Schubert varieties and generalizations

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Abstract. This contribution reviews the main results on Schubert varieties and their generalizations. It covers, more or less, the material of the lectures at the Seminar. These were partly expository, introducing material needed by other lecturers. In particular, Section 1 reviews 'classical' material, used in several of the other contributions.

Introduction.

The aim of this paper is to give a review of the main results on Schubert varieties and their generalizations. In the first section Schubert varieties (over $\mathbf{C}$) are introduced, in the setting of the theory of reductive groups and their Bruhat decomposition. Some geometric results are discussed. The Steinberg variety associated to a reductive group is also introduced. Most of the material of this section is 'classical'.
In section 2 examples are given of constructions of algebraic objects, based on the geometry discussed in Section 1. For example, 2.2 gives an elementary geometric construction of the Weyl group W of a reductive group G. It uses correspondences on the flag variety X of G. Using machinery from algebraic topology, a calculus of correspondences on X produces the Hecke algebra $\mathcal{H}$ of W. This is discussed in 2.4 and 2.5.
Section 3 discusses generalizations of Schubert varieties. These occur, for example in the context of spherical varieties. A closed subgroup H of G is spherical if a Borel subgroup B of G has finitely many orbits on G/H. Then G/H is a (homogeneous) *spherical variety*. The orbit closures generalize Schubert varieties (which one recovers for $H = B$). An important special case is the case of symmetric varieties, where H is the fixed point group of an involutorial automorphism of G.
The combinatorial properties of the set of orbits are discussed in 3.6. A calculus of correspondences gives rise to a representation of $\mathcal{H}$, discussed in 3.7. The last part of Section 3 reviews special features of the the case of symmetric varieties.

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1. Flag manifolds and Schubert varieties.

1.1. The origin of the Schubert varieties lies in the 'Schubert calculus', devised by H. Schubert at the end of the 19^{th} century, which gives recipes to determine -not always rigorously- numbers of solutions of geometric problems (see [Sch]).
A simple example of such a problem: determine the number of lines in $\mathbf{P}^3(\mathbf{C})$ intersecting 4 lines in general position (the answer is 2). A more general example: determine the number of d-planes in $\mathbf{P}^n(\mathbf{C})$ intersecting $(d+1)(n-d)$ planes of dimension $(n-d-1)$ in general position (the answer is

$$\frac{1!2!...d!((d+1)(n-d))!}{(n-d)!(n-d+1)!...n!}).$$

Let $G_{d,n}$ be the set of d-dimensional subspaces of $\mathbf{C}^n$. It is a projective algebraic variety, coordinatized by 'Plücker coordinates'. It is also the variety of $(d-1)$-planes in $\mathbf{P}^{n-1}(\mathbf{C})$.
Fix a basis $(e_1, ..., e_n)$ of $V = \mathbf{C}^n$ and let V_i be the subspace of V spanned by $(e_1, ..., e_i)$, with $V_0 = \{0\}$. Then $\mathcal{F} = (V_0, V_1, ..., V_{n-1}, V_n)$ is a complete flag in V. Let W be a d-dimensional subspace and put

$$J(W) = \{j \mid V_{j-1} \cap W \neq V_j \cap W\}.$$

This is an increasing sequence of d integers ≥ 1, the *jump sequence* of W. It determines the position of W relative to $\mathcal{F}$. For example, for a subspace 'in general position' we have $J(W) = (n-d+1, ..., n)$. Let Y_J be the set of $W \in Y = G_{d,n}$ with $J(W) = J$, a given sequence. Then Y is the disjoint union of the Y_J. Moreover, one shows that each Y_J is locally closed in Y, and is isomorphic to an affine space. After ordering the set of J componentwise, the closures (both in the Zariski topology and the complex topology) are described by

$$\overline{Y_J} = \bigcup_{J' \leq J} Y_{J'}.$$

These closures are the *Schubert varieties* in $G_{d,n}$.
In the Schubert calculus one deals with intersections of Schubert varieties and their multiplicities. This is best done in terms of the Chow ring of Y, spanned by equivalence classes of subvarieties of Y. This leads into the theory of symmetric functions. See [Fu, 14.7]. I shall not go into this.
I shall concentrate on the group theoretical aspects. The group $G = GL_n(\mathbf{C})$ acts algebraically on Y. The action is transitive, so Y is a homogeneous space of G and is of the form $Y = G/P$, where P is the parabolic subgroup of the $g = (g_{ij}) \in G$ with $g_{ij} = 0$ for $i > d$ and $j \leq d$. It is not hard to see that the Y_J are precisely the orbits of G.
We shall consider a more general situation, which will englobe the special case of Grassmannians.

1.2. Notations.
The notions and results from the theory of algebraic groups which we use without further reference can be found in [Bo2] or [Hu2]. For root systems and the Weyl group see [Bou].
G is a connected, reductive, linear algebraic group over $\mathbf{C}$ (one could work over an arbitrary algebraically closed field, but I won't do this). We fix a maximal torus T of G and a Borel subgroup $B \supset T$. Also, N is the normalizer of T and $W = N/T$ is the Weyl group. Fix a section $w \mapsto \dot{w}$ of W to N.
Let R be the root system of (G, T) and let R^+ be the system of positive roots defined by B. For $\alpha \in R$ we have a one parameter subgroup U_α of G, normalized by T. The unipotent radical U of B is generated by the U_α with $\alpha \in R^+$. For $w \in W$ let U_w be the subgroup of G generated by the subgroups U_α with $\alpha \in R^+,\ w^{-1}\alpha \notin R^+$.
For $\alpha \in R$ let $s_\alpha \in W$ be the reflection which it defines. Let D be the basis of R defined by R^+. The set S of simple reflections s_α $(\alpha \in D)$ generates W. The corresponding length function on W is l. We have dim $U_w = l(w)$.
Put $G_w = B\dot{w}B$. This is a locally closed subset of G, being an orbit of $B \times B$. (The underlying topology is the Zariski topology. One might also take the complex topology.)

1.3. Bruhat's lemma.
Proposition. *('Bruhat's lemma') (i)* $G = \coprod G_w$*;*
(ii) $(u, b) \mapsto u\dot{w}b$ *defines an isomorphism of algebraic varieties* $U_w \times B \to B\dot{w}B$.

In fact, (G, B, N, S) make up the ingredients of a *Tits system*, see [Hu1, no. 29]. This implies that for $w \in W,\ s \in S$

$$G_s G_w = \begin{cases} G_{sw} & \text{if} \quad l(sw) > l(w), \\ G_w \cup G_{sw} & \text{if} \quad l(sw) < l(w). \end{cases}$$

It follows that for $s \in S$

$$P_s = G_e \cup G_s$$

is a parabolic subgroup of G containing $B = G_e$. We have $P_s/B \simeq \mathbf{P}^1$. It also follows that if $\mathbf{s} = (s_1, ..., s_l)$ is a reduced decomposition of $w \in W$ (where $s_i \in S$, $l = l(w)$ we have

$$G_w = G_{s_1} G_{s_2} ... G_{s_l}.$$

Lemma. *$P_{s_1} P_{s_2} ... P_{s_l}$ is the closure $\overline{G_w}$.*
If Y and Z are varieties with a right (respectively, left) B-action we write $Y \times_B Z$ for the quotient of $Y \times Z$ by the B-action $b(y,z) = (yb^{-1}, bz)$. (It is presupposed that the quotient exists.) A similar notation is used for multiple products.
Put

$$Z = P_{s_1} \times_B P_{s_2} \times_B ... \times_B P_{s_l},$$

this is an irreducible variety. The product map of G induces a morphism $\pi : Z \to G$, which is proper (because all quotients P_{s_i}/B are projective lines). Hence Im π is closed and irreducible. Moreover $\pi^{-1} G_w$ is open and dense in Z and the restriction of π to this set maps it bijectively onto G_w. The lemma follows from these facts.

1.4. Bruhat order.
The closure $\overline{G_w}$ is a union of double cosets G_x. Define an order on W by $x \leq w$ if $G_x \subset \overline{G_w}$. This is the Bruhat order (originally introduced by Chevalley).
It follows from the lemma of 1.3 that there is the following combinatorial description of the Bruhat order. Let $\mathbf{s} = (s_1, ..., s_l)$ be a reduced decomposition of $w \in W$ (where $s_i \in S$, $l = l(w)$) and let $x \in W$. Then $x \leq w$ if and only if x is a subproduct of $s_1 ... s_l$. In fact, on any Coxeter group there exists an order with this description, see [Hu2, 5.9].

1.5. Schubert varieties.
The quotient $X = G/B$ is a *flag variety.* It is an irreducible, smooth, projective, homogeneous space for G. Let X_w be the image of G_w in X under the canonical map, this is a *Bruhat cell* in X. The *big cell* is X_{w_0}, where w_0 is the longest element of W. The Bruhat cell X_w is a locally closed subvariety of X, isomorphic to affine space $\mathbf{A}^{l(w)}$, as a consequence of 1.3 (ii). The big cell is open and dense in X.
By 1.3 (i), $(X_w)_{w \in W}$ is a 'paving' of X by affine spaces (or a 'cellular decomposition'). The X_w are the B-orbits (or U-orbits) on X.
A *Schubert variety* is a closure $S_w = \overline{X_w}$ $(w \in W)$. It is an (in general non-smooth) irreducible, projective variety on which B acts. By 1.3 (i) and 1.4 we have a paving $S_w = (X_x)_{x \leq w}$.

Example. Let $G = \mathbf{GL}_n$. It acts on $V = k^n$. A *flag* in V of length s is a sequence of distinct subspaces V_i $(0 \leq i \leq s)$ of V with $V_0 = \{0\}$, $V_0 \neq V_1 \subset V_2 \subset ... \subset V_s$. The flag is complete if $s = n$ (in which case $\dim V_i = i$ for all i). G acts on the set of flags and the parabolic subgroups of G are the stabilizers of flags. The Borel subgroups are the stabilizers of complete flags.
Let $(V_0, V_1, ..., V_n)$ be the complete flag of 1.1. Its stabilizer is the Borel group B of upper triangular matrices and G/B can be identified with the space of all complete flags. Let $P \supset B$ be the stabilizer of the flag (V_0, V_d, V_n). Then G/P is the Grassmannian $G_{d,n}$. The canonical morphism $G/B \to G/P$ maps a complete flag onto its d-dimensional ingredient. A 'classical' Schubert variety Y_J as in 1.1 is the image in G/P of a Schubert variety S_w in G/B, or the closure of a B-orbit in G/P. By Tits system theory these orbits are parametrized by the cosets of the Weyl group of G modulo the Weyl group of P (see [Bou, Ch. IV, p. 28]). In

the present case this means that the Schubert varieties in $G_{d,n}$ are indexed by the elements of $S_n/(S_d \times S_{n-d})$, i.e. by the d-element subsets of $\{1, 2, ..., n\}$. These are in bijection with the 'jump sequences' of 1.1.

