Lectures on Young Measure Theory and its Applications in Economics (final version) *

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1 Introduction

The first four sections of these notes form a quick, incisive introduction to the subject of Young measure theory. The term *Young measures* refers to transition probabilities that are studied in connection with a certain weak topology (i.e., the *narrow* topology for Young measures). This name honors L.C. Young, whose seminal work on generalized solutions in the calculus of variations in 1937 [98] formed the starting point of such considerations.

Our presentation involves very little functional analysis, and is largely based on a transfer of the classical theory of narrow convergence from the domain of probabilities (section 2) to the more general domain of transition probabilities (section 4) by means of K-*convergence* and an associated key Prohorov-type extension of Komlós' theorem (Theorem 3.7). Such an extension of Komlós' theorem applies, much more generally than displayed here, to certain classes of abstract-valued scalarly integrable functions [18, 19, 38]. However, in the Young measure context it is particularly effective to transfer narrow convergence properties. This is because *tightness*, a crucial condition for Theorem 3.7, is, under mild restrictions, an *automatic* feature of narrow convergence of *sequences* of Young measures [25]. The useful portmanteau and product convergence theorems for classical narrow convergence, as well as Prohorov's theorem (an important device for relative narrow compactness) and certain limiting support properties are thus made available for Young measures.

These results of section 4 form the point of departure for the second part of the notes, where lower closure (section 5), and variational inequalities and equilibria (section 6) are studied in connection with some existence questions in economics (viz., optimal growth, optimal consumption, Cournot-Nash equilibrium distributions and Nash equilibria in continuum games and games with incomplete information). To keep these notes within a reasonable size, the choice has been made to discuss those applications at a great level of generality, and with little regard for the economical context. However, adequate references are suggested to fill this gap.

Other surveys of Young measure theory include the account given in J. Warga's textbook [97] (largely control-theoretical and mostly limited to a compact image space, but going well beyond existence and lower closure issues), the study by H. Berliocchi and J.M. Lasry [42] (offering a locally compact image space and a Scorza-Dragoni-type connection with classical narrow convergence, but in many respects a very innovative study), M. Valadier's survey in [94] (presenting much of the material treated in sections 2 to 4 via a more functional-analytic approach and with rather different applications) and the present author's lecture notes [25], which cover more ground than the present paper, but do not address economical applications at the level of generality presented here. Let us also [67] for an apparently different approach to sequential narrow convergence on product spaces that can nevertheless be reduced to the present one [33], [61, p. 2]. For recent important applications in nonlinear analysis (that started with [91]) we refer to [83].

*Nine lectures given at the School on Measure Theory and Real Analysis (GNAFA, CNR) in Grado, Italy (15-26 September, 1997).

2 Narrow convergence of probability measures

This section recapitulates some results on narrow convergence of probability measures on a topological space; cf. [4, 43, 44, 55, 75, 82]. We discuss these results in two settings: (i) a metrizable one, for which the material presented is rather standard and (ii) a nonmetrizable one, which includes the situation where the topological space is completely regular and Suslin. As a rule, we extend from (i) to (ii) via tightness.

Let S be a completely regular topological space, whose topology we indicate by τ. Let $\mathcal{B}(S,\tau)$ be the Borel σ-algebra on (S,τ) and let $\mathcal{C}_b(S,\tau)$ be the set of all bounded τ-continuous functions on S. Throughout this paper we work with the following hypothesis:

Hypothesis 2.1 *There exist a separable metric space P and a continuous mapping $\sigma : P \to S$ such that $S = \sigma(P)$.*

Clearly, this hypothesis implies that the space (S,τ) is separable.

Proposition 2.2 *There exists an (at most) countable collection (c_i) in $\mathcal{C}_b(S,\tau)$ that separates the points of S (i.e., $x = x'$ if and only if $c_i(x) = c_i(x')$ for all $i \in \mathbf{N}$). Consequently, there exists a* weak *metric ρ on S whose topology τ_ρ is such that $\tau_\rho \subset \tau$.*

Proof. Since $P \times P$ is second countable, it has the the Lindelöf property. That is, every open subset of $P \times P$ has the countable subcover property. But then $S \times S$, being the continuous surjective image of $P \times P$, also has the Lindelöf property. In particular, the complement C of the diagonal in $S \times S$ has the countable subcover property. Now C is covered by the collection of all open sets $\{(x,x') \in S \times S : c(x) \neq c(x')\}$, $c \in \mathcal{C}_b(S)$. Hence, C is already covered by a countable subcollection, and this evidently corresponds to the fact that there is a countable subset (c_i) of $\mathcal{C}_b(S,\tau)$ separating the points of S. Setting $\rho(x,x') := \sum_{i=1}^{\infty} 2^{-i}(\sup_S |c_i|)^{-1}|c_i(x) - c_i(x')|$ then produces a metric on S, and the inclusion $\tau_\rho \subset \tau$ is trivial. QED

While we accept that the topologies τ and τ_ρ may be different, the associated Borel σ-algebras are required to be identical:

Hypothesis 2.3 *The metric ρ in Proposition 2.2 is such that*

$$\mathcal{B}(S,\tau_\rho) = \mathcal{B}(S,\tau) =: \mathcal{B}(S).$$

Two different sufficient conditions for Hypothesis 2.3 to hold are as follows:

Remark 2.4 *(i) If (S,τ) is a separable metric space, then it meets Hypotheses 2.1 and 2.3 trivially (let ρ be the postulated metric on S; then $\tau = \tau_\rho$).*
(ii) Let (S,τ) be completely regular and Suslin (i.e., a Hausdorff space that is the surjective image of a complete, separable metric space under a continuous mapping [55, 89]). Then Hypothesis 2.1 evidently holds, and Hypothesis 2.3 holds by a well-known property of Suslin spaces [89, Corollary 2, p. 101].

Many useful spaces, e.g., Euclidean spaces, compact metric spaces, separable Banach spaces with their strong or weak topology are completely regular and Suslin (observe that infinite-dimensional separable Banach spaces are *not* metrizable for their weak topology – this example explains why we are not just interested in the metrizable case).

Let $\mathcal{P}(S)$ be the set of all probability measures on $(S,\mathcal{B}(S))$. Let $\mathcal{C}_b(S,\rho)$ be the set of all τ_ρ-continuous bounded functions from S into $\mathbf{R}$.

Definition 2.5 A sequence (ν_n) in $\mathcal{P}(S)$ converges *narrowly* with respect to the topology τ_ρ to $\nu_0 \in \mathcal{P}(S)$ (notation: $\nu_n \overset{\rho}{\Rightarrow} \nu_0$) if $\lim_n \int_S c d\nu_n = \int_S c d\nu_0$ for every c in $\mathcal{C}_b(S,\rho)$.

The corresponding notion of τ-narrow convergence, denoted by " $\stackrel{\tau}{\Rightarrow}$ ", is defined by replacing $\mathcal{C}_b(S,\rho)$ in the above definition by $\mathcal{C}_b(S,\tau)$. Clearly, τ-narrow convergence implies τ_ρ-narrow convergence by Proposition 2.2, but in some interesting cases the two convergence modes will actually coincide.

A useful tool is the following so-called *portmanteau* theorem for τ_ρ-narrow convergence. Here $\mathcal{C}_u(S,\rho)$ stands for the set of all uniformly ρ-continuous and bounded functions from S into $\mathbf{R}$.

Theorem 2.6 *(i) Let (ν_n) and ν_0 be in $\mathcal{P}(S)$. The following are equivalent:*

(a) $\nu_n \stackrel{\rho}{\Rightarrow} \nu_0$.

(b) $\lim_n \int_S c d\nu_n = \int_S c d\nu_0$ for every $c \in \mathcal{C}_u(S,\rho)$.

(c) $\liminf_n \int_S q d\nu_n \geq \int_S q d\nu_0$ for every ρ-lower semicontinuous function $q : S \to (-\infty,+\infty]$ which is bounded from below.

(ii) Moreover, if (ν_n) is τ-tight, then the above are also equivalent to the following:

(d) $\nu_n \stackrel{\tau}{\Rightarrow} \nu_0$.

(e) $\liminf_n \int_S q d\nu_n \geq \int_S q d\nu_0$ for every sequentially τ-lower semicontinuous function $q : S \to (-\infty,+\infty]$ which is bounded from below.

Recall that τ-tightness in the above theorem can be defined in two equivalent forms:

Definition 2.7 A sequence (ν_n) in $\mathcal{P}(S)$ is *τ-tight* if either one of the following two *equivalent* statements is true:

(a) There exists a sequentially τ-inf-compact function $h : S \to [0,+\infty]$ (i.e., a function h for which all lower level sets $\{x \in S : h(x) \leq \beta\}$, $\beta \in \mathbf{R}$, are sequentially τ-compact) such that $\sup_n \int_S h d\nu_n < +\infty$.

(b) For every $\epsilon > 0$ there exists a sequentially τ-compact set $K_\epsilon \subset S$ such that $\sup_n \nu_n(S \backslash K_\epsilon) \leq \epsilon$.

Of course, the definition of τ_ρ-tightness goes likewise, simply by replacing the topology τ by τ_ρ, and clearly τ-tightness of a sequence implies its τ_ρ-tightness (notice in (a) that h is *a fortiori* ρ-inf-compact). Returning to τ-tightness itself, note further that h is also ρ-lower semicontinuous, whence $\mathcal{B}(S)$-measurable. Similarly, it follows that the K_ϵ in (b) belong to $\mathcal{B}(S)$. The equivalence of (a) and (b) in the above definition is a simple exercise [46, Exercise 10, p. 109] (see also the proof following Definition 3.3). To identify sets in S that are sequentially τ-compact, it is useful to observe that any τ-compact set $K \subset S$ is automatically sequentially τ-compact (use Proposition 2.2 and the fact that τ coincides with the metrizable topology τ_ρ on K).

PROOF OF THEOREM 2.6: Part (i) is classical and can be found in e.g. [4, 4.5.1], [43, Proposition 7.21] or [44, Theorem 2.1]. As for part (ii), we note the following:

$(d) \Rightarrow (a)$: This is *a fortiori*.

$(e) \Rightarrow (d)$: Evident by applying (e) to both c and $-c$.

$(d) \Rightarrow (e)$: Let h be as in Definition 2.7. For q as specified we notice that for any $\epsilon > 0$ the function $q_\epsilon := q + \epsilon h$ is sequentially τ-inf-compact, whence τ_ρ-inf-compact and thus also τ_ρ-lower semicontinuous. We may therefore apply (c) to q_ϵ, which gives $\liminf_n \int_S q d\nu_n + \epsilon \sup_n \int_S h d\nu_n \geq \int q d\nu_0$. Letting ϵ go to zero finishes the proof. QED

Remark 2.8 *The above proof also justifies the existence of the quasi-integrals $\int_S q d\nu_n$ in (e). This goes as follows: in the above notation, $\tilde{q} := \sup_k q_{1/k}$ is $\mathcal{B}(S)$-measurable. Clearly, $\tilde{q}$ coincides with q on the set $\{h < +\infty\}$ and it is equal to $+\infty$ on $\{h = +\infty\}$. It remains to notice that Definition 2.7 forces the set $\{h = +\infty\}$ to have ν_n-measure zero for each n.*

It turns out that tightness is a sufficient – and in a number of cases also necessary – condition for relative compactness in the narrow topology. Just as in Definition 2.7 we only state the sequential version of this result, even though there is also a fully topological analogue.

Theorem 2.9 (Prohorov) *(i) Let (ν_n) in $\mathcal{P}(S)$ be τ_ρ-tight. Then there exist a subsequence $(\nu_{n'})$ of (ν_n) and $\nu_* \in \mathcal{P}(S)$ such that $\nu_{n'} \stackrel{\rho}{\Rightarrow} \nu_*$.*

(ii) Moreover, if (ν_n) is τ-tight, then in fact $\nu_{n'} \stackrel{\tau}{\Rightarrow} \nu_*$ can be achieved in (i).

PROOF. Part (i) is Prohorov's classical result in sequential format [44, Theorem 6.1], [75, Theorem 12.3.A]. Part (ii) follows by Theorem 2.6(ii)). QED

As a necessity complement to the above result, we remark that tightness is known to be a necessary condition for relative sequential narrow compactness when S is complete separable metric or locally compact [44, Theorem 6.2], [89, Theorem 4, p. 381]. See also Theorem 2.19 below.

As a rule, in what follows the parts (i) of the above results, and also of those that still follow in this section, are essential for the transfer process. What is done in the parts (ii), all of which exploit τ-tightness to reduce the situation to that of the corresponding part (i), could also have been added *ad hoc*. However, it is hoped that the systematic inclusion of such parts (ii) underlines the harmony of the present approach.

Next, we study narrow convergence of product measures. The essence of the results that we require is already available if we just consider probability measures on the product $S \times \hat{\mathbf{N}}$. Here $\hat{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$ is the usual Alexandrov-compactification of the natural numbers. This is a compact metrizable space, which obviously satisfies Hypotheses 2.1, 2.3. From now on, let $\hat{\rho}$ be a fixed metric on $\hat{\mathbf{N}}$, let $\tilde{\rho}$ be any compatible product metric on $S \times \hat{N}$, and denote the topology $\tau \times \tau_{\hat{\rho}}$ by $\tilde{\tau}$. We denote $\tilde{\rho}$-narrow and $\tilde{\tau}$-narrow convergence in $\mathcal{P}(S \times \hat{\mathbf{N}})$ by " $\stackrel{\tilde{\rho}}{\Rightarrow}$ " and " $\stackrel{\tilde{\tau}}{\Rightarrow}$ " respectively. For $n \in \hat{\mathbf{N}}$, let $\epsilon_n \in \mathcal{P}(\hat{\mathbf{N}})$ stand for the Dirac measure concentrated at the point n.

Proposition 2.10 (i) *Let (ν_n) and ν_0 be in $\mathcal{P}(S)$. The following are equivalent:*

(a) $\frac{1}{N}\sum_{n=1}^N \nu_n \stackrel{\rho}{\Rightarrow} \nu_0$.

(b) $\frac{1}{N}\sum_{n=1}^N (\nu_n \times \epsilon_n) \stackrel{\tilde{\rho}}{\Rightarrow} \nu_0 \times \epsilon_\infty$.

(ii) *Moreover, if $(\frac{1}{N}\sum_{n=1}^N \nu_n)$ is τ-tight, then the above are also equivalent to the following:*

(c) $\frac{1}{N}\sum_{n=1}^N \nu_n \stackrel{\tau}{\Rightarrow} \nu_0$.

(d) $\frac{1}{N}\sum_{n=1}^N (\nu_n \times \epsilon_n) \stackrel{\tilde{\tau}}{\Rightarrow} \nu_0 \times \epsilon_\infty$.

PROOF. $(a) \Rightarrow (b)$: Let $c \in \mathcal{C}_u(S \times \hat{\mathbf{N}}, \tilde{\rho})$ and $\eta > 0$ be arbitrary. There exists $p \in \mathbf{N}$ such that $|c(x,n) - c(x,\infty)| < \eta/2$ for all $n > p$, uniformly in $x \in S$. Hence, the triangle inequality gives

$$\int_S |\frac{1}{N}\sum_{n=1}^N c(x,n) - c(x,\infty)|\nu_n(dx) \leq \frac{2p}{N}\sup_S |c| + \frac{(N-p)\eta}{2N},$$

where the right hand side is less than η for N sufficiently large. So now (b) follows easily by invoking Theorem 2.6(i).

$(b) \Rightarrow (a)$: Trivial, since any function c in $\mathcal{C}_b(S,\rho)$ can be identified with the function $\tilde{c}$ in $\mathcal{C}_b(S \times \hat{\mathbf{N}})$ given by $\tilde{c}(x,n) := c(x)$.

$(a) \Leftrightarrow (c)$: By Theorem 2.6(ii).

$(b) \Leftrightarrow (d)$: Also by Theorem 2.6(ii), since the sequence (π_N) is trivially $\tilde{\tau}$-tight. Here $\pi_N := \frac{1}{N}\sum_{n=1}^N (\nu_n \times \epsilon_n)$. Indeed, by hypothesis there exists $h : S \to [0,+\infty]$, sequentially τ-inf-compact, such that $s := \sup_N \frac{1}{N}\sum_{n=1}^N \int_S h d\nu_n < +\infty$. Then $\tilde{h}(x,n) := h(x)$ defines a function $\tilde{h} : S \times \hat{\mathbf{N}} \to [0,+\infty]$ that is sequentially $\tilde{\tau}$-inf-compact (by compactness of the space $\hat{\mathbf{N}}$), with $\sup_N \int \tilde{h} d\pi_N = s < +\infty$. QED

Corollary 2.11 (i) *Let (ν_n) and ν_0 be in $\mathcal{P}(S)$. The following are equivalent:*

(a) $\nu_n \stackrel{\rho}{\Rightarrow} \nu_0$

(b) $\nu_n \times \epsilon_n \stackrel{\tilde{\rho}}{\Rightarrow} \nu_0 \times \epsilon_\infty$.

(ii) *Moreover, if (ν_n) is τ-tight, then the above are also equivalent to the following:*

(c) $\nu_n \times \epsilon_n \stackrel{\tilde{\tau}}{\Rightarrow} \nu_0 \times \epsilon_\infty$.

PROOF. $(a) \Rightarrow (b)$: Suppose (b) were not true. Then there would exist $\epsilon > 0$, $\bar{c} \in \mathcal{C}_b(S \times \hat{\mathbf{N}}, \tilde{\rho})$ and a subsequence $(\nu_{n'})$ of (ν_n) such that $\int_S \bar{c} d(\nu_0 \times \epsilon_\infty) + \epsilon < \int_S \bar{c} d(\nu_{n'} \times \epsilon_{n'})$ for all n'. Set $\pi_N := \frac{1}{N}\sum_{n'=1}^N (\nu_{n'} \times \epsilon_{n'})$. Then evidently $\int_S \bar{c} d(\nu_0 \times \epsilon_\infty) + \epsilon < \int_S \bar{c} d\pi_N$ for all N. But $\nu_{n'} \stackrel{\rho}{\Rightarrow} \nu_0$

implies $\frac{1}{N}\sum_{n'=1}^N \nu_{n'} \overset{\rho}{\Rightarrow} \nu_0$, so $\pi_N \overset{\tilde\rho}{\Rightarrow} \nu_0 \times \epsilon_\infty$ by Proposition 2.10. In the limit this contradicts the above inequality for the π_N. The implication $(b) \Rightarrow (a)$ is evident (see the proof of the same implication in Proposition 2.10).

$(b) \Leftrightarrow (c)$: As in the proof of Proposition 2.10, it follows easily that under the additional hypothesis $(\nu_n \times \epsilon_n)$ is $\tilde\tau$-tight by compactness of the space $\hat{\mathbf{N}}$. So the result follows by Theorem 2.6. QED

Let (S', τ') be another completely regular topological space for which the analogue of Hypotheses 2.1, 2.3 holds; the associated metric on S' is denoted by ρ' (cf. Proposition 2.2). It is easy to see that the Hypotheses 2.1, 2.3 hold for $S \times S'$, which can either be equipped with the product metric $\rho \times \rho'$ or the product topology $\tau \times \tau'$.

Theorem 2.12 *(i) Let $\nu_n \overset{\rho}{\Rightarrow} \nu_0$ in $\mathcal{P}(S)$ and let $\nu'_n \overset{\rho'}{\Rightarrow} \nu'_0$ in $\mathcal{P}(S')$. Then $\nu_n \times \nu'_n \overset{\rho\times\rho'}{\Rightarrow} \nu_0 \times \nu'_0$ in $\mathcal{P}(S \times S')$.*
(ii) Moreover, if (ν_n) is τ-tight and (ν'_n) is τ'-tight, then in fact $\nu_n \times \nu'_n \overset{\tau\times\tau'}{\Rightarrow} \nu_0 \times \nu'_0$.

Proof. Let $c \in \mathcal{C}_u(S \times S', \rho \times \rho')$ be arbitrary. Define $\tilde c : S \times \hat{\mathbf{N}} \to \mathbf{R}$ as follows:

$$\tilde c(x,k) := \begin{cases} \int_{S'} c(x,x')\nu'_k(dx') & \text{if } k < \infty \\ \int_S c(x,x')\nu'_0(dx') & \text{if } k = \infty \end{cases}$$

Then $\tilde c$ is $\tilde\rho$-continuous, thanks to uniform continuity of c. Hence, the proof of part (i) is finished by invoking Corollary 2.11(i), since $\int_{S\times\hat{\mathbf{N}}} \tilde c d(\nu_n \times \epsilon_n) = \int_{S'\times S''} c d(\nu_n \times \nu'_n)$. Under the extra tightness conditions of part (ii), the sequence $(\nu_n \times \nu'_n)$ is clearly tight with respect to the product topology $\tau \times \tau'$ on $S \times S'$. So the desired result follows from part (i) by virtue of Theorem 2.6(ii). QED

After this, we study the support of the limit of a narrowly convergent sequence.

Definition 2.13 The *support* τ-supp ν of a probability measure $\nu \in \mathcal{P}(S)$ is defined by

$$\tau\text{-supp } \nu := \cap\{F : F \subset S, F \ \tau\text{-closed}, \nu(F) = 1\}.$$

The τ_ρ-support of a measure ν in $\mathcal{P}(S)$, denoted by τ_ρ-supp ν, is defined by replacing the topology τ by τ_ρ in the above formula; of course, τ-supp ν is always contained in τ_ρ-supp ν.

Proposition 2.14 *Every $\nu \in \mathcal{P}(S)$ is carried by its support, i.e., $\nu(\tau\text{-supp } \nu) = 1$.*

Proof. By Definition 2.13 the set $C := S\backslash\tau$-supp ν is the union of all τ-open sets G with $\nu(G) = 0$. By Hypothesis 2.1, C evidently has the countable subcover property (see the proof of Proposition 2.2). So C, being the union of a countable collection of ν-null sets, is a ν-null set itself. QED

Definition 2.15 The *sequential τ-limes superior* of a sequence of subsets (A_n) of S, denoted by τ-$\mathrm{Ls}_n A_n$, is the set of all $x \in S$ for which there exists a subsequence $(A_{n'})$ of (A_n), and corresponding elements $x_{n'} \in A_{n'}$, such that $x = \tau\text{-}\lim_{n'} x_{n'}$.

The definition of the τ_ρ-limes superior is of course completely analogous. However, the metrizable nature of τ_ρ causes an equivalent alternative formulation to be valid. The proof of this is an easy exercise, left to the reader.

Lemma 2.16 *Let (A_n) be a sequence of subsets of S. Then*

$$\tau_\rho\text{-}\mathrm{Ls}_n A_n := \cap_{p=1}^\infty \tau_\rho\text{-}cl \ \cup_{n\geq p} A_n.$$

Theorem 2.17 *(i) Let (ν_n) and ν_0 be in $\mathcal{P}(S)$ with $\frac{1}{N}\sum_{n=1}^N \nu_n \overset{\rho}{\Rightarrow} \nu_0$ in $\mathcal{P}(S)$ (this holds in particular when $\nu_n \overset{\rho}{\Rightarrow} \nu_0$). Then*

$$\tau_\rho\text{-supp } \nu_0 \subset \tau_\rho\text{-}\mathrm{Ls}_n \tau_\rho\text{-supp } \nu_n.$$

(ii) Moreover, if $(\frac{1}{N}\sum_{n=1}^N \nu_n)$ *is* τ*-tight then in fact*

$$\nu_0(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \nu_n) = 1$$

and, consequently,

$$\tau\text{-supp } \nu_0 \subset \tau\text{-cl } \tau\text{-Ls}_n \tau\text{-supp } \nu_n.$$

Recall that the τ*-sequential closure* τ-seq-cl A of a set A in S is defined as the intersection of all those τ-sequentially closed sets C in S for which $C \supset A$. Clearly, τ-seq-cl $A \subset \tau$-cl A. Given Hypothesis 2.1, it is easy to check that for any sequence (A_n) of subsets of S one has τ-seq-cl τ-$\text{Ls}_n A_n \subset \tau_\rho$-$\text{Ls}_n A_n$.

