Evolution towards Symmetry*

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Were the laws of the past those of today
and will the laws of tomorrow still be the same?
Henri Poincaré in 'The evolution of the laws', Dernières Pensées.

Abstract
The dynamics of time-dependent evolution towards symmetry in Hamiltonian systems poses a difficult problem as the analysis has to be global in phasespace. For one and two degrees of freedom systems this leads to the presence of one respectively two global adiabatic invariants and also the persistence of asymmetric features over a long time.

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1 Introduction

In studies in the natural sciences assumptions of symmetries abound: spherical symmetry, cylindrical symmetry, reflection symmetry etc. This is partly because such an assumption simplifies the mathematical analysis but it is also based on the observation that equilibrium states as the outcome of a transient, evolutionary process, usually display symmetries.

It is quite remarkable that in nature one can observe many systems with symmetries, for instance rotating planets or stars which usually are displaying axi-symmetry. A similar observation holds for galaxies which for a large part have a more or less regular structure like a rotating disk or ellipsoid. More down to earth: flow structures may display some rotational or axi-symmetry, mechanical structures like pendula are often characterized by reflection symmetry with respect to a plane.

Symmetry assumptions are causing degenerations which necessitate the calculation of higher order normal forms. In mathematics, the analysis is usually focused on generic cases but in fields of application like mechanical engineering or nonlinear wave mechanics, symmetric cases abound which are not generic at all in the mathematical sense. A survey of the consequences of symmetry assumptions for Hamiltonian systems by normal form methods is given by Verhulst (1998).

Here we consider the problem of *evolution towards symmetry*. Apart from the additional complications of time-dependence we have that analysis by normal forms is not sufficient: normalisation involves localisation in some sense while time-evolution is interesting when including global dynamics. We shall review a number of case studies with rather intriguing results.

In this paper we start with an evolution problem in a special formulation: tidal evolution. The surprise in this case is that the outcome of evolution is rather symmetric. This inspires us to assume that the subset of symmetric systems has an attracting set in the space of dynamical systems. We want to study the dynamics inside this attracting set, i.e. the evolution from an asymmetric system to a symmetric system.

For one and two degrees of freedom Hamiltonian systems this leads to the presence of one respectively two adiabatic invariants and also unexpectedly to the persistence of asymmetric features over a very long time.

2 Tidal evolution in the two-body problem

Consider the Newtonian two-body problem where the bodies are stars, star and planet or planet and satellite. We assume that the masses are outside the Roche limit which determines the break-up of a body by tidal forces. The bodies contain fluid (water, gases etc.) which is affected by gravitation, resulting in tidal bulges. The basic mass distribution of the bodies is spherical apart from the fluid bulges; of course the bulges in their turn influence the motion but this is a higher order effect.

In such a mechanical system we have energy dissipation of a form which depends strongly on the actual physical conditions: the location of continents, the nature of the fluid etc. Apart from energy dissipation the tidal bulges produce a torque which results in an exchange of orbital and spin angular momenta.
Because of tidal evolution the bodies will slowly spiral in and remarkably enough, we can predict the possible outcome of evolution in such a system *without* a more detailed specification of the physical mechanism. This is important as the long time behaviour of for instance the Solar System, cannot be understood without taking into account tidal evolution.

The analysis which we present is based on Counselman (1973) where the two-dimensional case is treated, and Hut (1980) which deals with the full three-dimensional case. For the slowly changing energy we have in relative coordinates

$$E = -G\frac{Mm}{2a} + \frac{1}{2}I_1|\mathbf{\Omega}|^2 + \frac{1}{2}I_2|\omega|^2$$

where G is the gravitational constant, M and m the masses of the bodies, a

the length of the (slowly changing) semi-major axis, I_1 and I_2 the moments of inertia, $\mathbf{\Omega}$ and ω the respective angular velocities of rotation. The total angular momentum $\mathbf{L}$ will be conserved

$$\mathbf{L} = \mathbf{h} + I_1\mathbf{\Omega} + I_2\omega$$

with $\mathbf{h}$ the orbital angular momentum.

To find the possible equilibrium states we determine the critical points of E under the constraint that $\mathbf{L}$ is constant in a six-dimensional subspace (obtained after some reductions) with nine parameters. The result is that we have

- no equilibrium state if $L < L_{crit}$ with $L = |\mathbf{L}|$ and L_{crit} a positive number depending on the masses and the moments of inertia.
- two equilibrium states if $L > L_{crit}$ which correspond with orbits which are coplanar, circular and corotating. A very symmetric end stage.

