“On the braided Fourier transform in the n-dimensional quantum space”

*Giovanna Carnovale**
Mathematisch Instituut
P.O.Box 80.010, 3508 TA
Utrecht, The Netherlands
email address: carnoval@math.uu.nl

Abstract: We work out in detail a theory of integrability on the braided covector Hopf algebra and the braided vector Hopf algebra of type A_n introduced in [Ma] and [KeMa]. Starting with a definition of braided Fourier transform very similar to that in [KeMa] we obtain n-dimensional analogous results to those in [Koo] expressing the correspondence between products of the q^2-Gaussian $g_{q^2}(\underline{x})$ times monomials, and products of the q^2-Gaussian $G_{q^2}(\underline{\partial})$ times q^2-Hermite polynomials under the transform. We invert the correspondence by finding a suitable inversion, different from that in [KeMa]. We show that with this transforms, whenever $n \geq 2$, the Plancherel measure will depend on the parity of the power series that we are transforming.

1.0. Introduction.

Recently Majid ([Ma], and references given there) has defined a generalisation of the concept of Hopf algebra, namely “braided Hopf algebras”. Hopf superalgebras and genuine Hopf algebras are examples of this objects, but there are more examples, associated to quantum groups, since the category of representations of a quasitriangular Hopf algebra is braided. This is also the subject of [HH] with different terminology.
Kempf and Majid introduced in [KeMa] an integration theory for a class of braided Hopf algebras arising from matrix solutions of the quantum Yang-Baxter equation, as “braided covector algebras”(see also [Ma]). They used this theory to define a formal braided Fourier transform and its inverse on these algebras. In their paper they also present the case of the braided line as an example, and the n-dimensional case in less details.
The main problem in their theory is that it is very difficult to find an explicit integral that behaves well enough. They provide powerful general results in a theoretical way, but the description in specific cases is often hard to handle. Besides, they do not provide a definition of convergence of an integral nor do they say in their article when an object is called integrable.
The purpose of this paper is to work out as far as we can the example of the n-dimensional quantum space of type A_n viewed as a braided Hopf algebra. We will provide a definition of an integral (similar to that in [KeMa]) accompanied by results on convergence of the integral based on extensions of representations of the braided covector and vector algebras. We define three different concepts of integrability, and provide examples and counterexamples. Using this facts, we show, similarly to [KeMa] that the given integral is invariant under translation, and we use it for the definition of two types of Fourier transform. Our transforms will be slightly different from that of [KeMa]. One of them looks more like

* This research is financially supported by the NWO-Stichting SWON, under project number 610-06-100

that of [Ch] since the integral does not have trivial braiding with elements in the braided Hopf algebra, and because the braided antipode appears in the definition. We also took inspiration from [Koo], who studied also an analogue of the Fourier transform for the case of the braided line, although his transform goes from an algebra to itself, while we are looking for a transform going from an algebra to its braided dual, as in [KeMa], [OR] and [Ch].

We find an n-dimensional analogue of the correspondence between products of q^2-Gaussians times monomials and other q^2-Gaussians times q^2-Hermite polynomials, similar to the results for the braided line in [Koo]. We give also inverses for our transforms that invert the correspondence mentioned above, similarly to what appears in [Koo]. The main tool for this inversion formula is the symmetry between the braided vector algebra and the braided covector algebra. Kempf and Majid had already defined an inversion formula in their article, but they used properties that our integral does not have. Other inversion formulae for the braided line are to be found in [OR] where the case of distributions is also treated.

One of the most interesting results is that whenever $n \geq 2$, there is a loss of symmetry, so that the Plancherel measure will no longer be the same in the whole space. Indeed there is a action of ${\mathbf{Z_2}}^n$ associted to the parity of the power series we are working with, and the Plancherel measure will be constant only on the subspaces of power series with constant parity. Therefore the transforms we define can also be seen as sine and cosine transform. A phenomenon of break of simmetry for q-integrals was also noted in [CHMW], where the authors were defining a calculus associated to a q-deformed Heisenberg Algebra.