PROOF. (i) By Proposition 2.10 we have $\pi_N := \frac{1}{N}\sum_{n=1}^N(\nu_n \times \epsilon_n) \stackrel{\tilde{\rho}}{\Rightarrow} \nu_0 \times \epsilon_\infty$. Define $q_0' : S \times \hat{\mathbf{N}} \to \{0, +\infty\}$ by

$$q_0'(x,k) := \begin{cases} 0 & \text{if } x \in \tau_\rho\text{-supp } \nu_k \text{ and } k < \infty, \\ 0 & \text{if } x \in \tau_\rho\text{-Ls}_n \tau_\rho\text{-supp } \nu_n \text{ and } k = \infty \\ +\infty & \text{otherwise.} \end{cases}$$

Then q_0' is $\tilde{\rho}$-lower semicontinuous in every point (x,k) of $S \times \hat{\mathbf{N}}$. Indeed, let $(x^j, k^j) \to (x,k)$ (note that sequential arguments suffice to verify lower semicontinuity). We must show that $\alpha := \liminf_n q_0'(x^j,k^j) \geq q_0'(x,k)$. If $k < \infty$, then eventually $k^j \equiv k$, so $\alpha \geq q_0'(x,k)$ follows by the fact that τ_ρ-supp ν_k is τ_ρ-closed (Lemma 2.16). If $k = \infty$, we distinguish two cases: if eventually $k^j \equiv \infty$, then $\alpha \geq q_0'(x,\infty)$ follows by closedness of τ_ρ-$\text{Ls}_n \tau_\rho$-supp ν_n, which in turn is an immediate consequence of Lemma 2.16. On the other hand, if $k^j < \infty$ infinitely often, then the same inequality follows directly from Definition 2.15. So we conclude that q_0' is indeed $\tilde{\rho}$-lower semicontinuous. Now $\int_{S\times\hat{\mathbf{N}}} q_0' d(\nu_n \times \epsilon_n) = \int_S q_0'(x,n)\nu_n(dx) = 0$ for every n (by Proposition 2.14). Hence, $\int_{S\times\hat{\mathbf{N}}} q_0' d\pi_N = 0$ for every N. Thus, Theorem 2.6(i) gives $\int_S q_0'(x,\infty)\nu_0(dx) = 0$, and the desired support properties for ν_0 follow.

(ii) Under the additional τ-tightness condition it follows that $\pi_N := \frac{1}{N}\sum_{n=1}^N(\nu_n \times \epsilon_n) \stackrel{\tilde{\tau}}{\Rightarrow} \nu_0 \times \epsilon_\infty$ by Proposition 2.10(ii). Let $q_0 : S \times \hat{\mathbf{N}} \to \{0, +\infty\}$ be given by

$$q_0(x,k) := \begin{cases} 0 & \text{if } x \in \tau\text{-supp } \nu_k \text{ and } k < \infty, \\ 0 & \text{if } x \in \tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \nu_n \text{ and } k = \infty \\ +\infty & \text{otherwise.} \end{cases}$$

With a little careful work, this function is seen to be $\tilde{\tau}$-sequentially lower semicontinuous on $S \times \hat{\mathbf{N}}$ (observe that, unlike the previous part, τ-$\text{Ls}_n \tau$-supp ν_n is not sequentially closed – hence the additional sequential closure operation has been added in the definition of q_0). By Proposition 2.14 and Theorem 2.6(ii) we find $\int_S q_0(x,\infty)\nu_0(dx) = 0$. The desired properties of ν_0 then follow with ease. QED

Remark 2.18 *If in Theorem 2.17 there exists a* τ*-compact set* K *containing* $\cup_n$supp ν_n*, then the set* τ-$\text{Ls}_n \tau$-supp ν_n *is* τ*-closed and the following simplification can be made:*

$$\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \nu_n = \tau\text{-Ls}_n \tau\text{-supp } \nu_n.$$

Indeed, on K *the topologies* τ *and* τ_ρ *coincide, which gives* τ-$\text{Ls}_n \tau$-supp $\nu_n = \tau_\rho$-$\text{Ls}_n \tau_\rho$-supp ν_n*, and the latter set is* τ_ρ*-closed, whence* τ*-closed (cf. Lemma 2.16).*

In order to connect narrow and K-convergence of Young measures in section 4, the following sufficient condition for τ_ρ-tightness is quite useful. Recall that a probability measure ν in $\mathcal{P}(S)$ is said to be τ_ρ*-Radon* if the singleton $\{\nu\}$ is τ_ρ-tight (cf. Definition 2.7). The set of all such Radon probability measures is denoted by $\mathcal{P}_{Radon}(S, \tau_\rho)$.

Theorem 2.19 *Let* (ν_n) *and* ν_0 *be in* $\mathcal{P}_{Radon}(S,\tau_\rho)$*. Then* $\nu_n \stackrel{\rho}{\Rightarrow} \nu_0$ *implies that* (ν_n) *is* τ_ρ*-tight.*

This is [44, Theorem 8, Appendix III] and [92, Theorem 9.3]; the proof depends critically on both the metric nature of τ_ρ and the fact that one considers only sequential narrow convergence.

3 K-convergence of Young measures

This section develops K-convergence, an auxiliary, nontopological convergence mode for Young measures introduced in [19, 20, 25]. This will be of great use in the next section when we transfer narrow convergence results of the previous section from probability measures to Young measures. Thus, the present section can be regarded as an intermediate stage in the transfer process. As in section 2, results are developed both in a metrizable and in a nonmetrizable setting.

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let us remark that much of what is done here extends without further ado to a σ-finite measure space [such a measure is equivalent to a finite one, and one can always premultiply the integrands below by the appropriate Radon Nikodym derivative and an appropriate extension of uniform integrability is also available]. Let (S, τ) be as in the previous section (i.e., a completely regular topological space satisfying Hypotheses 2.1, 2.3).

Let $\mathcal{R}(\Omega; S)$ be the set of all transition probabilities from $(\Omega, \mathcal{A})$ into $(S, \mathcal{B}(S))$ [81, III.2]. That is to say, $\mathcal{R}(\Omega; S)$ consists of all functions $\delta : \Omega \to \mathcal{P}(S)$ such that $\omega \mapsto \delta(\omega)(B)$ is $\mathcal{A}$-measurable for every $B \in \mathcal{B}(S)$. [Note that this notion subsumes that of probability measure: $\mathcal{P}(S)$ can be identified with the constant functions in $\mathcal{R}(\Omega; S)$; in fact, $\mathcal{R}(\Omega; S)$ coincides with $\mathcal{P}(S)$ when $\mathcal{A}$ is trivial, i.e., $\mathcal{A} = \{\emptyset, \Omega\}$.] In association with the central topology of these lecture notes (Definition 4.1), transition probabilities are also called *Young measures*, and we shall adopt this terminology (other names used for Young measures in the literature are, depending on the context: Markov kernels, randomized decision functions, relaxed control functions, etc.). For some elementary measure-theoretical properties of Young measures the reader is referred to [81, III.2] or [4, 2.6] (see also Appendix A). In particular, the product measure that is induced on $(\Omega \times S, \mathcal{A} \times \mathcal{B}(S))$ by μ and any $\delta \in \mathcal{R}(\Omega; S)$ (cf. [81, III.2]) is denoted by $\mu \otimes \delta$; cf. Theorem A.1. Let $\mathcal{L}^0(\Omega; S)$ be the set of all measurable functions from $(\Omega, \mathcal{A})$ into $(S, \mathcal{B}(S))$. A Young measure $\delta \in \mathcal{R}(\Omega; S)$ is said to be *Dirac* if it is a degenerate transition probability, i.e., if there exists a function $f \in \mathcal{L}^0(\Omega; S)$ such that for every ω in Ω

$$\delta(\omega) = \epsilon_{f(\omega)} := \text{Dirac measure at the point } f(\omega).$$

In this special case δ is denoted by ϵ_f and is called the Young measure *relaxation* of the function f. The set of all Dirac Young measures in $\mathcal{R}(\Omega; S)$ is denoted by $\mathcal{R}_{Dirac}(\Omega; S)$.

The fundamental idea behind Young measure theory is that, in some sense, $\mathcal{R}(\Omega; S)$ forms a *completion* of $\mathcal{L}^0(\Omega; S)$, when the latter is identified with $\mathcal{R}_{Dirac}(\Omega; S)$. In the context of the previous section, the much less fruitful analogue of this would be to view $\mathcal{P}(S)$ as an extension of S, because the latter can be identified with the set $\{\epsilon_x : x \in S\}$ of all Dirac measures, to which it is homeomorphic.

Let us agree to the following terminology: an *integrand* on $\Omega \times S$ is a function $g : \Omega \times S \to (-\infty, +\infty]$ such that for every $\omega \in \Omega$ the function $g(\omega, \cdot)$ on S is $\mathcal{B}(S)$-measurable. Moreover, such an integrand g is said to be *integrably bounded below* if there exists $\phi \in \mathcal{L}^1(\Omega; \mathbf{R})$ such that $g(\omega, x) \geq \phi(\omega)$ for all $\omega \in \Omega$ and $x \in S$. Further, a function $g : \Omega \times S \to (-\infty, +\infty]$ is said to be a (sequentially) *τ-lower semicontinuous [τ-continuous] [[τ-inf-compact]] integrand* on $\Omega \times S$ if for every $\omega \in \Omega$ the function $g(\omega, \cdot)$ on S is (sequentially) τ-lower semicontinuous [τ-continuous] [[τ-inf-compact]] respectively. Let g be an integrand on $\Omega \times S$. The following expression is meaningful for any $\delta \in \mathcal{R}(\Omega; S)$:

$$I_g(\delta) := \int_\Omega^* [\int_S g(\omega, x)\delta(\omega)(dx)]\mu(d\omega),$$

provided that the two integral signs are interpreted as follows: (1) for every fixed ω the integral over the set S of the function $g(\omega, \cdot)$, which is $\mathcal{B}(S)$-measurable by definition of the term integrand, is a *quasi-integral* in the sense of [81, p. 41] and Appendix B, (2) the integral over Ω is interpreted as an outer integral in the sense of Appendix B (note that outer integration comes down to quasi-integration when measurable functions are involved). The resulting functional $I_g : \mathcal{R}(\Omega; S) \to [-\infty, +\infty]$ is called the *Young measure integral functional* associated to g. Another integral functional associated to g, this time on the set $\mathcal{L}^0(\Omega; S)$ of all measurable functions from Ω into S, is given by the formula

$$J_g(f) := \int_\Omega^* g(\omega, f(\omega))\mu(d\omega) = I_g(\epsilon_f).$$

The following notion of convergence was introduced and studied in a more abstract context in [18, 19].

Definition 3.1 A sequence (δ_n) in $\mathcal{R}(\Omega; S)$ *K-converges* with respect to the topology τ to $\delta_0 \in \mathcal{R}(\Omega; S)$ (notation: $\delta_n \xrightarrow{K,\tau} \delta_0$) if for every subsequence $(\delta_{n'})$ of (δ_n)

$$\frac{1}{N}\sum_{n'=1}^N \delta_{n'}(\omega) \overset{\tau}{\Rightarrow} \delta_0(\omega) \text{ as } N \to \infty \text{ for a.e. } \omega \text{ in } \Omega.$$

Note that the exceptional null set is allowed to vary with the subsequence $(\delta_{n'})$. Of course, the short arrow " $\overset{\tau}{\Rightarrow}$ " above refers to τ-narrow convergence in $\mathcal{P}(S)$ in the sense of Definition 2.5. Unlike narrow convergence, K-convergence is *nontopological*. If in the above definition $\overset{\tau}{\Rightarrow}$, i.e., the mode of pointwise convergence mode, is replaced by $\overset{\rho}{\Rightarrow}$, we obtain a corresponding notion of K-convergence with respect to τ_ρ that is denoted by " $\overset{K,\rho}{\to}$ ". Since " $\overset{\rho}{\Rightarrow}$ " is implied by " $\overset{\tau}{\Rightarrow}$ ", it follows that " $\overset{K,\rho}{\to}$ " is implied by " $\xrightarrow{K,\tau}$ ".

Example 3.2 Let $(\Omega, \mathcal{A}, \mu)$ be $([0,1], \mathcal{L}([0,1]), \lambda_1)$ (i.e., the Lebesgue unit interval). Let (f_n) be the sequence of *Rademacher functions*, defined by $f_n(\omega) := \text{sign} \sin(2^n\pi\omega)$ (here $S := \mathbf{R}$, of course). Then $\epsilon_{f_n} \xrightarrow{K,\tau} \delta_0$, where $\delta_0 \in \mathcal{R}([0,1]; \mathbf{R})$ is the constant function $\delta_0(\omega) \equiv \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_{-1}$. In fact, here one could argue that the strong law of large numbers applies to the sequence (ϵ_{f_n}) of $\mathcal{P}(\mathbf{R})$-valued random variables, but one can also give a proof of the above by means of the standard (scalar) strong law of large numbers and scalarization, analogous to the proof of Theorem 3.7 below.

A crucial instrument for the transfer process of these notes is the following generalization of Definition 2.7.

Definition 3.3 A sequence (δ_n) in $\mathcal{R}(\Omega; S)$ is *τ-tight* if either one of the following two *equivalent* statements is true:

(a) There exists a nonnegative, sequentially τ-inf-compact integrand h on $\Omega \times S$ such that

$$\sup_n I_h(\delta_n) < +\infty.$$

(b) For every $\epsilon > 0$ there exists a multifunction $\Gamma_\epsilon : \Omega \to 2^S$, with $\Gamma_\epsilon(\omega)$ sequentially compact for every $\omega \in \Omega$, such that

$$\sup_n \int_\Omega^* \delta_n(\omega)(S \backslash \Gamma_\epsilon(\omega))\mu(d\omega) \leq \epsilon.$$

Recall from the previously given definition of integrands that a sequentially τ-inf-compact integrand h is simply a function on $\Omega \times S$ with the following property: for every $\omega \in \Omega$ the function $h(\omega, \cdot)$ is sequentially τ-inf-compact on S (i.e., all sets $\{x \in S : h(\omega, x) \leq \beta\}$, $\beta \in \mathbf{R}$, are sequentially τ-compact). As is by now usual, the alternative, weaker notion of τ_ρ-tightness of a sequence of Young measures is obtained by replacing the topology τ by τ_ρ in the above definition.

PROOF OF EQUIVALENCE OF (a) AND (b) IN DEFINITION 3.3 [68].

$(a) \Rightarrow (b)$: Let $s := \sup_n I_h(\delta_n)$; then $s \in \mathbf{R}_+$. For every $\epsilon > 0$, let $\Gamma_\epsilon(\omega)$ be the set of all $x \in S$ for which $h(\omega, x) \leq s/\epsilon$; then $\Gamma_\epsilon(\omega)$ is sequentially τ-compact for every ω. Also, for every n

$$\frac{s}{\epsilon}\int_\Omega^* \delta_n(\omega)(S \backslash \Gamma_\epsilon(\omega))\mu(d\omega) \leq I_h(\delta_n) \leq s,$$

and this proves that the definition as given in part (b) holds.

$(b) \Rightarrow (a)$: Let Γ_m be the given multifunction corresponding to $\epsilon = 3^{-m}$, $m \in \mathbf{N}$. With no loss of generality we may suppose that $\Gamma_m(\omega) \subset \Gamma_{m+1}(\omega)$ for every ω and m (otherwise, we could always take finite unions of the Γ_m). Now set $\Gamma_0 \equiv \emptyset$ and define

$$h(\omega, x) := \begin{cases} 2^m & \text{if } x \in \Gamma_m(\omega) \backslash \Gamma_{m-1}(\omega),\ m \in \mathbf{N} \\ +\infty & \text{if } x \notin \cup_m \Gamma_m(\omega) \end{cases}$$

Then $h(\omega, \cdot)$ is sequentially τ-inf-compact on S for every ω and $\sup_n I_h(\delta_n) \leq 6$. QED

Example 3.4 (a) Let E be a separable reflexive Banach space with norm $\|\cdot\|$. Let E' be the dual space of E. Observe that (E, τ) is a completely regular Suslin space for $\tau := \sigma(E, E')$. Suppose that $(f_n) \subset \mathcal{L}^1(\Omega; E)$ is bounded in $\mathcal{L}^1$-seminorm: $\sup_n \int_\Omega \|f_n(\omega)\| \mu(d\omega) < +\infty$. Then the corresponding sequence (ϵ_{f_n}) in $\mathcal{R}(\Omega; E)$ is τ-tight: just take $h(\omega, x) := \|x\|$ in Definition 3.3.

(b) Let E be a separable Banach space with norm $\|\cdot\|$. Then (E, τ) is a completely regular Suslin space for $\tau := \sigma(E, E')$. Suppose that $(f_n) \subset \mathcal{L}^1(\Omega; E)$ is bounded in $\mathcal{L}^1$-seminorm and that there exists a multifunction $R : \Omega \to 2^S$ such that for a.e. ω both $\{f_n(\omega) : n \in \mathbf{N}\} \subset R(\omega)$ and $R(\omega)$ is τ-ball-compact [i.e., the intersection of $R(\omega)$ with every closed ball in E is $\sigma(E, E')$-compact]. Then (ϵ_{f_n}) is τ-tight, as is seen by considering $h_R(\omega, x) := \|x\|$ if $x \in R(\omega)$, and $h_R(\omega, x) := +\infty$ otherwise. For notice that for every $\omega \in \Omega$ and $\beta \in \mathbf{R}_+$ the set of all $x \in E$ such that $h_R(\omega, x) \leq \beta$ coincides with the intersection of $R(\omega)$ and the closed ball with radius β around 0. This set is $\sigma(E, E')$-compact, whence sequentially $\sigma(E, E')$-compact by the Eberlein-Šmulian theorem.

Part (b) in the above example generalizes part (a): simply observe that in part (a) E itself is $\sigma(E, E')$-ball-compact (by reflexivity), so that there one can set $R(\omega) := E$ for all $\omega \in \Omega$.

A very important property of K-convergence for Young measures is the following Fatou-Vitali-type result:

Proposition 3.5 *(i) Let $\delta_n \stackrel{K,\rho}{\to} \delta_0$ in $\mathcal{R}(\Omega; S)$. Then $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ for every τ_ρ-lower semicontinuous integrand g on $\Omega \times S$ such that*

$$s(\alpha) := \sup_n \int_\Omega^* [\int_{\{g \leq -\alpha\}_\omega} g^-(\omega, x) \delta_n(\omega)(dx)] \mu(d\omega) \to 0 \text{ for } \alpha \to \infty. \tag{3.1}$$

(ii) Moreover, if (δ_n) is τ-tight, then in fact $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ for every sequentially τ-lower semicontinuous integrand g on $\Omega \times S$ such that (3.1) holds.

Here $g^- := \max(-g, 0)$ and $\{g \leq -\alpha\}_\omega$ denotes the set $\{x \in S : g(\omega, x) \leq -\alpha\}$.

Remark 3.6 *(i) If g is integrably bounded from below, then (3.1) holds automatically.*

(ii) In case $\delta_n = \epsilon_{f_n}$ for all $n \in \mathbf{N}$ (this specification does not include the limit δ_0) the condition (3.1) runs as follows:

$$\lim_{\alpha \to \infty} \sup_n \int_\Omega^* 1_{\{g(\cdot, f_n(\cdot)) \leq -\alpha\}} g^-(\omega, f_n(\omega)) \mu(d\omega) = 0.$$

Clearly, for every $\omega \in \Omega$ we have $g(\omega, f_n(\omega)) \leq -\alpha$ if and only if $g^-(\omega, f_n(\omega)) \geq \alpha$. This means that (3.1) simply comes down to uniform (outer) integrability of the sequence $(g^-(\cdot, f_n(\cdot)))$ in the case of a Dirac sequence. If g is $\mathcal{T} \times \mathcal{B}(S)$-measurable in addition, this coincides with the usual classical formulations of uniform integrability à la Vitali of the sequence of negative parts $(g^-(\cdot, f_n(\cdot)))$; cf. [65, 9].

Proof of Proposition 3.5. The proof of part (i) will be given in two steps.

Step 1: the case $g \geq 0$. Set $\beta := \liminf_n I_g(\delta_n)$; then there is a subsequence $(\delta_{n'})$ such that $\beta = \lim_{n'} I_g(\delta_{n'})$. Define $\psi_N(\omega) := \frac{1}{N} \sum_{n'=1}^N \int_S g(\omega, x) \delta_{n'}(\omega)(dx)$ and $\psi_0(\omega) := \int_S g(\omega, x) \delta_0(\omega)(dx)$. Then $\liminf_N \psi_N(\omega) \geq \psi_0(\omega)$ for a.e. ω by Theorem 2.6(i), because by Definition 3.1 $\frac{1}{N} \sum_{n'=1}^N \delta_{n'}(\omega)$

$\overset{\beta}{\Rightarrow}$ $\delta_0(\omega)$ in $\mathcal{P}(S)$ for a.e. ω. Thus, Fatou's lemma for outer integration (Proposition B.4) can be applied. This gives $\liminf_{N\to\infty} \int^*_\Omega \psi_N d\mu \geq \int^*_\Omega \psi_0 d\mu$. Here the right-hand side is equal to $I_g(\delta_0)$, and the left-hand side is at most β, by subadditivity of outer integrals (Lemma B.5) and by the choice of $(\delta_{n'})$.

Step 2: the general case. We essentially follow Ioffe [65] by pointing out that the simple inequality $g + 1_{\{g\leq -\alpha\}} g^- \geq g_\alpha := \max(g, -\alpha)$ on $\Omega \times S$ leads to

$$\int_S g(\omega,x)\delta_n(\omega)(dx) + \int_S 1_{\{g\leq -\alpha\}}(\omega,x) g^-(\omega,x)\delta_n(\omega)(dx) \geq \int_S g_\alpha(\omega,x)\delta_n(\omega)(dx).$$

After one more (outer) integration this gives, in the notation of (3.1), $I_g(\delta_n) + s(\alpha) \geq I_{g_\alpha}(\delta_n)$, where we use again the subadditivity of outer integration (Lemma B.5). Now observe that step 1 trivially extends to any g that is bounded from below, such as g_α. This gives

$$\liminf_n I_g(\delta_n) + s(\alpha) \geq \liminf_n I_{g_\alpha}(\delta_n) \geq I_{g_\alpha}(\delta_0) \geq I_g(\delta_0),$$

where the last inequality follows from $g_\alpha \geq g$. In view of (3.1), the proof of (i) is now finished by letting α go to infinity.

(ii) Let h be as in Definition 3.3 and denote $s := \sup_n I_h(\delta_n)$. We augment g, similar to the proof of Theorem 2.6(ii): For $\epsilon > 0$ define $g^\epsilon := g + \epsilon h$. Then $g^\epsilon \geq g$ and $g^\epsilon(\omega,\cdot)$ is τ_ρ-lower semicontinuous on S for every $\omega \in \Omega$ (see the proof of Theorem 2.6(ii)). Thus, part (i) gives $\liminf_n I_g(\delta_n) + \epsilon s \geq \liminf_n I_{g^\epsilon}(\delta_n) \geq I_{g^\epsilon}(\delta_0) \geq I_g(\delta_0)$ for any $\epsilon > 0$. Letting ϵ go to zero gives the desired inequality. QED

The following important Prohorov-type "relative sequential compactness criterion for K-convergence" (apostrophes are in order because K-convergence is nontopological) is a crucial tool for these notes. It extends Prohorov's classical Theorem 2.9 to K-convergence of Young measures and was first obtained in [19, Theorem 5.1] as a specialization of an abstract Komlós' theorem (i.e., an abstract version of Theorem 3.9 below) to Young measures.

Theorem 3.7 (i) *Let* (δ_n) *be a* τ_ρ*-tight sequence in* $\mathcal{R}(\Omega;S)$*. Then there exist a subsequence* $(\delta_{n'})$ *of* (δ_n) *and* $\delta_* \in \mathcal{R}(\Omega;S)$ *such that* $\delta_{n'} \overset{K,\rho}{\to} \delta_*$*.*
(ii) *Moreover, if* (δ_n) *is* τ*-tight, then in fact* $\delta_{n'} \xrightarrow{K,\tau} \delta_*$ *can be achieved in* (i)*.*

The following example, which extends Example 3.2, demonstrates the power of this result. Clearly, this brings K-convergence (for subsequences!) to settings where the law of large numbers stands no chance at all.

Example 3.8 Let $(\Omega, \mathcal{A}, \mu)$ be $([0,1], \mathcal{L}([0,1]), \lambda_1)$ (cf. Example 3.2). Let $f_1 \in \mathcal{L}^1([0,1];\mathbf{R})$ be arbitrary; it can be extended periodically from $[0,1]$ to all of $\mathbf{R}$. We define $f_{n+1}(\omega) := f_1(2^n\omega)$. Clearly, the sequence (ϵ_{f_n}) is tight in the sense of Definition 3.3 [e.g., use $h(\omega,x) := |x|$ to meet part (a) or $K_\epsilon \equiv [-1,1]$ to satisfy part (b)]. Therefore, by Theorem 3.7 there exist a subsequence $(f_{n'})$ of (f_n) and some $\delta_* \in \mathcal{R}([0,1];\mathbf{R})$ such that $\epsilon_{f_{n'}} \xrightarrow{K,\tau} \delta_*$. The precise nature of δ_* could now be determined by means of Proposition 3.5, but we shall defer this to Example 4.3 later on.

To prove Theorem 3.7 we use an outstanding theorem, due to J. Komlós [71].[1]

Theorem 3.9 (Komlós) *Let* (ϕ_n) *be a sequence in* $\mathcal{L}^1(\Omega;\mathbf{R})$ *such that*

$$\sup_n \int_\Omega |\phi_n| d\mu < +\infty.$$

[1] The original proof in [71] went by subtle truncation arguments and application of a martingale limit theorem. It is not hard to show that Komlós' theorem implies the strong law of large numbers. What is much more interesting is that, conversely, Theorem 3.9 also follows from the strong law of large numbers by invoking "subsequence principle theory" [1, 53].