For $w \in W$ put
$$\mathcal{O}_w = \{(x, y) \in X \times X \mid x^{-1}y \in G_w\}.$$
The $\mathcal{O}_w$ are the G-orbits on $X \times X$. There is a close connection with the Bruhat cells: the first projection $X \times X \to X$ defines a fibering $\mathcal{O}_w \to X$ with fibers X_w. Similarly for the closures $\overline{\mathcal{O}_w}$. It follows that
$$\overline{\mathcal{O}_w} = \bigcup_{x \leq w} \mathcal{O}_x.$$

1.6. The T-action.
The torus T acts on X and on all Schubert varieties. The fixed points of T in X are the images p_w of the $\dot{w}$ in X ($w \in W$), so their number is finite. The fixed points of T in the Schubert variety S_w are the p_x with $x \leq w$.
Let again w_0 be the longest element of W.

Lemma 1. *Let $x, w \in W$ and assume that $l(x) \leq l(w)$.*
(i) If the intersection $S_x \cap \dot{w}_0 S_{w_0 w}$ is non-empty then $x = w$ and the intersection is the point p_w;
(ii) X_w and $\dot{w}_0 X_{w_0 w}$ intersect transversally at p_w.
(iii) Let $x < w$. Then $(u\dot{x}B, gB) \mapsto ugB$ ($u \in U_x$) defines a T-equivariant isomorphism of
$$X_x \times (\dot{w}_0 X_{w_0 x} \cap S_w)$$
onto an open neighborhood of p_x in S_w.
The intersection in (i) is a T-stable projective variety and contains T-fixed points. If p_y is one then we have both $y \leq x$ and $w_0 y \leq w_0 w$, i.e. $y \geq w$. This implies that $y = x = w$. If the intersection had dimension > 1 it would contain at least two fixed points and (i) follows. Part (ii) is proved by considering tangent spaces.
Put $U^- = \dot{w}_0 U (\dot{w}_0)^{-1}$, this is the subgroup of G generated by the U_α with $\alpha \in -R^+$. Then $U^- B$ is open in G by 1.3 (ii). Hence xU^-B/B is an open neighborhood of p_x in X. Then (iii) follows by observing that
$$U^- = x^{-1} U_x x . \dot{w}_0 X_{w_0 x} (\dot{w}_0)^{-1}.$$

The variety $X_{x,w} = \dot{w}_0 X_{w_0 x} \bigcap S_w$ is a 'transverse slice' at p_x of X_x inside S_w. Let λ be a cocharacter of T (a one parameter multiplicative subgroup) such that $\langle w_0 \alpha, \lambda \rangle > 0$ for all $\alpha \in R^+$ with $x^{-1} w_0 \alpha \in -R^+$ (the brackets denote the pairing between characters and cocharacters). Then λ contracts $X_{x,w}$ to x, i.e. if $a \in X_{x,w}$ then $\lim_{t \to 0} \lambda(t).a = x$.

Let S be a subset of T and let $H = Z_G(S)$ be its centralizer. If S fixes a point up_w ($u \in U_w$) of X then S must centralize u, and since centralizers of semi-simple elements in connected solvable groups are connected, we can conclude that u lies in the connected centralizer H°. Similarly, the intersection of H with a Borel subgroup of G containing T is a Borel subgroup of H°. The irreducible components of the fixed point set X^S of S in X are H°-stable and it readily follows that each component is a homogeneous space for H°, isomorphic to the flag

manifold of H°. Similarly, one sees that the fixed point sets S_w^S must be stable under the Borel group $B \cap H^\circ$ of H° and it follows that each irreducible component of such a fixed point set is isomorphic to a Schubert variety for H°. Notice that if S is an algebraic subgroup of T of codimension d, the reductive group H° has semi-simple rank $\leq d$. In particular, if $d = 1$ the irreducible components of S_w^S are Schubert varieties for $\mathbf{SL}_2$, hence are points or T-stable projective lines. Conversely, a T-stable irreducible curve in S_w must be fixed pointwise by a codimension one subtorus of T and hence is a component of its fixed point set.
In particular, the number of T-stable curves in X is finite. They can be described explicitly.

Let Σ be the set of reflections in W. A reflection $\sigma \in \Sigma$ defines a 3-dimensional subgroup G_σ of G, generated by U_α and $U_{-\alpha}$, where $\sigma = s_\alpha = s_{-\alpha}$ (notations of 1.1). For $(w, \sigma) \in W \times \Sigma$ put $C_{w,\sigma} = G_\sigma \dot{w} B/B$.
Lemma 2. *(i) $C_{w,\sigma}$ is a T-stable curve. Any T-stable curve is of this form;*
(ii) The T-fixed points contained in $C_{w,\sigma}$ are p_w and $p_{\sigma w}$;
(iii) If $C_{w',\sigma'} = C_{w,\sigma}$ then either $(w', \sigma') = (w, \sigma)$ or $(w', \sigma') = (\sigma w, \sigma)$;
(iv) $C_{w,\sigma} \subset S_x$ if and only if $w \leq x, \sigma w \leq x$.
We have $G_\sigma = Z_G(\text{Ker } \alpha)^\circ$. The observations of the previous paragraph then imply that $C_{w,\sigma}$ is a T-stable curve.
A T-stable curve $C \subset X$ must contain a fixed point p_w. Then $\dot{w}U^-B/B = \dot{w}U^-(\dot{w}^{-1})B/B$ is a T-stable open neighborhood of p_w. Its intersection with C is a T-stable affine curve. One is reduced to finding such curves, or to finding T-stable curves in U through the identity element. It is not hard to see that the latter are the U_α contained in U. Then the second part of (i) follows. The argument also gives that $C_{w,\sigma} \subset S_w$, which implies (iv).

1.7. Geometric properties of Schubert varieties.
Schubert varieties tend to be singular. There is a useful 'resolution' of a Schubert variety S_w, which we now describe.
Let $\mathbf{s}$ be a reduced decomposition of w, as in 1.5. With the notations of 1.3, put

$$Z_{\mathbf{s}} = P_{s_1} \times_B P_{s_2} \times_B \dots \times_B (P_{s_l}/B).$$

This is a *Bott-Samelson variety.* It is an iterated $\mathbf{P}^1$-bundle, hence is irreducible and smooth. From the lemma of 1.3 we deduce the following.

Proposition 1. *There is a proper, surjective, birational morphism $\psi : Z_{\mathbf{s}} \to S_w$.*
The morphism ψ is induced by the product morphism in G.
ψ is not always a resolution of singularities of S_w in the usual sense, as ψ need not be bijective on the inverse image of the set of smooth points of S_w. For example, if R is irreducible and $w = w_0$ (so $S_w = X$) then ψ is bijective only if R is of type A_1.

Criteria for smoothness and rational smoothness of a Schubert variety were recently given by Kumar [Ku]. See also Brion's contribution [Bri2, no. 5], where these matters are discussed in the context of equivariant intersection theory. (Recall that an irreducible algebraic variety Z is rationally smooth at a point z if z has arbitarily small open neighborhoods which are 'homologically like open balls'. A formal definition is: the constant sheaf $\mathbf{Q}$ is its own Grothendieck-Verdier dual, up to a dimension shift.) See also [Bri3].
We discuss some more elementary results about smoothness of Schubert varieties, due to Carrell and Peterson (see [Ca]).

Lemma. *Let Y be a T-stable irreducible subvariety of X. If $y \in Y$ is a T-fixed point, the number of T-stable curves in Y passing through y is at least* $\dim Y$. *Equality holds if Y is smooth at y.*
The proof of the first part is quite elementary, and goes through in greater generality.

Proposition 2. *Let $x, w \in W$ with $x < w$ and denote by $a(x, w)$ the number of $\sigma \in \Sigma$ with $x < \sigma x \leq w$.*
(i) $a(x, w) \geq l(w) - l(x)$. Equality holds if S_w is smooth at p_x;
(ii) If $a(x, w) = l(w) - l(x)$ then S_w is rationally smooth at p_x.
(i) follows from the lemma. The proof of (ii) given in [loc. cit.] uses a computation with Kazhdan-Lusztig polynomials. For another proof see [Bri3, 2.1].
Peterson (unpublished) has proved that if the root system R is simply laced, one may replace in (ii) 'rationally smooth' by 'smooth'.

Example. Let $G = \mathbf{SL}_4$. Then $W = S_4$. The set S of generators of W consists of $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$. Take $w = s_2 s_1 s_3 s_2 = (13)(24)$, $x = s_2$. Then $l(w) - l(x) = 3$ and $a(x, w) = 4$, so S_w is not smooth.

Although Schubert varieties are in general not smooth they are always normal and Cohen-Macaulay (see [Ra]). These matters are also discussed in Littelmann's contribution [Li,] ref.Littelmann

1.8. Line bundles on X.

Denote by X^*T the character group of T. A character λ of T can be lifted to a character of B, denoted by the same symbol.
Let Y be a variety with a right B-action. For $\lambda \in X^*T$ we have a line bundle $\mathcal{L}(\lambda)$ on Y/B. Namely, $\mathcal{L}$ is the quotient of $Y \times \mathbf{A}^1$ by the B-action $b(y, a) = (yb^{-1}, \lambda(b)^{-1}a)$.
In particular, taking $Y = G$, we have line bundles $\mathcal{L}(\lambda)$ on the flag manifold X. They are G-equivariant, and are locally trivial for the Zariski topology.

1.9. Some related varieties.

Assume G to be semi-simple. Lie algebras will be denoted by gothic letters, so $\mathfrak{g}$ is the Lie algebra of G. Let $F(\, , \,)$ be the Killing form on $\mathfrak{g}$, a non-degenerate bilinear symmetric form invariant under the adjoint action Ad of G on $\mathfrak{g}$. For the matters to be discussed in this section see [CG, Ch. 3].
Consider the cotangent variety T^*X of the flag variety. The tangent space T_xX to X at $x = gB$ is $\mathfrak{g}/\mathrm{Ad}g(\mathfrak{b})$. Identifying the linear dual of $\mathfrak{g}$ with $\mathfrak{g}$ via F, the dual of T_xX is the subspace of $\mathfrak{g}$ orthogonal to $\mathrm{Ad}(g)(\mathfrak{b})$, which is $\mathrm{Ad}(g)\mathfrak{u}$. So T^*X is the set of pairs $(\xi, gB) \in \mathfrak{g} \times X$ with $\xi \in \mathrm{Ad}(g)\mathfrak{u}$. Let $\mathcal{N} \subset \mathfrak{g}$ be the variety of nilpotent elements of $\mathfrak{g}$. The group G acts on it via the adjoint action, with finitely many orbits (see [CG, 3.3.28]).
The first projection induces a morphism $\pi : T^*X \to \mathcal{N}$. In fact, π can be viewed as the moment map for the G-action on the symplectic variety T^*X (see Brylinski's contribution). ref.Brylinski

Theorem 1. *π is a resolution of singularities of $\mathcal{N}$.*
Recall that this means that T^*X is smooth, π is proper and that π induces an isomorphism $\pi^{-1}\mathcal{N}_{\mathrm{sm}} \to \mathcal{N}_{\mathrm{sm}}$, where $\mathcal{N}_{\mathrm{sm}}$ is the open subvariety of smooth points of $\mathcal{N}$. The first two properties are easy. To prove the third one has to use properties of the regular nilpotent elements of $\mathfrak{g}$. An element $x \in \mathcal{N}$ is *regular* if its G-orbit is open in $\mathcal{N}$. One proves that x is

regular if and only if one of the following holds:
(a) $\mathcal{N}$ *is smooth at* x,
(b) $\pi^{-1}\{x\}$ *consists of one point.*
The required property follows. (For a discussion of these matters and further references see [Slo, no. 3, p. 40].)

