(\varinjlim_I F_i) \tag{1}$$

is a monomorphism (Remark 3.4 (2)). We need to show that the additional condition on f is necessary and sufficient for θ to be epi, and hence an isomorphism.

Suppose first the condition is satisfied as stated. It is sufficient to show that θ is "locally surjective," in the sense that any map

$$E \xrightarrow{\alpha} f_*(\varinjlim_I F_i)$$

factors through θ on a cover of E. So fix such α and consider the transposed $\hat{\alpha}: f^*E \to \varinjlim_I F_i$. Construct for each index j the pullback along the colimit map $\nu_j: F_j \to \varinjlim_I F_i$

$$\begin{tikzcd}
P_j \arrow[r] \arrow[d] & F_j \arrow[d, "\nu_j"] \\
f^*E \arrow[r, "\hat{\alpha}"] & \varinjlim_I F_i .
\end{tikzcd}$$

Then, since the images of the maps $P_j \to f^*E$ form a directed epimorphic family of subobjects, we can by relative propriety of f and (i) find a cover of E by arrows $e: E' \to E$ for which there is a commutative square of the form

$$\begin{tikzcd}
B' \arrow[rr] \arrow[d, two heads] & & F_j \arrow[d, "\nu_j"] \\
f^*E' \arrow[r] & f^*E \arrow[r, "\hat{\alpha}"] & \varinjlim_I F_i ,
\end{tikzcd}$$

with the vertical arrow on the left epi as indicated. By replacing E by E', we may thus assume that $\hat{\alpha}: f^*E \to \varinjlim_I F_i$ can be composed with an epimorphism to give a factorisation

$$\begin{tikzcd}
B \arrow[r, dashed, "\beta"] \arrow[d, two heads] & F_{j_0} \arrow[d, "\nu_{j_0}"] \\
f^*E \arrow[r, "\hat{\alpha}"] & \varinjlim_I F_i
\end{tikzcd} \tag{4}$$

for some index j_0. Now construct for each transition map $\tau\colon F_{j_0} \to F_j$ in the system $\{F_j\}$ the equalizer of $\tau\beta\pi_0$ and $\tau\beta\pi_1$:

$$R_\tau \longrightarrow B \times_{f^*E} B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B \xrightarrow{\tau\beta} F_j \ .$$

Since the composite $B \to F_{j_0} \to \varinjlim_I F_i$ factors through f^*E as in (4), the equivalence relations R_τ cover $B \times_{f^*E} B$. Thus, the assumed property of f gives, after replacing E by a family of objects covering E, a "refinement"

$$\begin{tikzcd} A \arrow[rr, dashed] \arrow[dr] & & B \arrow[dl] \\ & f^*E & \end{tikzcd}$$

such that $A \times_{f^*E} A \to B \times_{f^*E} B$ factors through some R_τ. Thus, for this τ, the two composites

$$A \times_{f^*E} A \rightrightarrows A \longrightarrow B \longrightarrow F_{j_0} \xrightarrow{\tau} F_j$$

are equal. Since the epi $A \twoheadrightarrow f^*E$ is the coequaliser of its kernel pair, it follows that $\hat{\alpha}\colon f^*E \to \varinjlim_I F_i$ factors through $F_j \to \varinjlim_I F_i$. Thus, after having passed to a cover of E twice, we have shown that $\alpha\colon E \to f_*(\varinjlim_I F_i)$ factors through f_*F_j for some j, hence factors through θ as desired.

Conversely, assume that f is relatively tidy. Take any epi $\gamma\colon F \twoheadrightarrow f^*E$ and any directed union $\bigcup_i R_i = F \times_{f^*E} F$ by equivalence relations R_i. Form the coequalizers Q_i

$$\begin{tikzcd} R_i \arrow[r, shift left] \arrow[r, shift right] & F \arrow[r, two heads] \arrow[d, "\lambda"'] & Q_i \arrow[dl, dashed, "\lambda_i"] \\ & f^*E & \end{tikzcd}$$

so that $f^*E = \varinjlim_i Q_i$ because the R_i cover the kernel pair of λ. Since f is relatively tidy, $f_*f^*E = \varinjlim_i f_*(Q_i)$, so that by pullback along the unit $\eta\colon E \to f_*f^*E$ we obtain a colimit $E = \varinjlim_i E_i$,

$$\begin{tikzcd} E_i \arrow[r] \arrow[d] & f_*(Q_i) \arrow[d] \\ E \arrow[r, "\eta"] & f_*f^*E \ . \end{tikzcd}$$

By transposition we get a commutative diagram

$$\begin{tikzcd} f^*(E_i) \arrow[rr] \arrow[dr] & & Q_i \arrow[dl] \\ & f^*E & \end{tikzcd}$$

for each i. Now form the pullback

$$\begin{tikzcd} B_i \arrow[r] \arrow[d] & F \arrow[d] \\ f^*(E_i) \arrow[r] & Q_i \ . \end{tikzcd}$$

Then $B_i \times_{f^*(E_i)} B_i$ factors through R_i, since R_i is the kernel pair of $F \to Q_i$. This verifies the stated property of f, and completes the proof of the proposition. ■

§4 Relatively tidy morphisms of sites

In this section, we give a description of relative tidiness of a map $f: \mathcal{F} \to \mathcal{E}$ between toposes in terms of a morphism of sites inducing f, and use it to prove the non-trivial stability properties of relatively tidy maps. More precisely, we prove the following two theorems.

4.1. Theorem ("change of base"). *In a diagram of pullback squares*

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\theta"] \arrow[d, "f'"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}' \arrow[r, "\psi"] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{S}' \arrow[r, "\varphi"] & \mathcal{S} \ , \end{tikzcd}$$

suppose f is proper (resp. tidy) relative to $\mathcal{S}$. Then f' is proper (resp. tidy) relative to $\mathcal{S}'$, and the weak BCC (resp. BBC) is satisfied in the top square; that is, the induced natural transformation

$$\psi^* f_* \to f'_* \theta^*$$

is a monomorphism (resp. an isomorphism).