Then there exist a subsequence $(\phi_{n'})$ of (ϕ_n) and a function $\phi_* \in \mathcal{L}^1(\Omega;\mathbf{R})$ such that for every further subsequence $(\phi_{n''})$ of $(\phi_{n'})$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n''=1}^{N} \phi_{n''}(\omega) = \phi_*(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

Observe here that ϕ_* is universal with respect to the possible choices of a subsequence $(\phi_{n''})$ from $(\phi_{n'})$, but that the associated exceptional μ-null set in the limit statement is allowed to vary with the subsequence.

Lemma 3.10 *There exists a countable set $\mathcal{C}_0 \subset \mathcal{C}_u(S,\rho)$ such that for every (ν_n) and ν_0 in $\mathcal{P}(S)$*

$$\lim_n \int_S c d\nu_n = \int_S c d\nu_0 \text{ for all } c \in \mathcal{C}_0$$

if and only if $\nu_n \overset{\rho}{\Rightarrow} \nu_0$. In particular, $\mathcal{C}_0$ separates the points of $\mathcal{P}(S)$.

Proof. As was observed following Hypothesis 2.1, (S,τ) is separable. Hence, (S,τ_ρ) is a separable metric space (apply Proposition 2.2). Therefore, the result follows from [43, Proposition 7.19]. QED

Lemma 3.11 *Let (ν_n) in $\mathcal{P}(S)$ be τ_ρ-tight and let $\mathcal{C}_0 \subset \mathcal{C}_u(S,\rho)$ be as in Lemma 3.10. If*

$$\lim_n \int_S c d\nu_n \text{ exists for every } c \in \mathcal{C}_0,$$

then there exists $\nu_* \in \mathcal{P}(S)$ such that $\nu_n \overset{\rho}{\Rightarrow} \nu_*$.

Proof. By Theorem 2.9 there exist a subsequence $(\nu_{n'})$ of (ν_n) and $\nu_* \in \mathcal{P}(S)$ such that $\nu_{n'} \overset{\rho}{\Rightarrow} \nu_*$. Then $\int_S c d\nu_* = \alpha_c := \lim_n \int_S c d\nu_n$ for every $c \in \mathcal{C}_0$. Now if (ν_n) as a whole were not to converge to ν_*, there would exist $\tilde{c} \in \mathcal{C}_b(S,\rho)$ and $\epsilon > 0$ such that for some subsequence (ν_m) of (ν_n) one would have $|\int_S \tilde{c} d\nu_m - \int_S \tilde{c} d\nu_*| \geq \epsilon$ for all m. Since (ν_m) is τ_ρ-tight, there would then exist, by another application of Theorem 2.9, a subsequence $(\nu_{m'})$ and $\nu_{**} \in \mathcal{P}(S)$ such that $\nu_{m'} \overset{\rho}{\Rightarrow} \nu_{**}$. Just as above, this would entail $\int_S c d\nu_{**} = \alpha_c$ for all $c \in \mathcal{C}_0$, so $\nu_* = \nu_{**}$ by the point-separating property of $\mathcal{C}_0$. But since also $|\int_S \tilde{c} d\nu_{**} - \int_S \tilde{c} d\nu_*| \geq \epsilon$, a contradiction would follow. QED

Proof of Theorem 3.7. (i) By Lemma 3.10 there exists a countable subset $\mathcal{C}_0 = \{c_i : i \in \mathbf{N}\}$ of $\mathcal{C}_u(S,\rho)$ that separates the points of $\mathcal{P}(S)$. Clearly, $\sup_n \int_\Omega |\phi_{i,n}| d\mu < +\infty$ for every $i \in \mathbf{N}$, where we set $\phi_{i,n}(\omega) := \int_S c_i(x)\delta_n(\omega)(dx)$. Let h be as in Definition 3.3 (case of τ_ρ-tightness). By Lemma B.3 there exists for each $n \in \mathbf{N}$ a function $\phi_{0,n} \in \mathcal{L}^1(\Omega;\mathbf{R})$ such that $\phi_{0,n}(\omega) \geq \int_S h(\omega,x)\delta_n(\omega)(dx)$ for all $\omega \in \Omega$ and $\int_\Omega \phi_{0,n} d\mu = I_h(\delta_n)$. Applying the Komlós Theorem 3.9 in a diagonal extraction procedure, we obtain a subsequence $(\delta_{n'})$ of (δ_n) and functions $\phi_{i,*} \in \mathcal{L}^1(\Omega;\mathbf{R})$, $i \in \mathbf{N} \cup \{0\}$, such that $\lim_N \frac{1}{N}\sum_{n''=1}^N \phi_{i,n''} = \phi_{i,*}$ a.e. for every further subsequence $(\delta_{n''})$ and for all $i \in \mathbf{N} \cup \{0\}$. It follows therefore that for every such subsequence $(\delta_{n''})$ for a.e. ω in Ω

$$\lim_N \int_S h(\omega,x) \frac{1}{N} \sum_{n''=1}^{N} \delta_{n''}(\omega)(dx) = \phi_{0,*}(\omega) < +\infty, \tag{3.2}$$

$$\lim_N \int_S c_i(x) \frac{1}{N} \sum_{n''=1}^{N} \delta_{n''}(\omega)(dx) = \phi_{i,*}(\omega) \text{ for all } i \in \mathbf{N}. \tag{3.3}$$

Let us begin by considering $(\delta_{n'})$ itself as the subsequence in question. Fix ω outside the exceptional null set M, associated with this particular choice of a subsequence in (3.2)–(3.3). Then (3.2) implies that for a.e. ω the sequence $(\pi_{\omega,N})$ in $\mathcal{P}(S)$, defined by $\pi_{\omega,N} := \frac{1}{N}\sum_{n'=1}^N \delta_{n'}(\omega)$, is τ_ρ-tight in $\mathcal{P}(S)$ in the sense of Definition 2.7. Also, (3.3) implies that $\lim_N \int_S c_i d\pi_{\omega,N}$ exists for every i. By

Lemma 3.11(i), there exists $\nu_{\omega,*}$ in $\mathcal{P}(S)$ such that $\pi_{\omega,N} \overset{\rho}{\Rightarrow} \nu_{\omega,*}$. Define $\delta_*(\omega) := \nu_{\omega,*}$ for $\omega \in \Omega \backslash M$. Also, on M we define δ_* to be equal to an arbitrary, but fixed element from $\mathcal{P}(S)$. Then it is elementary, in view of Proposition A.2, that δ_* belongs to $\mathcal{R}(\Omega; S)$. Finally, the argument following (3.3) can be repeated if one starts out with an arbitrary subsequence $(\delta_{n''})$ of $(\delta_{n'})$, instead of $(\delta_{n'})$ itself. Except for the change in the exceptional null set M, for which the definition of K-convergence allows, nothing changes. This finishes the proof of part (i). Part (ii) then follows immediately by Theorem 2.6(ii), in view of the fact that for every subsequence $(\delta_{n''})$ of the above $(\delta_{n'})$ (3.2) implies that $(\pi'_{\omega,N})$ is τ-tight for a.e. ω, where $\pi'_{\omega,N} := \frac{1}{N}\sum_{n''=1}^{N} \delta_{n''}(\omega)$. QED

Remark 3.12 *From (3.2) in the above proof it is seen that the sequence $(\delta_{n'})$ in Theorem 3.7 is such that for every further subsequence $(\delta_{n''})$ the sequence $(\frac{1}{N}\sum_{n''=1}^{N} \delta_{n''}(\omega))$ in $\mathcal{P}(S)$ is either τ_ρ-tight for a.e. ω (part (i)) or even τ-tight for a.e. ω (part (ii)).*

As the final results in this intermediate section, we present direct consequences of Proposition 2.10 and Theorem 2.17 for K-convergence of Young measures. Such results first figured in [20]; they will be used in the next section.

Proposition 3.13 *(i) Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. The following are equivalent:*

(a) $\delta_n \overset{K,\rho}{\to} \delta_0$.

(b) $\delta_n \times \epsilon_n \overset{K,\tilde\rho}{\to} \delta_0 \times \epsilon_\infty$.

(ii) Moreover, if (δ_n) is τ-tight, then the following two equivalent statements are implied by the above:

(c) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \overset{K,\tau}{\longrightarrow} \delta_0$.

(d) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \times \epsilon_{n''} \overset{K,\tilde\tau}{\longrightarrow} \delta_0 \times \epsilon_\infty$.

Proof. $(a) \Leftrightarrow (b)$ follows by pointwise application of Proposition 2.10(i). In part (ii) $(c) \Leftrightarrow (d)$ follows by pointwise application of Proposition 2.10(ii), by taking into consideration Remark 3.12. Finally, the implication $(a) \Rightarrow (c)$ of part (ii) follows by pointwise application of Theorem 2.6, again taking into consideration Remark 3.12. QED

Theorem 3.14 *(i) Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. Then $\delta_n \overset{K,\rho}{\to} \delta_0$ implies*

$$\tau_\rho\text{-supp } \delta_0(\omega) \subset \tau_\rho\text{-Ls}_n\, \tau_\rho\text{-supp } \delta_n(\omega) \textit{ for a.e. } \omega \textit{ in } \Omega.$$

(ii) Moreover, if (δ_n) is τ-tight, then in fact

$$\delta_0(\omega)(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1 \textit{ for a.e. } \omega \textit{ in } \Omega,$$

so that in particular

$$\tau\text{-supp } \delta_0(\omega) \subset \tau\text{-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega) \textit{ for a.e. } \omega \textit{ in } \Omega.$$

Proof. Part (i) of this result follows directly from a pointwise application of Theorem 2.17(i). Part (ii) also follows by a pointwise application of Theorem 2.17(ii), in view of the tightness observation in Remark 3.12. QED

4 Narrow convergence of Young measures

In this section our program to transfer narrow convergence results for probability measures (section 2) to Young measures is completed. We use the same fundamental hypotheses as in the previous section: $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and (S, τ) is a topological space for which Hypotheses 2.1, 2.3 hold. We start out by giving the definition of narrow convergence for Young measures:

Definition 4.1 A sequence (δ_n) in $\mathcal{R}(\Omega; S)$ converges *τ-narrowly* to δ_0 in $\mathcal{R}(\Omega; S)$ (this is denoted by $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$) if for every $A \in \mathcal{A}$ and for every c in $\mathcal{C}_b(S, \tau)$

$$\lim_n \int_A [\int_S c(x)\delta_n(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c(x)\delta_0(\omega)(dx)]\mu(d\omega).$$

The obviously weaker notion *τ_ρ-narrow* convergence is defined by replacing τ by τ_ρ. Similar to section 2, the latter notion is denoted by " $\stackrel{\rho}{\Longrightarrow}$ ". In analogy to section 2, we shall see that for tight sequences of Young measures τ-narrow and τ_ρ-narrow convergence are actually the same. For further benefit, note carefully the difference in notation between the narrow convergences for probability measures (indicated by *short* arrows) and Young measures (indicated by *long* arrows).

In its above form the definition of narrow convergence is classical in statistical decision theory [96, 74]. It merges two completely different classical modes of convergence:

Remark 4.2 *Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. The following are obviously equivalent:*
(a) $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$ in $\mathcal{R}(\Omega; S)$.
(b) For every $A \in \mathcal{A}$ with $\mu(A) > 0$

$$[\mu \otimes \delta_n](A \times \cdot)/\mu(A) \stackrel{\tau}{\Rightarrow} [\mu \otimes \delta_0](A \times \cdot)/\mu(A) \text{ in } \mathcal{P}(S).$$

(c) For every $c \in \mathcal{C}_b(S, \tau)$

$$\int_S c(x)\delta_n(\cdot)(dx) \stackrel{*}{\rightharpoonup} \int_S c(x)\delta_0(\cdot)(dx) \text{ in } \mathcal{L}^\infty(\Omega; \mathbf{R}),$$

where "$\stackrel{*}{\rightharpoonup}$" denotes convergence in the topology $\sigma(\mathcal{L}^\infty(\Omega; \mathbf{R}), \mathcal{L}^1(\Omega; \mathbf{R}))$.

The following example continues the previous Examples 3.2 and 3.8.

Example 4.3 Let $(\Omega, \mathcal{A}, \mu)$ be $([0,1], \mathcal{L}([0,1]), \lambda_1)$ (cf. Example 3.2). As in Example 3.8, let $f_1 \in \mathcal{L}^1([0,1]; \mathbf{R})$ be arbitrary and extended periodically from $[0,1]$ to all of $\mathbf{R}$. We define $f_{n+1}(\omega) := f_1(2^n\omega)$. Then $\epsilon_{f_n} \stackrel{\tau}{\Longrightarrow} \delta_0$, where $\delta_0 \in \mathcal{R}([0,1]; \mathbf{R})$ is the constant function given by $\delta_0(\omega) \equiv \lambda_1^{f_1}$. Here $\lambda^{f_1} \in \mathcal{P}(\mathbf{R})$ is the image of λ_1 under the mapping f_1; i.e., $\lambda^{f_1}(B) := \lambda(f_1^{-1}(B))$. To prove the above convergence statement, let $c \in \mathcal{C}_b(\mathbf{R})$ be arbitrary, and let A be of the form $A = [0, \beta]$ with $\beta > 0$. Then a simple change of variable gives

$$\int_A c(f_{n+1}(\omega))d\omega = \int_0^\beta c(f_1(2^n\omega))d\omega = 2^{-n}\int_0^{2^n\beta} c(f_1(\omega'))d\omega',$$

and by periodicity of f_1 the latter expression equals $\beta \int_0^1 c(f_1(\omega'))d\omega' = \lambda_1(A)\int_{\mathbf{R}} c(x)\lambda_1^{f_1}(dx)$ in the limit. So it has been shown that

$$\lim_{n\to\infty} \int_A c(f_n(\omega))d\omega = \int_A [\int_{\mathbf{R}} c(x)\delta_0(\omega)(dx)]d\omega \tag{4.1}$$

for $A = [0, \beta]$. By subtraction, (4.1) continues to be valid for A's of the form $A = (\alpha, \beta]$, and, by summation, also for A's that are a finite disjoint union of such intervals. Finally, by [4, 1.3.11] for any $A \in \mathcal{A}$ and any $\epsilon > 0$ there exists a finite union A' of intervals $(\alpha, \beta]$ such that the symmetric difference of A and A' has Lebesgue measure at most ϵ. But then $|\int_A c(f_n) - \int_{A'} c(f_n)| \le \epsilon \sup_S |c|$, so, by letting ϵ go to zero, we conclude that (4.1) continues to hold in the general case.

The above example shows that δ_* in Example 3.8 is equal to the above δ_0, modulo a λ_1-null set. In fact, the narrow limit of a sequence of Young measures in $\mathcal{R}(\Omega; S)$ can only be essentially unique (that is to say, unique modulo a μ-null set). This follows immediately from the following general result:

Proposition 4.4 *For every δ, δ' in $\mathcal{R}(\Omega;S)$ the following are equivalent:*
(a) For every $A \in \mathcal{A}$ and $c \in \mathcal{C}_0$

$$\int_A [\int_S c(x)\delta(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c(x)\delta'(\omega)(dx)]\mu(d\omega).$$

(b) $\delta(\omega) = \delta'(\omega)$ for a.e. ω in Ω.

The essentially sequential setup chosen for these lecture notes leads to frequent use of a *semimetric $d_{\mathcal{R}}$* on $\mathcal{R}(\Omega;S)$, as defined in the next result. This allows us to use sequentially oriented approaches when we apply the narrow topology (the latter is of course defined by rereading Definition 4.1 with *generalized* sequences in mind).

Theorem 4.5 *Suppose that the σ-algebra $\mathcal{A}$ on Ω is countably generated. Then there exists a semimetric $d_{\mathcal{R}}$ on $\mathcal{R}(\Omega;S)$ such that for every (δ_n) and δ_0 in $\mathcal{R}(\Omega;S)$ the following are equivalent:*
(a) $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$.
(b) $\lim_n d_{\mathcal{R}}(\delta_n, \delta_0) = 0$.

PROOF. Define a semimetric on $\mathcal{R}(\Omega;S)$ by

$$d_{\mathcal{R}}(\delta,\delta') := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 2^{-i-j}|\int_{A_j}[\int_S c_i(x)\delta(\omega)(dx)]\mu(d\omega) - \int_{A_j}[\int_S c_i(x)\delta'(\omega)(dx)]\mu(d\omega)|/\mu(A_j).$$

Here (c_i) is an enumeration of the functions, conveniently normalized so as to give $\sup_S |c_i| = 1$ for each i, in the narrow convergence determining set $\mathcal{C}_0$ used in Lemma 3.10. Also, (A_j) is an at most countable algebra which generates $\mathcal{A}$.

$(a) \Rightarrow (b)$: By using the approximation result [4, 1.3.11] in the same way as in the above Example 4.3, it follows that

$$\lim_n \int_A [\int_S c_i(x)\delta_n(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c_i(x)\delta_0(\omega)(dx)]\mu(d\omega).$$

for every $A \in \mathcal{A}$ and every i. By the narrow convergence determining property of $\mathcal{C}_0$ in Lemma 3.10, this implies

$$[\mu \otimes \delta_n](A \times \cdot)/\mu(A) \stackrel{\rho}{\Rightarrow} [\mu \otimes \delta_0](A \times \cdot)/\mu(A) \text{ in } \mathcal{P}(S)$$

for every $A \in \mathcal{A}$ with $\mu(A) > 0$. By Remark 4.2 this implies $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$. The converse implication $(a) \Rightarrow (b)$ is very simple. QED

From Proposition 3.5 and its proof we immediately obtain that K-convergence implies narrow convergence:

Remark 4.6 *Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega;S)$. The following hold:*
(a) If $\delta_n \stackrel{K,\rho}{\to} \delta_0$, then $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$.
(b) If $\delta_n \stackrel{K,\rho}{\to} \delta_0$ and if (δ_n) is τ-tight, then $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$.
(c) If $\delta_n \stackrel{K,\tau}{\longrightarrow} \delta_0$, then $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$.

The implications in this remark cannot be reversed: the following example shows that a narrowly convergent sequence does not have to K-converge, even when S is the set of real numbers. Let us already mention that, nevertheless, in Theorem 4.13 below a partial converse will be achieved in terms of subsequences.

Example 4.7 Consider the sequence (f_n) of Rademacher functions from Example 3.2. Define the following sequence (f'_n) in $\mathcal{L}^1(\Omega;\mathbf{R})$: for each $m \in \mathbf{N}$ define $f'_n := f_m$ for $2^{m-1} \le n \le 2^m - 1$. From Examples 3.2 and 4.3 it is clear that $\epsilon_{f'_n} \stackrel{\tau}{\Longrightarrow} \delta_0$, where $\delta_0 \equiv \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_{-1}$ a.e. By Remark 4.6 we know that if $(\epsilon_{f'_n})$ were to K-converge to some Young measure, it would have to be δ_0 (modulo null sets). But it is easy to check the following: for $N = 2^m - 1$

$$\frac{1}{N}\sum_{n=1}^{N} \epsilon_{f'_n}(\omega) = \frac{1}{2^m - 1}\sum_{n=1}^{2^{m-2}-1} \epsilon_{f'_n}(\omega) + \frac{2^{m-2}}{2^m - 1}\epsilon_{f_{m-1}}(\omega) + \frac{2^{m-1}}{2^m - 1}\epsilon_{f_m}(\omega).$$

This shows that $\epsilon_{f'_n} \xrightarrow{K,\tau} \delta_0$ is not possible, since $2^{m-i}/(2^m - 1) \to 2^{-i}$ for $i = 1, 2$, and $\lambda_1(\{\omega \in \Omega : f_m(\omega) = f_{m-1}(\omega)\}) > 0$ for all $m \in \mathbf{N}$.

Concatenation of Theorem 3.7 and Remark 4.6 gives immediately a Prohorov-type result for narrow convergence of Young measures:

Theorem 4.8 *(i) Let (δ_n) be a τ_ρ-tight sequence in $\mathcal{R}(\Omega; S)$. Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and $\delta_* \in \mathcal{R}(\Omega; S)$ such that $\delta_{n'} \stackrel{\rho}{\Longrightarrow} \delta_*$.*
(ii) Moreover, if (δ_n) is τ-tight, then in fact $\delta_{n'} \stackrel{\tau}{\Longrightarrow} \delta_*$ can be achieved in (i).

Example 4.9 We continue with Example 3.4(*b*). By $\sigma(E, E')$-tightness of (ϵ_{f_n}) we get from Theorem 4.8 that there exist a subsequence $(f_{n'})$ of (f_n) and $\delta_* \in \mathcal{R}(\Omega; E)$ such that $\epsilon_{f_{n'}} \stackrel{\tau}{\Longrightarrow} \delta_*$.

(*a*) We now introduce a function $f_* \in \mathcal{L}^1_E$ that is "barycentrically" associated to δ_*, simply by inspecting the consequences of the tightness inequality $s := \sup_n I_{h_R}(\epsilon_{f_n}) < +\infty$ that was established there. For h_R is *a fortiori* a $\sigma(E, E')$-lower semicontinuous integrand, so Theorem 4.10(*e*) gives $I_{h_R}(\delta_*) \leq s < +\infty$, which implies $\int_S h_R(\omega, x)\delta_*(\omega)(dx) < +\infty$ for a.e. ω. So by the definition of h_R it follows that both $\delta_*(\omega)(R(\omega)) = 1$ and $\int_E \|x\|\delta_*(\omega)(dx) < +\infty$ for a.e. ω. By Theorem A.10(*i*) it follows that the barycenter $f_*(\omega) := \text{bar } \delta_*(\omega)$ of the probability measure $\delta_*(\omega)$ is defined for a.e. ω. Thus, if we set $f_* := 0$ on the exceptional null set, we obtain a function $f_* \in \mathcal{L}^0(\Omega; E)$. Finally we notice that, as announced, f_* is μ-integrable, i.e., $f_* \in \mathcal{L}^1(\Omega; E)$. This follows simply from $I_{h_R}(\delta_*) < +\infty$ by use of Jensen's inequality and the inequality $h_R(\omega, x) \geq \|x\|$.

(*b*) Suppose that in part (*a*) one has in addition that $(\|f_{n'}\|)$ is uniformly integrable in $\mathcal{L}^1(\Omega; \mathbf{R})$. Then $f_{n'} \xrightarrow{w} f_* \in \mathcal{L}^1(\Omega; E)$ (weak convergence in $\mathcal{L}^1(\Omega; E)$). This follows directly from another application of Theorem 4.10(*e*), namely, to all integrands g of the type $g(\omega, x) = \pm < x, b(\omega) >$, $b \in \mathcal{L}^\infty(\Omega, E')[E]$. The latter symbol denotes the set of all scalarly measurable bounded E'-valued functions on Ω; it forms the prequotient dual of $\mathcal{L}^1(\Omega; E)$. This yields $\lim_{n'} I_g(\epsilon_{f_{n'}}) = I_g(\delta_*)$, with $I_g(\epsilon_{f_{n'}}) = J_g(f_{n'}) = \int_\Omega < f_{n'}, b > d\mu$ and $I_g(\delta_*) = \int_\Omega < f_*, b > d\mu$ (cf. Theorem A.10(*i*)).

Part (*b*) in the above example implies that $f_n \xrightarrow{w} f_0$ in Example 4.3, where f_0 is the constant function given by $f_0(\omega) := \text{bar } \delta_0(\omega) = \int_{\mathbf{R}} f_1 d\lambda_1$ (apply [55, II.12]).

Proposition 3.5 and Theorem 3.7 imply the following transfer of the earlier portmanteau Theorem 2.6 to Young measures (see [16, Theorem 2.2] for other equivalences of this sort).

Theorem 4.10 *Suppose that (S, ρ) is Suslin. Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. The following are equivalent:*

(*a*) $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$.

(*b*) $\lim_n \int_A [\int_S c(x)\delta_n(\omega)(dx)]\mu(d\omega) = \int_A [\int_S c(x)\delta_0(\omega)(dx)]\mu(d\omega)$ *for every* $A \in \mathcal{A}$, $c \in \mathcal{C}_u(S, \rho)$.

(*c*) $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ *for every ρ-lower semicontinuous integrand g on $\Omega \times S$ such that*

$$\lim_{\alpha \to \infty} \sup_n \int_\Omega^* [\int_{\{g \leq -\alpha\}_\omega} g^-(\omega, x)\delta_n(\omega)(dx)]\mu(d\omega) = 0.$$

(ii) Moreover, if (δ_n) is τ-tight, then the above are also equivalent to the following:

(*d*) $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$.