We denote by Z the fibre product $T^*X \times_{\mathcal{N}} T^*X$. More concretely,

$$Z = \{(\xi, gB, hB) \in \mathcal{N} \times X \times X \mid \xi \in \mathrm{Ad}(g)\mathfrak{u} \cap \mathrm{Ad}(h)\mathfrak{u}\}.$$

This is the *Steinberg variety* of G. It is clear that G acts on it.
We have morphisms $\mu : Z \to \mathcal{N}$ and $\nu : Z \to X \times X$. For $w \in W$ put $Z_w = \nu^{-1}\mathcal{O}_w$, where $\mathcal{O}_w$ is as in 1.5. Then $\overline{Z_w}$ is the conormal bundle $T^*_{\mathcal{O}_w}$ (see [CG, Prop. 3.3.4]).
Let $\xi \in \mathcal{N}$ and let C, C' be two irreducible components of $\pi^{-1}\xi$. Then $Z_{\xi,C,C'} = G.(\{\xi\} \times C \times C')$ is an irreducible subset of Z.

Theorem 2. *Z has pure dimension $2\dim X$. Its irreducible components are the closures $\overline{Z_w}$ and also the closures $\overline{Z_{\xi,C,C'}}$.*
This is proved in [loc. cit., Ch. 3], using symplectic geometry.

For $\xi \in \mathfrak{g}$ let $Z_G(\xi) = Z(\xi)$ be its centralizer in G, and let r be the rank of G.
Corollary 1. *Let $\xi \in \mathcal{N}$. The fibre $\pi^{-1}\xi$ is connected. Its irreducible components have dimension $\frac{1}{2}(\dim Z(\xi) - r)$.*
See [loc. cit., 3.3.24].

Let Σ be a set of representatives of the nilpotent orbits. For $\xi \in \mathcal{N}$ let $\Gamma(\xi)$ be the set of irreducible components of $\pi^{-1}\xi$ and let $A(\xi)$ be the quotient of $Z(\xi)$ by its identity component $Z(\xi)^\circ$. This is a finite group, which acts on $\Gamma(\xi)$. Theorem 2 gives two descriptions of the components of Z.
Corollary 2. *There is a bijection $W \simeq \coprod_{\xi \in \Sigma} A(\xi)\backslash(\Gamma(\xi) \times \Gamma(\xi))$.*

Example. Let $G = \mathbf{SL_n}(\mathbf{C})$. Then Σ can be identified with the set of partitions of n, i.e. with the set of Young diagrams with n boxes. The groups $A(x)$ are all trivial. If $\xi \in \Sigma$ (a Young diagram) then $\Gamma(\xi)$ can be viewed as the set of standard tableaux with shape ξ. Corollary 2 then leads to the *Robinson-Schensted correspondence* between the symmetric group S_n and pairs of standard tableaux of the same shape. See [St2].

Corollary 2 also gives a map of W onto the set of nilpotent orbits in $\mathfrak{g}$.
The Steinberg variety provides a bridge between Schubert varieties and nilpotent elements.

2. Constructions of algebraic objects.

2.1. In the geometry of flag varieties, Schubert varieties and the Steinberg variety algebraic objects are hidden. To bring these to light various tools from algebraic topology are used. The following objects appear:
(a) The cohomology ring of a flag variety X. Closely related is the Chow ring of X. The -more general- T-equivariant cohomology and Chow ring of X are discussed in [Bri3].

(b) Cohomology of the Steinberg variety Z.
(c) G-equivariant sheaves on X, intersection cohomology of Schubert varieties.
(d) G-equivariant coherent sheaves on X.
(e) K-theory and equivariant K-theory of X and the Steinberg variety Z. See [CG, Ch. 7] and [Gi, no. 11].
I shall give some examples of constructions of algebraic objects.

2.2. An elementary construction of the Weyl group W.
Notations are as in 1.2. If $x, y \in W$ the product set $G_x.G_y$ is a union of finitely many G_z. Let $\Pi_{x,y} \subset W$ be the set of these z.
Proposition. $\Pi_{x,y}$ *contains a unique minimal element for the Bruhat order, namely xy.*
I sketch a proof. If $x \in S$ we have $\Pi_{x,y} = \{xy\}$ if $l(xy) > l(y)$ and $\Pi_{x,y} = \{x, xy\}$ if $l(xy) < l(y)$, whence the proposition in this case, and similarly in the case that $y \in S$.
We use induction on $l(x)$. We may assume that $l(x) > 0$. Choose $s \in S$ such that $l(xs) < l(x)$. Then $G_x = G_{xs}.G_s$ and

$$\Pi_{x,y} = \Pi_{xs,sy} \text{ if } l(sy) > l(y),$$

$$\Pi_{x,y} = \Pi_{xs,y} \cup \Pi_{xs,sy} \text{ if } l(sy) < l(y).$$

The proposition will follow if we show that in the last case $xsy > xy$. Now $xsy = xsx^{-1}.xy$. Let $x = s_1 \cdots s_{a-1}s_a$, $y = t_1t_2 \cdots t_b$ with $s_i,\ t_j \in S$ and $a = l(x)$, $b = l(y)$, $s_a = t_1 = s$. Then $xy = s_1 \cdots s_{a-1}t_2 \cdots t_b$. If $l(xsy) < l(xy)$ the 'strong exchange condition' of Coxeter groups [Hu2, p. 117] shows that x or y would have smaller length than a or b, a contradiction. Hence $l(xy) < l(xsy)$ and the strong exchange condition implies that $xy < xsy$.

Remark. The proposition could be viewed as a special case of the following result on the Hecke algebra $\mathcal{H}$ of a Coxeter group (W, S) (defined in [Hu2, Ch. 7]). Let $(e_x)_{x \in W}$ be the standard basis of $\mathcal{H}$ (denoted by (T_x) in [loc. cit.]. If $x, y \in W$ then all $z \in W$ such that e_z occurs in $e_x e_y$ with a non-zero coefficient are $\geq xy$ (for the Bruhat order on W) and $z = xy$ occurs (see [Sh]).

The proposition can be reformulated in terms of correspondences on X. For $x \in W$ the G-orbit $\mathcal{O}_x$ is a locally closed correspondence on X. If $*$ denotes the set-theoretical composite of correspondences we have

$$\mathcal{O}_x * \mathcal{O}_y = \bigcup_{z \in \Pi_{x,y}} \mathcal{O}_z.$$

It follows that $\mathcal{O}_{xy}$ is the orbit of lowest dimension in the composite. Also, $\mathcal{O}_{x^{-1}}$ is the inverse of the correspondence $\mathcal{O}_x$. So the Weyl group W is produced by the set-theoretical calculus of correspondences.
More formally, the product of correspondences is described as follows. Let π_{ij} $((ij) = (12), (13), (23)$ be the obvious maps $X \times X \times X \to X \times X$ and denote by $\Delta : X \times X \times X \to (X \times X) \times (X \times X)$ the map $(x, y, z) \to ((x, y), (y, z))$. If S, T are correspodences on X, i.e. subsets of $X \times X$ their product is

$$S \star T = \pi_{13}((\Delta)^{-1}(S \times T))).$$

A sheaf-theoretical version of this formalism will produce the Hecke algebra of W, see below in 2.4. Another version of the formalism in algebraic topology leads to convolution in

Borel-Moore homology, used in [CG, Ch. 3] for the Steinberg variety Z of 1.8 to construct the group algebra $\mathbf{Q}[W]$ [loc. cit., 3.4.1]). See also [Gi].

2.3. The Chow ring of X.
The Chow ring $A^*X = \bigoplus A^r X$, where $A^r X$ is a quotient of the free group generated by the irreducible subvarieties of X of dimension $\dim X - r$, see [Fu, p. 141]. For $w \in W$ let $\alpha_w \in A^{l(w_0)-l(w)}X$ be the image of the Schubert variety S_w.

Proposition. *(i) A^*X is a free abelian group with basis $(\alpha_w)_{w \in W}$;*
(ii) If $l(x) = l(w)$ we have $\alpha_w.\alpha_{w_0 x} = \delta_{x,w}\alpha_{w_0}$.
See [De1]. Part (ii) follows from lemma 1 of 1.6. It provides a duality pairing on A^*X.

In 1.8 we associated to a character λ of T a line bundle $\mathcal{L}(\lambda)$ on X. Let $c(\lambda) \in A^1X$ be its Chern class. Let S^* be the (graded) symmetric algebra of the character group X^*T, it is acted upon by the Weyl group W. For any root $\alpha \in R$ we define an endomorphism d_α of degree -1 of S^* by

$$d_\alpha u = \alpha^{-1}(u - s_\alpha.u).$$

If α is a simple root and $s = s_\alpha$ we put $d_s = d_\alpha$.
The maps d_α were introduced in [BGG, BGG] and in [De1]. They have the properties of the following lemma.

Lemma. *(i) $d_\alpha^2 = 0$;*
(ii) Let $\mathbf{s} = (s_1, ..., s_r)$ be a reduced decomposition of $w \in W$. Then $d_w = d_{s_1}...d_{s_r}$ depends only on w.
The lemma could be proved algebraically, but a proof also comes out of the proof of the next theorem.

We have a graded ring homomorphism $c : S^* \to A^*X$. Denote by I the ideal in $\mathbf{Q} \otimes S^*$ generated by the non-constant homogeneous W-invariant elements. Let $\epsilon : S^* \to \mathbf{Z}$ be the augmentation map.

Theorem. *(i) $c(u) = \sum_{w \in W} \epsilon(d_w.u)\alpha_{ww_0}$;*
(ii) c induces an isomorphism of graded algebras $(\mathbf{Q} \otimes S^*)/I \simeq \mathbf{Q} \otimes A^*X$.
In the proof of (i) given in [De1] first the Chow group of the Bott-Samelson variety associated to a reduced decomposition of w_0 is determined.

We have a cycle map from the Chow group of X to the cohomology of X, doubling degrees. Using it one obtains 'Borel's theorem':
Corollary. *$H^*(X, \mathbf{Q}) \simeq (\mathbf{Q} \otimes S^*)/I$.*
By the corollary, the Weyl group W acts on $H^*(X, \mathbf{Q})$. As a W-module, it is the regular representation of W.

The action of W on the cohomology $H^*(X, \mathbf{Q})$ can also be described in a more direct manner. The canonical map $G/T \to G/B = X$ makes G/T into a locally trivial vector bundle over X, and hence induces an isomorphism of cohomology groups

$$H^*(G/T, \mathbf{Q}) \simeq H^*(X, \mathbf{Q}).$$

Now W acts on G/T, via right action of the normalizer N of T. Hence W acts on the cohomology group of the left-hand side and hence on the one of the right-hand side. It can be shown that this action is the same as that of the theorem.