4.2. Theorem ("filtered inverse limits"). *Suppose $f\colon \mathcal{F} \to \mathcal{E}$ is the limit of a diagram of maps $\{f_i\colon \mathcal{F}_i \to \mathcal{E}_i\}$ over a base topos $\mathcal{S}$, indexed by a filtered category I:*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, "q_i"] \arrow[d, "f"'] & & \mathcal{F}_i \arrow[d, "f_i"] \\
\mathcal{E} \arrow[rr, "p_i"] \arrow[dr] & & \mathcal{E}_i \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd} \tag{1}$$

If each f_i is proper (resp. tidy) relative to $\mathcal{S}$, then so is f. Moreover, for any $i \in I$, the natural transformation

$$\varinjlim_\alpha p_j{}^* f_{j\,*} u_\alpha{}^* \to f_* q_i{}^*, \tag{2}$$

where $\alpha\colon j \to i$ varies over the category I/i and $u_\alpha\colon \mathcal{F}_j \to \mathcal{F}_i$ is the transition map induced by α, is then a monomorphism (resp. an isomorphism).

Before embarking on the technicalities of proof, we straightaway draw a needed conclusion from Theorem 4.2. First, note that it gives:

4.3. Corollary. *Suppose in (1) that for each $\alpha\colon j \to i$ in I, the canonical natural transformation $t_\alpha{}^* f_{i\,*} \to f_{j\,*} u_\alpha{}^*$ induced by the transition maps $t_\alpha\colon \mathcal{E}_j \to \mathcal{E}_i$ and $u_\alpha\colon \mathcal{F}_j \to \mathcal{F}_i$ is a monomorphism (resp. an isomorphism). Then if the maps f_i are proper (resp. tidy) relative to $\mathcal{S}$, the canonical natural transformation $p_i{}^* f_{i\,*} \to f_* q_i{}^*$ is a monomorphism (resp. an isomorphism) for each $i \in I$.* ■

4.4. Proposition ("localization lemma"). *Let $f\colon \mathcal{F} \to \mathcal{E}$ be a tidy morphism relative to (a base topos) $\mathcal{S}$, and let $p\colon \mathcal{S} \to \mathcal{E}$ be a point of $\mathcal{E}$. Then in the pullback square*

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "v"] \arrow[d, "g"'] & \mathcal{F} \arrow[d, "f"] \\
\mathrm{Loc}_p(\mathcal{E}) \arrow[r, "u"] & \mathcal{E}
\end{tikzcd}$$

the map g is again tidy relative to $\mathcal{S}$, and the Beck-Chevalley condition $g_* v^* \cong u^* f_*$ holds.

Proof. As explained in §1, the localization $\mathrm{Loc}_p(\mathcal{E})$ can be constructed as a filtered inverse limit

$$\mathrm{Loc}_p(\mathcal{E}) = \varprojlim_U \mathcal{E}/E$$

where U ranges over the (étale) neighbourhoods of p. It follows that

$$\mathcal{G} = \varprojlim_U \mathcal{F}/f^* U.$$

For any transition map $e: U \to V$ between étale neighbourhoods of p, we have a pullback

$$\begin{tikzcd}
\mathcal{F}/f^*U \arrow[r] \arrow[d, "f/U"] & \mathcal{F}/f^*V \arrow[r] \arrow[d, "f/V"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{E}/U \arrow[r, "e"] & \mathcal{E}/V \arrow[r] & \mathcal{E}
\end{tikzcd}$$

where the morphisms f/U and f/V are again tidy while the square satisfies the Beck-Chevalley condition (by the explicit construction of $(f/U)_*$ and $(f/V)_*$ from f_*). Thus g is tidy by Theorem 4.2, while the conditions of Corollary 4.3 are satisfied to yield the Beck-Chevalley condition $g_* v^* \cong u^* f_*$. ∎

The remainder of this section is devoted to proving Theorems 4.1 and 4.2. The material to follow generalizes that of (I §5) and (III §4) in an evident way. The proofs will therefore be appropriately terse where arguments are analogous.

First, we define relatively tidy morphism between pretopos sites. We shall need the following addition to the notation and terminology introduced in (I §4) for sites and morphisms between sites. Consider any morphism of sites $F: \mathbb{C} \to \mathbb{D}$ such that $\mathbb{D}$ has pullbacks. The morphisms F can be "sliced" at any $C \in \mathbb{C}$ to produce a morphism of sites $F/C: \mathbb{C}/C \to \mathbb{D}/F(C)$, and any arrow $C' \to C \in \mathbb{C}$ induces a "localization" morphism $\mathbb{D}/F(C) \to \mathbb{D}/F(C')$ by pullback along the arrow $F(C') \to F(C)$. We shall say that a condition which refers to an object $C \in \mathbb{C}$ and data in the site $\mathbb{D}/F(C)$ holds *locally* (at C) if there is a cover $\{C_j \to C \mid j \in J\}$ in $\mathbb{C}$ such that for each $j \in J$ the condition holds for the "localized" data in $\mathbb{D}/F(C_j)$. For example, given an arrow $\delta: D \to F(C)$, a family $\{D_i \to D \mid i \in I\}$ in $\mathbb{D}$ *locally* has a member which covers at C if we can find a cover of C as above such that, for each $j \in J$, the pulled back family $\{F(C_j) \times_{F(C)} D_i \to F(C_j) \times_{F(C)} D \mid i \in I\}$ has a single element which covers.

Given $C \in \mathbb{C}$ and $\delta: D \to F(C) \in \mathbb{D}$, we extend the terminology of (III §4) by calling a family of monomorphisms of the form $\{V_k \rightarrowtail D \times_{F(C)} D \mid k \in K\}$ *effective (at C)* if there exists a subobject $E \rightarrowtail D$ such that both $E \to F(C)$ and the induced family $\{(E \times_{F(C)} E) \cap V_k{}^{(n)} \rightarrowtail E \times_{F(C)} E \mid k \in K, n \in \mathbb{N}\}$ are covers of $\mathbb{D}$. A subobject $V \rightarrowtail D \times_{F(C)} D$ is effective at C if it is so as a singleton family.