To determine the stability of the equilibrium states one considers the second order variation of the energy E under the constraint $\mathbf{L}$ is constant. One finds that for stability the orbital angular momentum has to exceed a critical value; this means that the equilibrium state corresponding with the widest orbit is stable, the other one is unstable.

We note that it is quite remarkable that without specification of the mechanism of tidal dissipation we can determine the outcome of evolution. Also, that the equilibrium states clearly display a certain symmetry.

3 Evolution towards symmetry in one degree of freedom

Studies of the evolution of actual physical systems are difficult and so relatively rare. We propose therefore to ignore, at least for the time being, the physical mechanisms and to consider systems described by a simple looking Hamiltonian of the form

$$H(p, q, \varepsilon t) = \frac{1}{2}(p^2 + q^2) - \frac{1}{3}a(\varepsilon t)q^3 \tag{1}$$

where the asymmetric part is slowly vanishing as we put for $a(\varepsilon t)$ a smooth, monotonically decreasing function for which

$$a(0) = 1, \lim_{t \to \infty} a(\varepsilon t) = 0, 0 < \varepsilon \ll 1 \tag{2}$$

The problem was analysed by Huveneers and Verhulst (1997). Putting $p = \dot{x}$, $q = x$ we have the equation of motion

$$\ddot{x} + x = a(\varepsilon t) x^2 \tag{3}$$

We note that in the autonomous system (3), $\varepsilon = 0$, there are basically two regions: within the homoclinic solution the orbits are bounded, outside the homoclinic solution the orbits diverge to infinity (with the exception of the stable manifold and the saddle point itself). In system (3) for $\varepsilon > 0$ we have no fixed saddle point, still it turns out that we have two separate regions of initial values in which the orbits are bounded or diverge to infinity. It is instructive (though slightly wrong) to view system (3) as having a saddle point moving slowly towards infinity and having a slowly expanding homoclinic orbit. In this picture, an orbit can remain bounded in two ways, either by starting inside the homoclinic orbit, or by getting "captured" by the slowly expanding homoclinic orbit, which can only happen if the orbit starts sufficiently close to the stable manifold of the saddle point. To make these statements mathematically correct, one should use the concept of *normally hyperbolic motion*. One of the aims of this study is then to obtain adiabatic invariants characterizing the dynamics of the problem.

Two key ideas play a part in the paper by Huveneers and Verhulst (1997). First, by using a simple transformation a direct relation to dissipative mechanics is established. Secondly the subsequent analysis in this paper is based on averaging methods using elliptic and hypergeometric functions but, because of its relation to dissipative mechanics and "crossing of separatrix" aspects it clearly profits from the results by Haberman (1983), Robinson (1983), Haberman and Ho (1994,1995) and Bourland and Haberman (1990, 1991). Rand (1990) used a different approach (Jacobian elliptic functions) to study a similar class of dynamical systems. We also mention Neishtadt (1987) and a nice survey of the theory of adiabatic invariants by Henrard (1993).

The key step in analyzing system (3) is performing the transformation

$$y = a(\varepsilon t) x \tag{4}$$

The idea behind this transformation is to fix the normally hyperbolic motion of system (3). We want to study system (3) *for all time*; this time-dependent rescaling of the coordinates enables us to study a *bounded* domain, which simplifies the calculations considerably. This transformation has also disadvantages, in particular the loss of the Hamiltonian structure. The special choice $a(\varepsilon t) = e^{-\varepsilon t}$ will be used to show some of the more general results; for the general treatment see Huveneers and Verhulst (1997). With this choice for $a(\varepsilon t)$, system (3) becomes

$$\ddot{y} + y - y^2 = -2\varepsilon\dot{y} - \varepsilon^2 y \tag{5}$$

From the original equation describing evolution towards symmetry, we have now obtained a dissipative system. The region of attraction of system (5) is bounded by the stable manifold of the saddle point.

By averaging one determines an adiabatic invariant inside the homoclinic solution which takes a different form in different domains of the phase-plane. The location of the stable manifold of the saddle point of system (5) is calculated by considering the variation of the energy along the stable manifold. Since this variation is an $O(\varepsilon)$ effect, we may use the unperturbed stable manifold in this calculation, which involves elliptic functions. One of the conclusions is:
There exists a global adiabatic invariant inside the homoclinic orbit of the unperturbed system with the exclusion of an exponentially thin boundary layer, valid for all time.