Part (i) of the theorem solves (theoretically) the problem of Schubert calculus: to describe the intersection of Schubert varieties, i.e. to determine the multiplicative structure of A^*X. It follows from (ii) that $c(S^*)$ has finite index in A^*X. But c need not be surjective (see [De2]).

The equivariant cohomology of X is discussed in [Bri3].

2.4. A construction of the Hecke algebra of W.
Recall that the Hecke algebra $\mathcal{H}$ of W is a free module over $\mathbf{Z}[t, t^{-1}]$, with a basis $(e_w)_{w\in W}$. The multiplication is determined by the rules

$$e_s.e_w = e_{sw} \text{ if } l(sw) > l(w),$$

$$e_s.e_w = (t^2-1)e_{sw} + t^2 e_w \text{ if } l(sw) < l(w).$$

In particular, $e_s^2 = (t^2-1)e_s + t^2$. Specializing $t = 1$ we obtain the group algebra $\mathbf{Z}[W]$ (see [Hu2, Ch. 7]
I first sketch a sheaf-theoretical construction of $\mathcal{H}$ via correspondences on X (see [Sp1, no. 2]). One works with sheaves of $\mathbf{Q}$-vector spaces on an algebraic variety Y, which are constructible relative to some stratification (a finite decomposition into locally closed irreducible pieces $Y = \coprod Y_i$ i.e. sheaves which are locally constant along the strata Y_i and whose stalks are finite-dimensional). We work with $Y = X \times X$ and the stratification defined by the orbits $\mathcal{O}_w$. For $w \in W$ let A_w be the sheaf on $X \times X$ whose restriction to $\mathcal{O}_w$ is the constant sheaf $\mathbf{Q}$, and whose stalks at the points outside $\mathcal{O}_w$ are zero. (A_w will produce the basis element e_w of $\mathcal{H}$.)
To make the calculus of correspondence work we need, however, to enlarge the category of sheaves on a variety Y to the bounded derived category $D_c^b(Y)$ whose objects are complexes A of $\mathbf{Q}$-sheaves, such that the cohomology sheaves $H^i(A)$ are constructible (as before) and vanish if $|i|$ is large. We will not go into the definition of such categories, their morphisms, and the definitions of the functors between such categories associated to morphisms (see [Bo1, Ch. V] for these matters, see also [Gi, no. 3]).
We identify a sheaf S with the complex A such that $A^0 = S,\ A^i = 0$ for $i \neq 0$.

In our situation we denote by $E(X \times X)$ the full subcategory of $D_c^b(X \times X)$ whose objects have cohomology sheaves which are locally constant (and then they are in fact constant) along the G-orbits $\mathcal{O}_w$. For $A \in E(X \times X)$ define the element $h(A) \in \mathcal{H}$ by

$$h(A) = \sum_{w\in W} (\sum_i \dim H^i(A)_w t^i) e_w,$$

where $H^i(A)_w$ is the stalk of $H^i(A)$ at a point of $\mathcal{O}_w$ (these stalks are all isomorphic).
For example, $h(A_w) = e_w$. If $s \in S$ then $\overline{\mathcal{O}_s} = \mathcal{O}_s \cup \mathcal{O}_e$. Let A'_s be the sheaf which is $\mathbf{Q}$ on $\overline{\mathcal{O}_w}$ and 0 outside this set. Then $h(A'_s) = e_s + 1$.

For $A, B \in E(X \times X)$ define their convolution product $A \star B$, by

$$A \star B = \pi_{13}^{\star}(\Delta^*(A \odot B).$$

Here $A \odot B$ is the outer tensor product of A and B, Δ^* is the pull-back morphism associated to the morphism Δ of 2.2 and π_{13}^* is the direct image morphism associated to the proper morphism π_{13}. (It is to define such direct image functors that one needs complexes of sheaves.)

Proposition. *Let $s \in S$. Let $A \in E(X \times X)$ be such that $H^i(A) = 0$ for all even i or for all odd i. Then $A'_s \star A$ is a complex with the same properties and $h(A'_s \star A) = (e_s + 1)h(A)$.*
This is proved (by elementary means) in [Sp1, 2.6].
The proposition shows that $\mathcal{H}$, described in terms of its the generators $e_s + 1$ ($s \in S$) can be constructed via the sheaf theoretic calculus of correspondences on the flag variety. But a cleaner construction is obtained in the context of the theory of perverse sheaves (or intersection cohomology).

2.5. Construction of the Hecke algebra of W in intersection cohomology.
Let Y be an irreducible algebraic variety. In the category $D_c^b(Y)$ one has a duality functor D, with good properties, see [Bo1, V, §7], see also [Gi, no. 3].
A *perverse sheaf* on Y is a complex A in $D_c^b(Y)$ such that

$$\dim \operatorname{supp}(H^i(A)) \leq -i, \ \dim \operatorname{supp}(H^i(DA)) \leq -i.$$

For the theory of perverse sheaves we refer to [BBD], see also [Gi, no. 4]. We only mention a few essential points. The perverse sheaves on Y form an abelian category $\mathcal{P}(Y)$, all of whose objects have finite length. The irreducible objects are as follows. For each irreducible subvariety Z of Y and each irreducible local system on a smooth open piece U of Z, there is a unique irreducible perverse sheaf $I = IC(Y, \mathcal{L})$ supported by Y such that the restriction of I to U is the complex $\mathcal{L}[\dim Z]$ ($\mathcal{L}$ in dimension $-\dim Z$ and 0 in the other dimensions). This is the intersection cohomology complex for Y and $\mathcal{L}$. We have

$$\dim \operatorname{supp}(H^i(I)) < -i \text{ if } i > -\dim Z.$$

If $\mathcal{L}$ is the constant sheaf $\mathbf{Q}$ then the hypercohomology of I is the *intersection cohomology* of Z. It satisfies Poincaré duality.

If Z is smooth then $IC(Y, \mathbf{Q}) = \mathbf{Q}[\dim Z]$, and intersection cohomology coincides with ordinary cohomology.
A complex $A \in D_c^b(Y))$ is *semi-simple* if it is a direct sum of shifted irreducible perverse sheaves I_i,

$$A = \oplus I_i[n_i].$$

A powerful result is the *decomposition theorem*:
If $f : Y \to Z$ is a proper morphism and if $A \in \mathrm{Ob}(\mathcal{P}(Y)$ is irreducible then the direct image f_*A is semi-simple.
See [BBD, nos. 5, 6].

We return to the Hecke algebra. There is a ring automorphism $h \mapsto \overline{h}$ of $\mathcal{H}$ mapping t

to t^{-1} and such that $\overline{e_w} = e_{w^{-1}}^{-1}$. One shows that for $w \in W$ there is a unique element $c_w \in \mathcal{H}$ with $\overline{c_w} = c_w$, of the form

$$c_w = t^{-l(w)} \sum_{x \leq w} P_{x,w}(t^2) e_x,$$

where, the $P_{x,w}$ being polynomials with $P_{w,w} = 1$, 2deg $P_{x,w} < l(w) - l(x)$ if $x < w$. The c_w form a basis of $\mathcal{H}$, the *Kazhdan-Lusztig basis* (introduced in [KL1], see also [Hu2, II, 7]).

We shall now connect the Kazhdan-Lusztig elements with the intersection cohomology complexes $I_w = IC(\overline{\mathcal{O}_w}, \mathbf{Q})$ of the G-orbit closures in $X \times X$. For $s \in S$ we have $I_s = A'_s[1]$, where A'_s is as in 2.4. Also, h is as in 2.4.

Theorem. *Let $x, w \in W$.*
(i) $h(I_w) = c_w$,
(ii) $h(I_x \star I_w) = c_x c_w$.
See [Sp1, no. 2]. It follows, in particular, that

$$t^{-l(w)} P_{x,w}(t^2) = \sum \dim H^i(I_w)_x t^i,$$

from which we see that I_w satisfies the *parity condition*:

$$H^i(I_w) = 0 \text{ if } i \not\equiv l(w) \,(\mathrm{mod}\ 2).$$

The theorem shows that the Hecke algebra $\mathcal{H}$ can be recovered from the perverse sheaves I_w.

There is a variant of this construction, which works with the perverse sheaves on G. Consider the perverse sheaves on G which are $B \times B$-equivariant (the action being $(b, b').g \mapsto bg(b')^{-1}$). Denote by $\mathcal{K}$ the Grothendieck group of the category of these perverse sheaves. It has as basis the classes $\tilde{c}_w$ of the intersection cohomology complexes $\tilde{I}_w = IC(\overline{G_w}, \mathbf{Q})$ ($w \in W$). To a semi-simple complex $A = \bigoplus \tilde{I}_w[h_w]$ we associate the element

$$\tilde{h}(A) = \sum t^{h_w} \tilde{c_w} \in \mathbf{Z}[t, t^{-1}] \otimes \mathcal{K}.$$

The product map $G \times G$ induces a proper morphism $\pi : G \times_B G \to G$. Let $x, y \in W$. The outer tensor product of $\tilde{I}_x$ and $\tilde{I}_y$ is the pull back of a shifted perverse sheaf $\tilde{A}$ on $G \times_B G$. By the decomposition theorem the direct image complex $\pi_* \tilde{A}$ is semi-simple.
Define an algebra structure on $\tilde{\mathcal{H}} = \mathbf{Z}[t, t^{-1}] \otimes \mathcal{K}$ by

$$\tilde{c}_x.\tilde{c}_y = \tilde{h}(\pi_* A).$$

Proposition. *The $\mathbf{Z}[t, t^{-1}]$-algebra $\tilde{\mathcal{H}}$ is isomorphic to $\mathcal{H}$, the isomorphism sending $\tilde{c}_w$ to c_w.*
A proof of the proposition is contained in [MS1, 4.2]. In that proof it is shown that the inductive formulas of [KL] for the Kazhdan-Lusztig polynomials can be recovered from the morphism π (x being a simple reflection).

The construction using G can be generalized. Instead of $\tilde{I}_w$ one can work, more generally, with a perverse sheaf $IC(\overline{G_w}, \mathcal{L})$, where $\mathcal{L}$ is a $B \times B$-equivariant local system on the smooth open piece G_w of $\overline{G_w}$ (there exist non-constant local systems of this kind; this is not the case for $\mathcal{O}_w$, which is simply connected). One is led to a more general kind of Hecke algebra, see [loc. cit.].

2.6. The Steinberg variety.
In the Steinberg variety Z of 1.8 the representation theory of the Weyl group W is hidden. In [CG, Ch. 3] convolution on the top Borel-Moore homology of Z is used to reveal that representation theory. This is also discussed in [Gi, no. 6].
Another method to do this uses Lusztig's observation that there is an action of the Weyl group W on the direct image complex $\pi_* \mathbf{Q}$ (see e.g. [Sp1, no. 4]), where π is as in 1.9. It follows that there is an action of $W \times W$ on $\mu_* \mathbf{Q}$, where $\mu : Z \to \mathcal{N}$ is the morphism of 1.9. Hence $W \times W$ operates on the cohomology with compact support $H_c^*(\mathcal{N}, \mu_* \mathbf{Q}) = H_c^*(Z, \mathbf{Q})$. The irreducible components of Z are parametrized by the elements of W (see 1.9), they all have the same dimension $2d$, where $d = \dim X = l(w_0)$. The cohomology classes z_w which they determine span the top cohomology group $H_c^{4d}(Z, \mathbf{Q})$, which thus has dimension $|W|$.