(*e*) $\liminf_n I_g(\delta_n) \geq I_g(\delta_0)$ *for every sequentially τ-lower semicontinuous integrand g on $\Omega \times S$ such that*

$$\lim_{\alpha \to \infty} \sup_n \int_\Omega^* [\int_{\{g \leq -\alpha\}_\omega} g^-(\omega, x)\delta_n(\omega)(dx)]\mu(d\omega) = 0.$$

Observe that $(a) \Rightarrow (c)$ and $(d) \Rightarrow (e)$, which are the most powerful implications of the above theorem, constitute a very general theorem of Fatou-Vitali type for narrow convergence of Young measures. Results of this kind are usually obtained by means of approximation procedures for the lower semicontinuous integrands [48, 42, 7, 67, 9, 16, 94, 95]. In contrast to the present result, such procedures depend on approximation arguments requiring the measurable projection theorem and related Suslin conditions for S.

The following important lemma establishes that τ_ρ-narrow convergence implies τ_ρ-tightness when (S, ρ) is a Suslin space (note: this is the case in particular when (S, τ) itself is a Suslin space).

Lemma 4.11 *Suppose that (S, ρ) is Suslin. Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$ with $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$. Then (δ_n) is τ_ρ-tight.*

Proof. Set $\nu_n := [\mu \otimes \delta_n](\Omega \times \cdot)/\mu(\Omega)$; then $\nu_n \in \mathcal{P}_{Radon}(S, \tau_\rho)$ for every $n \in \mathbf{N} \cup \{0\}$, since $\mathcal{P}_{Radon}(S, \tau_\rho) = \mathcal{P}(S)$ by [55, III.69]. By Remark 4.2 it follows that $\nu_n \stackrel{\rho}{\Rightarrow} \nu_0$. Therefore, Theorem 2.19 implies that (ν_n) is τ_ρ-tight in $\mathcal{P}(S)$. By Definition 2.7(a), this means that there exists a τ_ρ-inf-compact function $h' : S \to [0, +\infty]$ such that $\sup_n \int_S h' d\nu_n < +\infty$. Now by definition of ν_n we have $\int_S h' d\nu_n = I_h(\delta_n)/\mu(\Omega)$ for every n, where $h(\omega, x) := h'(x)$. Thus $\sup_n I_h(\delta_n) < +\infty$, which demonstrates that (δ_n) is τ_ρ-tight. QED

Proof of Theorem 4.10. We start with the proof of part (i).

$(a) \Leftrightarrow (b)$: The equivalence follows immediately from the equivalence of (a) and (b) in Theorem 2.6 and Remark 4.2.

$(c) \Rightarrow (b)$: Obvious, for (b) follows by applying (c) to both $g(\omega, x) := 1_A(\omega)c(x)$ and $g'(\omega, x) := -1_A(\omega)c(x)$, with $A \in \mathcal{A}$ and $c \in \mathcal{C}_u(S, \rho)$.

$(a) \Rightarrow (c)$: For g as stated, let $\beta := \liminf_n I_g(\delta_n)$. Then $\beta = \lim_{n'} I_g(\delta_{n'})$ for a suitable subsequence $(\delta_{n'})$ of (δ_n). By Lemma 4.11 we have that (δ_n), whence $(\delta_{n'})$, is τ_ρ-tight, so by Theorem 3.7(i) there exists a subsequence $(\delta_{n''})$ of $(\delta_{n'})$ such that $\delta_{n''} \stackrel{K,\rho}{\to} \delta_*$ for some δ_* in $\mathcal{R}(\Omega; S)$. But in combination with (a) this implies $\delta_*(\omega) = \delta_0(\omega)$ a.e. (apply Remark 4.6 and Proposition 4.4), so in fact $\delta_{n''} \stackrel{K,\rho}{\to} \delta_0$. The desired Fatou-Vitali inequality $\beta \geq I_g(\delta_0)$ then follows from Proposition 3.5.

Next, we prove part (ii) of the theorem.

$(e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$: These all hold *a fortiori* (see also the proof of (i)).

$(a) \Rightarrow (e)$: The proof is virtually the same as the proof of $(a) \Rightarrow (c)$ that was given above. This time, tightness is forced *ab initio*; let h correspond to the condition of τ-tightness as in Definition 3.3. In the remainder of the proof of $(a) \Rightarrow (c)$ we now substitute $g^\epsilon := g + \epsilon h$, which is certainly a τ_ρ-lower semicontinuous integrand (see the proof of Proposition 3.5). Letting ϵ go to zero then gives (e). QED

Remark 4.12 *Note that in the above proof the Suslin space hypothesis for S (in the shape of Lemma 4.11) was only used one time, namely for the proof of the implication $(a) \Rightarrow (c)$.*

From Remark 4.6 we already know that K-convergence implies narrow convergence of Young measures. The above proof of Theorem 4.10 enables us now to characterize narrow convergence completely in terms of K-convergence:

Theorem 4.13 *(i) Suppose that (S, ρ) is Suslin. Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. The following are are equivalent:*

(a) $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$.

(b) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \stackrel{K,\rho}{\to} \delta_0$.

(ii) Moreover, if (δ_n) is τ-tight, then the above are also equivalent to the following:

(c) $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$.

(d) Every subsequence $(\delta_{n'})$ of (δ_n) contains a further subsequence $(\delta_{n''})$ such that $\delta_{n''} \stackrel{K,\tau}{\longrightarrow} \delta_0$.

In parts (b) and (d) the use of subsequences cannot be replaced by the use of the entire sequence (δ_n) itself, because of Example 4.7. Observe also that in part (ii) the Suslin space hypothesis is actually not needed by Remark 4.12.

Theorem 4.14 *(i) Suppose that (S, ρ) is Suslin. Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$. The following are are equivalent:*

(a) $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$.

(b) $\delta_n \times \epsilon_n \stackrel{\tilde{\rho}}{\Longrightarrow} \delta_0 \times \epsilon_\infty$.

(ii) Moreover, if (δ_n) is τ-tight, then the above are also equivalent to the following:

(c) $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$.

(d) $\delta_n \times \epsilon_n \stackrel{\tilde{\tau}}{\Longrightarrow} \delta_0 \times \epsilon_\infty$.

This result, which is the Young measure analogue of Corollary 2.11, follows simply from Proposition 3.13 by Theorem 4.13. Observe once more that in part (ii) the Suslin space hypothesis is actually not needed by Remark 4.12. The transfer of the support Theorem 2.17 to Young measures is now immediate because of the intermediate support Theorem 3.14 and Theorem 4.13:

Theorem 4.15 *(i) Suppose that (S, ρ) is Suslin. Let (δ_n) and δ_0 be in $\mathcal{R}(\Omega; S)$ with $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$. Then*

$$\tau_\rho\text{-supp } \delta_0(\omega) \subset \tau_\rho\text{-}\mathrm{Ls}_n\, \tau_\rho\text{-supp } \delta_n(\omega) \textit{ for a.e. } \omega \textit{ in } \Omega.$$

(ii) Let (δ_n), δ_0 be in $\mathcal{R}(\Omega; S)$, with $\delta_n \stackrel{\tau}{\Longrightarrow} \delta_0$ and (δ_n) τ-tight. Then

$$\delta_0(\omega)(\tau\text{-seq-cl } \tau\text{-}\mathrm{Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1 \textit{ for a.e. } \omega \textit{ in } \Omega.$$

As before, in part (ii) the Suslin space hypothesis is actually not needed (Remark 4.12).

Next, we examine narrow convergence when it is restricted to the set $\mathcal{R}_{Dirac}(\Omega; S)$. Recall first that a sequence (f_n) in $\mathcal{L}^0(\Omega; S)$ is defined to converge *in measure* to $f_0 \in \mathcal{L}^0(\Omega; S)$ (we denote this as $f_n \stackrel{\mu}{\rightarrow} f_0$) if for every $\epsilon > 0$

$$\lim_n \mu(\{\omega \in \Omega : \rho(f_n(\omega), f_0(\omega)) > \epsilon\}) = 0.$$

Recall also that for any $f \in \mathcal{L}^0(\Omega; S)$ the *image measure* μ^f of μ under f is defined by $\mu^f(B) := \mu(f^{-1}(B))$, $B \in \mathcal{B}(S)$; by $\epsilon_f(\omega)(B) = 1_B(f(\omega))$ this implies $[\mu \otimes \epsilon_f](\Omega \times \cdot) = \mu^f(\cdot)$.

Proposition 4.16 *Suppose that (S, ρ) is Suslin. Let (f_n) and f_0 be in $\mathcal{L}^0(\Omega; S)$. Then the following are equivalent:*

(a) $\epsilon_{f_n} \stackrel{\rho}{\Longrightarrow} \epsilon_{f_0}$ *in* $\mathcal{R}_{Dirac}(\Omega; S)$.

(b) $f_n \stackrel{\mu}{\rightarrow} f_0$ *in* $\mathcal{L}^0(\Omega; S)$.

Proof. $(a) \Rightarrow (b)$: Let $\epsilon > 0$ be arbitrary. Define a lower semicontinuous integrand on $\Omega \times S$ by

$$g(\omega, x) := \begin{cases} -1 & \text{if } \rho(x, f_0(\omega)) \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.11 and Theorem 4.10(i) we have $\liminf_n J_g(f_n) \geq J_g(f_0) = 0$; i.e., $\limsup_n \mu(\{\omega \in \Omega : \rho(f_n(\omega), f_0(\omega)) \geq \epsilon\}) = 0$.

$(b) \Rightarrow (a)$: Let $A \in \mathcal{A}$, $c \in \mathcal{C}_b(S, \rho)$ be arbitrary. It is enough to prove that $\beta = \int_A c(f_0) d\mu$ for $\beta := \liminf_n \int_A c(f_n) d\mu$ (for the same argument applies to $-c$). Clearly, there exists a subsequence $(f_{n'})$ such that $\beta = \lim_{n'} \int_A c(f_{n'}) d\mu$. By (b), $(\rho(f_{n'}, f_0))$ certainly converges in measure to zero in $\mathcal{L}^0(\Omega; \mathbf{R})$. So by [4, Theorem 2.5.3] $(f_{n'})$ has a subsequence $(f_{n''})$ that ρ-converges a.e. to f_0. The desired identity for β thus follows from the dominated convergence theorem. QED

Next, Theorem 2.12 is transferred to tensor products of Young measures. Let $(\Omega', \mathcal{A}', \mu')$ be another finite measure space and let (S', τ') be another topological space for which the obvious analogues of Hypotheses 2.1, 2.3 hold; we denote the associated metric on S' by ρ' (observe that the topological space $S \times S'$ then also meets the analogue of Hypotheses 2.1, 2.3). The *tensor product* $\delta \otimes \delta'$ of $\delta \in \mathcal{R}(\Omega; S)$ and $\delta' \in \mathcal{R}(\Omega'; S')$ is defined by

$$(\delta \otimes \delta')(\omega, \omega') := \delta(\omega) \times \delta'(\omega'),$$

i.e., $(\delta \otimes \delta')(\omega, \omega')$ is the product of the two probability measures $\delta(\omega)$ and $\delta'(\omega')$. It is clear that $\delta \otimes \delta'$, thus defined, is a transition probability from $(\Omega \times \Omega', \mathcal{A} \times \mathcal{A}')$ into $S \times S'$; hence, it belongs to $\mathcal{R}(\Omega \times \Omega'; S \times S')$. We now present a continuity result for the tensor product with respect to narrow convergence. There is also a fully topological analogue; see [16] where these results were first introduced (see also [97, Ch. IX]).

Theorem 4.17 *(i) Let $\delta_n \stackrel{\rho}{\Longrightarrow} \delta_0$ in $\mathcal{R}(\Omega; S)$ and let $\delta'_n \stackrel{\rho'}{\Longrightarrow} \delta'_0$ in $\mathcal{R}(\Omega'; S')$. Then*

$$\delta_n \otimes \delta'_n \stackrel{\rho \times \rho'}{\Longrightarrow} \delta_0 \otimes \delta'_0 \text{ in } \mathcal{R}(\Omega \times \Omega'; S \times S').$$

(ii) Moreover, if (δ_n) is τ-tight and (δ'_n) is τ'-tight, then

$$\delta_n \otimes \delta'_n \stackrel{\tau \times \tau'}{\Longrightarrow} \delta_0 \otimes \delta'_0 \text{ in } \mathcal{R}(\Omega \times \Omega'; S \times S').$$

Lemma 4.18 *For every $\tilde{A} \in \mathcal{A} \times \mathcal{A}'$ and every ϵ there exist finitely many disjoint measurable rectangles $A_i \times A'_i$ in $\mathcal{A} \times \mathcal{A}'$, $i = 1, \ldots, m$, such that the symmetric difference of $\tilde{A}$ and $\cup_{i=1}^m A_i \times A'_i$ has $\mu \times \mu'$-measure at most ϵ.*

Proof. The algebra consisting of finite disjoint unions of measurable rectangles generates $\mathcal{A} \times \mathcal{A}'$; hence, the result follows by [4, 1.3.11]. QED

Proof of Theorem 4.17. (i) Let $\tilde{A} \in \mathcal{A} \times \mathcal{A}'$ and $c \in \mathcal{C}_b(S \times S', \rho \times \rho')$, and set $g(\omega, \omega', x, x') := 1_{\tilde{A}}(\omega, \omega') c(x, x')$. Since uniform limits of finite sums of continuous functions are continuous, the result obtained in Lemma 4.18 enables us to just consider the case $\tilde{A} = A \times A'$, with $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$. We may also suppose $\mu(A) > 0$, $\mu'(A') > 0$. Then $I_g(\delta_n \otimes \delta'_n) = \mu(A)\mu'(A') \int_{S \times S'} c d(\nu_n \times \nu'_n)$, where $\nu_n := [\mu \otimes \delta_n](A \times \cdot)/\mu(A)$ and $\nu'_n := [\mu \otimes \delta'_n](A' \times \cdot)/\mu'(A')$ satisfy $\nu_n \stackrel{\rho}{\Rightarrow} \nu_0$ and $\nu'_n \stackrel{\rho'}{\Rightarrow} \nu'_0$, in view of Remark 4.2. By Theorem 2.12(i) this gives $I_g(\delta_n \otimes \delta'_n) \to I_g(\delta_0 \otimes \delta'_0)$. This finishes the proof of part (i). Part (ii) directly follows by Theorem 4.8(ii), since $(\delta_n \otimes \delta'_n)$ is evidently tight for $\tau \times \tau'$. Alternatively, it can be obtained as above by using Theorem 2.12(ii) this time. QED

As shown by the following counterexample, Theorem 4.17 need not hold when the measure on $(\Omega \times \Omega', \mathcal{A} \times \mathcal{A}')$ is not a product measure, even when μ and μ' are its marginals.

Example 4.19 Take for $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ the space $([0,1], \mathcal{B}([0,1]))$. Let (f_n) be the sequence of Rademacher functions on Ω and let (f'_n) be the sequence of Rademacher functions on Ω' (see Example 3.2). Equip $\tilde{\Omega} := [0,1]^2$ with $\tilde{\mathcal{A}} := \mathcal{B}([0,1]^2)$ and $\tilde{\mu}$, defined to be the uniform measure concentrated on the diagonal of $[0,1]^2$. Equip $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ each with the Lebesgue measure. Then by Example 3.2, we have $\epsilon_{f_n} \stackrel{\tau}{\Longrightarrow} \delta_0$ in $\mathcal{R}(\Omega; \mathbf{R})$ and $\epsilon_{f'_n} \stackrel{\tau}{\Longrightarrow} \delta_0$ in $\mathcal{R}(\Omega'; \mathbf{R})$, but $(\epsilon_{f_n} \otimes \epsilon_{f'_n})$ does *not* narrowly converge to $\delta_0 \otimes \delta_0$ in $\mathcal{R}(\tilde{\Omega}; \mathbf{R}^2)$. To see the latter, apply Definition 4.1 with $A := \tilde{\Omega}$ and $c(x, x') := xx'$; then in Definition 4.1 the limit on the left equals 1, but the expression on the right is equal to 0.

5 Lower closure

Let $(\Omega, \mathcal{A}, \mu)$ be as in section 4 and let (S, τ) be a completely regular Suslin space (cf. Remark 2.4(ii)). In this section we combine the main results from section 4 in the form of so-called *lower closure* results. As an abstract starting point for lower closure we have the following immediate consequence of Theorems 4.8, 4.14 and 4.15:

Theorem 5.1 *Let (δ_n) be a τ-tight sequence in $\mathcal{R}(\Omega; S)$. Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and $\delta_* \in \mathcal{R}(\Omega; S)$ such that*

$$\delta_{n'} \times \epsilon_n \stackrel{\tilde{\tau}}{\Longrightarrow} \delta_* \times \epsilon_\infty.$$

Besides, δ_* has the following pointwise support property:

$$\delta_*(\omega)(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1 \textit{ for a.e. } \omega \textit{ in } \Omega.$$

Somewhat more concretely Theorem 5.1 can be stated as follows. Let (D, d_D) be an arbitrary metric space.

Theorem 5.2 *Let (δ_n) in $\mathcal{R}(\Omega; S)$ be τ-tight and let $d_n \xrightarrow{\mu} d_0$ in $\mathcal{L}^0(\Omega; D)$ (convergence in measure). Then there exist a subsequence $(\delta_{n'})$ of (δ_n) and δ_* in $\mathcal{R}(\Omega; S)$ such that*

$$\liminf_{n'} \int_\Omega^* [\int_S \ell(\omega, x, d_{n'}(\omega))\delta_{n'}(\omega)(dx)]\mu(d\omega) \geq \int_\Omega^* [\int_S \ell(\omega, x, d_0(\omega))\delta_*(\omega)(dx)]\mu(d\omega)$$

for every sequentially $\tau \times \tau_{d_D}$-sequentially lower semicontinuous integrand ℓ on $\Omega \times (S \times D)$ such that

$$s'(\alpha) := \sup_n \int_\Omega^* [\int_{\{\ell \leq -\alpha\}_{\omega,n}} \ell^-(\omega, x, d_n(\omega))\delta_n(\omega)(dx)]\mu(d\omega) \to 0 \textit{ for } \alpha \to \infty. \tag{5.1}$$

Besides, δ_* has the following pointwise support property:

$$\delta_*(\omega)(\tau\text{-seq-cl } \tau\text{-Ls}_n \tau\text{-supp } \delta_n(\omega)) = 1 \textit{ for a.e. } \omega \textit{ in } \Omega.$$

Here $\{\ell \leq -\alpha\}_{\omega,n}$ stands for the set of all $x \in S$ for which $\ell(\omega, x, d_n(\omega)) \leq -\alpha$.

PROOF. Theorem 4.8 and well-known facts about convergence in measure ([4, Theorem 2.5.3]) imply the existence of a subsequence $(\delta_{n'}, d_{n'})$ of (δ_n, d_n) and existence of a $\delta_* \in \mathcal{R}(T; S)$ such that $\delta_{n'} \stackrel{\tau}{\Longrightarrow} \delta_*$ and $d_D(d_{n'}(\omega), d_0(\omega)) \to 0$ for a.e. ω. By Theorem 4.15 this implies the stated pointwise support property for δ_*. By Theorem 4.14 this gives $\tilde{\delta}_{n'} \stackrel{\tilde{\tau}}{\Longrightarrow} \tilde{\delta}_*$ in $\mathcal{R}(\Omega; \tilde{S})$, with $\tilde{S} := S \times \hat{\mathbf{N}}$, $\tilde{\delta}_n := \delta_n \times \epsilon_n$ and $\tilde{\delta}_* := \delta_* \times \epsilon_\infty$. Rather than to renumber, we suppose without loss of generality that (n') enumerates *all* the numbers in $\mathbf{N}$. Let ℓ be as stated. We define $g_\ell : \Omega \times \tilde{S} \to (-\infty, +\infty]$ by

$$g_\ell(\omega, \tilde{x}) := \begin{cases} \ell(\omega, x, d_k(\omega)) & \text{if } k < \infty \\ \ell(\omega, x, d_0(\omega)) & \text{if } k = \infty \end{cases}$$

Then g_ℓ is a $\tilde{\tau}$-lower semicontinuous integrand, modulo an insignificant null set (note that for $k = \infty$ lower semicontinuity of $g_\ell(\omega, \cdot)$ at (x, ∞) follows from $d_{n'}(\omega) \to d_0(\omega)$ and the lower semicontinuity of $\ell(\omega, \cdot, \cdot)$ at $(x, d_0(\omega))$). Since (5.1) coincides with (3.1) for $g = g_\ell$, we may apply Theorem 4.10 to g_ℓ. This gives $\liminf_{n'} I_{g_\ell}(\tilde{\delta}_{n'}) \geq I_{g_\ell}(\tilde{\delta}_*)$. Since the following identities hold elementarily for each n' and ω:

$$\int_{\tilde{S}} g_\ell(\omega, \tilde{x})\tilde{\delta}_{n'}(\omega)(d\tilde{x}) = \int_S \ell(\omega, x, d_{n'}(\omega))\delta_{n'}(\omega)(dx),$$

$$\int_{\tilde{S}} g_\ell(\omega, \tilde{x})\tilde{\delta}_*(\omega)(d\tilde{x}) = \int_S \ell(\omega, x, d_0(\omega))\delta_*(\omega)(dx),$$

the main inequality of the theorem has also been proven. QED

Remark 5.3 *Let h be the nonnegative, sequentially τ-inf-compact integrand h on $\Omega \times S$ that corresponds as in Definition 3.3 to the τ-tight sequence (δ_n) in Theorem 5.2; i.e., with $s := \sup_n I_h(\delta_n) < +\infty$. Then the uniform integrability condition (5.1) applies whenever the integrand ℓ has the following growth property with respect to h: for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^1(\Omega; \mathbf{R})$ such that for every $n \in \mathbf{N}$*

$$\ell^-(\omega, x, d_n(\omega)) \leq \epsilon h(\omega, x) + \phi_\epsilon(\omega) \textit{ on } \Omega \times \times S.$$

Indeed, we can observe that the set $\{\ell \leq -\alpha\}_{\omega,n}$ in (5.1) is contained in the union of $\{\phi_\epsilon < \epsilon h\}$ and $\{\phi_\epsilon \geq \alpha/2\}$, which gives $s'(\alpha) \leq 3\epsilon s + \int_{\{\phi_\epsilon \geq \alpha/2\}} \phi_\epsilon \, d\mu$, whence $s'(\alpha) \to 0$ for $\alpha \to \infty$, as claimed.

Let us show that the so-called fundamental theorem for Young measures in [40] follows from Theorem 5.2. To this end, let L be a locally compact space that is countable at infinity; its usual Alexandrov compactification is denoted by $\hat{L} := L \cup \{\infty\}$. Although it could be avoided by the additional introduction of transition subprobabilities (see the comments below), the Alexandrov compactification $\hat{L}$ of L figures explicitly in the result. The space $\hat{L}$ is metrizable, and its metric is denoted by $\hat{d}$. On L we use the natural restriction of $\hat{d}$, and denote it by d. Let $\mathcal{C}_0(L)$ be the usual space of continuous functions on L that converge to zero at infinity. Also, below ν denotes a σ-finite measure on $(\Omega, \mathcal{A})$.

Corollary 5.4 *(i) Let (f_n) in $\mathcal{L}^0(\Omega; L)$ and the closed set $C \subset L$ be such that $\lim_n \nu(f_n^{-1}(L\backslash G)) = 0$ for every open G, $C \subset G \subset L$. Then there exist a subsequence $(f_{n'})$ of (f_n) and δ_* in $\mathcal{R}(\Omega; \hat{L})$ such that*

$$\lim_n \int_\Omega \phi(\omega) c(f_{n'}(\omega))\nu(d\omega) = \int_\Omega [\int_L \phi(\omega) c(x) \delta_*(\omega)(dx)]\nu(d\omega)$$

for every $\phi \in \mathcal{L}^1(\Omega; \mathbf{R})$ and every $c \in \mathcal{C}_0(L)$. Besides, we have $\delta_*(\omega)(L\backslash C) = 0$ for a.e. ω in Ω.
(ii) Moreover, if for that subsequence $(f_{n'})$ there exists a sequence (K_r) of compact sets in L such that $\lim_{r\to\infty} \sup_{n'} \nu(\{\omega \in \Omega : f_{n'}(\omega) \notin K_r\} = 0$ then $\delta_*(\omega)(\{\infty\}) = 0$ for a.e. ω in Ω and

$$\lim_n \int_A \phi(\omega) c(f_{n'}(\omega))\nu(d\omega) = \int_A [\int_L \phi(\omega) c(x) \delta_*(\omega)(dx)]\nu(d\omega)$$

for every $A \in \mathcal{A}$, $\phi \in \mathcal{L}^1(A; \mathbf{R})$ and $c \in \mathcal{C}(L)$ for which $(1_A c(f_{n'}))$ is relatively weakly compact in $\mathcal{L}^1(A; \mathbf{R})$.