An explicit calculation of the adiabatic invariant and transforming back to the original $x, \dot{x}$ variables produces an interesting result. The level curves of the adiabatic invariant *for a fixed time* "resemble" ellipses, of which the long axis and the short axis differ by an $O(\varepsilon)$ amount. and which are rotated around the origin, causing asymmetry.

This behaviour persists when t tends to infinity. Put in other words, when t goes to infinity, our dynamical system (3) becomes symmetric (with respect to x and $\dot{x}$), but the level curves of the adiabatic invariant remain asymmetric. So we have reached the following conclusion:
The evolution of an ensemble of phase points towards a symmetric potential will show significant (i.e. $O(\varepsilon)$) traces of its asymmetrical past, for all time.

So there is a sort of hysteresis effect present: although the system becomes symmetric, it still "knows" that it was asymmetric in the past. One can demonstrate this phenomenon visually by taking ε not too small and choosing for instance $a(\varepsilon t) = e^{-\varepsilon t}$ to obtain level curves of the adiabatic invariant. The asymmetric effect is clearly present.

4 Evolution towards symmetry in two degrees of freedom

A natural question is how to extend the preceding results to Hamiltonian systems with two degrees of freedom. It turns out that this is quite difficult; however, with some restrictive assumptions, Huveneers (1997) obtained results which we shall review and discuss. The difficulties can be summarized as follows:

- There is very little experience with a global (i.e. global in the energy) analysis of Hamiltonian systems, time-dependent or not.
- For two and more degrees of freedom there are resonance manifolds in phase space which pose special obstructions.
- The Hamiltonian may be integrable, near-integrable or non-integrable which requires different treatments.
- In the integrable case we should employ action-angle variables but these are very difficult to construct globally.

We start again with

$$H = H_{sym} + a(\varepsilon t) H_{asym}, 0 < \varepsilon \ll 1, \tag{6}$$

where H_{sym} is the symmetric part of the Hamiltonian, H_{asym} the asymmetric part and a decreases again monotonically and smoothly from 1 to 0. The condition on ε ensures adiabatic evolution to symmetry.

A second degree of freedom introduces an important complication: *resonance*. It is clear from normal form theory that in order to have an interesting system, we must choose the linear part to be in low order resonance, like 1:2, 1:1 or 1:3. Here the 1:2 resonance has been chosen, since the first resonant nonlinear terms appear already at third order in this case. Actually the calculations for the 1:1 resonance are easier (for various mathematical reasons), although one has to take fourth order terms into account. The calculations for the 1:3 resonance are more elaborate, on the other hand.

4.1 The unperturbed Hamiltonian

We will concentrate on the case in which the unperturbed system, $\varepsilon = 0$,is integrable:

$$H = \frac{1}{2}p_1^2 + 2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}x_2^2 + a(\varepsilon t)\left\{2x_1^3 + x_1x_2^2\right\} \tag{7}$$

for which the unperturbed system ($\varepsilon = 0$)

$$H_0 = \frac{1}{2}p_1^2 + 2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}x_2^2 + 2x_1^3 + x_1x_2^2 \tag{8}$$

possesses the two first integrals

$$E_1 = \frac{1}{2}p_1^2 + 2x_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}x_2^2 + \left(2x_1^3 + x_1x_2^2\right) \tag{9}$$

$$E_2 = x_2p_1p_2 + x_1x_2^2 - x_1p_2^2 + \left(\frac{1}{4}x_2^4 + x_1^2x_2^2\right) \tag{10}$$

Note that the unperturbed potential is discrete symmetric with respect to x_2, but asymmetric with respect to x_1. The corresponding equations of motion are

$$\begin{cases} \dot{x}_1 &= p_1 \\ \dot{p}_1 &= -4x_1 - x_2^2 - 6x_1^2 \\ \dot{x}_2 &= p_2 \\ \dot{p}_2 &= -x_2 - 2x_1x_2 \end{cases}$$

Identifying $2x_1^2 + \frac{1}{2}x_2^2 + 2x_1^3 + x_1x_2^2$ as the potential energy of the unperturbed system, one can draw the equipotential curves which give a crude qualitative description of the dynamics.