Proposition. *(i) $H_c^*(Z, \mathbf{Q})$ is isomorphic to the induced graded $W \times W$-module*

$$\mathrm{Ind}_W^{W \times W}(H^*(X, \mathbf{Q})[-2d]);$$

(ii) In particular, the $W \times W$-module $H_c^{4d}(Z, \mathbf{Q})$ is isomorphic to $\mathbf{Q}[W]$, under two-sided action.
In (i) the W-action on $H^*(X, \mathbf{Q})$ is as in 2.3. For a proof see [Sp3, no. 3]. A concrete description of the $W \times W$-action on the basis (z_w) of $H_c^{4d}(Z, \mathbf{Q})$ does not seem to be known.

Equivariant K-theory on Z is used to construct the affine Hecke algebra associated to W and its representations. We shall not go into this. See [CG, Ch. 7,8] and [Gi, no. 11].

2.7. Coherent cohomology.
Assume G to be semi-simple and simply connected. The character group X^*T is the weight lattice P of R. Let $P^+ \subset P$ be the set of dominant weights. For $\lambda \in P$ we have the line bundle $\mathcal{L}(\lambda)$ of 1.8.
Theorem. *Let $w \in W$, $\lambda \in P^+$. Then $H^i(S_w, \mathcal{L}(\lambda)) = 0$ for $i > 0$.*
See [Ra, 3.2] (where more general results are discussed). See also Littelmann's contribution. ref. Littelmann

In the situation of the theorem, $H^0(S_w, \mathcal{L}(\lambda))$ is a finite dimensional complex vector space V on which B acts. In particular, T acts. Let $\mathbf{Z}[P]$ be the group ring of P. It has a basis $e(\chi)$ ($\chi \in P$), with $e(\chi + \psi) = e(\chi)e(\psi)$, and the Weyl group W acts on it.
Put $\mathrm{char}(V) = \sum_{\chi \in P} (\dim V_\chi) e(\chi)$, where V_χ is a weight space for T. This element of $\mathbf{Z}[P]$ is the *character* of V.
Let α be a simple root and put $s = s_\alpha$. Define an endomorphism D_s of $\mathbf{Z}[P]$ by

$$D_s u = \frac{u - s.u}{1 - e(\alpha)}.$$

Lemma. *(i) $D_s^2 = D_s$;*
(ii) Let $\mathbf{s} = (s_1, ..., s_r)$ be a reduced decomposition of $w \in W$. Then $D_w = D_{s_1} ... D_{s_r}$ depends only on W.

This is a multiplicative analogue of the lemma of 2.3. With these definitions we have *Demazure's character formula.*

Proposition. *If* $\lambda \in P^+$ *then* $\mathrm{char}(H^0(S_w, \mathcal{L}(\lambda)) = e(\rho)D_w(e(-\lambda-\rho))$.
Here ρ is half the sum of the positive roots of R. See [De1], [Ra, p. 505] and Littelmann's contribution. ref. Littelmann

For $w = w_0$ we have $S_w = X$. In that case $H^0(X, \mathcal{L}(\lambda))$ is G-module, which is irreducible with highest weight $-w_0\lambda$. The proposition then gives Weyl's character formula.

3. Generalizations of Schubert varieties.

3.1. Schubert varieties associated to Kac-Moody algebras.

Let $\mathfrak{g}$ be a Kac-Moody algebra. It is an infinite dimensional Lie algebra, associated to a generalized Cartan matrix M (see [Ka]). M defines a Coxeter group (W, S), the *Weyl group* of $\mathfrak{g}$.
One can associate to $\mathfrak{g}$ Schubert varieties S_w, indexed by the elements of W. They have properties similar to those of the ordinary Schubert varieties: they are projective algebraic varieties, of dimension $l(w)$ (the length of w relative to S), and if $x \leq w$ then S_x is a closed subset of S_w.
In this generality, Schubert varieties are defined and studied in [Ma], in arbitrary characteristics. The definition involves representation theory of $\mathfrak{g}$. These Schubert varieties are normal projective varieties. In [loc. cit.] a generalization is proved of the theorem of 2.7, and of Demazure's character formula. See also Littelmann's contribution. ref.Littelmann
The Cartan matrix M defines an 'infinite dimensional group' B, an analog of the Borel group of 1.2. In [loc. cit.] analogues are constructed of the closures $\overline{G_w}$ of 1.4. These are (non-noetherian) affine schemes $B(w)$ over $\mathbf{C}$, with a two-sided $B \times B$-action, as in the case of $\overline{G_w}$. B operates locally freely on the right, and $S_w = B(w)/B$.
I will not go further into this general case. But to make things more concrete I shall briefly discuss one example in a special case. The associated Kac-Moody algebra would be an affine one, however it will not appear in the example.

3.2. An example.

Let $A = \mathbf{C}[[t]]$ be the ring of formal power series and $F = \mathbf{C}((t))$ its quotient field. Let G be a semi-simple, simply connected linear algebraic group over $\mathbf{C}$. The notations are as in 1.2. We denote by $\mathcal{W}$ the affine Weyl group of R (the semi-direct product of W and the lattice of coroots).
Put $\mathcal{G} = G(F)$, the group of F-valued points of G. The obvious homomorphism $A \to \mathbf{C}$ induces a group homomorphism of the group $G(A)$ of A-valued points to G. Let $\mathcal{B}$ be the inverse image of B under this homomorphism. This is an *Iwahori subgroup* of $\mathcal{G}$. As in 1.2 we have subsets $\mathcal{G}_w = \mathcal{B}\dot{w}\mathcal{B}$ of $\mathcal{G}$, where $w \in \mathcal{W}$. We have a Bruhat decomposition

$$\mathcal{G} = \coprod_{w \in \mathcal{W}} \mathcal{G}_w,$$

by the Bruhat-Tits theory (see e.g. [Ti, p. 51].)
Put $\mathcal{X} = \mathcal{G}/\mathcal{B}$, $\mathcal{X}_w = \mathcal{G}_w/\mathcal{B}$ and $\mathcal{S}_w = \bigcup_{x \leq w} \mathcal{X}_w$. Then $\mathcal{S}_w$ is the underlying set of a Schubert

variety. It can be given the structure of projective algebraic variety. We shall indicate how this can be done for $G = \mathbf{SL}_n$ (following [KL2, §5]).

Let $G = \mathbf{SL}_n$, so $\mathcal{G} = \mathbf{SL}_n(F)$. We take $\mathcal{B}$ to be the subgroup of $\mathbf{SL}_n(A)$ whose elements specialize to an upper triangular matrix for $t = 0$ (the upper triangular group is a Borel subgroup of G).
Let $V = F^n$. Then $\Lambda = A^n$ is a lattice in V over A, i.e. a free A-submodule of V containing a basis. The group $\mathcal{G}$ operates on the set of lattices. We identify $\mathcal{G}/\mathcal{B}$ with the set of sequences of lattices

$$\Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_n = t\Lambda_0$$

such that $\dim_{\mathbf{C}} \Lambda_j/\Lambda_{j+1} = 1$ $(0 \leq j \leq n-1)$, and that $\Lambda_0 \in \mathcal{G}.\Lambda$.
For $i \geq 0$ let $\mathcal{X}_i$ be the set of such sequences with

$$t^{-i}\Lambda \supset \Lambda_0 \supset \cdots \supset \Lambda_n \supset t^{i+1}\Lambda.$$

Then $(\mathcal{X}_i)$ is an increasing sequence of subsets of $\mathcal{X}$ with union $\mathcal{X}$ (to see this use that for any lattice Λ' there exist $a, b \geq 0$ with $t^a\Lambda \subset \Lambda' \subset t^{-b}\Lambda$).
Fix i and put $Z = t^{-i}\Lambda/t^{i+1}\Lambda$, this is a finite dimensional vector space on which t acts as a nilpotent endomorphism. Then $\mathcal{X}_i$ can be identified with the set of sequences of t-stable subspaces

$$Z_0 \supset Z_1 \supset \ldots \supset Z_n = tZ_0$$

of Z such that $\dim Z_j/Z_{j+1} = 1$, $Z_n = tZ_0$. This set has a structure of projective variety (being a closed subset of a product of Grassmannians), whence such a structure on $\mathcal{X}_i$.
Then $\mathcal{X}$ is an inductive limit of projective varieties. Any $\mathcal{X}_w$ is contained in an $\mathcal{X}_i$, whence a structure of projective variety on the Schubert varieties.

3.3. Spherical varieties.
The notations are as in 1.2. A *spherical variety* (for G) is an algebraic variety X with a G-action such that the Borel group B has finitely many orbits (see [Bri1]).
In particular, G has finitely many orbits. A G-orbit is a *spherical homogeneous space* G/H, where H is a closed subgroup such that the double coset space $B\backslash G/H$ is finite. Such a subgroup is called *spherical*. It is clear that H is a spherical subgroup if and only if H acts on $X = G/B$ with finitely many orbits.

Examples. (1) $X = G/B$. By Bruhat's lemma (1.3) B is a spherical subgroup. The B-orbits are indexed by W, and the Schubert varieties in X are the orbit closures.
(2) The symmetric case (see [Sp2]). Let θ be an involution (automorphism of order two) of the algebraic group G. The fixed point group K is reductive and $X = G/K$ is spherical. It is an affine algebraic variety, called a *symmetric variety*.
(3) Assume that $H = G \times G$, and let θ be the permutation automorphism: $(x, y) \mapsto (y, x)$ $(x, y \in G)$. The fixed point group K of θ is G, imbedded diagonally in H, and $H/K \simeq G$. $B \times B$ is a Borel subgroup of H. It acts on $H/K = G$ by $(b, b').g = bg(b')^{-1}$ By Bruhat's lemma, the $B \times B$-orbits on H/K are the G_w of 1.3. We recover example 1.
(4) The unipotent part U of B is another example of a spherical subgroup (by Bruhat's lemma), and G/U gives an example of a spherical variety which is not symmetric (as U is not reductive). Another such example is provided by $H = T.(U, U)$, where (U, U) is the

commutator subgroup of U. That H is spherical follows from Bruhat's lemma, and the observation that T acts by conjugation on $U/(U,U)$ with finitely many orbits (the quotient being isomorphic to $\prod_{\alpha \in D} U_\alpha$.)
(5) An instructive example of a symmetric variety is $G = \mathbf{SL}_2$, with θ the inner automorphism $\mathrm{Int}(i,-i)$. Let T be the diagonal torus and B the upper triangular subgroup. Then $K = T$ is spherical: in this case G/B is the projective line $\mathbf{P}^1$, and T acts on it with three orbits, namely the closed orbits $\{0\}$, $\{\infty\}$ of dimension 0 and the open orbit $\mathbf{P}^1 - \{0,\infty\}$ of dimension 1.
Let $G' = \mathbf{PSL}_2$ and let θ' is the automorphism of G' induced by θ. The fixed point group K' of θ' is non-connected, it is the image in G' of the normalizer N of T. B' denoting the image of B, we have $G'/B' = \mathbf{P}^1$.
K' has two orbits on $\mathbf{P}^1$, viz. the reducible orbit $\{0,\infty\}$ and its complement.