Other definitions of analogues of the Fourier, on genuine Hopf algebras, quantum spaces, or commutative algebras appeared before [KeMa] in [MMNNU], [KooSw] and [LyuMa].

1.1. Notation

In this paper a complex algebra has, unless otherwise stated, always a unit. q is a real number $0 < q < 1$. For a positive integer m, and for any $q \neq 1$ we write $[m]_q = \frac{q^m-1}{q-1}$ and $[m]_q! = \prod_{j=1}^{m} [j]_q$.

We will also use the $q-$shifted factorial. For any $a \in \mathbf{R}$ and for any $k \in \mathbf{Z}_{\geq 0}$, we will then put $(a;q)_k = \prod_{l=0}^{k-1}(1-aq^l)$. If $0 < q < 1$ we will also write $(a;q)_\infty = \lim_{k\to\infty}(a;q)_k$ and for r real numbers $a_1, \ldots, a_r$ we will put $(a_1, \ldots, a_r; q)_\infty = \prod_{j=1}^{r}(a_j;q)_\infty$. Finally, for $a \geq b$ with a and b both in $\mathbf{Z}_{\geq 0}$ in we will use the $q-$binomial coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q!}{[b]_q![a-b]_q!} = \frac{(q;q)_a}{(q;q)_b(q;q)_{a-b}}$

Whenever for any capital character E we have a multi-index $E = (e_1, \ldots, e_n)$ we will put: $E_i = \sum_{j=1}^{i-1} e_j$ and $E^i = \sum_{j=i+1}^{n} e_j$. Hence $|E| = e_i + E_i + E^i$ for every i.

We identify the set $\{+, -\}$ with the field $\mathbf{Z_2}$, letting $+$ correspond to $\bar{0}$ and $-$ correspond to $\bar{1}$, so that n-tuples in $\{+, -\}^n$ can be identified with vectors in ${\mathbf{Z_2}}^n$. By means of this identification we define the map $A\colon \mathbf{Z}^n \to \mathbf{Z}_2^n \to \{+, -\}^n$ by reducing modulo 2 first. I.e. $A(b) = +$ if b is even and $-$ otherwise. We will also denote by $B\colon \{+, -\}^n \to \{0, 1\}^n \subset \mathbf{Z}^n$ the map sending each "even" entry to 0 and each "odd" entry to 1.

Given an operator on the two-fold tensor product of an $n-$dimensional vector space V, we identify this operator with the $n^2 \times n^2$ matrix R and we denote its entries by R^{ab}_{cd} where a, b

are the row entries and c, d are the column entries. For such an R, the operator acting on the $p-$fold tensor product of V (for $p \geq 2$) as R on the i^{th} and j^{th} component and as the identity elsewhere will be denoted by R_{ij}. For summation we will use Einstein convention. For any vector space V, we will denote by P the operator $P: V \otimes V \to V \otimes V$ sending $v \otimes w$ to $w \otimes v$.

2. Braided covector algebra and braided vector algebra.

We begin with the definition of braided bialgebras and Hopf algebras, and we give then the definition of the braided covector algebra and braided vector algebra as given in [Ma].

Definition. *A braided Hopf algebra over a field K is an associative algebra A with multiplication m and a coassociative coalgebra with comupltiplication Δ and counit ε together with an invertible linear map $\Phi: A \otimes A \to A \otimes A$ called braiding, and a linear map $S: A \to A$ called braided antipode such that the following properties hold:*

$$\begin{aligned}
&\Phi(m \otimes \mathrm{id}) = (\mathrm{id} \otimes m)(\Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi)\\
&\Phi(\mathrm{id} \otimes m) = (m \otimes \mathrm{id})(\mathrm{id} \otimes \Phi)(\Phi \otimes \mathrm{id})\\
&(\mathrm{id} \otimes \Delta) \circ \Phi = (\Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi)(\Delta \otimes \mathrm{id})\\
&(\Delta \otimes \mathrm{id}) \circ \Phi = (\mathrm{id} \otimes \Phi)(\Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\\
&1\varepsilon = m(\mathrm{id} \otimes S)\Delta = m(S \otimes \mathrm{id})\Delta\\
&\Delta\, m = (m \otimes m)(\mathrm{id} \otimes \Phi \otimes \mathrm{id})(\Delta \otimes \Delta)\\
&\varepsilon \circ m = \varepsilon \otimes \varepsilon\\
&\Delta(1) = 1 \otimes 1
\end{aligned}$$