4.5. Definition. A morphism $F: \mathbb{C} \to \mathbb{D}$ between pretopos sites is said to be

(i) *relatively proper* if for any $C \in \mathbb{C}$, any directed cover $\{D_i \rightarrowtail F(C)\}$ in $\mathbb{D}$ locally has a member which covers;

(ii) *relatively tidy* if F is relatively proper and, moreover, for any $C \in \mathbb{C}$ and covering map $\delta: D \to F(C)$, any directed cover $\{R_i \hookrightarrow D \times_{F(C)} D\}$ of monomorphisms in $\mathbb{D}$ locally has a member which is effective at C.

4.6. Proposition. *Let $f: \mathcal{F} \to \mathcal{E}$ be a map of toposes and suppose f is induced by a morphism $F: \mathbb{C} \to \mathbb{D}$ where $\mathbb{C}$ and $\mathbb{D}$ are pretopos sites. Then f is relatively proper (resp. relatively tidy) precisely when F is.*

PROOF. We have a commutative square

$$\begin{tikzcd} \mathbb{D} \arrow[r, "h"] & \mathcal{F} \\ \mathbb{C} \arrow[u, "F"] \arrow[r, "h"] & \mathcal{E} \arrow[u, "f^*"'] \end{tikzcd}$$

of pretopos morphisms, in which the canonical functors denoted h preserve and reflect covers, and moreover, any (directed) cover of an object in the image of h is refined by the image under h of a (directed) cover in the site. Combining these facts with Proposition 3.7 gives the result (compare the proofs of (I 5.3) and (III 4.4)). ∎

The next "induction" lemma adapts (I 5.4) to the context of relative propriety:

4.7. Lemma. *Let $F: \mathbb{C} \to \mathbb{D}$ be a morphism between pretopos sites, and suppose F comes equipped with a stable (under "localization" in $\mathbb{C}$) system $\{M(C)\}_{C \in \mathbb{C}}$ of distinguished covering monomorphisms of $\mathbb{D}$ the form $U \rightarrowtail F(C)$ at $C \in \mathbb{C}$ such that*

(i) *The trivial cover $F(C) \xrightarrow{\mathrm{id}} F(C) \in M(C)$.*
(ii) *If $W \rightarrowtail V \rightarrowtail F(C)$, then $V \rightarrowtail F(C) \in M(C)$ whenever $W \rightarrowtail F(C) \in M(C)$.*
(iii) *For any basic S-cover $\{V_i \rightarrowtail V\}$ in $\mathbb{D}$, if $V \rightarrowtail F(C) \in M(C)$ then the family $\{V_i \rightarrowtail F(C)\}$ locally has a member in $M(C)$.*

Then for $C \in \mathbb{C}$, any cover of the form $U \rightarrowtail F(C)$ in $\mathbb{D}$ is locally in $M(C)$, and the morphism $F: \mathbb{C} \to \mathbb{D}$ is relatively proper.

PROOF. By induction on covers, property (iii) extends to any generated S-cover $\{U_i \rightarrowtail U\}$ in $\mathbb{D}$, and then by (I 4.6) and (ii) to any directed cover. Since M contains identity covers of the form $F(C) \rightarrowtail F(C)$, the result clearly follows. ∎

4.8. Corollary. *A morphism $F: \mathbb{C} \to \mathbb{D}$ between pretopos sites is relatively proper iff the conditions of (4.7) are satisfied by the sheaf M on $\mathbb{C}$ of "covering subobjects of 1" in $\mathbb{D}$ (in other words, the sheaf having covers of the form $U \rightarrowtail F(C)$ in $\mathbb{D}$ as sections at $C \in \mathbb{C}$).* ∎

4.9. Lemma. *Let $\varphi: \mathcal{S}' \to \mathcal{S}$ be a morphism of toposes, and suppose $F: \mathbb{C} \to \mathbb{D}$ is a relatively proper morphism between pretopos sites in $\mathcal{S}$. Then the morphism $\varphi^*F: \varphi^*\mathbb{C} \to \varphi^*\mathbb{D}$ is relatively proper in $\mathcal{S}'$. Moreover, if $M: \mathbb{C}^{\mathrm{op}} \to \mathcal{S}$ is the internal sheaf on $\mathbb{C}$ of covering subobjects of 1 in $\mathbb{D}$ as in (4.8), then the corresponding sheaf on $\varphi^*\mathbb{C}$ in $\mathcal{S}'$ is the sheafification of φ^*M.*

PROOF. The conditions of (4.7) satisfied by M are "geometric" and hence also hold for φ^*M. Thus, applying Lemma 4.7 in $\mathcal{S}'$, we deduce that $\varphi^*F: \varphi^*\mathbb{C} \to \varphi^*\mathbb{D}$ is relatively proper, and that the definition of M is preserved "up to sheafification." ∎

Next, we formulate an appropriate version of (III 4.5) for dealing with relative tidiness:

4.10. Lemma. *Let $F: \mathbb{C} \to \mathbb{D}$ be a morphism between pretopos sites, and suppose $\mathbb{D}$ comes equipped with a stable system $\{N(C)\}_{C \in \mathbb{C}}$ of distinguished effective subobjects of $\mathbb{D}$ at $C \in \mathbb{C}$, such that for $\delta: D \to F(C)$:*

(i) *If the arrow δ is a cover, then the trivial effective subobject $D \times_{F(C)} D \xrightarrow{\mathrm{id}} D \times_{F(C)} D$ is locally in $N(C)$.*
(ii) *For monomorphisms $E \rightarrowtail D$ and $W \rightarrowtail V \rightarrowtail E \times_{F(C)} E \rightarrowtail D \times_{F(C)} D$, if $W \rightarrowtail E \times_{F(C)} E \in N(C)$ then $V \rightarrowtail D \times_{F(C)} D \in N(C)$.*
(iii) *$V^{(n)} \rightarrowtail D \times_{F(C)} D \in N(C)$ only if $V \rightarrowtail D \times_{F(C)} D \in N(C)$.*
(iv) *For any basic S-cover $\{V_i \rightarrowtail V\}$ in $\mathbb{D}$, if $V \rightarrowtail D \times_{F(C)} D \in N(C)$ then the family $\{V_i \rightarrowtail D \times_{F(C)} D\}$ locally has a member in $N(C)$.*

Then any effective subobject at $C \in \mathbb{C}$ in $\mathbb{D}$ is locally in $N(C)$, and the morphism $F: \mathbb{C} \to \mathbb{D}$ is relatively tidy.