3.4. The set of orbits.
Let $X = G/H$ be spherical homogeneous. Denote by V the set of B-orbits in X. This is a finite set with combinatorial properties resembling those of a Weyl group.
(a) V carries a (partial) order.
If $v, w \in V$ define $v \leq w$ if $\bar{v} \subset \bar{w}$ (compare with the Bruhat order of W, see 1.4). Since X is irreducible V has a unique a unique maximal element $v_{\max}$, the open B-orbit.
(b) The minimal elements of V are the closed orbits. They are of the form BgH/H, where BgH is closed in G. Hence $Hg^{-1}B/B$ is closed in G/B, which means that $H \cap g^{-1}Bg$ is a Borel subgroup of H. This implies that all closed orbits have the same dimension. We define a length function l on V by $l(v) = \dim v - \dim v_0$, where v_0 is minimal. Then l is strictly monotonic, and $l(v) = 0$ if and only if v is minimal.
There may be several minimal orbits, as example (5) of 3.3 shows.

We may identify V with the set of K-orbits in G/B. The order and the length of V can also be defined via the latter set, in the same way.

To analyze further the combinatorial structure of V we use the parabolic subgroups P_s of G, as we did in 1.3 to establish the combinatorial description of the Bruhat order of W. We have to study the map $P_s \times v \to X$ defined by the G-action.

3.5. A basic construction.
Let $s \in S$ be a simple reflection. We have the parabolic subgroup

$$P = P_s = G_e \cup G_s.$$

Let $v \in V$ then $P.v$ is an irreducible locally closed subvariety of our variety X which is a finite union of B-orbits. There is one orbit which is open in $P.v$ and has maximal dimension. We denote it by $m(s).v$.
The P-action on X defines a morphism

$$\mu : P \times_B v \to P.v.$$

Lemma 1. *We have the following possibilities.*
I. $P.v = v$.

IIa. $P.v = v \cup m(s).v$, *v is closed in $P.v$, μ is generically bijective and* $\dim m(s).v = \dim v + 1$.
IIb. $P.v = v \cup v'$, *v is open in $P.v$,* $\dim v' = \dim v - 1$ *and v' is in case IIa.*
IIIa. $P.v = v \cup v' \cup m(s).v$, $v \neq v'$, *both v and v' are closed in $P.v$ and* $\dim m(s).v = \dim v + 1 = \dim v' + 1$.
IIIb. $P.v = v \cup v' \cup v''$, *both v' and v'' are closed in $P.v$,* $v' \neq v''$, $\dim v' = \dim v'' = \dim v - 1$ *and v', v'' are in case IIIa.*
IVa. As in case IIa, but μ is generically a double cover.
IVb. As in case IIb, but v' is in case IV a.
Let $x \in v$ and let P_x be the isotropy group of x in P. Then $P.v \simeq P/P_x$. There is a bijection of the finite set of B-orbits in $P.v$ onto the set of P_x-orbits on $B \backslash P \simeq \mathbf{P}^1$. So we are led to subgroups of the automorphism group $\mathbf{PGL}_2$ of $\mathbf{P}^1$ which have finitely many orbits. Analysis of the possible cases leads to the lemma. See [MS2, 4.1].
In the case that $X = G/B$ the cases III and IV do not occur, which simplifies matters considerably.

For $v \in V$, $s \in S$ we define $s.v = m(s).v$ in case IIa, $s.v = v'$ in cases IIb and $IIIa$ and $s.v = v$ in all other cases. The notation is suggestive of a W-action. We shall see below that there is indeed a W-action on V such that the simple reflections act as described.

Denote by π the canonical map $B \to B/U$. The restriction of π to T is bijective. We identify T with B/U.

For $x \in X$ let B_x be its isotropy group. It is easy to see that for all $x \in v$ the image πB_x is the same. Write T_v for this image. Let $s = s_\alpha$ ($\alpha \in D$) be as before. Denote by $\alpha^\vee$ the coroot of α (a homomorphism of $\mathbf{GL}_1$ into a one-dimensional subgroup of T. Recall that α is a non-trivial character of T.
The following lemma describes the relations between T_v and $T_{m(s).v}$, in the cases of the previous lemma where $m(s).v > v$, i.e. the cases IIa, IIIa, IVa.
Lemma 2. *Assume* $m(s).v > v$.
(i) (Case IIa) $T_{m(s).v} = s.T_v$;
(ii) (Case IIIa) $T_{m(s).v} = T_v \cap \mathrm{Ker}\ \alpha$ *and* $T = T_v.\mathrm{Ker}\ \alpha$;
(iii) (Case IVa) $\mathrm{Im}\ \alpha^\vee \subset T_v$, $\alpha(T_{m(s).v}) = \{\pm 1\}$ *and*

$$T_v \cap \mathrm{Ker}\ \alpha = T_{m(s).v} \cap \mathrm{Ker}\ \alpha.$$

See [MS2]. The lemma implies that in case IIa $\dim T_{m(s).v} = \dim T_v$ and that $\dim T_{m(s).v} = \dim T_v - 1$ in the cases IIIa and IVa.

3.6. The monoid $M(W)$.

Let $M = M(W)$ be the monoid with elements $m(w)$ indexed by the elements $w \in W$ such that

$$m(s)m(w) = m(sw) \text{ if } l(sw) > l(w),\ m(s)m(w) = m(w) \text{ if } l(sw) < l(w).$$

The existence of M follows from [Hu2, Theorem, p. 146]. M is generated by the $m(s)$ ($s \in S$) subject to the relations $m(s)^2 = m(s)$ and the 'braid relations' of the Weyl group

$$m(s)m(t)m(t) \cdots = m(t)m(s)m(t) \cdots \ (s, t \in S),$$

the number of factors on both sides being the order of st.
If $\mathbf{s} = (s_1, ..., s_r)$ is a reduced decomposition of $w \in W$ we put $m(w) = m(s_1) \cdots m(s_r)$. This

is well-defined.

Lemma. *The map $(s, v) \mapsto m(s).v$ extends to an action of $M(W)$ on V.*
This follows from $P_s.P_s = P_s$ and the braid relations $P_sP_tP_s \cdots = P_tP_sP_t \cdots$.
Remark. The D_w of 2.6 provide another example of an M- action.

We have the following properties.
Proposition 1. *Let $v, w \in V$, $s \in S$.*
(a) $v \leq m(s).v$ and $l(m(s).v) \leq l(v) + 1$.
(b) If $m(s).v = m(s).w$ and $l(v) = l(w)$, $v \neq w$ then $w = s.v$. Moreover $v < m(s).v$ and we have case IIIa;
(c) If $v \leq w$ then $m(s).v \leq m(s).w$.
(d) Put $V_{\leq v} = \{x \in V \mid x \leq v\}$. If $m(s).v \neq v$ then

$$V_{\leq m(s).v} = \bigcup_{x \leq v} \{m(s).x, x, s.x\}.$$

(a) is a consequence of lemma 1 of 3.5 and (c), (d) follow from the equality $\overline{P_s.v} = Ps.\bar{v}$.
(b) also follows from lemma 1 of 3.5. By (a) we may assume that $v < m(s).v$, $w < m(s).w$. The lemma then shows that $v \subset P_s(m(s).v)$. Looking at the possible cases (b) follows.

A *reduced decomposition* of $v \in V$ is a pair $(\mathbf{v}, \mathbf{s})$ of a sequence $\mathbf{v} = (v_0, v_1, ..., v_r)$ of distinct elements in V and a sequence $\mathbf{s} = (s_1, ...s_r)$ in S such that v_0 is minimal, $v_r = v$ and $v_i = m(s_i).v_{i-1}$ $(1 \leq i \leq r)$.
In the symmetric case (example (2) of 3.3) all v have a reduced decomposition (see below in 3.9). Examples show that this is not generally true.
The symmetric case is analyzed in [RS]. Some of the results carry over to the general case. I mention the following generalization of the combinatorial description of the Bruhat order of 1.4.
Let $(\mathbf{v}, \mathbf{s})$ be a reduced decomposition of v. A *subexpression* of $(\mathbf{v}, \mathbf{s})$ is a sequence $\mathbf{x} = (x_0, x_1, ..., x_r)$ in V with $x_0 = v_0$, such that for $1 \leq i \leq r$ we have one of the following alternatives: (α) $x_{i-1} = x_i$, (β) $x_i \neq x_{i-1}$, $l(x_i) = l(x_{i-1})$ and $x_i = s_i.x_{i-1}$, (γ) $l(x_i) = l(x_{i-1}) + 1$ and $x_i = m(s_i).x_{i-1}$. We call x_r the *final term* of $\mathbf{x}$.

Proposition 2. *Let $(\mathbf{v}, \mathbf{s})$ be a reduced decomposition of v. If $x \leq v$ there is a subexpression of $(\mathbf{v}, \mathbf{s})$ with final term x.*
This is a consequence of property (d), see [RS, 6.4, 6.5].

The existence of a reduced decomposition of $v \in V$ can be formulated as follows: there exist $w \in W$ and a minimal element $v_0 \in V$ with $m(w).v_0 = v$, $l(w) = l(v)$. Part (i) of the following proposition is a 'dual' result which is generally true.
Proposition 3. *(i) Let $v \in V$. There is $w \in W$ with $m(w).v = v_{\max}$, $l(w) = l(v_{\max}) - l(w)$;*
(ii) $v_{\max}$ has a reduced decomposition.
As in 3.4., $v_{\max}$ is the maximal element of V.
Assume that $v \neq v_{\max}$. Then $\overline{v} \neq \overline{v_{\max}} = G.v$. Since G is generated by the parabolic subgroups P_s $(s \in S)$ there must be $s \in S$ with $\dim P_s.v > \dim v$, i.e. with $m(s).v > v$. Then (i) follows by descending induction on $l(v)$. Applying (i) with v minimal we obtain (ii).

Let $(\mathbf{v},\mathbf{s})$ be a reduced decomposition of $v_{\max}$. By proposition 2 we have for any $v \in V$ a subexpression of $(\mathbf{v},\mathbf{s})$ with final term v. We obtain a weak sort of reduced decomposition of v.

Finally, we mention another example of a set V with an M-action, where we also have the properties of proposition 1. Namely, the set V of involutions in the Weyl group W. The action of M is defined by the $m(s).v$ for $s \in S$, $v \in V$. These are as follows: $m(s).v = svs$ if $l(svs) > l(v)$, $m(s).v = sv$ if $l(sv) > l(v)$ and $sv = vs$, and $m(s).v = v$ in all other cases (l is the length function on W). This example (slightly generalized) is discussed in [RS, Compl. no. 1].