i.e. Δ is an algebra map $A \to A \otimes A$ where $A \otimes A$ has the product $(m \otimes m)(\mathrm{id} \otimes \Phi \otimes \mathrm{id})$ that is associative and coassociative because of the properties of the braiding, and ε is an algebra map $A \to K$. If there is no braided antipode, A is called "braided bialgebra".

S. Majid who first introduced the concept, defines braided Hopf algebras as Hopf algebras living in a braided category. The definition we give here is equivalent to Majid's definition if the objects in the category are vector spaces. For the categorical approach and an extensive description of the properties and importance of such objects, see [Ma] and references given there. For a definition similar to the one appearing here, see [Du]. In the latter one can find a more general definition than here because the counit is not required to be an algebra homomorphism. [Du] shows that the two definitions coincide if and only if ε is an algebra homomorphism and he describes a few properties and equivalences. In particular we recall that in a braided Hopf algebra there holds

$$\begin{aligned}
&\Phi(1 \otimes a) = a \otimes 1, \quad \Phi(a \otimes 1) = 1 \otimes a \quad \forall a \in A\\
&(\varepsilon \otimes \mathrm{id})\Phi = \mathrm{id} \otimes \varepsilon, \quad (\mathrm{id} \otimes \varepsilon)\Phi = (\varepsilon \otimes \mathrm{id})\\
&\Phi = (m \otimes m)(S \otimes \Delta \otimes S)(\mathrm{id} \otimes m \otimes \mathrm{id})(\Delta \otimes \Delta)\\
&(\Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi)(\Phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \Phi)(\Phi \otimes \mathrm{id})(\mathrm{id} \otimes \Phi)\\
&S(1) = 1, \quad \varepsilon \circ S = \varepsilon\\
&\Delta\, S = \Phi(S \otimes S)\Delta, \quad S\, m = m(S \otimes S)\Phi\\
&\Phi(S \otimes \mathrm{id}) = (\mathrm{id} \otimes S)\Phi, \quad \Phi(\mathrm{id} \otimes S) = (S \otimes \mathrm{id})\Phi
\end{aligned}$$

In particular we observe that the operator Φ must satisfy the braid relation. On the other hand whenever we have an operator T satisfying the braid relations, it is well known that the operator $R = TP$ satisfies the quantum Yang-Baxter equation in the form $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. Hence one can build the dual quasitriangular bialgebra $A(R)$ by means of the FRT construction (see for instance [FRT]). One can find in [Ma, Chapter 9] and references in there that the category of right comodules for $A(R)$ is braided, and that R defines a braiding between every pair of right comodules. In particular for any two comodule algebras U and W, there is an algebra structure on $U \otimes W$ where the product is defined by means of the braiding and the products in U and W, such that $U \otimes W$ turns out to be again a comodule algebra for $A(R)$. This can be found in [Ma, Chapter 9]. Moreover, for a sufficiently well behaved braided category, on can always find a dual quasitriangular Hopf algebra H such that the category we have is the category of right comodules for H.
We will work with the following example of braided Hopf algebras:

Definition-Proposition [Ma, Theorem 10.2.1]. *Let R be an invertible $n^2 \times n^2$ matrix obeying the quantum Yang-Baxter equation. Suppose also that there is another invertible $n^2 \times n^2$ matrix Q for which $Q_{12}R_{13}R_{23} = R_{23}R_{13}Q_{12}$ and $R_{12}R_{13}Q_{23} = Q_{23}R_{13}R_{12}$, and $(PR + 1)(PQ - 1) = 0$. Then the "braided covector algebra" $\hat{V}(Q, R)$ (resp. "braided vector algebra" $V(Q, R)$) defined by the generators $x_1, \ldots, x_n$ (resp. $\partial^1, \ldots \partial^n$) and the relations $x_i x_j = Q^{ab}_{ij} x_b x_a$ (resp. $\partial^i \partial^j = Q^{ij}_{ab} \partial^b \partial^a$) form a braided bialgebra with $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ (resp. $\Delta(\partial^i) = \partial^i \otimes 1 + 1 \otimes \partial^i$), $\varepsilon(x_i) = 0$ (resp. $\varepsilon(\partial^i) = 0$) and braiding $\Phi(x_i \otimes x_j) = x_b \otimes x_a R^{ab}_{ij}$ (resp. $\Phi(\partial^i \otimes \partial^j) = R^{ij}_{ab} \partial^b \otimes \partial^a$). Moreover, if $PRPQ = PQPR$ we have a braided Hopf algebra with braided antipode $S(x_i) = -x_i$ (resp. $S(\partial^i) = -\partial^i$).*

Proof: We sketch the idea of the proof. The quantum Yang-Baxter equation for R ensures that the braid relations hold for Φ on the generators. The relations involving Q_{ij} and R_{kl} ensure that Φ can be extended on monomials as the axioms require. The fact that $(PR+1)(PQ-1) = 0$ implies that Δ can be extended as an algebra homomorphism. Finally, the fact that PR commutes with PQ imples that the antipode S defined on generators can be extended using $S\,m = m(S \otimes S)\Phi$ and $\Phi(S \otimes \mathrm{id}) = (\mathrm{id} \otimes S)\Phi$. This and similar results are also to be found in [HH]. □

We observe that the relations involving Q_{ij} and R_{kl} ensure that $\hat{V}(Q, R)$ and $V(Q, R)$ are comodule algebras for $A(R)$. Hence if $A(R)$ can be extended or quotiented to a dual quasitriangular Hopf algebra $H(R)$ with dual quasitriangular structure consistent with that of $A(R)$, and if $\hat{V}(Q, R)$ and $V(Q, R)$ are comodule algebras for $H(R)$, then all possible vector spaces obtained by tensoring $\hat{V}(Q, R)$ and $V(Q, R)$ can be provided of an algebra structure such that they are again comodule algebras for $H(R)$. The product is then defined by means of the braidings and the product in $\hat{V}(Q, R)$ and $V(Q, R)$.

We want to study the case in which R is (a multiple of) the matrix solution of the quantum Yang-Baxter equation associated to type A_n (see [FRT]) with real entries because of our choice of q. We regard the braided covector and vector algebras as complex algebras. Since R is Hecke, after some rescaling we have:
$R^{jj}_{jj} = q^2$, $R^{ij}_{ij} = q$ if $i \neq j$, $R^{ij}_{ji} = q^2 - 1$ if $i < j$, and all the other entries equal to 0.

We choose $Q = q^{-2}R$, i.e. $Q^{jj}_{jj} = 1$, $Q^{ij}_{ij} = q^{-1}$ if $i \neq j$, $R^{ij}_{ji} = 1 - q^{-2}$ if $i < j$, and all the other entries equal to 0. It is well known that in this case the matrix R is bi-invertible, i.e. R and R^{t_2} are both invertible, where t_2 stands for transposing only with respect to the second component in the tensor product, i.e. there exists an $n^2 \times n^2$ matrix $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$ such that $(\tilde{R})^{aj}_{ib} R^{id}_{cj} = R^{aj}_{ib} (\tilde{R})^{id}_{cj} = \delta^a_c \delta^d_b$.
Then the relations and the braiding for $\hat{V}(Q, R)$ are $x_i x_j = q x_j x_i$ if $i > j$ so this is the $n-$dimensional quantum space. We know that this algebra has a basis given by (increasing) ordered monomials $x_1^{e_1} \cdots x_n^{e_n}$. The braiding is given on the generators by

$$\Phi(x_i \otimes x_j) = \begin{cases} q x_j \otimes x_i & \text{if } i < j \\ q^2 x_i \otimes x_i & \text{if } i = j \\ (q^2 - 1) x_i \otimes x_j + q x_j \otimes x_i & \text{if } i > j \end{cases}$$