PROOF. By induction on covers, property (iv) extends to any generated S-cover $\{U_i \rightarrowtail U\}$ in $\mathbb{D}$, and then by (I 4.6) and (ii) to any directed cover. But then, if $\delta: D \to F(C)$ is a cover, any directed cover $\{S_i \rightarrowtail D \times_{F(C)} D\}$ locally has a member in $N(C)$, since locally the identity $D \times_{F(C)} D \to D \times_{F(C)} D \in N(C)$. This shows that $F: \mathbb{C} \to \mathbb{D}$ is relatively tidy.

To prove that all effective subobjects are locally in N, consider any such subobject at $C \in \mathbb{C}$, say $V \rightarrowtail D \times_{F(C)} D$ for $\delta: D \to F(C)$, and let $E \rightarrowtail D$ be a monomorphism such that $E \to F(C)$ and the family $\{(E \times_{F(C)} E) \cap V^{(n)} \rightarrowtail E \times_{F(C)} E \mid n \in \mathbb{N}\}$ are covers in $\mathbb{D}$. Then, by what we have just shown, some $(E \times_{F(C)} E) \cap V^{(n)} \rightarrowtail E \times_{F(C)} E$ is locally in $N(C)$, whence $V \rightarrowtail D \times_{F(C)} D$ is locally in $N(C)$ by conditions (ii) and (iii). ∎

4.11. Corollary. *A relatively proper morphism $F: \mathbb{C} \to \mathbb{D}$ of sites is relatively tidy iff the conditions of* (4.10) *are satisfied by the sheaf N on $\mathbb{C}$ of effective subobjects in $\mathbb{D}$ (that is, the sheaf having all effective subobjects of $\mathbb{D}$ at C as sections at $C \in \mathbb{C}$).* ∎

4.12. Lemma. *Let $\varphi: \mathcal{S}' \to \mathcal{S}$ be a morphism of toposes, and suppose $F: \mathbb{C} \to \mathbb{D}$ is a relatively tidy morphism between pretopos sites in $\mathcal{S}$. Then the morphism $\varphi^* F: \varphi^* \mathbb{C} \to \varphi^* \mathbb{D}$ is relatively tidy in $\mathcal{S}'$. Moreover, if $L: \mathbb{C}^{\mathrm{op}} \to \mathcal{S}$ is the internal sheaf having covers $D \to F(C)$ in $\mathbb{D}$ as sections at $C \in \mathbb{C}$ (using set-notation in $\mathcal{S}$) and N is the sheaf on $\mathbb{C}$ of effective subobjects in $\mathbb{D}$ as in* (4.11)*, then the corresponding sheaves on $\varphi^* \mathbb{C}$ in $\mathcal{S}'$ are the sheafifications of $\varphi^* L$ and $\varphi^* N$ respectively.*

PROOF. We know by Lemma 4.9 that $\varphi^* F: \varphi^* \mathbb{C} \to \varphi^* \mathbb{D}$ is relatively proper and (writing as if $\mathcal{S}'$ were the category of sets) that for $C \in \varphi^* \mathbb{C}$, the image of any cover $\delta: D \to (\varphi^* F)(C)$ is locally in $\varphi^* M(C)$ for M as in (4.8). It follows that $\delta: D \to (\varphi^* F)(C)$ is locally in $\varphi^* L(C)$, which shows that the sheafification of $\varphi^* L$ gives the sheaf of all such δ in $\mathcal{S}'$. This implies in particular that the presheaf $\varphi^* N$ on $\varphi^* \mathbb{C}$ as a system of "distinguished" effective subobjects satisfies condition (i) of Lemma 4.10. Since the remaining conditions (ii), (iii) and (iv) are evidently "geometric," hence inherited by $\varphi^* N$ from N as they stand, we can apply Lemma 4.10 in $\mathcal{S}'$ to conclude that the sheaf on $\varphi^* \mathbb{C}$ of all effective subjects of $\varphi^* \mathbb{D}$ is given by the sheafification of $\varphi^* N$, and that $\varphi^* F: \varphi^* \mathbb{C} \to \varphi^* \mathbb{D}$ is relatively tidy. ∎

Proof of Theorem 4.1. We reduce to the case $\mathcal{S} \equiv \mathbf{Set}$ by arguing constructively.

Let $F: \mathbb{C} \to \mathbb{D}$ be a morphism of pretopos sites inducing $f: \mathcal{F} \to \mathcal{E}$. If f is relatively proper (resp. relatively tidy), then F is relatively proper (resp. relatively tidy) as a morphism of pretopos sites, and so is $\varphi^*F: \varphi^*\mathbb{C} \to \varphi^*\mathbb{D}$ in $\mathcal{S}'$ by Lemma 4.9. It follows that f' is proper (resp. tidy) relative to $\mathcal{S}'$, being induced by φ^*F.