It is clear that the neighbourhood of the origin contains bounded solutions which live on compact tori due to the Liouville-Arnold theorem. Near the two saddle (hyperbolic) points the energy manifolds loses compactness, the tori break open, resulting in unbounded and less interesting dynamics. With this picture in mind, we can give a crude qualitative description of the perturbed system, for which the coefficient a decreases slowly to zero. Due to the slow decrease in a, the hyperbolic points are moving slowly towards infinity, resulting in capturing of nearby orbits; see for comparison similar behaviour in the one-degree-of-freedom problem discussed above. The unperturbed system possesses a heteroclinic orbit connecting the two hyperbolic

points. The heteroclinic orbit is important for our analysis, since it may be regarded as a periodic orbit with an infinitely large period, so averaging is due to fail near this heteroclinic orbit.
To do any computations, one must know the range of values which (E_1, E_2) can take (in the unperturbed problem). The range of E_1 is easily determined from the compactness requirement to be $\left[0, \frac{1}{4}\right]$. Determining the range for E_2 is more subtle since it depends on E_1. It is worthwile to note that at $E_{2,min}$ and $E_{2,max}$ there is only one value of x_1 at which the orbit is allowed to intersect the hyperplane $x_2 = 0$. This implies that these orbits are *periodic*. The one-parameter (E_1) family of these orbits constitutes a *manifold of relative equilibria* , see Derks and Valkering (1992), Derks (1992) and Zeegers (1993), see also (Mitropolsky, 1963).

There are two more sources of periodic orbits. The first one are the resonant tori and the second one is the normal mode $x_2 = p_2 = 0$, i.e. $E_2 = 0$ which is unstable (hyperbolic) for any $E_1 \leq \frac{1}{4}$.

4.2 Integrability and dissipation

A difficulty in analysing the perturbed system (7) is that the interesting dynamics takes place on a slowly expanding subset of phase space. Analogous to the 1-degree of freedom case (Huveneers and Verhulst, 1997) this difficulty is removed by rescaling the space variables adiabatically:

$$\left\{ \begin{array}{lcl} x_1 & = & \frac{1}{a(\varepsilon t)} X_1 \\ x_2 & = & \frac{1}{a(\varepsilon t)} X_2 \end{array} \right. \tag{11}$$

The conjugated momentum variables (p_1, p_2) are mixed with (x_1, x_2) and transformed into (P_1, P_2) defined by $P_i = \dot{X}_i$.

Note that this is *not* a canonical transformation. By performing this transformation the interesting part of phase space remains bounded for all time, but as a penalty we lose the Hamiltonian structure of our system. Also note that this transformation only works if the perturbation is homogeneous of a certain degree, as is the case in our system.

After applying (11) we arrive at the following system

$$\begin{cases} \dot{X}_1 &= P_1 \\ \dot{P}_1 &= -4X_1 - X_2^2 - 6X_1^2 + 2\varepsilon \frac{a'(\varepsilon t)}{a(\varepsilon t)} P_1 + \varepsilon^2 \left\{ \frac{a''(\varepsilon t)}{a(\varepsilon t)} - 2\left(\frac{a'(\varepsilon t)}{a(\varepsilon t)}\right)^2 \right\} X_1 \\ \dot{X}_2 &= P_2 \\ \dot{P}_2 &= -X_2 - 2X_1X_2 + 2\varepsilon \frac{a'(\varepsilon t)}{a(\varepsilon t)} P_2 + \varepsilon^2 \left\{ \frac{a''(\varepsilon t)}{a(\varepsilon t)} - 2\left(\frac{a'(\varepsilon t)}{a(\varepsilon t)}\right)^2 \right\} X_2 \end{cases} \tag{12}$$

which we will study. So our phasespace transformation (11) has transformed the perturbed system (7) into the unperturbed system (8) with an additional small (i.e. $O(\varepsilon)$) dissipation added. The terms of order ε^2 can be neglected on a $\frac{1}{\varepsilon}$ timescale, as long as the terms between curly braces remain bounded. Therefore we demand that both $\frac{a''(\xi)}{a(\xi)}$ and $\frac{a'(\xi)}{a(\xi)}$ are bounded for all ξ. These conditions are satisfied for many monotonically decreasing functions, and in particular for $a(\varepsilon t) = e^{-\varepsilon t}$ for which (12) reduces to an *autonomous* system.

This explicit equivalence of adiabatic Hamiltonian perturbation and small dissipation is very helpful both in understanding the dynamics and in doing calculations, since we can use ideas and techniques from both fields.