3.7. Hecke algebra representation associated to a spherical variety.
The notations are as before. The ideas of the constructions of the Hecke algebra of W with tools from algebraic topology, discussed in 2.4, can also be used in the context of spherical varieties. Write now Y for the flag variety, and as before X for a spherical variety. Then G acts with finitely many orbits in $Y \times X$, which are parametrized by our set V. Via a calculus of correspondences one is led to consider complexes M in $D_c^b(Y \times X)$ whose cohomology is locally constant along the G-orbits in $Y \times X$. If $A \in D_c^b(Y \times Y)$ is as in 2.4, a complex with cohomology constant along the G-orbits $\mathcal{O}_w$, one defines as in 2.4 a product $A \star M \in D_c^b(Y \times X)$.
We are thus led to a representation of $\mathcal{H}$ on a free $\mathbf{Z}[t,t^{-1}]$-module $\mathcal{M}$, which has a basis $(e_{v,\chi})$ indexed by pairs of an orbit $v \in V$ and a B-equivariant local system χ on V (in the sitiuation of 2.4 local systems did not appear, since we were dealing with simply connected orbits).
Also, in the present situation we do not have available the proposition of 2.4. To circumvent this difficulty one passes to a situation over a finite field, where the Frobenius action can be exploited. This was first carried out for the case of symmetric varieties by Lusztig and Vogan in [LV]. The case of spherical varieties is dealt with in [MS2].
Constructions in intersection cohomology, as those of 2.5, can also be carried out in the context of spherical varieties, see [loc. cit.]. I now describe the structure of the representation of $\mathcal{H}$ which one obtains. The results are for the situation over $\mathbf{C}$. But, as already pointed out, they are obtained via a passage to finite fields.

First some results on B-equivariant local systems on an orbit v. These are are classified by the character group Γ_v of the finite group $B_x/(B_x^0)$, where $x \in V$. Since $B_x \cap U$ is a unipotent algebraic group over $\mathbf{C}$ it is connected, and it follows that $B_x/B_x^\circ \simeq T_v/T_v^0$, hence is a finite abelian group, uniquely determined by v. We describe the connection between Γ_v and $\Gamma_{m(s).v}$ in the cases IIa, IIIa, IVa of 3.5.
Lemma 1. *(i) In case IIa there is an isomorphism $\xi \mapsto s.\xi$ of Γ_v onto $\Gamma_{m(s).v}$;*
(ii) In case IIIa there is an injection $\phi_{v,s} : \Gamma_v \to \Gamma_{m(s).v}$;
(iii) In case IVa there is an injection $\phi_{v,s}$ of Γ_v in a quotient $\Gamma'_{m(s).v}$ of $\Gamma_{m(s).v}$ by a subgroup of order 2.
This follows from lemma 2 of 3.5. For case IIa this is clear. In case IIIa we have by part (ii) of that lemma $\alpha(T_v) = \mathbf{G}_m$, whence $\alpha(T_v) = \alpha(T_v^0)$. So

$$T_v/T_v^0 = T_v \cap \mathrm{Ker}\ \alpha . T_v^0/T_v^0,$$

and there is a surjective homomorphism $T_{m(s).v}/T^0_{m(s).v} \to T_v/T_v^0$. This implies (ii). The proof of (iii) is similar.

The action of $\mathcal{H}$ on $\mathcal{M}$ is described by giving the products $e_s.e_{v,\chi}$ ($v \in V$, $\chi \in \Gamma_v$), of a generator of $\mathcal{H}$ and a basis element of $\mathcal{M}$. They are listed in the next proposition. The notations are explained below. The cases are as in lemma 1 of 3.5.

Proposition. *There is a representation of $\mathcal{H}$ on $\mathcal{M}$ such that the products $e_s.e_{v,\chi}$ are as follows.*
Case I. $t^2 e_{v,\xi}$;
Case IIa. $e_{s.v,s.\xi}$;
Case IIb. $(t^2-1)e_{v.\xi} + t^2 e_{s.v,s.\xi}$;
Case IIIa. $e_{s.v,s.\xi} + e_{m(s).v,\xi'}$;
Case IIIb. $(t^2-2)e_{v,\xi} + (t^2-1)(e_{v',\xi'} + e_{v'',\xi''})$ *or* $-e_{v,\xi}$;
Case IVa. $e_{v,\xi} + e_{m(s).v,\xi'} + e_{m(s).v,\xi''}$;
Case IVb. $(t^2-1)e_{v,\xi} - e_{v.\xi'} + (t^2-1)e_{v',\xi''}$ *or* $-e_{v,\xi}$.
In case IIIa we have written $\xi' = \phi_{v,s}\xi$ and in case IVa, ξ' and ξ'' are the elements of $\Gamma_{m(s).v}$ which project onto the element $\phi_{v,s}\xi$ of $\Gamma'_{v,s}$.
In case IIIb we have by the previous lemma a injections $\phi_{v',s}$, $\phi_{v",s}$ into Γ_v. The first alternative prevails if and only if there are $\xi' \in \Gamma_{v'}$, $\xi'' \in \Gamma_{v''}$ with $\xi = \phi_{v',s}\xi' = \phi_{v'',s}\xi''$. In case IVb, ξ and ξ' are the elements of Γ_v projecting on an element $\phi'_{v',s}\xi''$,, if such an ξ'' exists. Otherwise we have the second alternative.
The proposition is a consequence of the results of [MS2, 4.3]. There similar formulas are established, in a more general situation (for more general local systems on the orbits). However, in [loc. cit.] one works not over $\mathbf{C}$ but over the algebraic closure of a finite field $\mathbf{F}_q$ over which everything is defined. The Frobenius action on cohomology comes into play. In order to deduce the proposition one passes by a well-known reduction procedure (see e.g. [BBD, no. 6]) from $\mathbf{C}$ to a suitable finite field $\mathbf{F}_q$, to which the results of [MS2] can be applied. They imply that we have a representation of the 'specialized' Hecke algebra $\mathcal{H}_q$, where t^2 is replaced by q on the (similarly) specialized module $\mathcal{M}_q$. The same is true with q replaced by q^n, for all $n \geq 1$. Now observe that the assertion of the proposition is equivalent to the vanishing of a number of polynomials in t^2, which expresses the defining relations of $\mathbf{H}$. By [loc. cit.] these polynomials have infinitely many zeros, namely all powers q^n. Hence they vanish, and the proposition follows.

As a consequence of the proposition we prove the existence of the Weyl group action on V, announced in 3.5.

Corollary. *There exists an action of W on V such that, with the notations of lemma 1 of 3.5, we have $s.v = m(s).v$ in case IIa, $s.v = v'$ in cases IIb and IIIa and $s.v = v$ in all other cases.*
Specializing t to 1 in the proposition we obtain a representation of W in a free $\mathbf{Z}$-module $\mathcal{F}$ spanned by basis elements $f_{v,\xi}$ ($v \in V, \xi \in \Gamma_v$). Let $\mathcal{F}_r$ be the submodule of $\mathcal{F}$ spanned by the $f_{v,\xi}$ with $\dim T_v \leq r$, where T_v is as in 3.5. They define a filtration of $\mathcal{F}$ and it follows from lemma 2 of 3.5 and the formulas of the proposition that W stabilizes the filtration. Let $\overline{\mathcal{F}}$ be the associated graded module. It carries a representation of W. The basis $f_{v,\xi}$ defines a basis $\overline{f}_{v,\xi}$ of $\overline{\mathcal{F}}$ and the formulas of the proposition show that for $s \in S$ we have

$$s.\overline{f}_{v,\xi} = \pm \overline{f}_{s.v,\xi'},$$

for some ξ'. This implies that if $\mathbf{s} = (s_1, ..., s_l)$ is a reduced decomposition of $w \in W$, $w.\overline{f}_{v,\xi}$

is of the form

$$\pm\overline{f}_{s_1....s_l.v,*}.$$

It follows that $w.v = s_1.....s_l.v$ defines an action of W on V which is as asserted.
The W-action is due to Knop ([Kn], the results established there are more general).

The proof of the corollary also gives that for $w \in W$, $w.\overline{f}_{v,\xi}$ is of the form $\epsilon_v(w)\overline{f}_{w.v,*}$, with $\epsilon_v(w) = \pm 1$ (independent of ξ). We have for $x, w \in W$, $v \in V$

$$\epsilon_v(xw) = \epsilon_{w.v}(x)\epsilon_v(w).$$

Let W_v be the isotropy group of v in W. The preceding formula shows that the restriction of ϵ_v to W_v is a character of W_v with values in $\{\pm 1\}$. If $s \in S \cap W_v$ then we are in one of the cases IVa, IIIb, IVb, and by the proposition $\epsilon_v(s)$ equals 1 in the first case and -1 in the other cases.

3.8. Symmetric varieties.
From now on we assume that our spherical variety X is symmetric (see example (2) of 3.3). So there is an involution θ of G, with fixed point group K, such that $X = G/K$. By results of Steinberg [St2, p. 51] we may assume that B and T are fixed by θ. Then θ operates on R and W.
The set V of B-orbits in $X = G/K$ can now be described more concretely. Put $\mathcal{V} = \{x \in G \mid x(\theta x)^{-1} \in N\}$. This set is acted upon by N (and in particular by T) on the left and by K on the right.

Lemma 1. *(i) The map $x \mapsto BxK/K$ induces a bijection of $T\backslash\mathcal{V}/K$ onto V;*
(ii) The N-action on $\mathcal{V}$ induces the W-action on V of 3.5.
(iii) The map $x \mapsto x(\theta x)^{-1}T$ induces a map $\phi : V \to W$ whose image lies in the set of twisted involutions $\mathcal{I}_\theta = \{w \in W \mid \theta w = w^{-1}\}$.
For (i) and (ii) see [Sp2, §4]. The finiteness of V also follows. For (iii) see [RS, no. 2]. That the action of W is the one of 3.5 follows from [RS, 4.3].
In the sequel we identify V and $T\backslash\mathcal{V}/K$.

3.9. Combinatorial questions.
We describe the cases of the lemma 1 of 3.5 in terms of roots. Let $v \in V$, let α be a simple root and put $s = s_\alpha$. Put $\phi(v) = w$. Consider the cases of the lemma. In cases IIa (IIb) we have $w\theta\alpha \in R^+ - \{\alpha\}$ (respectively, -$w\theta\alpha \in R^+ - \{\alpha\}$). We then say that α is *complex* relative to v. In cases IIIa and IVa, $w\theta\alpha = \alpha$. Then α is *imaginary*. In the cases IIIb and IVb, $w\theta\alpha = -\alpha$, and α is *real*.
For $\beta \in R$ let G_β be the three dimensional subgroup of G generated by U_β and $U_{-\beta}$. In cases I, III and IV, θ stabilizes $x^{-1}G_\alpha x$, where $x \in \mathcal{V}$ represents v. Case I and III are distinguished by the property that θ acts trivially (respectively, non-trivially) on $x^{-1}G_\alpha x$. Then α is *compact imaginary* (respectively, *non-compact imaginary*). These definitions can be given for all roots of R, not necessarily simple (the notions are current in the theory of real Lie groups). We use the same terminology for the corresponding reflections.

The $\beta \in R$ which are imaginary (relative to v), i.e. satisfying $w\theta\beta = \beta$, form a closed subsystem R_i of R. Define $\epsilon : R_i \to \mathbf{Z}/2\mathbf{Z}$ by $\epsilon(\beta) = 0$ if and only α is compact imaginary. Then ϵ is a grading mod 2 of R_i, i.e. $\epsilon(-\beta) = \epsilon(\beta)$ and $\epsilon(\beta+\gamma) = \epsilon(\beta)+\epsilon(\gamma)$ if $\beta, \gamma, \beta+\gamma \in R_i$.