For the BCC, consider any object V of $\mathcal{F}$, represented by a sheaf B on $\mathbb{D}$. The corresponding sheaf for θ^*V made in the topos $\mathcal{S}'$, is given by the sheafification $B' \equiv (\varphi^*B)^{++}$ in $\mathcal{S}'$ of the presheaf φ^*A on $\varphi^*\mathbb{D}$. Thus, writing in the internal language of $\mathcal{S}'$, an element $y \in B'(D)$ for $D \in \varphi^*\mathbb{D}$ is given by a cover $\{D_j \to D\}$ in $\varphi^*\mathbb{D}$ and a "locally compatible" family of elements $y_j \in (\varphi^*B)(D_j)$. Two such families give the same y if they agree on a common refinement.

In similar fashion, the sheaf A on $\mathbb{C}$ representing f_*V, viz. the restriction of B along the functor $F^{\mathrm{op}}: \mathbb{C}^{\mathrm{op}} \to \mathbb{D}^{\mathrm{op}}$, produces a presheaf φ^*A on $\varphi^*\mathbb{C}$, the sheafification A' of which represents ψ^*f_*V. For $C \in \varphi^*\mathbb{C}$, an element $x \in A'(C)$, given by a cover $\{C_i \to C\}$ in $\varphi^*\mathbb{C}$ and a locally compatible family $x_i \in (\varphi^*A)(C_i)$, can be turned into a member $\eta_C(x)$ of $B'((\varphi^*F)(C))$ via the isomorphism $(\varphi^*A)(C_i) \cong (\varphi^*B)(\varphi^*F(C_i))$. This defines a morphism of sheaves $\eta: A' \to B' \circ (\varphi^*F)^{\mathrm{op}}$, which represents the canonical map $\varphi^*f_*V \to f'_*\psi^*V$. We are asked to prove that η is mono if f is relatively proper, and furthermore epi if f is relatively tidy.

Suppose f is proper. To deduce that η is mono, it is clearly enough to show for any $C \in \varphi^*\mathbb{C}$ and sections $y, y' \in (\varphi^*B)((\varphi^*F)(C))$ that if y and y' agree on a cover in $\varphi^*\mathbb{D}$, then this cover can be chosen to lie in the image of φ^*F. Since φ^*B remains a sheaf for P-covers of $\varphi^*\mathbb{D}$, we can assume that the cover $\{D_j \to (\varphi^*F)(C)\}$ on which y and y' agree is directed. By propriety of F and Lemma 4.9, there exists a cover $\{C_i \to C\}$ of $\varphi^*\mathbb{C}$ and for each i some j such that, writing $D_{ji} \rightarrowtail (\varphi^*)(C_i)$ for the pullback of $D_j \rightarrowtail (\varphi^*F)(C)$ along $(\varphi^*F)(C_i) \to (\varphi^*F)(C)$, we have $D_{ji} \rightarrowtail (\varphi^*)(C_i) \in M(C_i)$ while $y{\restriction}D_{ji} = y'{\restriction}D_{ji}$. Here M is the sheaf on $\mathbb{C}$ as defined in (4.8). Since φ^*B still satisfies the sheaf property for the covers in φ^*M, y and y' then agree on the cover $\{(\varphi^*)(C_i) \to (\varphi^*)(C)\}$, which is of the required form.

Suppose now that f is in fact tidy. To show that η is epi, consider $C \in \varphi^*\mathbb{C}$, a cover $\{D_j \to (\varphi^*F)(C)\}$ in $\varphi^*\mathbb{D}$ and a "locally compatible" family of elements $y_j \in (\varphi^*B)(D_j)$. We need to prove that we can replace this family by an equivalent one for which the corresponding covers involved lie in the image of φ^*F. Using propriety of φ^*F, we can reduce to the case of a singleton family $y \in (\varphi^*B)(D)$ for a single covering arrow $\delta: D \to (\varphi^*F)(C)$, and further assume that the restrictions of y along the projections $D \times_{(\varphi^*F)(C)} D \rightrightarrows D$ agree on a directed cover of monomorphisms (here we used again the fact that φ^*B satisfies the sheaf property for P-covers). By tidiness of F and Lemma 4.12, we find a cover $\{C_i \to C\}$ in $\mathbb{C}$ such that for each i, writing $\delta_i: D_i \to (\varphi^*F)(C_i)$ for the pullback of δ across $(\varphi^*F)(C_i) \to (\varphi^*F)(C)$ and y_i for $y{\restriction}D_i$, δ_i lies in $(\varphi^*L)(C_i)$ while the restriction of y_i along the projections $D_i \times_{(\varphi^*F)(C_i)} D_i \rightrightarrows D_i$ agree on some effective subobject $V_i \to D_i \times_{(\varphi^*F)(C_i)} D_i \in \varphi^*N(C_i)$. Here L and N are the sheaves on $\mathbb{C}$ defined in (4.12) and (4.11) respectively. But then, in terms of a "geometric" property of φ^*B inherited from B as in (III 4.8), we can find for each i a unique $z_i \in B((\varphi^*F)(C_i))$ such that $z_i{\restriction}D_i = y_i$. This new family $\{z_i\}$ is easily seen to be compatible, hence is a family equivalent to y in the image of η_C.

This completes the proof of the theorem. ∎

Proof of Theorem 4.2. We can assume that $\mathcal{S} \equiv \mathbf{Set}$ and argue constructively. Let $\{F_i\colon \mathbb{C}_i \to \mathbb{D}_i\}$ be a diagram of pretopos site morphisms inducing $\{f_i\colon \mathcal{F}_i \to \mathcal{E}_i\}$, and let $F\colon \mathbb{C} \to \mathbb{D}$ be a morphism inducing the limit $f\colon \mathcal{F} \to \mathcal{E}$ as in (I 4.9). Denote the canonical functors associated with an arrow $\alpha\colon j \to i \in I$ as indicated in the commutative diagram

$$
\begin{tikzcd}
 & \mathbb{D} \arrow[dd, "k"] & & \mathbb{D}_j \arrow[ll, "Q_j"'] \arrow[dd, "k_j"] & & \mathbb{D}_i \arrow[ll, "U_\alpha"'] \arrow[dd, "k_i"] \\
\mathbb{C} \arrow[ur, "F"] \arrow[dd, "h"'] & & \mathbb{C}_j \arrow[ll, "P_j"'] \arrow[ur, "F_j"] \arrow[dd, "h_j"] & & \mathbb{C}_i \arrow[ll, "T_\alpha"'] \arrow[ur, "F_j"] \arrow[dd, "h_i"] & \\
 & \mathcal{F} & & \mathcal{F}_j \arrow[ll, "{q_j}^*"'] & & \mathcal{F}_i \arrow[ll, "{u_\alpha}^*"'] \\
\mathcal{E} \arrow[ur, "f^*"] & & \mathcal{E}_j \arrow[ll, "{p_j}^*"'] \arrow[ur, "{f_j}^*"] & & \mathcal{E}_i \arrow[ll, "{t_\alpha}^*"'] \arrow[ur, "{f_i}^*"] &
\end{tikzcd}
$$