4.3 Action-Angle transformation

We will construct the action variables following the proof of the Liouville-Arnold theorem, i.e. by evaluating the contour integral

$$\oint \vec{P} \cdot d\vec{X}$$

along two irreducible circuits on the torus determined by fixed values of E_1 and E_2. The first circuit is of course easy to find, since it is to a high degree arbitrary. So we choose it in such a way that the contour integral becomes as simple as possible. This is done by intersecting the torus with the $X_2 = 0$ hyperplane. The contour integral reduces to

$$\oint P_1 dX_1$$

so we have to evaluate the area enclosed by the contour after projection onto the (P_1, X_1) plane. Using the results from the preceding subsection, we know that the contour is given by

$$P_1(X_1) = \pm\sqrt{2E_1 - 4X_1^2 + \frac{E_2}{X_1} - 4X_1^3}$$

From this relation the lower and upper limits of X_1 are easily calculated (numerically), taking into account that $X_1 > 0$ for $E_2 < 0$ and $X_1 < 0$ for $E_2 > 0$. After that, any sophisticated numerical integrator can calculate the value of the first action-variable I_1. We used the built-in numerical integrator of Mathematica 2.2.

There is a discontinuity at $E_2 = 0$ which has no physical relevance. In fact, the invariant normal mode plane $X_2 = P_2 = 0$ separates two independent parts of phase space. Orbits can never pass through this discontinuity. If necessary (which it is not) we could apply an additional transformation to get rid of this discontinuity.

Calculating the second action variable I_2 is more difficult, primarily because of the 1:2 resonance, which prohibits us to define the contour as the intersection of the torus with a hyperplane (we tried by extensive calculations). We must choose a contour that traverses the entire torus. In order to do so, we first project the torus onto the (X_2, P_2) plane. It is easy to show that this projection is symmetric with respect to reflection in the P_2 and X_2 axes and that there always is a neighbourhood of the origin which is not covered by the projection. To insure irreducibility and independence of the first circuit, we choose the second circuit to follow the inner boundary of the projection (the outer boundary gives raise to major numerical difficulties). Note that we still need to evaluate $\oint P_1 dX_1$ (parametrized by X_2) as well as $\oint P_2 dX_2$. The only use of the projection is that it defines a unique irreducible circuit independent of the first circuit.

The calculation of $\oint P_1 dX_1$ and $\oint P_2 dX_2$ is still not easy; for an efficient calculation see Huveneers (1997).

Now that I_1 and I_2 are known it is easy to determine the corresponding conjugated angles ϕ_1 and ϕ_2 by following the flow induced by the Hamiltonian on the torus. In the sequel we do not need ϕ_1 and ϕ_2 since they will vanish in the averaging process. So we proceed directly to determining the location of the resonances.

4.4 Resonances

In the next section we will show how averaging can be applied to the perturbed system (12). The concept of *resonant tori* (tori for which the angular frequencies are rationally dependent) plays an important role in this analysis, because on these tori the averaging process breaks down (better: needs a separate treatment).

Locating the resonant tori is straightforward, since we are able to compute the action-variables. We start with Hamilton's equations for the unperturbed system (8) in action-angle coordinates

$$\left\{ \begin{array}{lcl} \dot{I}_1 & = & 0 \\ \dot{I}_2 & = & 0 \\ \dot{\phi}_1 & = & \frac{\partial E_1}{\partial I_1} =: \omega_1 \\ \dot{\phi}_2 & = & \frac{\partial E_1}{\partial I_2} =: \omega_2 \end{array} \right.$$

The problem is how to compute ω_1 and ω_2, given some arbitrary but fixed values of E_1 and E_2, using only the transformation $(E_1, E_2) \rightarrow (I_1, I_2)$ and *not* its expensive inverse. This can be accomplished by applying the implicit function theorem, which results in

$$\omega_1 = \frac{\frac{\partial I_2}{\partial E_2}}{\frac{\partial I_2}{\partial E_2}\frac{\partial I_1}{\partial E_1} - \frac{\partial I_1}{\partial E_2}\frac{\partial I_2}{\partial E_1}}$$

$$\omega_2 = -\frac{\frac{\partial I_1}{\partial E_2}}{\frac{\partial I_2}{\partial E_2}\frac{\partial I_1}{\partial E_1} - \frac{\partial I_1}{\partial E_2}\frac{\partial I_2}{\partial E_1}}$$

So to compute ω_1 and ω_2 we only need to evaluate $\frac{\partial I_2}{\partial E_2}$, $\frac{\partial I_1}{\partial E_1}$, $\frac{\partial I_1}{\partial E_2}$ and $\frac{\partial I_2}{\partial E_1}$, which can be done by applying a numerical differentiator to the action-variables $I_{1,2}(E_1, E_2)$ as defined in the previous section. Again we used the built-in numerical differentiator of Mathematica 2.2.