In [40] both L and Ω are Euclidean, and the K_r's are closed balls around the origin with radius r. As was done in [40], the result could be equivalently restated in terms of the transition *sub*probability δ'_* from $(\Omega, \mathcal{A})$ into $(L, \mathcal{B}(L))$, defined by obvious restriction to L, i.e., $\delta'_*(\omega)(B) := \delta_*(\omega)(B \cup \{\infty\})$, $B \in \mathcal{B}(L)$. In this connection the tightness condition in part (ii) guarantees that δ_* is an authentic transition probability (Young measure). Rather than via (i), part (ii) can also be derived directly from Theorem 3.7 or 5.2.

PROOF. (i) By σ-finiteness of ν, there exists a finite measure μ that is equivalent to ν. Let $\tilde{\phi}$ be a version of the Radon-Nikodym density $d\nu/d\mu$. Now (δ_n), defined by $\delta_n := \epsilon_{f_n} \in \mathcal{R}(\Omega, ; \hat{L})$, is trivially tight by compactness of $\hat{L}$ (set $h \equiv 0$). By Theorem 4.8 or 5.2 there exist a subsequence $(f_{n'})$ of (f_n) and $\delta_* \in \mathcal{R}(\Omega; \hat{L})$ for which $\epsilon_{f_{n'}} \stackrel{\rho}{\Longrightarrow} \delta_*$. Every $c \in \mathcal{C}_0(L)$ has a canonical extension $\hat{c} \in \mathcal{C}_b(\hat{L})$ by setting $\hat{c}(\infty) = 0$. Now $\phi\tilde{\phi}$ is μ-integrable for any $\phi \in \mathcal{L}^1(\Omega, \mathcal{A}, \nu; \mathbf{R})$, and Theorem 4.10 (or 5.2) can be applied to $g : \Omega \times \hat{L} \to \mathbf{R}$ given by $g(\omega, x) := \pm\phi(\omega)\tilde{\phi}(\omega)\hat{c}(x)$. This gives the desired equality, because of the identity $\int_\Omega \phi\tilde{\phi} \int_L \hat{c}(x)\delta_*(\cdot)(dx)d\mu = \int_\Omega \phi \int_L c(x)\delta_*(\cdot)(dx)d\nu$.

Next, let C be as stated. For any $i \in \mathbf{N}$ the set F_i, consisting of all $x \in L$ with d-dist$(x, C) \leq i^{-1}$, is closed in L. Note already that $\cap_i F_i = C$, by the given τ_d-closedness of C in L. Further, $\hat{F}_i := F_i \cup \{\infty\}$ is closed in $\hat{L}$. Set $\hat{g}_i(\omega, x) := \tilde{\phi}(\omega)1_{\hat{L}\backslash\hat{F}_i}(x)$. This defines a nonnegative lower semicontinuous integrand $\hat{g}_i$ on $\Omega \times \hat{L}$. Hence, $I_{\hat{g}_i}(\delta_*) \leq \beta_i := \liminf_{n'} I_{\hat{g}_i}(\epsilon_{f_{n'}})$ by Theorem 4.10(c). By $\hat{L}\backslash\hat{F}_i = L\backslash F_i$ the definitions of $\hat{g}_i$ and $\epsilon_{f_{n'}}$ give $I_{\hat{g}_i}(\epsilon_{f_{n'}}) = \nu(f_{n'}^{-1}(L\backslash F_i))$. So $\beta_i = \liminf_{n'} \nu(f_{n'}^{-1}(L\backslash F_i)) \leq \nu(f_{n'}^{-1}(L\backslash G_i))$, where G_i, $G_i \subset F_i$, is the τ_d-open set of all $x \in L$ with d-dist$(x, C) < i^{-1}$. Since $G_i \supset C$, the hypotheses imply $0 = \beta_i \geq I_{\hat{g}_i}(\delta_*) = \int_\Omega \delta_*(\cdot)(L\backslash F_i)d\nu$. Hence $\delta_*(\omega)(L\backslash C) = 0$ ν-a.e. because of $\cap_i F_i = C$, which was demonstrated above.

(ii) The additional condition is then a tightness condition for (ϵ_{f_n}), when viewed as a subset of $\mathcal{R}(\Omega; L)$ (take $\Gamma_\epsilon \equiv K_r$ for large enough r in Definition 3.3(b)). Hence, there is a τ_ρ-inf-compact integrand h on $\Omega \times L$ with $s := \sup_n I_h(\delta_n) < +\infty$. Define the inf-compact integrand $\hat{h}$ on $\Omega \times \hat{L}$ by $\hat{h}(\omega, x) := h(\omega, x)$ if $x \in L$ and $\hat{h}(\omega, \infty) := +\infty$. Since $\hat{h}$ is in particular a lower semicontinuous integrand on $\Omega \times \hat{L}$, we have $I_{\hat{h}}(\delta_*) \leq \liminf_{n'} I_{\hat{h}}(\epsilon_{f_{n'}})$ by Theorem 4.10. Trivially, $I_{\hat{h}}(\epsilon_{f_{n'}}) = I_h(\epsilon_{f_{n'}})$, so we get $I_{\hat{h}}(\delta_*) \leq s < +\infty$. The latter shows that $\delta_*(\omega)(\{\infty\}) = 0$ for μ-a.e. ω in Ω, whence for ν-a.e. ω. So δ_* can also be viewed as an element of $\mathcal{R}(\Omega; L)$, for which we then get $\epsilon_{f_{n'}} \stackrel{d}{\Longrightarrow} \delta_*$ in $\mathcal{R}(\Omega; L)$ by the above. To conclude, observe that for any $A \in \mathcal{A}$ with $\nu(A) < +\infty$

Theorem 4.10 applies to $g(\omega, x) := \pm 1_A(\omega)\phi(\omega)\tilde{\phi}(\omega)c(x)$, which is a continuous integrand on $\Omega \times L$ that is μ-integrably bounded. In view of part (i), this gives the desired limit statement if A has finite measure. If $\nu(A) = +\infty$ and A is as stated, there exists, by ν's σ-finiteness, a sequence (A_j) of subsets of A with finite ν-measure, with $A_j \uparrow A$. The previous result applies to each of the A_j and the weak relative compactness hypothesis implies uniform σ-additivity, i.e., $\sup_n \int_{A \setminus A_j} |c(f_n)| d\nu \downarrow 0$ [47]. So also in this case the desired limit statement follows. QED

If in the above lower closure Theorem 5.2 additional conditions are imposed upon the Young measures (δ_n), then extra "barycentric" information about δ_* may become available in terms of its marginals. In this way, Theorem 5.2 will be turned into a very general lower closure result "with convexity". Let E and F be separable Banach spaces, each of which is equipped with a locally convex Hausdorff topology, respectively denoted by τ_E and τ_F, that is not weaker than the weak topology and not stronger than the norm topology. As usual, $\mathcal{L}^1(\Omega; E)$ denotes the space of all Bochner integrable E-valued functions (here this is precisely the space of all $e \in \mathcal{L}^0(\Omega; E)$ such that $\|e(\cdot)\|_E$ is μ-integrable). Let (D, d_D) be a metric space. Functions that are "barycentrically" associated to Young measures can play a special role in lower closure and existence results. This is demonstrated by our proof of the following result.

Theorem 5.5 *Let $d_n \xrightarrow{\mu} d_0$ in $\mathcal{L}^0(\Omega; D)$ (convergence in measure), $e_n \xrightarrow{w} e_0$ in $\mathcal{L}^1(\Omega; E)$ (weak convergence), and let (f_n) in $\mathcal{L}^1(\Omega; F)$ satisfy $\sup_n \int_\Omega \|f_n\|_F d\mu < +\infty$. Suppose that there exist τ_E- and τ_F-ball-compact multifunctions $R_E : \Omega \to 2^E$ and $R_F : \Omega \to 2^F$ (cf. Example 3.4) such that*

$$\{(e_n(\omega), f_n(\omega)) : n \in \mathbf{N}\} \subset R_E(\omega) \times R_F(\omega) \ \mu\text{-a.e.}$$

Then there exist a subsequence $(d_{n'}, e_{n'}, f_{n'})$ of (d_n, e_n, f_n) and $f_* \in \mathcal{L}^1(\Omega; F)$ such that

$$\liminf_{n'} \int_\Omega^* \ell(\omega, e_{n'}(\omega), f_{n'}(\omega), d_{n'}(\omega))\mu(d\omega) \geq \int_\Omega^* \ell(\omega, e_0(\omega), f_*(\omega), d_0(\omega))\mu(d\omega)$$

for every sequentially $\tau_E \times \tau_F \times \tau_D$-lower semicontinuous integrand ℓ on $\Omega \times (E \times F \times D)$ such that the following hold:

$$(\ell^-(\cdot, e_n(\cdot), f_n(\cdot), d_n(\cdot))) \text{ is uniformly (outer) integrable}$$

(see Remark 3.6(ii)) and

$$\ell(\omega, \cdot, \cdot, d_0(\omega)) \text{ is convex on } E \times F \text{ for a.e. } \omega.$$

Besides, the functions e_0 and f_* can be localized as follows: [2]

$$(e_0(\omega), f_*(\omega)) \in \text{cl co-w-Ls}_n\{(e_n(\omega), f_n(\omega))\} \text{ for a.e. } \omega \text{ in } \Omega.$$

Observe, as was already done following Example 3.4, that the ball-compactness condition involving R_E and R_F is *automatically* satisfied in case the Banach spaces E and F are reflexive.

PROOF. To apply Theorem 5.2 we set $S := E \times F$, $\tau := \tau_E \times \tau_F$ and $\delta_n := \epsilon_{(e_n, f_n)}$. Observe that S is a separable Banach space for the product norm $\|\cdot\|_S$, so (S, τ) is a Suslin space, and by the Hahn-Banach theorem (S, τ) is completely regular. Next, we note that $(\|e_n\|)$ in $\mathcal{L}^1(\Omega; \mathbf{R})$ is uniformly integrable; this follows from the weak convergence hypothesis (apply [47, Theorem 1] and [81, Proposition II.5.2]). In particular, this implies $\sup_n \int_\Omega \|(e_n, f_n)\|_S d\mu < +\infty$. By τ-ball-compactness of $R := R_E \times R_F$ this proves that (δ_n) is τ-tight, in view of Example 3.4. We can now apply Theorem 5.2. Let the subsequence $(\delta_{n'}, d_{n'})$ of (δ_n, d_n) and δ_* in $\mathcal{R}(\Omega; S)$ be as guaranteed by that theorem, i.e., with $\delta_{n'} \stackrel{\tau}{\Longrightarrow} \delta_*$. Then it is elementary to establish from Definition 4.1 that, "E-marginally", $\epsilon_{e_{n'}} \stackrel{\tau}{\Longrightarrow} \delta_*^E$ and, "F-marginally", $\epsilon_{f_{n'}} \stackrel{\tau}{\Longrightarrow} \delta_*^F$. Here $\delta_*^E(\omega) := \delta_*(\omega)(E \times \cdot)$ and $\delta_*^F(\omega) := \delta_*(\omega)(\cdot \times F)$. So E-marginally we then have the situation of Example 4.9(b), which gives that bar $\delta_*^E = e_0$ a.e. Also, F-marginally we have the more primitive situation of Example 4.9(a),

[2] In case E and F are finite-dimensional one may replace here "cl co" by "co".

which gives existence of $f_* \in \mathcal{L}^1(\Omega; F)$ such that $f_* = \text{bar } \delta_*^F$ a.e. (note that τ_E- and τ_F -ball-compactness imply $\sigma(E, E')$- and $\sigma(F, F')$-ball-compactness respectively). Recombining the above two marginal cases, we find bar $\delta_* = (e_0, f_*)$ a.e. (note that barycenters decompose marginally).

We now finish the proof. For an integrand ℓ of the stated variety Theorem 5.2 gives

$$\liminf_{n'} \int_\Omega^* \ell(\omega, e_{n'}(\omega), f_{n'}(\omega), d_{n'}(\omega))\mu(d\omega) \geq \int_\Omega^* [\int_{E\times F} \ell(\omega, x, y, d_0(\omega))\delta_*(\omega)(d(x,y))]\mu(d\omega)$$

(see also Remark 3.6(ii)). In the inner integral above, the convexity of $\ell(\omega, \cdot, \cdot, d_0(\omega))$ gives

$$\int_{E\times F} \ell(\omega, x, y, d_0(\omega))\delta_*(\omega) \geq g(\omega, \text{bar } \delta_*(\omega), d_0(\omega)) = g(\omega, e_0(\omega), f_*(\omega), d_0(\omega))$$

for a.e. ω, by Jensen's inequality and our previous identity bar $\delta_* = (e_0, f_*)$ a.e. The desired inequality thus follows. QED

The above lower closure result "with convexity" is quite general: it further extends the results in [9, 14], which in turn already generalize several lower closure results in the literature, including those for orientor fields (cf. [52]). See [22] for another development, not covered by the above result. Results of this kind are very useful in the existence theory for optimal control and optimal growth theory. Corollaries of Theorem 5.5 are so-called weak-strong lower semicontinuity results for integral functionals in the calculus of variations and optimal growth theory; cf. [45, 52, 65]. Recently, similar-spirited versions that employ quasi-convexity in the sense of Morrey have been derived from Theorem 5.2 in [72, 90] (these have for e_n the gradient function of d_n and depend on a characterization of so-called gradient Young measures [83]). Another result that is generalized by the above theorem is as follows.

Corollary 5.6 *Let $f_n \xrightarrow{w} f_0$ in $\mathcal{L}^1(\Omega; \mathbf{R}^d)$ (weak convergence). Then*

$$f_0(\omega) \in \text{co-Ls}_n\{f_n(\omega)\} \text{ for a.e. } \omega \text{ in } \Omega.$$

This result is due to Z. Artstein [3, Proposition C]. It is obtained from Theorem 5.5 by setting $E := \mathbf{R}^d$ and activating the footnote in its statement. We turn briefly to an extension of the Dunford-Pettis theorem (sufficiency part); this comes from [25, 32] and generalizes [49] and [36, Lemma 4.3]. Again E denotes a separable Banach space.

Theorem 5.7 *Let (f_n) in $\mathcal{L}^1(\Omega; E)$ be uniformly integrable and such that for every $\epsilon > 0$ there is a multifunction $\Gamma_\epsilon : \Omega \to 2^E$, having norm-compact values with $\mu^*(\{\omega \in \Omega : f_n(\omega) \notin \Gamma_\epsilon(\omega)\}) \leq \epsilon$ for all n. Then there exist a subsequence $(f_{n'})$ of (f_n) and $f_* \in \mathcal{L}^1(\Omega; E)$ such that $\lim_{n'} \| \int_A f_{n'} d\mu - \int_A f_* d\mu\| = 0$ for every $A \in \mathcal{A}$.*

Above μ^* stands for outer μ-measure. Obviously, when E is finite-dimensional, the tightness condition in the above result holds automatically and we get the Dunford-Pettis theorem (sufficiency part).

PROOF. We set $S := E$ and $\tau :=$ norm-topology. By Definition 3.3(b), the sequence (ϵ_{f_n}) is τ-tight. Also, by uniform integrability, (f_n) is of course bounded in $\mathcal{L}^1$-seminorm. Theorem 3.7 gives existence of a subsequence $(f_{n'})$ and $\delta_* \in \mathcal{R}(\Omega; E)$ such that $\epsilon_{f_{n'}} \overset{\tau}{\Longrightarrow} \delta_*$. Because of $\sigma(E, E') \subset \tau$, Example 4.9($b$) implies $f_{n'} \xrightarrow{w} f_* := \text{bar } \delta_*$. But more can be said. Let $A \in \mathcal{A}$ be arbitrary and set $\alpha := -\limsup_{n'} \| \int_A (f_{n'} - f_*)\| d\mu$. Without loss of generality we may suppose $-\| \int_A (f_{n'} - f_*) d\mu\| \to \alpha \leq 0$. By the Hahn-Banach theorem, there exists a sequence $(x'_{n'})$ in the unit sphere of the dual space E' such that

$$-\| \int_A (f_{n'} - f_*) d\mu\| = < \int_A (f_{n'} - f_*) d\mu, x'_{n'} > = \int_A < f_{n'} - f_*, x'_{n'} > d\mu$$

for every n'. By the Alaoglu-Bourbaki theorem it then follows that a subsequence of $(x'_{n'})$ converges in the weak star topology to some x'_∞ in the closed unit ball of E' (note that this ball is metrizable);

we may suppose without loss of generality that the entire sequence $(x'_{n'})$ converges to x'_∞. Since τ is the norm-topology, a semicontinuous integrand ℓ on $\Omega \times (\hat{\mathbf{N}} \times S)$ is defined by $\ell(\omega, n', x) := 1_A(\omega) < x - f_*(\omega), x'_{n'} >$. Then Theorem 5.2 gives $\alpha \geq \int_A < f_* - f_*, x'_\infty > d\mu = 0$, and we get $\alpha = 0$. QED

More obvious corollaries of Theorem 5.5 (namely, where the space F is completely absent) are so-called weak-strong lower semicontinuity results for integral functionals in the calculus of variations and optimal growth theory [45, 52, 65],

The following example is intended to indicate the usefulness of Theorem 5.5 for the study of existence in optimal growth. Notwithstanding its modesty, it already covers quite some models used in optimal growth theory (this point is elaborated in [41]). More general and more complex existence results, with infinite horizon and a recursive discount term in the objective integrand can be found in [22, 41]. Such applications require a slight extension of Theorem 5.5 to the situation where F is $[0, +\infty]$ (i.e., a non-vector space).

Example 5.8 Consider the following optimal growth problem.

$$(P): \text{ minimize } J(y) := \int^*_{[0,1]} g^0(t, y(t), \dot{y}(t)) dt$$

over all $y \in \mathcal{Y}$, where $\mathcal{Y}$ is the set of all absolutely continuous functions $y \in \mathcal{AC}([0,1]; \mathbf{R}^n)$ that satisfy both the differential inclusion

$$\dot{y}(t) \in U(t, y(t)) \text{ a.e. in } [0,1]$$

and the boundary condition $y(t) \in A(t)$ for all $t \in [0,1]$. Here $[0,1]$ is equipped with the usual Lebesgue structure and $A(t) \subset \mathbf{R}^n$ is compact for $t = 0$ and closed for all other t. Also, $U : [0,1] \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is a multifunction whose values are compact and convex, and for every $t \in [0,1]$ the multifunction $U(t, \cdot)$ is upper semicontinuous. We suppose that there exists $\phi \in \mathcal{L}^1([0,1]; \mathbf{R})$ such that every y in $\mathcal{Y}$ satisfies $|\dot{y}(t)| \leq \phi(t)$ a.e. Further, $g^0 : [0,1] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ is such that $g^0(t, \cdot, \cdot)$ is lower semicontinuous for every $t \in [0,1]$ and $g^0(t, d, \cdot)$ is convex for every $(t, d) \in [0,1] \times \mathbf{R}^n$. Then an optimal solution for (P) exists, provided $\mathcal{Y} \neq \emptyset$.

To let this existence result follow from the above, we take a minimizing sequence (y_k) in $\mathcal{Y}$. By the condition involving ϕ, the collection $(\dot{y}_k)$ is uniformly integrable, so by compactness of $A(0)$ we have that (y_k) is equi-continuous and bounded. Hence, by applying in succession the Arzela-Ascoli theorem and the Dunford-Pettis theorem, we get existence of a subsequence (y_n) of (y_k) and functions y_* in $\mathcal{C}([0,1]; \mathbf{R}^n)$ and e_* in $\mathcal{L}^1([0,1]; \mathbf{R}^n)$ such that $y_n \to y_*$ uniformly on $[0,1]$ and $\dot{y}_n \xrightarrow{w} e_*$ weakly in $\mathcal{L}^1([0,1]; \mathbf{R}^n)$. This immediately implies $y_*(t) \in A(t)$ for every t, and also, by $y_n(t) = y_n(0) + \int_0^t \dot{y}_n$ for every n, we get $y_*(t) = y_*(0) + \int_0^t e_*$. Hence, $\dot{y}_* = e_*$ a.e.

We now apply Theorem 5.5 with the following substitutions: $D := E := \mathbf{R}^n$, $d_n := y_n$, $d_0 := y_*$ and $e_n := \dot{y}_n$, $e_0 := \dot{y}_*$. Also, for g we take:

$$g(t, x, d) := \begin{cases} g^0(t, d, x) & \text{if } x \in U(t, d), \\ +\infty & \text{otherwise.} \end{cases}$$

Let us verify that $\alpha := \liminf_j g(t, x^j, d^j) \geq g(t, \bar{x}, \bar{d})$ whenever $(x^j, d^j) \to (\bar{x}, \bar{d})$ in $\mathbf{R}^{2n}$. If $\alpha = +\infty$, there is nothing to verify. Otherwise, we may suppose without loss of generality that $g(t, x^j, d^j) \to \alpha$ and that $g(t, x^j, d^j) < +\infty$ for all j. This gives $x^j \in U(t, d^j)$ and $g(t, x^j, d^j) = g^0(t, d^j, x^j)$, whence in the limit $\bar{x} \in U(t, \bar{d})$ (by upper semicontinuity of $U(t, \cdot)$) and $g^0(t, \bar{d}, \bar{x}) \leq \alpha$ (by lower semicontinuity of $g^0(t, \cdot, \cdot)$). We therefore conclude $g(t, \bar{x}, \bar{d}) \leq \alpha$, as was desired. It is evident that $g(t, \cdot, d_0(t))$ is convex for every t, so all the conditions of Theorem 5.5 are met. Since $d_* = \dot{y}_*$ a.e. and $g \geq g^0$ we get

$$\liminf_n \int^*_{[0,1]} g(t, \dot{y}_n(t), y_n(t)) dt \geq \int^*_{[0,1]} g^0(t, y_*(t), \dot{y}_*(t)) dt = J(y_*).$$

But recall that (y_n) is a subsequence of a minimimizing sequence of (P); this implies $\inf(P) = \lim_n J(y_n)$. Also, (y_n) is in $\mathcal{Y}$, which implies $g(t, \dot{y}_n(t), y_n(t)) = g^0(t, y_n(t), \dot{y}_n(t))$ for a.e. t for every $n \in \mathbf{N}$. Combination of the preceding gives $\inf(P) \geq J(y_*)$. Now either $\inf(P) = +\infty$ or $\inf(P) < +\infty$. The first possibility means that $J \equiv +\infty$ on $\mathcal{Y}$, in which case every $y \in \mathcal{Y}$ is an optimal solution of (P). The second possibility implies, by the inequality $\inf(P) \geq J(y_*)$, that for a.e. t we have $g(t, \dot{y}_*(t), y_*(t)) < +\infty$, i.e., $\dot{y}_*(t) \in U(t, y_*(t))$ (and $g(t, \dot{y}_*(t), y_*(t)) = g^0(t, y_*(t), \dot{y}_*(t))$). This proves that y_* belongs to $\mathcal{Y}$, so the conclusion is that y_* is an optimal solution of (P).

The following lower closure result "without convexity" comes from [9, 10]; it is a "Fatou-Vitali lemma in several dimensions" that subsumes the result given in [3] and the original "Fatou lemma in several dimensions" due to Schmeidler [87]. This kind of Fatou lemma has played a role as a technical tool to obtain equilibrium existence results; e.g., cf. [62]. See [37] for further generalizations of the result, involving multifunctions with unbounded values and associated asymptotic correction terms.

Theorem 5.9 *Let (f_n) in $\mathcal{L}^1(\Omega; \mathbf{R}^d)$ be such that*

$$a := \lim_n \int_\Omega f_n d\mu \text{ exists (in } \mathbf{R}^d)$$

and

$$(\max(0, -f_n^i))_n \text{ is uniformly integrable for } i = 1, \ldots, d.$$

Then there exists $f_* \in \mathcal{L}^1(\Omega; \mathbf{R}^d)$ such that $\int_\Omega f_* d\mu \leq a$ (i.e., componentwise) and

$$f_*(\omega) \in \mathrm{Ls}_n\{f_n(\omega)\} \text{ for a.e. } \omega \text{ in } \Omega.$$

Observe how, in contrast to Corollary 5.6, the convex hull operator has disappeared from the last statement of the theorem. We prepare the proof as follows. First, state Lyapunov's theorem in the following convenient form for Young measure theory, where (S, τ) is a completely regular Suslin space.

Theorem 5.10 *Suppose that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. Let $g := (g_1, \ldots g_d) : \Omega \times S \to \mathbf{R}^d$ be $\mathcal{A} \times \mathcal{B}(S)$-measurable and let $\delta \in \mathcal{R}(\Omega; S)$ be such that*

$$\int_\Omega [\int_S |g(\omega, x)| \delta(\omega)(dx)] \mu(d\omega) < +\infty.$$

Then there exists $f \in \mathcal{L}^0(\Omega; S)$ such that

$$J_{g_i}(f) = I_{g_i}(\delta), i = 1, \ldots, d, \text{ and } f(\omega) \in \mathrm{supp}\ \delta(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

In terminology of decision theory, the above result is a *purification* result. It immediately also implies a general denseness property of $\mathcal{R}_{Dirac}(\Omega; S)$ in $\mathcal{R}(\Omega; S)$ with respect to the τ-narrow topology; cf. [97] and [12, 29]. For $S := \mathbf{R}^d$ and $g_i(\omega, x) := x^i$ (i-th coordinate function) Theorem 5.10 yields the following corollary.