For each $i \in I$, let M_i and N_i be the sheaves on $\mathbb{C}_i$ which has for sections at $C_i \in \mathbb{C}_i$, respectively, covers of the form $U_i \rightarrowtail F_i(C_i)$ in $\mathbb{D}_i$ (as in Corollary 4.8) and effective subobjects at C_i in $\mathbb{D}_i$ (as in Corollary 4.11). Let M and N be the sheaves on $\mathbb{C}$ of corresponding data for F which (up to isomorphism) lifts to some F_i. Then it is straightforward to check that M inherits the conditions of Lemma 4.7 from the M_i if each f_i is relatively proper, and that N inherits the conditions of Lemma 4.10 from the N_i if, furthermore, each f_i is relatively tidy. It therefore follows by applying these lemmas that f is relatively proper (resp. relatively tidy) if the f_i are.

To show the second part for given $i \in I$, consider first any $E \in \mathcal{E}_i$, $C \in \mathbb{C}$ and a "C-element" $x\colon h(C) \to p_i^* E$. A "lifting" of x is given by the data (α, C', x') where $\alpha\colon j \to i \in I$, $C' \in \mathbb{C}_j$ such that $P_j(C') = C$ and x' is a C'-element $h_j(C') \to {t_\alpha}^* E$ in $\mathcal{E}_j$ such that $x = {p_j}^* x'$. Two such liftings (α_0, C'_0, x'_0) and (α_1, C'_1, x'_1) are "eventually equal" if there exists a commutative diagram

$$
\begin{tikzcd}
k \arrow[r, "\beta_1"] \arrow[d, "\beta_0"'] & j_1 \arrow[d, "\alpha_1"] \\
j_0 \arrow[r, "\alpha_0"] & i
\end{tikzcd}
\tag{3}
$$

in I such that $P_{\alpha_0}(C'_0) = P_{\alpha_1}(C'_1)$ in $\mathbb{C}_k$ and ${t_{\alpha_0}}^* x'_0 = {t_{\alpha_1}}^* x'_1$ in $\mathcal{E}_k$.

Any generating element $x\colon h(C) \to p_i^* E$ as above can be lifted "locally and locally compatibly," that is to say, on a cover of C and compatibly so (in the sense of "eventual equality") up to covers. To see this, let $A\colon \mathbb{C}_i^{\mathrm{op}} \to \mathbf{Set}$ be the canonical sheaf representing E, $A(C') \equiv$ the family of C'-elements $h_i(C') \to E$ for $C' \in \mathbb{C}_i$. The corresponding sheaf for ${p_i}^* E$ is the sheafification of the left Kan-extension $(P_i * A)\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ of A along $P_i\colon \mathbb{C}_i \to \mathbb{C}$. By directedness of I and the lifting property of finite commutative diagrams in $\mathbb{C}$, there exists $\alpha_0\colon j_0 \to i \in I$ and $C_0 \in \mathbb{C}_{j_0}$ such that $P_{j_0}(C_0) \cong C$. Rearranging the explicit construction then gives $(P_i * A)(C) \cong \varinjlim_\beta (T_\beta * A)(C_0)$ (where $\beta\colon j \to j_0$ runs over the category I/j_0), thus proving our claim.

Consider now any $V \in \mathcal{F}_i$. Any commutative triangle of the form

$$\begin{tikzcd} j' \arrow[rr, "\beta"] \arrow[dr, "\alpha'"'] & & j \arrow[dl, "\alpha"] \\ & i & \end{tikzcd} \tag{4}$$

in I induces a commutative triangle of canonical maps

$$\begin{tikzcd} P_{j'}{}^* f_{j'}{}_* u_{\alpha'}{}^* \arrow[dr, "s_{\alpha'}"'] & & P_j{}^* f_j{}_* u_\alpha{}^* \arrow[ll, "c_\beta"'] \arrow[dl, "s_\alpha"] \\ & f_* q_i{}^* & \end{tikzcd} \tag{5}$$

in $\mathcal{E}$. When the maps $f_i: \mathcal{F}_i \to \mathcal{E}_i$ are relatively proper, we need to show for any arrow $\alpha: j \to i$ in I and pair of generating elements $x_0, x_1: h(C) \to p_i{}^* f_j{}_* u_\alpha{}^* V$ for which $s_\alpha \circ x_0 = s_\alpha \circ x_1$, that there exists, locally in C, a commutative triangle (4) such that $c_\beta \circ x_0 = c_\beta \circ x_1$. When the maps $f_i: \mathcal{F}_i \to \mathcal{E}_i$ are relatively tidy, we need to show, furthermore, that any generating element $y: h(C) \to f_* q_i{}^* V$ is locally of the form $s_\alpha \circ x$ for some $\alpha: j \to i$ and $x: h(C) \to p_i{}^* f_j{}_* u_\alpha{}^* V$.