Often one is only interested in the value of $\frac{\omega_1}{\omega_2}$, in which case one only needs to evaluate $\frac{\partial I_2}{\partial E_2}$ and $\frac{\partial I_1}{\partial E_2}$

$$\frac{\omega_1}{\omega_2} = \frac{\frac{\partial I_2}{\partial E_2}}{\frac{\partial I_1}{\partial E_2}} \tag{13}$$

Using this approach we have calculated the position of the resonant tori for many integral values of $\frac{\omega_1}{\omega_2}$. For instance, for small values of E_2, the ratio $\frac{\omega_1}{\omega_2}$ becomes arbitrary large. This corresponds to motion close to the normal mode plane ($\omega_2 = 1$) with an occasional excursion in the X_2 direction ($\omega_1 \ll 1$).

For $E_2 > 0$ there are no resonant tori with $\frac{\omega_1}{\omega_2} = 2$, demonstrating the strong non-linearity. Moreover, the limit of $\frac{\omega_1}{\omega_2}$ towards the origin of phasespace ($E_1 = 0$) does not exist. This corresponds to one of the phase-variables becoming a *slow* variable (i.e. slowly dependent on time) which requires a different type of averaging. To study the neighbourhood of the origin we can use normal form theory instead.

For $E_2 < 0$ there is a small region where $\frac{\omega_1}{\omega_2}$ is close to two. These tori are close to the heteroclinic orbit connecting the two rightmost saddle-points.

Next we will need the property that $\frac{\omega_1}{\omega_2}$ is monotonically decreasing with respect to E_1 and E_2 (this is a necessary condition for averaging near resonant 2-tori). This property is hard to prove, since we have no analytic expression for $\frac{\omega_1}{\omega_2}$. That $\frac{\omega_1}{\omega_2}$ decreases monotonically with respect to E_2 is clear, since for increasing E_2 the torus moves away from the normal mode plane which is a hyperbolic fixed point in the Poincaré section $x_1 = 0$. Near this hyperbolic fixed point ω_1 becomes arbitrary small, since orbits pass this fixed point slowly (stay close to the normal mode plane) and occasionally make excursions into phasespace.

It is harder to show the monotonic decrease with respect to E_1. We have been unable to find a mathematical or physical argument for this behaviour and it might well be particular to the potential we are studying. We made however many numerical cross-sections $E_2 = constant$, the results of which firmly support the claim of monotonic decrease in E_1. Although we realize we have not proved this claim, we feel very confident about it and will assume it is true in the sequel.

At the maximum and minimum values of E_2 (the relative equilibria) the ratio $\frac{\omega_1}{\omega_2}$ approaches a limit value (for fixed E_1). Although it is tempting to assign the value $\frac{\omega_1}{\omega_2} = 2$ to the relative equilibria, we should assign the limit value instead, since on these degenerate tori I_1 equals zero. This implies that ω_1 is not defined on these tori, so we must take the above limit to assign a value to ω_1 on these tori.

Near the normal mode plane (and in particular, near the origin), the ratio

$\frac{\omega_1}{\omega_2}$ in (13) becomes arbitrary large. This has two important consequences:

- If we fix ε, we cannot extend our calculations arbitrary close to the normal mode plane (and thus the origin), since the timescale parameter $\frac{\omega_1}{\omega_2}$ becomes larger than $\frac{1}{\varepsilon}$. So we must be careful when taking limits. The closer we want to be to the normal mode plane, the smaller ε has to be.

- There is no smooth limit from the global asymptotic analysis (determined by $\frac{\omega_1}{\omega_2}$) to the normal form analysis around the origin, where the non-linear terms manifest themselves as a second perturbation and the unperturbed system is in exact 1:2 resonance.

The ratio $\frac{\omega_1}{\omega_2}$ has non-trivial minimum values for $E_2 > 0$ and for $E_2 < 0$. These values are easily calculated numerically using (13). We get

$$\left\{ \begin{array}{lcl} \min_{E_2>0} \frac{\omega_1}{\omega_2} & = & 2. \\ \min_{E_2<0} \frac{\omega_1}{\omega_2} & = & 1. \end{array} \right.$$

These ratios hold for the tori close to the two heteroclinic orbits connecting the two rightmost saddle-points.