Corollary 5.11 *Suppose that $(\Omega, \mathcal{A}, \mu)$ is nonatomic. Let $\delta \in \mathcal{R}(\Omega; \mathbf{R}^d)$ be such that*

$$\int_\Omega [\int_{\mathbf{R}^d} |x| \delta(\omega)(dx)] \mu(d\omega) < +\infty.$$

Then there exists $f \in \mathcal{L}^1(\Omega; \mathbf{R}^d)$ such that

$$\int_\Omega f d\mu = \int_\Omega \mathrm{bar}\ \delta d\mu \text{ and } f(\omega) \in \mathrm{supp}\ \delta(\omega) \text{ for a.e. } \omega \text{ in } \Omega.$$

Proof of Theorem 5.10. Denote $\Gamma(\omega) := \mathrm{supp}\ \delta(\omega)$. By Proposition 2.14 and Theorem A.10(iii) we have

$$p(\omega) := \int_S (|g(\omega, x)|, g(\omega, x)) \delta(\omega)(dx) \in \mathrm{co}\ \{(|g(\omega, x)|, g(\omega, x)) : x \in \Gamma(\omega)\} \text{ for a.e. } \omega \text{ in } \Omega.$$

The closed-valued multifunction $\Gamma : \Omega \to 2^S$ is measurable in the standard sense [50, III.9, III.10]), because for any open $U \subset S$ the set of all ω with $\Gamma(\omega) \cap U \neq \emptyset$ is precisely $\{\omega \in \Omega : \delta(\omega)(U) \neq 0\} \in \mathcal{A}$. So by Carathéodory's theorem and an obvious application of the implicit measurable selection theorem [50, Theorem III.38] there exist $\mathcal{A}$-measurable functions $\alpha_1, \ldots, \alpha_{d+2} : \Omega \to [0,1]$, with $\sum_{i=1}^{d+2} \alpha_i(\omega) = 1$ for all ω, and $\mathcal{A}$-measurable selections $s_1, \ldots, s_{d+2} : \Omega \to S$ of Γ such that

$$p(\omega) = \sum_{i=1}^{d+2} \alpha_i(\omega)(|g(\omega, s_i(\omega))|, g(\omega, s_i(\omega))) \text{ for a.e. } \omega \text{ in } \Omega.$$

Integration over ω in the first component of this identity gives $\int_\Omega \sum_i \alpha_i |g(\cdot, s_i(\cdot))| < +\infty$. Hence, by the extended Lyapunov Theorem A.9 there exists a measurable partition $B_1, \ldots, B_{d+2}$ of Ω such that each $g(\cdot, s_i(\cdot))$ is integrable over B_i and

$$\int_\Omega \sum_i \alpha_i(|g(\cdot, s_i(\cdot))|, g(\cdot, s_i(\cdot))) = \sum_i \int_{B_i} (|g(\cdot, s_i(\cdot))|, g(\cdot, s_i(\cdot))).$$

We define $f \in \mathcal{L}^1(\Omega; S)$ by setting $f := s_i$ on B_i, $i = 1, \ldots, d+2$. Then, f is evidently an a.e. selection of Γ and if we integrate over ω in the last d coordinates of the above identity for $p(\omega)$ we obtain

$$\int_\Omega \sum_i \alpha_i g(\cdot, s_i(\cdot)) = \sum_i \int_{B_i} g(\cdot, s_i(\cdot)) = \int_\Omega g(\cdot, f(\cdot)).$$

This is the desired identity, for its right hand side equals $(J_{g_1}(f), \ldots, J_{g_d}(f))$ and by the definition of $p(\omega)$ the left hand side is equal to $(I_{g_1}(\delta), \ldots, I_{g_d}(\delta))$. QED

Proof of Theorem 5.9. By Proposition A.6, $(\Omega, \mathcal{A}, \mu)$ can be decomposed in a nonatomic part Ω^{na} and a purely atomic part that is the union of at most countably many μ-atoms A_j. It is easy to see from the conditions that the sequence (f_n) is bounded in $\mathcal{L}^1$-seminorm. Since every function f_n is a.e. equal to some constant $c_n^j \in \mathbf{R}^d$ on the atom A_j, it follows from this $\mathcal{L}^1$-boundedness that (c_n^j) is relatively compact for every fixed j. Hence, an obvious diagonal extraction argument gives that there exist a subsequence (f_m) of (f_n) and a function $f_* : \cup_j A_j \to \mathbf{R}^d$, constant on each atom A_j, such that $f_m(\omega) \to f_*(\omega)$ for a.e. $\omega \in \cup_j A_j$. We can now apply Theorem 5.2 to the sequence (δ_m) in $\mathcal{R}(\Omega; \mathbf{R}^d)$, with $\delta_m := \epsilon_{f_m}$ (here $S := \mathbf{R}^d$). Notice that the central tightness condition of that theorem holds, because obviously $\sup_m \int_\Omega |f_m| d\mu < +\infty$ (cf. Example 3.4). By Theorem 5.2 there exist a subsequence $(\delta_{m'})$ and $\delta_* \in \mathcal{R}(\Omega; \mathbf{R}^d)$ for which the statements of the theorem hold. In particular, the pointwise support property for δ_* gives

$$\text{supp } \delta_*(\omega) \subset \text{Ls}_m\{f_m(\omega)\} \text{ for a.e. } \omega \text{ in } \Omega^{na}$$

and

$$\text{supp } \delta_*(\omega) = \{f_*(\omega)\} \text{ for a.e. } \omega \text{ in } \cup_j A_j.$$

Now we apply the Fatou-Vitali inequality of Theorem 5.2 to the continuous integrands $\ell_i : (\omega, n, x) \mapsto x^i$, $i = 1, \ldots, d$ (observe that $\ell_i(\omega, m, f_m(\omega)) \geq \phi_m(\omega) := -\max(0, -f_m^i)$ for each i, with (ϕ_m) uniformly integrable). This gives

$$a^i \geq \int_\Omega [\int_{\mathbf{R}^d} x^i \delta_*(\omega)(dx)]\mu(d\omega) = \int_\Omega (\text{bar } \delta_*)^i d\mu$$

(note that $\int_\Omega f_m^i(\omega)\mu(d\omega)$ equals $\int_\Omega^* [\int_{\mathbf{R}^d} x^i \epsilon_{f_m}(\omega)(dx)]\mu(d\omega)$). Additionally, applying the same sort of inequality to $\ell : (\omega, n, x) \mapsto 1_{\Omega^{na}}(\omega)|x|$ gives

$$\int_{\Omega^{na}} [\int_{\mathbf{R}^d} |x| \delta_*(\omega)(dx)]\mu(d\omega) < +\infty.$$

By Corollary 5.11, there is an integrable function $f_* : \Omega^{na} \to \mathbf{R}^d$ such that $f_*(\omega) \in \text{supp } \delta_*(\omega)$ a.e. and

$$\int_{\Omega^{na}} f_* d\mu = \int_{\Omega^{na}} \text{bar } \delta_* d\mu.$$

Concatenating f_* with the function f_* defined earlier on $\cup_j A_j$, we obtain the desired $f_* \in \mathcal{L}^1(\Omega; \mathbf{R}^d)$ (recall that $a^i \geq \int_\Omega (\text{bar } \delta_*)^i d\mu$ for each i). QED

We finish this section by two applications of lower closure "without convexity". The first of these concerns an existence problem whose origins lie in mathematical economics (cf. [6]):

Example 5.12 In [6] the following optimization problem was considered:

$$(P): \text{ maximize } J(f) := \int_{[0,1]} U(t, f(t)) dt$$

over all functions $f \in \mathcal{L}^1([0,1]; \mathbf{R}^p)$ with $f(t) \in \mathbf{R}^p_+$ a.e. and $\int_{[0,1]} f = b$. Here $b \in \mathbf{R}^p_+$ is fixed, and the utility integrand $U : [0,1] \times \mathbf{R}^p_+ \to [-\infty, +\infty)$ is $\mathcal{A} \times \mathcal{B}(\mathbf{R}^p_+)$-measurable, with $U(t, \cdot)$ upper semicontinuous and (coordinatewise) nondecreasing on $\mathbf{R}^p_+$ for every $t \in [0,1]$. In this form (P) need not have an optimal solution [e.g., consider $p = 1$, $U(t,x) := x^2$ and $b > 0$]. However, as shown in [6], (P) has an optimal solution if U has the following growth property: for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^1([0,1]; \mathbf{R})$, $\phi_\epsilon \geq 0$, such that for every $t \in [0,1]$

$$U(t,x) \leq \epsilon |x| \text{ for all } x \in \mathbf{R}^p_+ \text{ with } |x| \geq \phi_\epsilon(t).$$

We show that the principal existence result of [6] follows from Theorem 5.9. We first claim, following [42, p. 157], that the growth poperty of [6] implies the following growth property: for every $\epsilon > 0$ and every $t \in [0,1]$

$$U(t,x) \leq \epsilon |x| + \sqrt{p}\, \psi_\epsilon(t) \text{ for all } x \in \mathbf{R}^p_+,$$

where $\psi_\epsilon := \phi_1 + \phi_\epsilon$. Indeed, note that $U(t, \hat{x}(t)) \leq |\hat{x}(t)|$, where $\hat{x}(t)$ is the vector all of whose components equal $\psi_\epsilon(t)$; hence, by the given monotonicity of $U(t,\cdot)$, it follows that $U(t,x) \leq |\hat{x}(t)| = \sqrt{p}\,\psi_\epsilon(t)$ whenever $x^i \leq \psi_\epsilon(t)$ for all i, $1 \leq i \leq p$. And if $x^i > \psi_\epsilon(t)$ for any i, then $|x| \geq \phi_\epsilon(t)$, so that $U(t,x) \leq \epsilon|x|$ holds by the hypothesis. Hence, the claim has been proven. Let $s := \sup(P)$ and observe, by the growth property in its new form, that we have $s < +\infty$. Without loss of generality we may suppose $s \in \mathbf{R}$. Let (f_n) be any maximizing sequence for (P), i.e., (f_n) is a sequence of nonnegative functions in $\mathcal{L}^1([0,1]; \mathbf{R}^p)$ with $\int_{[0,1]} f_n = b$ and $J(f_n) \to s$ (note that (P) is feasible for elementary reasons). To apply Theorem 5.9 we take for $(\Omega, \mathcal{A}, \mu)$ the unit interval *cum* Lebesgue structure. Also, we define a sequence $(\tilde{f}_n)$ by

$$\tilde{f}_n(t) := (-U(t, f_n(t)), f_n(t)).$$

Then without loss of generality $(\tilde{f}_n) \subset \mathcal{L}^1([0,1]; \mathbf{R}^{p+1})$ and $\int_{[0,1]} \tilde{f}_n \to (-s, b)$. Also, $(\max(0, -\tilde{f}^i_n))_n$ is uniformly integrable for $i = 0, \ldots, p$. For $i = 1, \ldots, p$ this is trivial (by $f_n \geq 0$), and for $i = 0$ it follows from the growth property, in the form above, that for every $\epsilon > 0$ and every measurable subset A of $[0,1]$

$$\int_A U(t, f_n(t)) dt \leq \epsilon \int_A |f_n| + \sqrt{p} \int_A \psi_\epsilon \leq \epsilon \sum_{i=1}^p b^i + \sqrt{p} \int_A \psi_\epsilon.$$

This implies equi-integrability, so the sequence $(-\tilde{f}^0_n)$ is uniformly integrable [81, II.5.2]. Therefore, all the conditions of Theorem 5.9 hold. It follows that there exists $\tilde{f}_* \in \mathcal{L}^1([0,1]; \mathbf{R}^{p+1})$ such that $\int_{[0,1]} \tilde{f}^0_* \leq -s$, $\int_{[0,1]} f_* \leq b$ (here $f_* := (\tilde{f}^1_*, \ldots, \tilde{f}^p_*)$) and for a.e. t $\tilde{f}_*(t) \in \mathrm{Ls}_n \tilde{f}_n(t)$, i.e., for a.e. t there exists a subsequence (n'_t) of (n), possibly t-dependent, such that

$$\lim_{n'_t} -U(t, f_{n'_t}(t)) = \tilde{f}^0_*(t) \text{ and } \lim_{n'_t} f_{n'_t}(t) = f_*(t).$$

By upper semicontinuity of $U(t,\cdot)$, the above directly leads to $\tilde{f}^0_*(t) \geq -U(t, f_*(t))$, whence $s \leq \int_{[0,1]} U(t, f_*(t)) dt$. Now define $f_{**}(t) := f_*(t) + b - \int_{[0,1]} f_*$; then $\int_{[0,1]} f_{**} = b$ and $J(f_{**}) \geq s$, as a consequence of $f_{**} \geq f_*$ and the monotonicity property of U. This shows that f_{**} is an optimal solution of (P).

The second example concerns existence of an optimal control function in a problem with no explicit convexity properties; see [11] for more involved applications that are also based on Theorem 5.9.

Example 5.13 Consider the optimal control problem.

$$(P): \text{ minimize } J(f) := \int_{[0,1]} g^0(t, f(t))dt + e(y_f(1))$$

over all control functions $f \in \mathcal{L}^0([0,1]; \mathbf{R}^p)$ with $f(t) \in F(t)$ a.e. Here $[0,1]$ is equipped with the Lebesgue σ-algebra $\mathcal{A}$ and the Lebesgue measure $\mu := \lambda_1$. Also, $F : [0,1] \to 2^{\mathbf{R}^p}$ is a compact and nonempty-valued multifunction with $\mathcal{A} \times \mathcal{B}(\mathbf{R}^p)$-measurable graph. The latter is denoted by M. Further, the cost rate function $g^0 : M \to [0, +\infty]$ is product measurable, and $g^0(t, \cdot)$ is lower semicontinuous on $F(t)$ for every $t \in [0,1]$. The final time cost term $e : \mathbf{R}^m \to (-\infty, +\infty]$ is supposed to be lower semicontinuous and bounded from below. The dynamical system corresponding to (P) is as follows. To each control function f there corresponds the absolutely continuous functions $y_f \in \mathcal{AC}([0,1]; \mathbf{R}^m)$, defined as the solution y of

$$\dot{y}(t) = A(t)y(t) + g(t, f(t)) \text{ for a.e. } t \text{ in } [0,1],$$

with initial condition $y(0) = y_0$, where $y_0 \in \mathbf{R}^m$ is fixed. Here A belongs to $\mathcal{L}^1([0,1]; \mathbf{R}^{m \times m})$ and $g : M \to \mathbf{R}^m$ is measurable and such that $g(t, \cdot)$ is continuous on $F(t)$ for every $t \in [0,1]$. Moreover, we suppose that there exists $\phi \in \mathcal{L}^1([0,1]; \mathbf{R})$ such that $\sup_{x \in F(t)} |g(t,x)| \leq \phi(t)$ for every t. By the structure of the dynamical system, the trajectory y_f corresponding to a control function f can be expressed explicitly as follows [97, II.4.8]:

$$y_f(t) = \Lambda(t)y_0 + \Lambda(t) \int_0^t \Lambda(t')^{-1} g(t', f(t'))dt'.$$

Here $\Lambda \in \mathcal{AC}([0,1]; \mathbf{R}^{m \times m})$ is the *fundamental solution*, determined by $\dot{\Lambda} = A\Lambda$ and $\Lambda(0) = m \times m$-identity matrix. Using Theorem 5.9, we prove that (P) has an optimal solution. Observe that (P) is feasible, since F has a measurable a.e. selection. Let $\iota := \inf(P)$. We may suppose without loss of generality $\iota < +\infty$; hence, $\iota \in \mathbf{R}$. Let (f_n) be a minimizing sequence of control functions, i.e., with $J(f_n) \to \iota$. Let $\iota' := \liminf_n e(f_n)$; then $\iota' \in \mathbf{R}$. Rather than concentrating on a suitable subsequence, we may suppose without loss of generality that $e(f_n) \to \iota'$. Also, by integrability of ϕ, it follows easily that $(y_{f_n}(1))$ is a bounded sequence in $\mathbf{R}^m$. Hence, rather than taking a suitable subsequence, we can also suppose that $(y_{f_n}(1))$ converges to some $b \in \mathbf{R}^m$. Note already that $\iota' \geq e(b)$, by lower semicontinuity of e. Let us define $\tilde{f}_n \in \mathcal{L}^1([0,1]; \mathbf{R}^{2m+1})$ by

$$\tilde{f}_n(t) := (g^0(t, f_n(t)), \Lambda(1)\Lambda(t)^{-1}g(t, f_n(t)), -\Lambda(1)\Lambda(t)^{-1}g(t, f_n(t))).$$

Observe that $\int_{[0,1]} \tilde{f}_n \to (\iota - \iota', b - \Lambda(1)y_0, \Lambda(1)y_0 - b)$ and that $(\max(0, -\tilde{f}_n^i))$ is obviously uniformly integrable for each index i. Hence, by Theorem 5.9 there exists $\tilde{f}_*$ in $\mathcal{L}^1([0,1]; \mathbf{R}^{2m+1})$ such that $\int_{[0,1]} \tilde{f}_*^0 \leq \iota - \iota'$, $\int_{[0,1]} \hat{f} \leq b - \Lambda(1)y_0$, $\int_{[0,1]} \check{f} \leq \Lambda(1)y_0 - b$, and such that for a.e. t there exists a subsequence (n'_t) of (n), possibly t-dependent, with

$$\tilde{f}_*^0(t) = \lim_{n'_t} g^0(t, f_{n'_t}(t)), \hat{f}(t) = \lim_{n'_t} \Lambda(1)\Lambda(t)^{-1}g(t, f_{n'_t}(t)), \check{f}(t) = \lim_{n'_t} -\Lambda(1)\Lambda(t)^{-1}g(t, f_{n'_t}(t)).$$

Here $\hat{f} := (\tilde{f}_*^1, \ldots, \tilde{f}_*^m)$ and $\check{f} := (\tilde{f}_*^{m+1}, \ldots, \tilde{f}_*^{2m})$. From the above limit expressions it follows that $\hat{f} = -\check{f}$ a.e., which leads to $\int_{[0,1]} \hat{f} = b - \Lambda(1)y_0$. Also, in the above limits, each (n'_t) has a further subsequence (n''_t) such that $(f_{n''_t})$ converges to some vector x_t in $F(t)$ (this is by compactness of the set $F(t)$). Thus, we get for a.e. t

$$\tilde{f}_*^1(t) \geq g^0(t, x_t), \hat{f}(t) = -\check{f}(t) = \Lambda(1)\Lambda^{-1}(t)g(t, x_t)$$

by (semi)continuity of $g^0(t,\cdot)$ and $g(t,\cdot)$. By Theorem A.4 we get the existence of $f_* \in \mathcal{L}^0([0,1];\mathbf{R}^p)$ such that $f_*(t) = x_t$ for a.e. t. From the above we now conclude that f_* is a control function such that $\int_{[0,1]} \tilde{f}^0_* \leq \iota - \iota'$ and $\int_{[0,1]} \Lambda(1)\Lambda^{-1}(t)g(t,f_*(t))dt = b - \Lambda(1)y_0$. The latter simply states $y_{f_*}(1) = b$ and, by the previous inequality $\iota' \geq e(b)$, the former implies $J(f_*) \leq \iota := \inf(P)$. This concludes the argument.

A more general approach to the subject of existence without convexity can be found in [23]. There the dynamical system is also semilinear, as above, but the objective integrand g_0 is allowed to depend in the state variable in a special way, involving concavity. Problems of this kind were first investigated in [79]; see also [51, 76, 86, 80]. This approach uses a Bauer-type extremum principle [54] that is applied to a relaxation of the optimal control problem, i.e., a reformulation in terms of Young measures. Use of this extremum principle is based on the fact that in general the set $\mathcal{R}_D(\Omega;S)$ of Dirac Young measures forms the extreme point boundary of $\mathcal{R}(\Omega;S)$.

6 Nash equilibria

Instead of a lower closure result for Young measures, as formed by Theorem 5.2, we can also give existence results for variational inequalities in terms of Young measures. As shown in [21, 26, 30], such results can be used to obtain existence results of a more classical nature in game theory and economics. As in the previous section, we suppose that (S,τ) is a completely regular Suslin space and refer to Remark 2.4 in this connection.

Theorem 6.1 *Let h be a nonnegative, sequentially τ-inf-compact integrand on $\Omega \times S$ and let $\mathcal{R}_h$ be the set of all $\delta \in \mathcal{R}(\Omega;S)$ with $I_h(\delta) \leq 1$; suppose that $\mathcal{R}_h$ is nonempty. Let $g : \Omega \times S \times \mathcal{R}_h \to \mathbf{R}$ be $\mathcal{A} \times \mathcal{B}(S) \times \mathcal{B}(\mathcal{R}_h)$-measurable and such that $g(\omega,\cdot,\cdot)$ is lower semicontinuous on $S \times \mathcal{R}_h$ for every $\omega \in \Omega$, and $g(\omega,x,\cdot)$ is narrowly continuous on $\mathcal{R}_h$ for every $(\omega,x) \in \Omega \times S$. Moreover, g is supposed to have the following growth property with respect to h: for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^1(\Omega;\mathbf{R})$ such that*

$$|g(\omega,x,\delta)| \leq \epsilon h(\omega,x) + \phi_\epsilon(\omega) \text{ on } \Omega \times S \times \mathcal{R}_h.$$

Then there exists $\delta_* \in \mathcal{R}_h$ such that

$$\inf_{\delta \in \mathcal{R}_h} \int_\Omega [\int_S g(\omega,x,\delta_*)\delta(\omega)(dx)]\mu(d\omega) = \int_\Omega [\int_S g(\omega,x,\delta_*)\delta_*(\omega)(dx)]\mu(d\omega).$$

Of course, in this result the set $\mathcal{R}_h$ is equipped with the (relative) narrow topology and the corresponding Borel σ-algebra.