For the first, it is clearly enough to treat the case where the C-elements x_0, x_1 have liftings to $\mathcal{E}_j$, say $x_0', x_1': h_j(C') \to f_j{}_* u_\alpha{}^* V$ respectively, where $P_j(C') = C$. Let $y_0', y_1': f_j{}^* h_j(C') \to u_\alpha{}^* V$ be the respective adjoints of these under $f_j{}^* \dashv f_j{}_*$. Then $q_j{}^* y_0', q_j{}^* y_1'$ are the respective adjoints of $s_\alpha \circ x_0, s_\alpha \circ x_1$, so that $q_j{}^* y_0' = q_j{}^* y_1' = y$ say. It follows that there is a cover of $F(C)$ in $\mathbb{D}$ over which the two liftings y_0', y_1' of y are eventually equal, and by relative propriety (via Lemma 4.7), we may assume that locally in C, this cover consists of a single monomorphism (an element of the sheaf M) which lifts. It follows that, after passing to a cover of C in $\mathbb{C}$, we can find a commutative triangle (4) such that $u_\beta{}^* y_0 = u_\beta{}^* y_1$, or equivalently, such that $c_\beta \circ x_0 = c_\beta \circ x_1$ as required.

For the second, let $z: f^* h(C) \to q_i{}^* V$ be the adjoint of $y: h(C) \to f_* q_i{}^* V$. If z has a lifting of the form $z': k_j(F_j(C')) \cong f_j{}^* h_j(C') \to u_\alpha{}^* V$ to $\mathcal{F}_j$, then we would be done, for then the adjoint $x': h_j(C') \to f_j{}_* u_\alpha{}^* V$ would produce the required x by letting $x = p_j{}^* x'$. But we know that z lifts over a cover in $\mathbb{D}$; we therefore need to modify this to a cover in the image of $F: \mathbb{C} \to \mathbb{D}$, that is, we need to lift z locally in C. Using relative propriety, we may reduce to the case where there exists a singleton cover $\delta: D \to F(C)$ which lifts to $\mathbb{D}_j$, say $\delta': D' \to F_j(C')$ such that $Q_j(\delta') = \delta$, and a D'-element $s': k_j(D') \to u_\alpha{}^* V$ such that $z \upharpoonright D = q_j{}^* s'$. Then, by passing to a (further) cover in $\mathbb{C}$ if necessary, we may use relative tidiness (via Lemma 4.10) to reduce to having an effective subobject $U \rightarrowtail D \times_{F(C)} D$ of $\mathbb{D}$ at C in $N(C)$, one which lifts all the way to $\mathbb{D}_j$ as $U' \rightarrowtail D' \times_{F(C')} D'$, and for which the restrictions of s' along the projections $U' \rightrightarrows D'$ agree. This data is easily seen to produce $z': f_j{}^* h_j(C') \to u_\alpha{}^* V$ satisfying $q_j{}^* z' = z$, as required.

This finishes the proof of the theorem. ∎

§5 The main theorem

We are now ready to deduce our main result, Theorem 5.1 below, as a straighforward consequence of the formal properties of relatively tidy maps derived in the previous sections.

5.1. Theorem. *Let*

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E}
\end{tikzcd} \tag{1}$$

be a lax pullback of toposes over a base topos $\mathcal{S}$. If the morphism f is tidy relative to $\mathcal{S}$, then d_0 is tidy. Moreover, in this case the Beck-Chevalley condition holds, i.e. the canonical natural transformation

$$g^* f_* \to {d_0}_* {d_1}^*$$

induced by τ is an isomorphism.

Proof. Recall (2.4) that the lax pullback can be built up in stages, as indicated by the diagram

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d"] \arrow[dd, "d_0"'] & (\mathcal{E} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(2)"] & \mathcal{F} \arrow[d, "f"] \\
(3) & P(\mathcal{E}) \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "(1)"] & \mathcal{E} \arrow[d, "\mathrm{id}"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E} \arrow[r, "\mathrm{id}"] & \mathcal{E} .
\end{tikzcd}$$

By Proposition 1.3 the map ∂_0 in the "universal" lax pullback square (1) is tidy (that is, relative to $\mathcal{E}$). Also, the Beck-Chevalley condition holds for this square, that is, the 2-cell ${\partial_0}_* {\partial_1}^* \Rightarrow \mathrm{id}$ is an isomorphism.

We shall prove separately, in Lemma 5.2 below, that the map a is also tidy relative to $\mathcal{E}$, where $P(\mathcal{E})$ is viewed as an $\mathcal{E}$-topos via $\partial_0 \colon P(\mathcal{E}) \to \mathcal{E}$, and that the square (2) also saitisfies the Beck-Chevalley condition, i.e. $a_* b^* \cong {\partial_1}^* f_*$. It then follows by Proposition 3.6 that the composition $\partial_0 \circ a$ is tidy. Moreover, the Beck-Chevalley condition for the squares (1) and (2) compose to give the Beck-Chevalley condition $(\partial_0 a)_* b^* \cong f_*$.

But now we can apply pullback-stability of tidy maps (III 4.6), and conclude that c is tidy since $\partial_0 a$ is, while the left-hand square (3) again satisfies the Beck-Chevalley condition $g^*(\partial_0 a)_* \cong c_* d^*$. By composing this Beck-Chevalley condition with the one $(\delta_0 a)_* b^* \cong f_*$ already obtained, we find that the outer square satisfies the Beck-Chevalley condition $g^* f_* \cong c_*(bd)^*$, which is the one stated in the theorem.

Thus, the following lemma completes the proof Theorem 5.1.

5.2. Lemma. *Suppose $f\colon \mathcal{F} \to \mathcal{E}$ is tidy relative to $\mathcal{S}$. Then in the pullback square*

$$\begin{tikzcd}
(\mathcal{E} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "b"] \arrow[d, "a"'] & \mathcal{F} \arrow[d, "f"] \\
P(\mathcal{E}) \arrow[r, "\partial_1"] & \mathcal{E}
\end{tikzcd}$$

the map a is tidy relative to $\mathcal{E}$ via $\partial_0\colon P(\mathcal{E}) \to \mathcal{E}$ and the Beck-Chevalley condition $a_*b^* \cong {\partial_1}^* f_*$ holds.