4.5 Averaging

Now that we are able to compute the angular frequencies ω_i we can prove the existence of two adiabatic invariants, using averaging combined with a theorem due to Neihstadt (1987). The main problem in averaging systems with more than one degree of freedom is the presence of resonant tori, since near these tori a linear combination of the two angular coordinates becomes slowly dependent on time. This normally prohibits averaging over all angular coordinates near these resonant tori.

However, for the case of two degrees of freedom a more accurate result is known. Due to the perturbation the coordinates (I_1, I_2) (which determine the tori) will slowly change in time. If this flow crosses the main resonant tori (we will make this more precise below) transversally everywhere, one is still allowed to apply averaging. The idea behind this statement is that although orbits cross resonant tori occasionally, they are only near a resonant torus during a "short" (i.e. $O(\frac{1}{\sqrt{\varepsilon}})$) time-interval, which gives only an $O(\sqrt{\varepsilon})$

contribution to the total error. This is the only effect of the resonant tori: instead of an $O(\varepsilon)$ approximation on a $\frac{1}{\varepsilon}$ timescale, we now get an $O(\sqrt{\varepsilon})$ approximation on a $\frac{1}{\varepsilon}$ timescale. This is sufficient to give a constructive prove of the existence of two adiabatic invariants.

So we must first show that the perturbation in our system (12) induces a flow in the (I_1, I_2) plane which is transversal to the equi-angular-frequencies-ratio lines. We will do this by showing that both $\frac{d}{dt}E_1^2$ and $\frac{d}{dt}E_2^2$ are negative definite when averaged over the angular frequencies. We will omit the $O(\varepsilon^2)$ perturbation since it introduces only an $O(\varepsilon)$ error on a $\frac{1}{\varepsilon}$ timescale.

The first one, $\frac{d}{dt}E_1^2$, is trivial since the transformation (11) turned our perturbation into a friction, which guarantees a strict monotonic decrease in the unperturbed energy (E_1). This is also clear from

$$\frac{d}{dt}E_1^2 = 4\varepsilon\frac{a'(\varepsilon t)}{a(\varepsilon t)}\left(P_1^2 + P_2^2\right)E_1$$

The second one, $\frac{d}{dt}E_2^2$, is more difficult. We will first give the proof for the case that $E_2 > 0$. So we must show that $\left\langle \frac{d}{dt}E_2 \right\rangle < 0$, where $\langle\cdot\rangle$ denotes averaging. Using the definition of E_2 (10) and the perturbed system (12) it is easy to show that the following relations hold

$$\begin{aligned} \left\langle \tfrac{d}{dt}E_2 \right\rangle &= -2\varepsilon\tfrac{a'(\varepsilon t)}{a(\varepsilon t)}\left\langle \left(X_2P_1 - X_1P_2\right)P_2\right\rangle \\ &= -2\varepsilon\tfrac{a'(\varepsilon t)}{a(\varepsilon t)}\left\langle -E_2 + X_1X_2^2 + a(\varepsilon t)\left(\tfrac{1}{4}X_2^4 + X_1^2X_2^2\right)\right\rangle \end{aligned} . \qquad (14)$$

To calculate $\langle (X_2P_1 - X_1P_2)P_2\rangle$ we must average over the angle coordinates (the surface of a 2-torus). Except for the resonant tori, this is equivalent to evaluating the *time-average*

$$\frac{1}{T}\int_0^T \left(X_2P_1 - X_1P_2\right)P_2 dt$$

for arbitrary large T, where X_i and P_i are now understood to represent an orbit on the same torus. After partial integration and applying the *unperturbed* equations of motion and taking the limit $T \to \infty$ we arrive at

$$\left\langle \left(X_2P_1 - X_1P_2\right)P_2\right\rangle = -\left\langle 3X_1X_2^2 + 4a(\varepsilon t)\left(\frac{1}{4}X_2^4 + X_1^2X_2^2\right)\right\rangle \qquad (15)$$

Combining (14) and (15) we get

$$\left\langle X_1 X_2^2 \right\rangle = \left\langle \frac{1}{4}E_2 - \frac{5}{4}a(\varepsilon t)\left(\frac{1}{4}X_2^4 + X_1^2X_2^2\right)\right\rangle$$

and substituting this back in (14) we get the desired result

$$\left\langle \frac{d}{dt}E_2 \right\rangle = -2\varepsilon\frac{a'(\varepsilon t)}{a(\varepsilon t)}\left\langle -\frac{3}{4}E_2 - \frac{1}{4}a(\varepsilon t)\left(\frac{1}{4}X_2^4 + X_1^2X_2^2\right)\right\rangle < 0$$

This completes the proof for $E_2 > 0$. For $E_2 < 0$ the arguments are similar.