Proof. There exists, by Proposition A.11, a countably generated sub-σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that g is also $\mathcal{A}_0 \times \mathcal{B}(S) \times \mathcal{B}(\mathcal{R}_h)$-measurable. Hence, we may suppose without loss of generality that $\mathcal{A}$ itself is countably generated [note in particular that this also holds with respect to the nonemptiness issue – augment by the σ-algebra that is generated by any fixed $\delta \in \mathcal{R}_h \neq \emptyset$]. We set $C := \mathcal{R}_h$ and define $\pi : \mathcal{R}_h \times \mathcal{R}_h \to \mathbf{R}$ in the following way:

$$\pi(\delta,\eta) := I_{g_\delta}(\eta) - I_{g_\delta}(\delta),$$

where the integrand g_δ on $\Omega \times S$ is defined by $g_\delta(\omega,x) := g(\omega,x,\delta)$. By Theorem 5.2 we have that $\mathcal{R}_h$ is compact in the vector space generated by $\mathcal{R}(\Omega;S)$ (the narrow topology obviously extends to the latter). By that same theorem we also have that $(\delta,\eta) \mapsto I_{g_\delta}(\eta)$ is lower semicontinuous. Indeed, by Theorem 4.5 it is enough to check sequential lower semicontinuity, so if we let (δ_n,η_n) converge narrowly to (δ_0,η_0) we can define, in a by now well-known way, $\ell(\omega,n,x) := g(\omega,x,\delta_n)$ and $\ell(\omega,x,\infty) := g(\omega,x,\delta_0)$ to form an integrand g on $\Omega \times (S \times \mathcal{R}_h)$ that meets the conditions of Theorem 5.2. The corresponding lower semicontinuity statement in Theorem 5.2 then amounts precisely to $\liminf_n I_{g_{\delta_n}}(\eta_n) \geq I_{g_{\delta_0}}(\eta_0)$. Also, it follows, directly by Fatou's classical lemma, that for every $\eta \in \mathcal{R}_h$ the functional $\delta \mapsto I_{g_\delta}(\eta)$ is upper semicontinuous. Taken together, this shows that π meets the lower semicontinuity condition of Theorem A.3, and all of the remaining conditions hold trivially. An appeal to Theorem A.3 can thus be made, and this finishes the proof. QED

Observe in the proof above that measurability of $g(\omega, x, \delta)$ in the variable δ only serves to fulfill the requirements of Proposition A.11. Hence, if one works *a priori* with a countably generated sub-σ-algebra $\mathcal{A}$, there is no need for such measurability in δ. In a special, quite relevant situation the variational inequality statement of Theorem 6.1 can be sharpened considerably [21, 26, 30]:

Corollary 6.2 *Let $\Sigma : \Omega \to 2^S$ be a nonempty, τ-compact-valued multifunction with $\mathcal{A} \times \mathcal{B}(S)$-measurable graph G. Let $\mathcal{R}_\Sigma$ be the set of all $\delta \in \mathcal{R}(\Omega; S)$ for which*

$$\delta(\omega)(\Sigma(\omega)) = 1 \text{ for a.e. } \omega \in \Omega.$$

Let $g : G \times \mathcal{R}_\Sigma \to [-\infty, +\infty]$ be $((\mathcal{A} \times \mathcal{B}(S) \cap G) \times \mathcal{B}(\mathcal{R}_\Sigma)$-measurable and such that $g(\omega, \cdot, \cdot)$ is lower semicontinuous on $\Sigma(\omega) \times \mathcal{R}_\Sigma$ for every $\omega \in \Omega$, and $g(\omega, x, \cdot)$ is narrowly continuous on $\mathcal{R}_\Sigma$ for every $(\omega, x) \in G$. Then there exists $\delta_* \in \mathcal{R}_\Sigma$ such that

$$\delta_*(\omega)(\mathrm{argmin}_{x \in \Sigma(\omega)} g(\omega, x, \delta_*)) = 1 \text{ for a.e. } \omega \text{ in } \Omega.$$

PROOF. Define

$$h(\omega, x) := \begin{cases} 0 & \text{if } x \in \Sigma(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then h satisfies the conditions of Theorem 6.1 and $\mathcal{R}_h = \mathcal{R}_\Sigma$; also, the von Neumann-Aumann measurable selection theorem [50, Theorem III.22] implies that $\mathcal{R}_\Sigma$ is nonempty. Let $\tilde{g} := \arctan g$. Then $\tilde{g}$ possesses the same (semi-)continuity properties as g, and in addition it is bounded. Thus, by Theorem 6.1 there exists $\delta_* \in \mathcal{R}_\Sigma$ such that

$$\inf_{\delta \in \mathcal{R}_\Sigma} \int_\Omega [\int_{\Sigma(\omega)} \tilde{g}(\omega, x, \delta_*)\delta(\omega)(dx)]\mu(d\omega) = \int_\Omega [\int_{\Sigma(\omega)} \tilde{g}(\omega, x, \delta_*)\delta_*(\omega)(dx)]\mu(d\omega).$$

But here the left side can be processed further: it is certainly not larger than the corresponding infimum over $\mathcal{R}_{Dirac}(\Omega; S) \cap \mathcal{R}_\Sigma$. Hence,

$$\inf_{f \in \mathcal{L}^0_\Sigma} \int_\Omega \tilde{g}(\omega, f(\omega), \delta_*)\mu(d\omega) \geq \int_\Omega [\int_{\Sigma(\omega)} \tilde{g}(\omega, x, \delta_*)\delta_*(\omega)(dx)]\mu(d\omega),$$

where $\mathcal{L}^0_\Sigma$ stands for the set of all $f \in \mathcal{L}^0(\Omega; S)$ with $f(\omega) \in \Sigma(\omega)$ a.e. By another application of the measurable selection theorem in this specific context ([15, Theorem B.1] – see also [50, VII.7]) and by using obvious modifications of functions measurable with respect to the completion of $\mathcal{A}$, it follows that

$$\inf_{f \in \mathcal{L}^0_\Sigma} \int_\Omega \tilde{g}(\omega, f(\omega), \delta_*)\mu(d\omega) = \int_\Omega \inf_{x \in \Sigma(\omega)} \tilde{g}(\omega, x, \delta_*)\mu(d\omega).$$

So we conclude that

$$\int_\Omega \inf_{x \in \Sigma(\omega)} \tilde{g}(\omega, x, \delta_*)\mu(d\omega) \geq \int_\Omega [\int_{\Sigma(\omega)} \tilde{g}(\omega, x, \delta_*)\delta_*(\omega)(dx)]\mu(d\omega),$$

and, obviously, the converse inequality must hold as well. It follows now immediately that for a.e. ω the probability measure $\delta_*(\omega)$ is carried by the set $\mathrm{argmin}_{\Sigma(\omega)}\tilde{g}(\omega, \cdot, \delta_*)$, which is clearly identical to $\mathrm{argmin}_{\Sigma(\omega)} g(\omega, \cdot, \delta_*)$ by strict monotonicity of the arctangent function. QED

Corollary 6.3 *Suppose that μ is a probability measure on $(\Omega, \mathcal{A})$. Let $\Sigma : \Omega \to 2^S$ be a multifunction with nonempty τ-compact values and with a $\mathcal{A} \times \mathcal{B}(S)$-measurable graph G. Let $\mathcal{R}_\Sigma$ be as defined in Corollary 6.2. Let $U : G \times \mathcal{P}(S) \to [-\infty, +\infty]$ be $((\mathcal{A} \times \mathcal{B}(S) \cap G) \times \mathcal{B}(\mathcal{P}(S))$-measurable and such that $U(\omega, \cdot, \cdot)$ is upper semicontinuous on $\Sigma(\omega) \times \mathcal{P}(S)$ for every $\omega \in \Omega$, and $U(\omega, x, \cdot)$ is narrowly continuous on $\mathcal{P}(S)$ for every $(\omega, x) \in G$. Then there exists $\delta_* \in \mathcal{R}_\Sigma$ such that*

$$\delta_*(\omega)(\mathrm{argmax}_{x \in \Sigma(\omega)} U(\omega, x, [\mu \otimes \delta_*](\Omega \times \cdot))) = 1 \text{ for a.e. } \omega \text{ in } \Omega.$$

This is a specialization of Corollary 6.2. It generalizes the main results of [70, 77]; cf. [21, 26]. See [35] for further improvements, including a unification of the above results with two separate existence results given in [85]. Above, $(\Omega, \mathcal{A}, \mu)$ functions as a measure space of players, $U(\omega, \cdot, \cdot)$ stands for the *payoff* (or utility) function of a player ω, and the product probability measure $\mu \otimes \delta_*$ constitutes a so-called Cournot-Nash *equilibrium distribution* for the game.

PROOF. Apply Corollary 6.2 by setting $g(\omega, x, \delta) := -U(\omega, x, [\mu \otimes \delta](\Omega \times \cdot))$. By Remark 4.2 the mapping $\delta \mapsto [\mu \otimes \delta](\Omega \times \cdot)$ is continuous from $\mathcal{R}(\Omega; S)$ to $\mathcal{P}(S)$, so g easily meets the conditions imposed in Corollary 6.2. QED

Corollary 6.4 *Suppose that S is a separable Banach space, equipped with the weak topology τ. Let $\Sigma : \Omega \to 2^S$ be a multifunction with nonempty, τ-compact and convex values, integrably bounded and with a $\mathcal{A} \times \mathcal{B}(S)$-measurable graph G. Let $\mathcal{L}^1_\Sigma$ be the set of all $f \in \mathcal{L}^1(\Omega; S)$ with $f(\omega) \in \Sigma(\omega)$ a.e., equipped with the weak topology. Let $U : G \times \mathcal{L}^1_\Sigma \to [-\infty, +\infty]$ be $((\mathcal{A} \times \mathcal{B}(S) \cap G) \times \mathcal{B}(\mathcal{L}^1_\Sigma)$-measurable and such that $U(\omega, \cdot, \cdot)$ is upper semicontinuous on $\Sigma(\omega) \times \mathcal{L}^1_\Sigma$ for every $\omega \in \Omega$, $U(\omega, x, \cdot)$ is weakly continuous on $\mathcal{L}^1_\Sigma$ for every $(\omega, x) \in G$ and $U(\omega, \cdot, f)$ is quasi-concave on $\Sigma(\omega)$ for every $(\omega, f) \in \Omega \times \mathcal{L}^1_\Sigma$. Then there exists $f_* \in \mathcal{L}^1_\Sigma$ such that*

$$f_*(\omega) \in \mathrm{argmax}_{x \in \Sigma(\omega)} U(\omega, x, f_*) \text{ for a.e. } \omega \text{ in } \Omega.$$

PROOF. First, we apply Corollary 6.2 by setting $g(\omega, x, \delta) := -U(\omega, x, \mathrm{bar}\ \delta)$. Note that for $\delta \in \mathcal{R}_\Sigma$ the barycentric function $\omega \mapsto \mathrm{bar}\ \delta(\omega)$ (or at least an a.e.-modification of it) belongs to $\mathcal{L}^1_\Sigma$; cf. the proof of Theorem 5.5. Recall that the dual of $\mathcal{L}^1(\Omega; S)$ can be identified with the (prequotient) space $\mathcal{L}^\infty(\Omega; S^*)[S]$ of all uniformly bounded and scalarly measurable S^*-valued functions on $(\Omega, \mathcal{A}, \mu)$; cf. [66, IV]. For any $b \in \mathcal{L}^\infty(\Omega; S^*)[S]$ the identity

$$\int_\Omega < \mathrm{bar}\ \delta, b > d\mu = \int_\Omega [\int_S < x, b(\omega) > \delta(\omega)(dx)] \mu(d\omega)$$

shows that $\delta \mapsto \mathrm{bar}\ \delta$ from $\mathcal{R}_\Sigma$ into $\mathcal{L}^1_\Sigma$ is a narrowly continuous mapping. Hence, the conditions of Corollary 6.2 are met, and we conclude that there exists $\delta_* \in \mathcal{R}_\Sigma$ such that

$$\delta_*(\omega)(\mathrm{argmax}_{x \in \Sigma(\omega)} U(\omega, x, f_*)) = 1 \text{ for a.e. } \omega \text{ in } \Omega,$$

where we set $f_* := \mathrm{bar}\ \delta_*$ (by the above, this is an integrable function). By the given quasi-concavity the "argmax" set is convex in the above expression. Since it is also weakly closed (in fact weakly compact) it follows (Hahn-Banach) that it is strongly closed. Therefore, the desired statement follows directly from Theorem A.10(ii). QED

The above existence result for Nash equilibria generalizes [88, Theorem 2.1] and [69, Theorem 7.1]. Recently, a more general existence result was obtained in [31]; see also [35] for further extensions. This involves a new topology, called the *feeble* topology, which dispenses with integrable boundedness of Σ by extending the above weak topology on $\mathcal{L}^1_\Sigma$ to the set $\mathcal{L}^0_\Sigma$ of measurable a.e. selections of Σ. Also, this result includes (partial) purification by nonatomicity (so as to avoid quasi-concavity and convexity assumptions), and for instance the main result Theorem 4.7.3 in [64] follows from it as well. The analogy should be clear to the reader: just as barycentric techiques for lower closure "with convexity" were useful above, so can techniques for lower closure "without convexity" lead to parallel (or combined, via a partition of the measure space, as in the case of Theorem 5.9 and [31]) existence results for equilibria.

Finally, we present an existence result for Bayesian Nash equilibrium in games with incomplete information [63, 16, 39, 78]. In such games each player i privately observes the i-th component of a random outcome $\omega = (\omega_1, \ldots, \omega_m)$, as generated by some (probability) measure μ on $\Omega = \Pi_{i=1}^m \Omega_i$, and acts accordingly. However, player i's payoff function U_i depends upon the entire realization ω ("incomplete information").

Theorem 6.5 *Suppose that $S = \Pi_{i=1}^m S_i$ and that $\Omega = \Pi_{i=1}^m \Omega_i$, where (S_i, τ_i) is a completely regular Suslin space for $i = 1, \ldots, m$ and where $(\Omega_i, \mathcal{A}_i, \mu_i)$ is a finite measure space. Suppose that τ is* *the product of the topologies τ_i and that μ is absolutely continuous with respect to the product of the measures μ_i. For $i = 1, \ldots, m$, let h_i be a nonnegative, sequentially τ-inf-compact integrand on $\Omega_i \times S_i$, and, corespondingly, let $\mathcal{R}_{h_i}$ be the set of all $\delta_i \in \mathcal{R}(\Omega_i; S_i)$ with $I_{h_i}(\delta_i) \leq 1$. Suppose that $\mathcal{R}_{h_1}, \ldots, \mathcal{R}_{h_m}$ are nonempty. For $i = 1, \ldots, m$, let $U_i : \Omega \times S \to \mathbf{R}$ be $\mathcal{A} \times \mathcal{B}(S)$-measurable and such that $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \mapsto U_i(\omega, x)$ is continuous on $\Pi_{j \neq i} S_j$ for every $(\omega, x_i) \in \Omega \times S_i$ and $U_i(\omega, \cdot)$ is upper semicontinuous on S for every $\omega \in \Omega$. Moreover, U_i is supposed to have the following growth property: for every $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{L}^1(\Omega; \mathbf{R})$ such that*

$$|U_i(\omega, x)| \leq \epsilon \sum_{j=1}^{m} h_j(\omega, x_j) + \phi_\epsilon(\omega) \text{ on } \Omega \times S.$$

Then for $i = 1, \ldots, m$ there exist δ_{*i} in $\mathcal{R}_{h_i}$ such that

$$\sup_{\delta_i \in \mathcal{R}_{h_i}} I_{U_i}(\delta_{*1} \otimes \cdots \delta_{*i-1} \otimes \delta_i \otimes \delta_{*i+1} \otimes \cdots \otimes \delta_{*m}) = I_{U_i}(\delta_{*1} \otimes \cdots \otimes \delta_{*m}).$$

PROOF. Rather than premultiplying all integrands by the Radon-Nikodym derivative of μ with respect to the measures μ_i, we shall suppose that μ itself has this product structure (as required in Theorem 4.17) without loss of generality. Observe also that by Proposition A.11 the σ-algebra $\mathcal{A}$ may be supposed countable (see the proof of Theorem 6.1). We shall apply Theorem A.3 to $C := \Pi_i \mathcal{R}_{h_i}$ and to $\pi : C \times C \to \mathbf{R}$, defined as follows:

$$\pi((\delta_1, \ldots, \delta_m), (\eta_1, \ldots, \eta_m)) := \sum_{i=1}^{m} I_{U_i}(\delta_1 \otimes \cdots \otimes \eta_i \otimes \cdots \delta_m) - I_{U_i}(\delta_1 \otimes \cdots \otimes \delta_m).$$

Using Theorem 4.10, in the style of the proof of Theorem 5.2, and Theorem 4.17, one can see that $(\delta_1, \cdots, \delta_m) \mapsto I_{U_i}(\delta_1 \otimes \cdots \otimes \eta_i \otimes \cdots \delta_m)$ is narrowly continuous on C for every $\eta_i \in \mathcal{R}_{h_i}$, $i = 1, \ldots, m$. Moreover, by the same sort of argument $(\delta_1, \cdots, \delta_m) \mapsto I_{U_i}(\delta_1 \otimes \cdots \otimes \delta_m)$ is narrowly upper semicontinuous on C. Hence, it follows that $\pi(\cdot, (\eta_1, \ldots, \eta_m))$ is lower semicontinuous on C for every $(\eta_1, \ldots, \eta_m)$ in C On the other hand, $\pi((\delta_1, \ldots, \delta_m), \cdot)$ is trivially affine on C for every $(\delta_1, \ldots, \delta_m) \in C$. Also, C is trivially convex, and it is narrowly relatively compact by Theorem 4.8 and narrowly closed by Theorem 4.10. Hence, all conditions of Ky Fan's Theorem A.3 hold. The existence result then follows with ease. QED

A Auxiliary results

We recall and derive some results from measure theory and convex analysis which play a role in the main text. Our first result is a Fubini-type theorem from [81, III.2] (see also [4, 2.6]). As in the main text, $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and S a topological space.

Theorem A.1 *For any $\delta \in \mathcal{R}(\Omega; S)$ the formula*

$$[\mu \otimes \delta](A \times B) := \int_A \delta(\omega)(B)\mu(d\omega)$$

defines a unique product measure $\mu \otimes \delta$ on $(\Omega \times S, \mathcal{A} \times \mathcal{B}(S))$. Moreover, for every $\mathcal{A} \times \mathcal{B}(S)$-measurable function $g : \Omega \times S \to [0, +\infty]$

$$\omega \mapsto \int_S g(\omega, x)\delta(\omega)(dx) \text{ is } \mathcal{A}\text{-measurable}$$

and

$$\int_{\Omega \times S} g d(\mu \otimes \delta) = \int_\Omega [\int_S g(\omega, x)\delta(\omega)(dx)]\mu(d\omega).$$

Proposition A.2 *Let $\delta : \Omega \to \mathcal{P}(S)$. The following are equivalent:*
(a) $\delta \in \mathcal{R}(\Omega; S)$.
(b) δ is measurable with respect to $\mathcal{A}$ and the narrow Borel σ-algebra on $\mathcal{P}(S, \tau_\rho)$.

PROOF. $(a) \Rightarrow (b)$: For every $c \in \mathcal{C}_b(S, \rho)$ the mapping $\omega \mapsto \int_S c(x)\delta(\omega)(dx)$ is $\mathcal{A}$-measurable by Theorem A.1. Since $\mathcal{P}(S, \tau_\rho)$ is separable and metrizable for the narrow convergence topology ([43, Proposition 7.20], [55, III.60]), (b) follows elementarily.

$(b) \Rightarrow (a)$: For any τ_ρ-open set $G \subset S$ there exists a nondecreasing sequence (c_n) in $\mathcal{C}_b(S, \rho)$ such that $\lim_n c_n(x) = 1_G(x)$ for every $x \in S$ ([4, A6], [43, Lemma 7.7]). Hence, $\delta(\cdot)(G)$ is $\mathcal{A}$-measurable by an application of the monotone convergence theorem. Since finite intersections of open sets are open, (a) follows by an application of a well-known σ-additive class result [4, 4.1.2], in view of the identity $\mathcal{B}(S, \tau) = \mathcal{B}(S, \tau_\rho)$ by Hypothesis 2.3. QED

The following result is due to Ky Fan ([59, Lemma 1], [5, Theorem 5, p. 330]). This result remains valid in a non-Hausdorff setting, because, as observed in [57, pp. 500-501], Ky Fan's proof does not require the Hausdorff property.

Theorem A.3 (Ky Fan) *Let C be a compact convex and nonempty subset of a topological vector space (possibly non-Hausdorff). Let $\pi : C \times C \to [-\infty, +\infty]$ be such that*

$$\pi(\cdot, y) \text{ is lower semicontinuous for every } y \in C,$$

$$\pi(x, \cdot) \text{ is quasiconcave for every } x \in C,$$

$$\pi(x, x) \leq 0 \text{ for every } x \in C.$$

Then there exists $x^* \in C$ such that $\pi(x^*, y) \leq 0$ for all $y \in C$.

The following implicit measurable function result is taken from [50, Theorem III.38].

Theorem A.4 *Let $(V, \mathcal{V})$ be a measurable space, S a Suslin space, and $\Theta : \Omega \to 2^V$ a multifunction whose graph*

$$\text{gph } \Theta := \{(\omega, v) \in \Omega \times V : v \in \Theta(\omega)\}$$

belongs to $\mathcal{A} \times \mathcal{B}(V)$. Let $g : \Omega \times S \to V$ be measurable with respect to $\mathcal{A} \times \mathcal{B}(S)$ and $\mathcal{V}$ such that $g(\omega, S) \cap \Theta(\omega) \neq \emptyset$ for a.e. ω. Then there exists $f \in \mathcal{L}^0(\Omega; S)$ such that $g(\omega, f(\omega)) \in \Theta(\omega)$ for a.e. ω in Ω.

Next, we give some Lyapunov-type results which lead up to the instrumental Theorem A.9.

Definition A.5 An *atom* of $(\Omega, \mathcal{A}, \mu)$ is a set $A \in \mathcal{A}$, $\mu(A) > 0$, for which there exists no $B \in \mathcal{A}$, $B \subset A$, such that $0 < \mu(B) < \mu(A)$.

Note that as atoms we only accept nonnull sets. It is elementary to check that any $\mathcal{A}$-measurable function must be a.e. *constant* on any atom of $(\Omega, \mathcal{A}, \mu)$.

Proposition A.6 *There exists an at most countable collection $(\bar{A}_j)$ of atoms of $(\Omega, \mathcal{A}, \mu)$, such that $\Omega^{na} := \Omega \backslash \cup_j \bar{A}_j$ contains no atoms.*

PROOF. For each $i \in \mathbf{N}$ there can be at most i atoms whose μ-measure is at least $\mu(\Omega)/i$. This gives the desired collection $(\bar{A}_j)$. QED

If $\Omega = \Omega^{na}$ then $(\Omega, \mathcal{A}, \mu)$ is said to be *nonatomic*. The most important property of nonatomic measure spaces is as follows [50, p. 118 ff.].

Theorem A.7 (Lyapunov) *Let $q \in \mathbf{N}$ and let $f \in \mathcal{L}^1(\Omega; \mathbf{R}^q)$. If Ω is nonatomic, then*

$$C := \{\int_A f d\mu : A \in \mathcal{A}\} = \{\int_\Omega f \alpha d\mu : \alpha \in \mathcal{L}^\infty(\Omega; \mathbf{R}), 0 \leq \alpha \leq 1\}.$$

Corollary A.8 *Let $m, r \in \mathbf{N}$, let $f_1, \cdots, f_r$ be functions in $\mathcal{L}^1(\Omega; \mathbf{R}^m)$ and let $\alpha_1, \cdots, \alpha_r$ be nonnegative functions in $\mathcal{L}^\infty(\Omega; \mathbf{R})$, with $\sum_{i=1}^r \alpha_i = 1$. If Ω is nonatomic, then there exists a measurable partition $B_1, \cdots, B_r$ of Ω such that*

$$\int_\Omega \sum_{i=1}^r \alpha_i f_i d\mu = \sum_{i=1}^r \int_{B_i} f_i d\mu.$$

PROOF. We use induction for r. For $r = 1$ the result holds trivially. Suppose it is true for $r = k-1$. Denote $\sum_1^{k-1} \alpha'_i f_i$ by g, where $\alpha'_i(\omega) := \alpha_i(\omega)/(1-\alpha_k(\omega))$ if $\alpha_k(\omega) < 1$ and $\alpha'_i(\omega) := 0$ if $\alpha_k(\omega) = 1$. By Theorem A.7, there exists $A \in \mathcal{A}$ for which $\int_A (g, f_k) = \int_\Omega \alpha_k (g, f_k)$. This gives $\int_{\Omega \setminus A} g = \int_\Omega (1-\alpha_k) g$, so now the result follows by the induction step applied to the functions $f_i 1_{\Omega \setminus A}$. QED

The next result is [24, Proposition 3.2], which extends Corollary A.8: the important fact to observe is that the participating functions are no longer supposed integrable.

Theorem A.9 (extended Lyapunov theorem) *Let $m, r \in \mathbf{N}$, let $f_1, \cdots, f_r$ be functions in $\mathcal{L}^0(\Omega; \mathbf{R}^m)$ and let $\alpha_1, \cdots, \alpha_r$ be nonnegative functions in $\mathcal{L}^\infty(\Omega; \mathbf{R})$, with $\sum_{i=1}^r \alpha_i = 1$, such that*

$$\int_\Omega \sum_{i=1}^r \alpha_i |f_i| d\mu < +\infty.$$

If Ω is nonatomic, then there exists a measurable partition $B_1, \cdots, B_r$ of Ω such that for $i = 1, \ldots, r$ the function f_i is integrable over B_i and

$$\int_\Omega \sum_{i=1}^r \alpha_i f_i d\mu = \sum_{i=1}^r \int_{B_i} f_i d\mu.$$

PROOF. Define for every $p \in \mathbf{N}$ the set Ω_p to consist of all ω for which $\max_i |f_i(\omega)|$ belongs to $(p, p-1]$. Then the Ω_p are disjoint and on each Ω_p we can apply Corollary A.8. For every p this gives the existence of a measurable partition $B_{1,p}, \cdots, B_{r,p}$ of Ω_p such that

$$\int_{\Omega_p} \sum_{i=1}^r \alpha_i (|f_i|, f_i) = \sum_{i=1}^r \int_{B_{i,p}} (|f_i|, f_i),$$

By Beppo Levi's theorem we then get $\sum_{i=1}^r \int_{\cup_p B_{i,p}} |f_i| = \int_\Omega \sum_{i=1}^r \alpha_i |f_i| < +\infty$, by summing over p and noting that for each i the $B_{i,p}$ are disjoint. This implies that each f_i is integrable over $B_i := \cup_p B_{i,p}$. It is now elementary to conclude that, by the above,

$$\int_\Omega \sum_{i=1}^r \alpha_i f_i = \sum_p \int_{\Omega_p} \sum_{i=1}^r \alpha_i f_i = \sum_{i=1}^r \int_{\cup_p B_{i,p}} f_i,$$

which proves the result. QED

Theorem A.10 *Let E be a separable Banach space; let $\nu \in \mathcal{P}(E)$ be such that $\int_E \|x\| \nu(dx) < +\infty$.*
(i) A unique point in E, the barycenter *of ν, is defined by*

$$\text{bar } \nu := \int_E x \nu(dx).$$

(ii) If $C \subset E$ is closed and convex with $\nu(C) = 1$ then bar ν *belongs to C.*

(iii) If $E = \mathbf{R}^d$ and if $C \subset \mathbf{R}^d$ is convex – possibly nonmeasurable – with outer measure $\nu^*(C) = 1$ then bar ν *belongs to C.*

Part (i) follows from elementary facts about Bochner integration in separable Banach spaces [56, 97]. Part (ii) follows directly from the Hahn-Banach theorem [97, I.6.13], and part (iii) is proven by induction for the dimension of E (see [84, Lemma] and [60, Lemma 3, p. 74]).