So our orbit v gives rise to algebraic objects: an involution $\iota = w\theta$ of the root system R and a grading mod 2 on the corresponding imaginary roots. The connection of orbits with such algebraic objects was first pointed out by Vogan in [Vo] (see also [Sp6]).

Let $v \in V$. Then v has a reduced decomposition (see [RS, 7.9], the proof uses a reduced decomposition in W of $\phi(v)$. Let

$$((v_0, ..., v_r), (s_1, ..., s_r))$$

be one. For $1 \leq i \leq r$ we have $m(s_i).v_{i-1} = v_i > v_{i-1}$ (see 3.6). For each i we have one of the cases IIa, IIIa, IVa of lemma 1 of 3.4. Let a, b, c be the respective numbers of such i, so $a + b + c = r = l(v)$.
Lemma 2. *(i)* $a = l(\phi(v)) - l(v)$*;*
(ii) a*,* b *and* c *are independent of the choice of the reduced decomposition of* v*.*
ϕ is as in lemma 1. (i) shows that a is independent of the reduced decomposition. (i) follows from [RS, 3.8, 3.9]. The independence of b and c was proved by Richardson (unpublished).

Finally, we mention the following result, proved in [RS, 2.7].
Proposition. *There is a bijection of the set of* W*-orbits in* V *onto the set of* K*-orbits of* θ*-stable maximal tori of* G*.*
In [loc. cit., no. 9] this is made more precise.

3.10. Geometric questions.
We shall now write X_v for the orbit v. The closures $S_v = \bar{v}$ ($v \in V$) are generalizations of Schubert varieties. We briefly review some facts about their geometry. First notice that the X_v need not be isomorphic to an affine space, and need not be simply connected (in contrast to the case of Bruhat cells). This one sees already in example (5) of 3.3 (with $G = \mathbf{SL}_2$), where the open orbit is isomorphic to $\mathbf{C} - \{0\}$.

The Bott-Samelson varieties of 1.7 have an analogue in the symmetric case. Let $(\mathbf{v}, \mathbf{s})$ be a reduced decomposition of $v \in V$ (recall that in the symmetric case all elements of V possess reduced decompositions). With notations as in 1.7 define

$$Z_{(\mathbf{v},\mathbf{s})} = P_{s_r} \times_B P_{s_{r-1}} \times_B ... \times_B P_{s_1} \times_B X_{v_0}.$$

Proposition. *There is a proper, surjective morphism* $\psi : Z_{(\mathbf{v},\mathbf{s})} \to S_v$*. Its degree is* $2^{c(v)}$*.*
$c(v)$ is the integer c of lemma 2 of 3.9. The proof is similar to the proof of propositon 1 of 1.7.

There is a bijection of the set V of B-orbits on $X = G/K$ onto the set of of K-orbits in $\tilde{X} = B\backslash G$, K acting on the right (see 3.4). We write these orbits as $\tilde{X}_v$ and their closures as $\tilde{S}_v$. The latter are projective varieties. With the notations of 1.6 we put $\tilde{C}_{w,\sigma} = B\backslash B\dot{w}G_\sigma$. Lemma 2 of 1.6 implies that these are the T-stable curves in $\tilde{X}$, T acting on the right.

Now assume that θ is an inner automorphism $\mathrm{Int}(t)$, with $t \in T$. Then T is a maximal torus of K and T operates on the varieties X_v. In [Sp4] the smoothness criterion of Carrell and Peterson (see the lemma of 1.7) is applied in the present situation. The result is as follows.

Let $\sigma \in \Sigma$ be a non-compact imaginary reflection. The group G_σ (see 1.6) is acted upon non-trivially by θ (see 3.9). Let $\alpha \in R$ be a root with $\sigma = r_\alpha$. Then $T_\sigma = \mathrm{Im}\ \alpha^\vee$ is a maximal torus in G_σ; let $n \in G_\sigma$ be an element in the normalizer of T_σ. There is $x \in G_\sigma$ with $x(\theta x)^{-1} = n$. Then x lies in the set $\mathcal{V}$ of 3.8 and defines an element $v(\sigma)$ of V. In fact $v(\sigma)$ is determined by the property that $S_{w.v(\sigma)}$ intersects $\tilde{C}_{w,\sigma}$ in an open subset. Let $v_0 = B\backslash B.K$; this is a closed orbit.
Proposition. *Let $v \in V$, $w \in W$ be such that $w.v_0 \leq v$. The number of non-compact imaginary $\sigma \in \Sigma$ with $w.v(r) \leq v$ is at least $l(v)$. Equality holds if S_v is smooth in $B\dot{w}$.*
Since $\tilde{S}_v$ is smooth if and only if S_v is, the proposition gives a smoothness criterion for the latter (in the case that the involution is inner).
In [Bri3] more general results are established, as an application of results from equivariant cohomology. See [loc. cit., 2.4].

In contrast to Schubert varieties the S_v are not always normal. For a counterexample (in $\mathbf{Sp}_8$) see [BE, 6.9].

Let $v \in V$ and let $\xi \in \Gamma_v$ (see 3.7). The definition shows that in the present case we have $\Gamma_v \simeq \mathrm{Ker}(w\theta - 1, T)/\mathrm{Im}(w\theta + 1, T)$, which shows that now Γ_v is an elementary abelian 2-group.
$\xi \in \Gamma_v$ is a one dimensional local system on X_v. Let $I_{v,\xi} = IC(S_v, \xi)$ be the intersection cohomology complexes (irreducible perverse sheaves) determined by (v, ξ) (whose definition was recalled in 2.5).
Theorem. *$I_{v,\xi}$ satisfies the parity condition, i.e.*

$$H^i(I_{v,\xi}) = 0 \text{ if } i \not\equiv \dim S_v \pmod 2.$$

This was first proved in [LV], using representation theory. A geometric proof follows from the results of [MS1, no. 6] (where this is established over the algebraic closure of a finite field). An important geometric ingredient is the existence, in the case of symmetric varieties, of a 'transverse slice' at a point of X_x inside S_v if $x < v$ (see [MS1, 6.4], compare with part (iii) of 1.6, lemma 1).

The theorem leads to generalized Kazhdan-Lusztig polynomials in the present situation. Let $x \in V$, $x \leq v$. The restriction of $H^{2i-l(v)}(I_{v,\xi})$ to X_x is a direct sum of local systems in Γ_x. For $\eta \in \Gamma_x$ let $m^i_{x,\eta;v,\xi}$ be the multiplicity of η in that restriction. The polynomials are given by

$$P_{x,\eta;v,\xi}(t^2) = \sum_{i \geq 0, x, \eta} m^i_{x,\eta;v,\xi} t^{2i}.$$

The elements

$$c_{v,\xi} = t^{-\dim v} \sum_{x \leq v, \eta \in \Gamma_x} P_{x,\eta;v,\xi}(t^2) e_{x,\eta}$$

of the module $\mathcal{M}$ of 3.7. are analogues of the elements c_w of the Hecke algebra $\mathcal{H}$ of 2.4.

3.10. A generalization of the Steinberg variety.

We still assume $X = G/K$ to be symmetric, with G semi-simple and simply connected. The involution θ acts on the Lie algebra $\mathfrak{g}$. Let $\mathfrak{p}$ be the -1-eigenspace. It is non-zero.

As in 1.9, let $\mathcal{N} \subset \mathfrak{g}$ be the the variety of nilpotent elements of $\mathfrak{g}$. We have the resolution $\pi : T^*(G/B) \to \mathcal{N}$ (theorem 1 of 1.9). Put $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}$, $Z_\theta = Z = \pi^{-1}\mathcal{N}_\theta$. So

$$Z = \{(\xi, gB) \in \mathcal{N}_\theta \times G/B \mid \xi \in \mathrm{Ad}(g)\mathfrak{u}\}.$$

If $G = H \times H$ with θ permuting the factors, Z is the Steinberg variety of H.
We have morphisms $\mu : Z \to \mathcal{N}_\theta$ and $\nu : Z \to G/B$. For $v \in V$ put $Z_v = \nu^{-1}(Kx^{-1}B/B)$, where $x \in \mathcal{V}$ represents v. Let $\xi \in \mathcal{N}_\theta$ and let C be a component of $\pi^{-1}\xi$. Put $Z_{\xi,C} = K.(\{\xi\} \times C)$. We have the following generalization of the proposition of 1.8.

Proposition 1. *Z is of pure dimension $d = \dim G/B$. Its irreducible components are the closures $\overline{Z_v}$ and also the closures $\overline{Z_{\xi,C}}$.*
The proof uses the following lemma, due to Kostant and Rallis [Ko, Prop. 5, p. 770].

Lemma. $\dim Z_G(\xi) - 2\dim Z_K(\xi)$ *is independent of ξ, for $\xi \in \mathfrak{p}$.*

Remarks (1) $\overline{Z_v}$ is (isomorphic to) the conormal bundle of G/B along $Kx^{-1}B/B$.
(2) Let Σ be a set of representatives of the K-orbits in $\mathcal{N}_\theta$, it is finite. As in 1.9, denote by $\Gamma(\xi)$ the set of irreducible components of $\pi^{-1}\xi$ ($\xi \in \mathcal{N}$). Now let $A(\xi)$ be the quotient $Z_K(\xi)/Z_K(\xi)^0$. We obtain a 'generalized Robinson-Schensted correspondence', a bijection

$$V \simeq \coprod_{\xi \in \Sigma} A(\xi)\backslash\Gamma(\xi).$$

In particular, we obtain a map of V to the set of nilpotent K-orbits in $\mathfrak{p}$.

As in 2.6 we have an action of the Weyl group on the cohomology $H_c^*(Z, \mathbf{Q})$. A full generalization of the proposition of 2.5 does not seem to be known. Only the action of W on the top cohomology $H_c^{2d}(Z, \mathbf{Q})$ has been described (in [Sp2, 4.7], via a passage to finite fields). This goes back to Rossmann, who proved similar results by analytic methods, in the context of real Lie groups (see [Ro, 3.2, 3.3]). To describe the W-module structure of $H^{2d}(Z, \mathbf{Q})$ we need some more notation.
Let $v \in V$ and let $x \in \mathcal{V}$ represent it. Then $x^{-1}Tx$ is a θ-stable maximal torus in G. Put $W_v = (xKx^{-1} \cap N)/(xKx^{-1} \cap T)$. This is a finite subgroup of W, uniquely determined by v. Let Φ be a set of representatives of the K-orbits in V (by the proposition of 3.9 these orbits are in bijection with the K-conjugacy classes of θ-stable maximal tori in G). For $v \in V$ let again W_v be its isotropy group in W.

Proposition 2. *There exist characters ϵ_v of W_v, with values in $\{\pm 1\}$, such that the W-module $H_c^{2d}(Z, \mathbf{Q})$ is isomorphic to*

$$\bigoplus_{v \in \Phi} \mathrm{Ind}_{W_v}^W(\epsilon_v).$$

The ϵ_v are described in [Sp2, p. 609] (actually, they coincide with the characters introduced at the end of 3.7, but this is not discussed in [loc. cit.]).

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