In conclusion, we are now allowed to apply Arnold's theorem which states that averaging the perturbed equations for I_1 and I_2 produces an $O(\sqrt{\varepsilon})$ approximation on a $\frac{1}{\varepsilon}$ timescale.

This implies that we are also allowed to average the equations for $\dot{E}_1$ and $\dot{E}_2$ over the 2-tori in (X_1, X_2, P_1, P_2) space provided we use the correct measure

$$\mu = \frac{1}{\left|\overrightarrow{\nabla}E_1\right|\left|\overrightarrow{\nabla}E_2\right|}$$

in evaluating the phasespace integrals. This opens a route for implementing the averaging process numerically in an efficient way.

An important remark is that (like in the one degree of freedom case) the averaged equations are of the following form

$$\left\{\begin{array}{rcl} \dot{I}_1 &=& \varepsilon\frac{a'(\varepsilon t)}{a(\varepsilon t)}A_1(I_1, I_2) \\ \dot{I}_2 &=& \varepsilon\frac{a'(\varepsilon t)}{a(\varepsilon t)}A_2(I_1, I_2) \end{array}\right.$$

This implies that the adiabatic invariants are given by the initial conditions of these equations, or alternatively by the initial conditions of the corresponding equations for $\dot{E}_1$ and $\dot{E}_2$ in (X_1, X_2, P_1, P_2) space. Moreover, the slow time-dependence can be removed from these equations by a suitable time rescaling

$$\left\{\begin{array}{rcl} \tau &=& -\frac{1}{\varepsilon}\log(a(\varepsilon t)) \\ a(\varepsilon t) &=& e^{-\varepsilon\tau} \end{array}\right. \tag{16}$$

So the way in which a decays to zero does not affect the dynamics qualitatively. It only changes the timescale on which the dynamics takes place. We might just as well set $a(\xi) = e^{-\xi}$ by which our perturbed system (12) becomes autonomous.

4.6 Discussion

- Close to the origin we can rescale to produce an unperturbed Hamiltonian ($\varepsilon = 0$) consisting of two independent harmonic oscillators, the solutions of which are readily expressed in terms of goniometric functions. We can normalize the system. For our system (7) these calculations have been done by Van den Broek (1988).

  The resulting system posesses two integrals of motion corresponding to two adiabatic invariants of the general Hamiltonian. An easy generalization of this result shows that close to the origin *even the non-integral Hamiltonian posesses two adiabatic invariants.*

- A different analysis is necessary near the two saddle points of the energy manifold (which have energy $E_1 = \frac{1}{4}$) where we are studying the neighbourhood of the degenerate torus (heteroclinic orbit) connecting the two saddle points; here both ω_1 and ω_2 become arbitrary small (although their ratio has a nice limit). The basic idea is the same as for the one degree of freedom study: if we consider a torus at a distance δ from the homoclinic orbit, the orbits on this torus pass the two saddle points during a time-interval of the order $-\log\delta$. This implies that ω_2 is of the order $(-\log\delta)^{-1}$. Since ω_1 is of the same order, this introduces a multiplicative error of order $-\log\delta$ in the averaging procedure (i.e. when averaging over both angle coordinates). It is easy to show that the approximation produced by averaging remains valid on a $\frac{1}{\varepsilon}$ timescale. Decreasing δ only makes the approximation less accurate.

  We conclude that the averaging method works as long as $-\varepsilon\log\delta$ is small ($o(1)$), i.e. outside an exponentially small $\left(e^{-\frac{1}{\varepsilon}}\right)$ neighbourhood of the homoclinic orbit.

- It is relatively easy to show that the phase flow of our Hamiltonian (7) is volume-preserving in the (x_i, p_i) space , although it is time-dependent. This implies that our picture of saddle points slowly moving away from the origin with a slowly expanding heteroclinic orbit is not complete. As the saddle points are slowly moving (at speed ε) some orbits are captured by entering the expanding set of points with energy less than the saddle points, since it is *positive invariant* and volume preserving. The only way an orbit can enter this set is along the stable manifold of the two rightmost saddle points.

- There are many open problems left. We mention the more general problem of evolution towards symmetry of a nonintegrable system which poses a different type of averaging problem. There is also the important case of more than two degrees of freedom. In the last type of problem one may expect new phenomena; see also Verhulst (1998) or Verhulst and Hoveijn (1992).

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