The following result forms a trick to reduce arguments involving at most countably many product measurable sets or functions to a countably generated situation; cf. [50, p. 78] and [93, Appendix].

Proposition A.11 *Let $(V, \mathcal{V})$ be a measurable space and let $g : \Omega \times V \to [-\infty, +\infty]$ be $\mathcal{A} \times \mathcal{V}$-measurable. Then there exists a countably generated sub-σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that g is also $\mathcal{A}_0 \times \mathcal{V}$-measurable. Moreover, if $(\Omega, \mathcal{A}, \mu)$ is nonatomic, then $\mathcal{A}_0$ can be chosen in such a way as to make $(\Omega, \mathcal{A}_0, \mu)$ nonatomic.*

PROOF. If $g = 1_G$, $G \in \mathcal{A} \times \mathcal{V}$, then it suffices to observe that the union of all σ-algebra's $\mathcal{A}_0 \times \mathcal{V}$, $\mathcal{A}_0$ a countably generated sub-σ-algebra of $\mathcal{A}$, is a σ-algebra which must coincide with $\mathcal{A} \times \mathcal{V}$. The usual approximation by a sequence of simple functions then finishes the argument for general g. In addition, if $\mathcal{A}$ is nonatomic, then let $(\bar{A}_j)$ be an enumeration of the atoms of $\mathcal{A}_0$, just as in Proposition A.6. By nonatomicity of $\mathcal{A}$, for each $m \in \mathbf{N}$ each $\mathcal{A}_0$-atom $\bar{A}_j$ can be partitioned as $\bar{A}_j = \cup_{i=1}^m B_i^{m,j}$, with $\mu(B_i^{m,j}) \leq \mu(\bar{A}_j)/m$, $1 \leq i \leq m$. Now let $\mathcal{A}_1$ be the σ-algebra generated by $\mathcal{A}_0$ and all $B_i^{m,j}$. Suppose that A is an atom of $\mathcal{A}_1$. Of course, we can only have $\mu(A \cap (\Omega \backslash \cup_j \bar{A}_j)) > 0$ if $\mu(A) = \mu(A \cap (\Omega \backslash \cup_j \bar{A}_j))$. But this implies that, modulo a null set, the $\mathcal{A}_1$- and $\mathcal{A}_0$-atom A is contained in $\Omega \backslash \cup_j \bar{A}_j$, which is the nonatomic part of $(\Omega, \mathcal{A}_0, \mu)$ (cf. Proposition A.6). Therefore, it follows that $\mu(A \cap (\Omega \backslash \cup_j \bar{A}_j)) = 0$, i.e., A is essentially contained in $\cup_j \bar{A}_j$. Hence, for every $m \in \mathbf{N}$ there must be j and i, $1 \leq i \leq m$, with $\mu(A \cap B_i^{m,j}) > 0$. But since A is an atom this implies then $\mu(A) = \mu(B_i^{m,j}) \leq \mu(\bar{A}_j)/m \leq \mu(\Omega)/m$. So $\mu(A) = 0$, in contradiction to our Definition A.5. QED

B Outer integration

We recapitulate some standard facts concerning outer integrals; e.g., see [43] for a slightly different treatment.

Definition B.1 Let $\psi : \Omega \to [-\infty, +\infty]$ be arbitrary (possibly nonmeasurable). Then the *outer integral* $\int_\Omega^* \psi d\mu$ is defined by

$$\int_\Omega^* \psi d\mu := \inf\{\int_\Omega \phi d\mu : \phi \in \mathcal{L}^1(\Omega; \mathbf{R}), \phi \geq \psi \text{ on } \Omega\},$$

where the infimum over the empty set is set equal to $+\infty$.

Lemma B.2 *Let $\psi : \Omega \to [-\infty, +\infty]$ be $\mathcal{A}$-measurable. Then*

$$\int_\Omega^* \psi d\mu = \int_\Omega \psi d\mu := \int_\Omega \psi^+ d\mu \dot{-} \int_\Omega \psi^- d\mu,$$

where $\psi^+ := \max(\psi, 0)$, $\psi^- := \max(-\psi, 0)$ and $\dot{-}$ is as ordinary subtraction, but with the additional convention $(+\infty) \dot{-} (+\infty) := +\infty$.

PROOF. If $\int_\Omega \psi^+ = +\infty$, the result is immediate (the infimum in Definition B.1 is then taken over the empty set).

So suppose $\int_\Omega \psi^+ < +\infty$. Note that $\int_\Omega \phi \geq \int_\Omega \psi$ for every ϕ participating in the infimum in Definition B.1. Hence, $\int_\Omega^* \psi \geq \int_\Omega \psi$. Now if $\int_\Omega \psi^- < +\infty$, then $\psi \in \mathcal{L}^1(\Omega; \mathbf{R})$, so Definition B.1 implies that $\int_\Omega^* \psi \leq \int_\Omega \psi$, which finishes the argument. And if $\int_\Omega \psi^- = +\infty$, then an obvious argument with the sequence $\phi_n := \psi^+ - \min(\psi^-, n)$ shows that $\int_\Omega^* \psi = -\infty = \int_\Omega \psi$. QED

Lemma B.3 *Let $\psi : \Omega \to [-\infty, +\infty]$ (possibly nonmeasurable) and $\phi \in \mathcal{L}^1(\Omega; \mathbf{R})$ be such that $\psi \geq \phi$ on Ω and $\int_\Omega^* \psi d\mu < +\infty$. Then there exists $\tilde{\phi} \in \mathcal{L}^1(\Omega; \mathbf{R})$, $\tilde{\phi} \geq \phi$, such that $\int_\Omega \tilde{\phi} d\mu = \int_\Omega^* \psi d\mu$.*

PROOF. By Definition B.1 there exists a sequence (ϕ_k) in $\mathcal{L}^1(\Omega;\mathbf{R})$ such that $\phi_k \geq \psi$ and $\int_\Omega \phi_k \leq \int_\Omega^* \psi + k^{-1}$ for all k. Define $\tilde{\phi} := \inf_k \phi_k$; then $\phi_1 \geq \tilde{\phi} \geq \psi \geq \phi$ (hence $\tilde{\phi} \in \mathcal{L}^1(\Omega;\mathbf{R})$) and $\int_\Omega \tilde{\phi} \leq \int_\Omega^* \psi$. The converse inequality is trivial. QED

Proposition B.4 (Fatou-Vitali) *Let (ψ_n) be a sequence of (possibly nonmeasurable) functions $\psi_n : \Omega \to [-\infty,+\infty]$ such that there exists a uniformly integrable sequence (ϕ_n) in $\mathcal{L}^1(\Omega;\mathbf{R})$ for which for every $n \in \mathbf{N}$*

$$\psi_n(\omega) \geq \phi_n(\omega) \text{ for all } \omega \in \Omega.$$

Then

$$\liminf_n \int_\Omega^* \psi_n(\omega)\mu(d\omega) \geq \int_\Omega^* \liminf_n \psi_n(\omega)\mu(d\omega).$$

PROOF. By Lemma B.3, for each n there exists $\tilde{\phi}_n \in \mathcal{L}^1(\Omega;\mathbf{R})$ such that $\tilde{\phi}_n \geq \psi_n \geq \phi_n$ and $\int_\Omega \tilde{\phi}_n = \int_\Omega^* \psi_n$. By uniform integrability of (ϕ_n), the classical Fatou-Vitali lemma [4, 7.5.2] applies. This gives

$$\liminf_n \int_\Omega^* \psi_n = \liminf_n \int_\Omega \tilde{\phi}_n \geq \int_\Omega \liminf_n \tilde{\phi}_n.$$

Since $\liminf_n \tilde{\phi}_n \geq \liminf_n \psi_n$, Definition B.1 gives $\int_\Omega^* \liminf_n \tilde{\phi}_n \geq \int_\Omega^* \liminf_n \psi_n$. Since $\liminf_n \tilde{\phi}_n$ is $\mathcal{A}$-measurable, Lemma B.2 applies, and the result follows. QED

Lemma B.5 *Let $\psi, \psi' : \Omega \to [-\infty,+\infty]$ be arbitrary (possibly nonmeasurable). Then*

$$\int_\Omega^* \psi d\mu \dot{+} \int_\Omega^* \psi' d\mu \geq \int_\Omega^* (\psi \dot{+} \psi')d\mu,$$

where $\dot{+}$ is defined just as ordinary addition, but with $(-\infty)\dot{+}(+\infty) := +\infty$ as an additional convention.

PROOF. If either term on the left is equal to $+\infty$, the result is trivially true. So suppose that $\int_\Omega^* \psi d\mu < +\infty$ and $\int_\Omega^* \psi' d\mu < +\infty$ (hence both ψ and ψ' are a.e. not equal to $+\infty$). By Definition B.1, there exist sequences (ϕ_n) and (ϕ'_n) in $\mathcal{L}^1(\Omega;\mathbf{R})$ such that $\int_\Omega \phi_n \to \int_\Omega^* \psi$ and $\int_\Omega \phi'_n \to \int_\Omega^* \psi'$, with $\phi_n \geq \psi$ and $\phi'_n \geq \psi'$. But then simple work with $(\phi_n + \phi'_n)$ gives the inequality immediately. QED

Lemma B.6 *Let $\psi : \Omega \to [-\infty,+\infty]$ be arbitrary and let $\phi \in \mathcal{L}^1(\Omega;\mathbf{R})$. Then*

$$\int_\Omega^* (\psi + \phi)d\mu = \int_\Omega^* \psi d\mu + \int_\Omega \phi d\mu.$$

PROOF. An elementary consequence of Definition B.1. QED

References

[1] Aldous, D.J. Limit theorems for subsequences of arbitrarily-dependent sequences of random variables. *Z. Wahrscheinlichkeitsth. Verw. Geb.* **40** (1977), 59-82.

[2] Artstein, Z. On a variational problem. *J. Math. Anal. Appl.* **45** (1974), 404-415.

[3] Artstein, Z. A note on Fatou's lemma in several dimensions. *J. Math. Econom.* **6** (1979), 277-282.

[4] Ash, R.B. *Real Analysis and Probability.* Academic Press, New York, 1972.

[5] Aubin, J.-P. and Ekeland, I. *Applied Nonlinear Analysis.* Wiley, New York, 1984.

[6] Aumann, R.J. and Perles, M. A variational problem arising in economics. *J. Math. Anal. Appl.* **11** (1965), 488-503.

[7] Balder, E.J. On a useful compactification for optimal control problems. *J. Math. Anal. Appl.* **72** (1979), 391-398.

[8] Balder, E.J. Mathematical foundations of statistical decision theory: A modern viewpoint I, Preprint No. 199, Mathematical Institute, University of Utrecht, 1981.

[9] Balder, E.J. A general approach to lower semicontinuity and lower closure in optimal control theory. *SIAM J. Control Optim.* **22** (1984), 570-598.

[10] Balder, E.J. A unifying note on Fatou's lemma in several dimensions. *Math. Oper. Res.* **9** (1984), 267-275.

[11] Balder, E.J. Existence results without convexity conditions for general problems of optimal control with singular components. *J. Math. Anal. Appl.* **101** (1984), 527-539.

[12] Balder, E.J. A general denseness result for relaxed control theory. *Bull. Austral. Math. Soc.* **30** (1984), 463-475.

[13] Balder, E.J. Lower closure for orientor fields by lower semicontinuity of outer integral functionals. *Ann. Mat. Pura Appl.* (IV) **139** (1985), 349-360.

[14] Balder, E.J. An extension of Prohorov's theorem for transition probabilities with applications to infinite-dimensional lower closure problems. *Rend. Circ. Mat. Palermo* **34** (1985), 427-447.

[15] Balder, E.J. On seminormality of integral functionals and their integrands. *SIAM J. Control Optim.* **24** (1986), 95-121.

[16] Balder, E.J. Generalized equilibrium results for games with incomplete information. *Math. Oper. Res.* **13** (1988), 265-276.

[17] Balder, E.J. On Prohorov's theorem for transition probabilities. *Sém. Analyse Convexe* **19** (1989), 9.1-9.11.

[18] Balder, E.J. Unusual applications of a.e. convergence. In: *Almost Everywhere Convergence.* (G.A. Edgar and L. Sucheston, eds.) Academic Press, New York, 1989, pp. 31-53.

[19] Balder, E.J. New sequential compactness results for spaces of scalarly integrable functions. *J. Math. Anal. Appl.* **151** (1990), 1-16.

[20] Balder, E.J. On equivalence of strong and weak convergence in L_1-spaces under extreme point conditions. *Israel J. Math.* **75** (1991), 21-48.

[21] Balder, E.J. On Cournot-Nash equilibrium distributions for games with differential information and discontinuous payoffs. *Econ. Theory* **1** (1991), 339-354.

[22] Balder, E.J. Existence of optimal solutions for control and variational problems with recursive objectives. *J. Math. Anal. Appl.* **178** (1993), 418-437.

[23] Balder, E.J. New existence results for optimal controls in the absence of convexity: the importance of extremality. *SIAM J. Control Optim.* **32** (1994), 890-916.

[24] Balder, E.J. A unified approach to several results involving integrals of multifunctions. *Set-valued Anal.* **2** (1994), 63-75.

[25] Balder, E.J. *Lectures on Young Measures.* Cahiers de Mathématiques de la Décision **9517**. CEREMADE, Université Paris-Dauphine, Paris, 1995.

[26] Balder, E.J. A unifying approach to existence of Nash equilibria. *Intern. J. Game Theory* **24** (1995), 79-94.

[27] Balder, E.J. On the existence of optimal contract mechanisms for incomplete information principal-agent models. *J. Econ. Theory* 68 (1996), 133-148.

[28] Balder, E.J. Comments on the existence of equilibrium distributions. *J. Math. Econom.* 25 (1996), 307-323.

[29] Balder, E.J. Consequences of denseness of Dirac Young measures. *J. Math. Anal. Appl.* 207 (1997), 536-540.

[30] Balder, E.J. Young measure techniques for existence of Cournot-Nash-Walras equilibria. in: Fields Institute Communications (Conference in honor of R.J. Aumann), in press.

[31] Balder, E.J. On the existence of Cournot-Nash equilibria in continuum games. Preprint no. 993, Department of Mathematics, University of Utrecht, 1997. *J. Math. Econom.*, in press.

[32] Balder, E.J. An extension of the Dunford-Pettis theorem with applications to a Bocce-type oscillation restriction criterion. Preprint no. 1021, Department of Mathematics, University of Utrecht, 1997.

[33] Balder, E.J. Report on the doctoral thesis of Florence Galdéano, Département de Mathématiques, Université de Perpignan, Académie de Montpellier, 1997.

[34] Balder, E.J. New fundamentals of Young measure convergence. Proceedings of "Calculus of Variations and Related Topics", Haifa, 1998. Pitman Research Notes in Mathematics, in press.

[35] Balder, E.J. Unifying Cournot-Nash equilibirum existence results for continuum games, Preprint, Department of Mathematics, University of Utrecht, 1999.

[36] Balder, E.J., Girardi, M. and Jalby, V. From weak to strong types of $\mathcal{L}^1_E$-convergence by the Bocce-criterion. *Studia Math.* **111** (1994), 241-262.

[37] Balder, E.J. and Hess, C. Fatou's lemma for multifunctions with unbounded values. *Math. Oper. Res.* **20** (1995), 175-188.

[38] Balder, E.J. and Hess, C. Two generalizations of Komlós' theorem with lower closure-type applications. *J. Convex Anal.* **3** (1996), 25-44.

[39] Balder, E.J. and Rustichini, A. An equilibrium result for games with private information and infinitely many players. *J. Econ. Theory* **62** (1994), 385-393.

[40] Ball, J.M. A version of the fundamental theorem for Young measures. In: *PDEs and Continuum Models of Phase Transitions* (M. Rascle, D. Serre and M. Slemrod, eds.) Lecture Notes in Physics **344**, Springer-Verlag, Berlin, 1984, pp. 207-215.

[41] Becker, R.A., Boyd, J.H. and Sung, B.Y. Recursive utility and optimal capital accumulation. I. Existence. *J. Econ. Theory* **47** (1989), 76-100.

[42] Berliocchi, H. and Lasry, J.-M. Intégrandes normales et mesures paramétrées en calcul des variations. *Bull. Soc. Math. France* **101** (1973), 129-184.

[43] Bertsekas, D.P and Shreve, S.E. *Stochastic Optimal Control: the Discrete Time Case.* Academic Press, New York, 1978.

[44] Billingsley, P. *Convergence of Probability Measures.* Wiley, New York, 1968.

[45] Bottaro, G. and Opezzi, G. Semicontinuità inferiore di un funzionale integrale dipendente da funzionali di classe L^p a valori in un spazio di Banach. *Ann. Mat. Pura Appl.* (IV) **123** (1980), 16-26.

[46] Bourbaki, N. *Intégration*, Chap. IX. Hermann, Paris, 1969.

[47] Brooks, J. and Dinculeanu, N. Weak compactness in spaces of Bochner integrable functions and applications. *Adv. in Math.* **24** (1977), 172-188.

[48] Castaing, C. A propos de l'existence des sections séparément mesurables et séparément continues d'une multiapplication séparément mesurable et séparément semicontinue inférieurement. *Sém. Analyse Convexe* **6** (1976).

[49] Castaing, C. Un résultat de compacité lié á la propriété des ensembles Dunford-Pettis dans $L^1_F(\Omega, \mathcal{A}, \mu)$. *Sém. Analyse Convexe* **9** (1979), 17.1-17.7.

[50] Castaing, C. and Valadier, M. *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. **580**, Springer-Verlag, Berlin, 1977.

[51] Cellina, A. and Colombo, G. On a classical problem of the calculus of variations without convexity assumptions. *Ann. Inst. H. Poincaré, Analyse nonlinéaire* **7** (1989), 97-106.

[52] Cesari, L. *Optimization – Theory and Applications.* Springer-Verlag, Berlin, 1983.

[53] Chatterji, S.D. A subsequence principle theory. *Jber. d. Dt. Math.-Verein.* **87** (1985), 91-107.

[54] Choquet, G. *Lectures on Analysis.* Benjamin, Reading, Mass., 1969.

[55] Dellacherie, C. and Meyer, P.-A. *Probabilités et Potentiel.* Hermann, Paris, 1975 (English translation: North-Holland, Amsterdam, 1978).

[56] Diestel, J. and Uhl, J. J. *Vector Measures.* Math. Surveys **15**, American Mathematical Society, Providence, RI, 1977.

[57] Ding, X.P., and Tan, K.-K. Generalized variational inequalities and generalized quasi-variational inequalities. *J. Math. Anal. Appl.* **148** (1990), 497-508.

[58] Ekeland, I. and Temam, R. *Convex Analysis and Variational Problems.* North-Holland, Amsterdam, 1977.

[59] Fan, Ky. A generalization of Tychonoff's fixed point theorem. *Math. Ann.* **142** (1961), 305-310.

[60] Ferguson, T. *Mathematical Statistics : a Decision Theoretic Approach.* Academic Press, New York, 1967.

[61] Galdéano, F. and Truffert, A. On the narrow convergence of measures defined on a product space. Preprint, Département de Mathématiques, Université de Perpignan, 1999.

[62] Greenberg, J., Shitovitz, B. and Wieczorek, A. Existence of equilibria in atomless production economies with price dependent preferences. *J. Math. Econom.* **6** (1979), 31-41.

[63] Harsanyi, J.C. Games with incomplete information played by 'Bayesian' players. *Management Sci.* **14**, 159-182.

[64] Ichiishi, T. *Game Theory for Economic Analysis.* Academic Press, New York, 1983.

[65] Ioffe, A.D. On lower semicontinuity of integral functionals, I, II. *SIAM J. Control Optim.* **15** (1977), 521-538, 991-1000.

[66] Ionescu Tulcea, A. and C. *Topics in the Theory of Lifting.* Springer-Verlag, Berlin, 1969.

[67] Jacod, J. and Mémin, J. Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité. *Séminaire de Probabilités* XV (J. Azéma and M. Yor, eds.) Lecture Notes in Math. **986**, Springer-Verlag, Berlin, 1983, pp. 529-546.

[68] Jawhar, A. Mesures de transition et applications. *Sém. Analyse Convexe* **14** (1984), 13.1-13.62.

[69] Khan, M.A. On extensions of the Cournot-Nash theorem. in: *Advances in Equilibrium Theory* (C.D. Aliprantis, O. Burkinshaw and N.J. Rothman, eds.), Lecture Notes Econ. Math. Syst. **244** (1985), Springer-Verlag, Berlin, pp. 79-106.

[70] Khan, M.A. and Rustichini, A. Cournot-Nash equilibrium distributions for games with differential information. in: *Fixed Point Theory and Applications* (M.A. Théra and J.B. Baillon, eds.), Pitman Research Notes **252** (1991), pp. 245-259.

[71] Komlós, J. A generalisation of a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar.* **18** (1967), 217-229.

[72] Kristensen, J. Lower semicontinuity in spaces of weakly differentiable functions. Preprint, Mathematical Institute, University of Oxford, England.

[73] LeCam, L. Convergence in distribution of stochastic processes. *Univ. California Publ. Statist.* **2**, No. 11 (1957), 207-236.

[74] LeCam, L. An extension of Wald's theory of statistical decision functions. *Ann. Math. Stat.* **26** (1955), 69-81.

[75] Loève, M. *Probability Theory I*, Springer-Verlag, Berlin, 1977.

[76] Mariconda, C. A generalization of the Cellina-Colombo theorem for a class of non-convex variational problems. *J. Math. Anal. Appl.* **175** (1993), 514-522.

[77] Mas-Colell, A. On a theorem of Schmeidler. *J. Math. Econom.* **13** (1984), 201-206.

[78] Milgrom, P.R. and Weber, R.J. Distributional strategies for games with incomplete information. *Math. Oper. Res.* **10** (1985), 619-632.

[79] Morduchovich, B.S. The existence of optimal controls in a problem with a free right terminal. *Diff. Equations* **8** (1972), 1545-1548.

[80] Morduchovich, B.S. Existence theorems in nonconvex optimal control. to appear in: Proceedings of "Calculus of Variations and Related Topics", Haifa, 1998.

[81] Neveu, J. *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Fransisco, 1965.

[82] Parthasarathy, K.R. *Probability Measures on Metric Spaces.* Academic Press, New York, 1967.

[83] Pedregal, P. *Parametrized Measures and Variational Principles.* Birkhäuser Verlag, Basel, 1997.

[84] Pfanzagl, J. Convexity and conditional expectations. *Ann. Prob.* **2** (1974), 490-494.

[85] Rath, K.P. Existence and upper hemicontinuity of equilibrium distributions of anonymous games with discontinuous payoffs. *J. Math. Econom.* **26** (1996), 305-324.

[86] Raymond, J.-P. Existence theorems in optimal control theory without convexity assumptions. *J. Optim. Theory Appl.* **67** (1990), 109-132.

[87] Schmeidler, D. Fatou's lemma in several dimensions. *Proc. Amer. Math. Soc.* **24** (1970), 300-306.

[88] Schmeidler, D. Equilibrium points of non-atomic games. *J. Statist. Phys.* **7** (1973), 295-300.

[89] Schwartz, L. *Radon Measures.* Oxford University Press, 1975.

[90] Sychev, M. Young measure approach to characterization of behaviour of integral functionals on weakly convergent sequences by means of their integrands. *Ann. Inst. H. Poincaré (Analyse Nonlinéaire)*, to appear.

[91] Tartar, L. Compensated compactness and aplications to partial differential equations. In: *Nonlinear Analysis and Mechanics* (R. Knops, ed.) Res. Notes Math. **39**, Pitman, Boston, 1979, pp. 136-212

[92] Topsœ, F. *Topology and Measure.* Lecture Notes in Math. **133**, Springer-Verlag, Berlin, 1970.

[93] Valadier, M. Some bang-bang theorems. in: *Multifunctions and Integrands* (G. Salinetti, ed.) Lecture Notes in Math. **1091**, Springer-Verlag, Berlin, 1984, pp. 225-234.

[94] Valadier, M. Young measures. in: *Methods of Nonconvex Analysis* (A. Cellina, ed.), Lecture Notes in Math. **1446**, Springer-Verlag, Berlin, 1990, pp. 152-188.

[95] Valadier, M. A course on Young measures. *Rend. Istit. Mat. Univ. Trieste* **24** (1994), supplemento, 349-394.

[96] Wald, A. *Statistical Decision Functions.* Wiley, New York, 1950.

[97] Warga, J. *Optimal Control of Differential and Functional Equations*, Academic Press, New York, 1972.

[98] Young, L.C. Generalized curves and the existence of an attained absolute minimum in the calculus of variations, *C. R. Sci. Lettres Varsovie*, C III **30** (1937), 212-234.