PROOF. We decompose the above square into two smaller squares, the left-hand one of which can be viewed as a square over the topos $\mathcal{E}$ as base, as indicated below

$$\begin{tikzcd}
(\mathcal{E} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "e"] \arrow[d, "a"'] & \mathcal{E} \times \mathcal{F} \arrow[r, "\pi_2"] \arrow[d, "\mathrm{id} \times f"] & \mathcal{F} \arrow[d, "f"] \\
P(\mathcal{E}) \arrow[r, "(\partial_0, \partial_1)"] \arrow[dr, "\partial_0"'] & \mathcal{E} \times \mathcal{E} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathcal{E} \\
& \mathcal{E}\,, &
\end{tikzcd}$$

where $e = (\partial_0 a, b)$.

First, consider the right-hand square: since f is tidy relative to $\mathcal{S}$, stability under change of base (4.1) yields that $\mathrm{id} \times f$ is tidy relative to $\mathcal{E}$, together with the Beck-Chevalley condition $(\mathrm{id} \times f)_* {\pi_2}^* \cong {\pi_2}^* f_*$ for this square.

Next, since $\partial_0\colon P(\mathcal{E}) \to \mathcal{E}$ is the localization of $(\mathcal{E} \times \mathcal{E} \xrightarrow{\pi_1} \mathcal{E})$ at the "generic point" $\Delta\colon \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ (see §1), the "localization lemma" (4.4) yields that a is tidy over $\mathcal{E}$ together with the Beck-Chevalley condition $(\partial_0, \partial_1)^*(\mathrm{id} \times f)_* \cong a_* e^*$. Composing the Beck-Chevalley conditions for these two squares yields the condition ${\partial_1}^* f_* \cong a_* b^*$, and proves the lemma. ∎

§6 Applications to lax descent

For a morphism $f\colon \mathcal{F} \to \mathcal{E}$ over a base topos $\mathcal{S}$, one obtains by iterated lax pullbacks a diagram

$$\begin{tikzcd}
\mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & \mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathcal{F} \arrow[r, "f"] & \mathcal{E}\,. \\
& \mathcal{F}\,, \arrow[u, "\delta"] & &
\end{tikzcd}$$

and one can extend the definitions concerning descent for sheaves (specializing those given for locales in (I §7)) to the "lax" case in the evident way. Thus, *lax descent data* on an object F of $\mathcal{F}$ is a morphism $\theta: d_0{}^*F \to d_1{}^*F$ satisfying the evident cocycle condition in $\mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F}$ and unit condition in $\mathcal{F}$. The category of such pairs (F, θ) is denoted $\mathrm{LDes}(f)$. For any object $E \in \mathcal{E}$, the natural transformation $\tau: d_0{}^* f^* \to d_1{}^* f^*$ that comes with the lax pullback $\mathcal{F} \Rightarrow_{\mathcal{E}} \mathcal{F}$ provides f^*E with lax descent data τ_E. This construction defines a functor

$$T: \mathcal{E} \to \mathrm{LDes}(f), \qquad E \mapsto (E, \tau_E).$$

The morphism $f: \mathcal{F} \to \mathcal{E}$ is said to be of effective lax descent (for sheaves) if this functor T is an equivalence of categories.

The notion of effective lax descent is a weakening of that of effective descent: any morphism which is of effective descent (for sheaves) is also of effective lax descent. Some important classes of morphisms are only of effective lax descent, for example *essential* surjections of toposes [].

The following result is a direct application of Theorem 5.1.

6.1. Theorem. *Any relatively tidy surjection $f: \mathcal{F} \to \mathcal{E}$ is of effective lax descent for sheaves.*

Proof. To begin with, let us recall that by a well-known application of the "tripleability theorem" (see [,]), the codomain $\mathcal{E}$ of any surjection $f: \mathcal{F} \to \mathcal{E}$ betrween toposes is equivalent to the category of coalgebras for the comonad $C_f = f^* f_*$ on $\mathcal{F}$:

$$\mathcal{E} \cong \mathrm{Coalg}(C_f). \tag{1}$$

If the lax pullback square

$$\begin{tikzcd}
(\mathcal{G} \Rightarrow_{\mathcal{E}} \mathcal{F}) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\tau}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{G} \arrow[r, "g"] & \mathcal{E}
\end{tikzcd}$$

satisfies the Beck-Chevalley condition $f^* f_* \cong d_{0*} d_1{}^*$, arrows $c: E \to f^* f_* E$ are in bijective correspondence to arrows $E \to d_{0*} d_1{}^* E$, or by adjunction, to arrows $\theta: d_0{}^* E \to d_1{}^* E$. A well-known theorem [] asserts that, in this way, coalgebra structures c correspond precisely to lax descent data θ, so that we obtain an equivalence of categories

$$\mathrm{Coalg}(C_f) \cong \mathrm{LDes}(f). \tag{2}$$

The equivalence $T: \mathcal{E} \to \mathrm{LDes}(f)$ of the theorem is obtained by composing the two equivalences (1) and (2). ∎

By Example 3.2, we obtain as a special case:

6.2. Corollary. *Any coherent surjection between coherent toposes is of effective lax descent for sheaves.* ∎

This corollary implies Zawadowski's theorem [] for pretoposes. Indeed, if $f: \mathcal{F} \to \mathcal{E}$ is a surjection between coherent toposes, an object E of $\mathcal{E}$ is coherent iff f^*E is coherent. Thus, the equivalence $\mathcal{E} \cong \mathrm{LDes}(f)$ restricts in this case to an equivalence between the corresponding pretoposes of coherent objects. We should point out, however, that for the special case of Corollary 6.2 the results in this chapter on relatively tidy maps are either not necessary or can be proved in a much easier fashion for coherent toposes. In this way, one obtains a direct proof of Corollary 6.2 and hence of Zawadowski's result, which is more straightforward than the proof of our general descent theorem (6.1), and much easier than Zawadowski's original treatment, which was based on Makkai's Stone duality [,,]. The interested reader is referred to [].

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