so that, for $i < j$, it follows that $\Phi(x_i^a \otimes x_j^b) = q^{ab} x_j^b \otimes x_i^a$. We use this, the properties and the q-binomial formula to see that the comultiplication of an arbitrary element of the basis equals:

$$\Delta(x_1^{e_1} \cdots x_n^{e_n}) = \sum_{j_1=0}^{e_1} \cdots \sum_{j_n=0}^{e_n} \left(\prod_{j=1}^{n} \begin{bmatrix} e_i \\ j_i \end{bmatrix}_{q^2} \right) q^{\sum_v J_v (e_v - j_v)} x_1^{e_1 - j_1} \cdots x_n^{e_n - j_n} \otimes x_1^{j_1} \cdots x_n^{j_n}$$

The antipode is given by $S(x_1^{e_1} \cdots x_n^{e_n}) = q^{|E|^2 - |E|} (-1)^{|E|} x_1^{e_1} \cdots x_n^{e_n}$ as one can easily prove by induction.

For convenience we put $\partial_j := \partial^j$. The relations for $V(Q, R)$ are given by $\partial_i \partial_j = q \partial_j \partial_i$ if $i < j$. The braiding is given on generators by

$$\Phi(\partial_i \otimes \partial_j) = \begin{cases} q \partial_j \otimes \partial_i & \text{if } i > j \\ q^2 \partial_i \otimes \partial_i & \text{if } i = j \\ (q^2 - 1) \partial_i \otimes \partial_j + q \partial_j \otimes \partial_i & \text{if } i < j \end{cases}$$

and can be extended on monomials according to the axioms. Again we know that ordered monomials provide a basis for $V(Q, R)$. In this case we fix the basis given by ordered monomials where the order is *decreasing*.
One can observe that for every choice of nonzero constants c_j, for $j = 1, \ldots, n$, there is an algebra isomorphism ψ between $\hat{V}(Q, R)$ and $V(Q, R)$ mapping x_j to $c_j \partial_{n+1-j}$, such that $\Delta_{V(Q,R)} \circ \psi = (\psi \otimes \psi) \circ \Delta_{\hat{V}(Q,R)}$ and $S_{V(Q,R)} \circ \psi = S_{\hat{V}(Q,R)}$. In particular, for $c_j = q^{-j + \frac{1}{2}(n-1)}$ for every j, then we also have $\Phi_{V(Q,R),V(Q,R)}(\psi \otimes \psi) = \Phi_{\hat{V}(Q,R),\hat{V}(Q,R)}$.
Using this property, or by computations similar as for $\hat{V}(Q, R)$, one finds the comultiplication and the antipode on basis elements:

$$\Delta(\partial_n^{e_n} \cdots \partial_1^{e_1}) = \sum_{j_n=0}^{e_n} \cdots \sum_{j_1=0}^{e_1} \left(\prod_{j=1}^{n} \begin{bmatrix} e_i \\ j_i \end{bmatrix}_{q^2} \right) q^{\sum_u J^u (e_u - j_u)} \partial_n^{e_n - j_n} \cdots \partial_1^{e_1 - j_1} \otimes \partial_n^{j_n} \cdots \partial_1^{j_1}$$

$$S(\partial_n^{e_n}\cdots\partial_1^{e_1}) = q^{|E|^2-|E|}(-1)^{|E|}\partial_n^{e_n}\cdots\partial_1^{e_1}$$

$A(R)$ can either be quotiented to the algebra of functions on quantum SL_n or localized at the quantum determinant in order to obtain the algebra of functions on quantum GL_n, and both are Hopf algebras. Hence by Majid's theory (see Corollary 9.2.14 and Proposition 10.3.6 in [Ma]) we recover the braiding between $\hat{V}(Q,\,R)$ and $V(Q,\,R)$ and between $V(Q,\,R)$ and $\hat{V}(Q,\,R)$. They are given by $\Phi(\partial_i\otimes x_j) = \sum_{r,s} x_r\otimes\partial_s(R^{-1})^{ir}_{sj}$ and $\Phi(x_i\otimes\partial_j) = \sum_{r,s}(\tilde{R})^{rj}_{is}\partial_s\otimes x_r$. By the simplicity of the case A_n, one can also check directly that these define braidings such that the two braided tensor products are again braided algebras (i.e. algebras with a braiding with the necessary compatibility). We compute R^{-1} and $\tilde{R}$ and we see that:

$$\Phi_{V(Q,\,R),\hat{V}(Q,\,R)}(\partial_i\otimes x_j) = \begin{cases} q^{-1}x_j\otimes\partial_i & \text{if } i\neq j\\ q^{-2}x_j\otimes\partial_j + \sum_{r>j}(q^{-2}-1)x_r\otimes\partial_r & \text{if } i=j\end{cases}$$

and

$$\Phi_{\hat{V}(Q,\,R),V(Q,\,R)}(x_i\otimes\partial_j) = \begin{cases} q^{-1}\partial_j\otimes x_i & \text{if } i\neq j\\ \sum_{r<j}(q^{-2}-1)q^{-2(j-r)}\partial_r\otimes x_r + q^{-2}\partial_j\otimes x_j & \text{if } i=j\end{cases}$$

We observe that for the algebra isomorphism ψ described above, with $c_j = q^{-j+\frac{1}{2}(n-1)}$ we have:

$$(\psi^{-1}\otimes\psi)\circ\Phi_{\hat{V}(Q,\,R),V(Q,\,R)} = \Phi_{V(Q,\,R),\hat{V}(Q,\,R)}\circ(\psi\otimes\psi^{-1})$$

and

$$(\psi\otimes\psi^{-1})\circ\Phi_{V(Q,\,R),\hat{V}(Q,\,R)} = \Phi_{\hat{V}(Q,\,R),V(Q,\,R)}\circ(\psi^{-1}\otimes\psi)$$

However, ψ cannot be considered as a morphism in the braided category since it is *not true* that $(\psi\otimes\mathrm{id})\circ\Phi_{\hat{V}(Q,\,R),V(Q,\,R)} = \Phi_{\hat{V}(Q,\,R),\hat{V}(Q,\,R)}\circ(\mathrm{id}\otimes\psi)$, as one can easily see by computing lthe actions of the left hand side and of the right hand side on $(\partial_2\otimes x_1)$ for $n=2$.

Majid introduces also a theory of differentiation. Namely, every ∂_j acts on $\hat{V}(Q,\,R)$ as braided partial differentiation. Majid defines this action as a sort of analogue of the linear approximation of a function. Namely, for a basis element $x_1^{e_1}\cdots x_n^{e_n}$, one computes $\Delta(x_1^{e_1}\cdots x_n^{e_n})\in\hat{V}(Q,\,R)\otimes\hat{V}(Q,\,R)$ and one expresses the latter as sum of elements as "basis elements of $\hat{V}(Q,\,R)$ tensor elements of $\hat{V}(Q,\,R)$". Then $\partial_i(x_1^{e_1}\cdots x_n^{e_n})$ is the term appearing as a "coefficient" of $x_i\otimes 1$. One can easily check that this defines an action of $V(Q,\,R)$ on $\hat{V}(Q,\,R)$ and one can compute that for every $f(\underline{x})\in\hat{V}(Q,\,R)$ this turns out to be:

$$\partial_j f(\underline{x}) = x_j^{-1}\left[\frac{f(q^2x_1,\,\ldots,\,q^2x_j,\,x_{j+1},\ldots,\,x_n) - f(q^2x_1,\,\ldots,\,q^2x_{j-1},\,x_j,\,x_{j+1},\ldots\,x_n)}{(q^2-1)}\right]$$

where the inverse of x_j is only formal, and "apparent". However, if the reader doesn't like this, he can work in the Ore localization of $\hat{V}(Q,\,R)$ at x_j, since the set $\{x_j^k\mid k\geq 0\}$ satisfies Ore conditions.

In particular, one has $\partial_j(x_1^{e_1}\cdots x_n^{e_n}) = [e_j]_{q^2} q^{E_j} x_1^{e_1}\cdots x_{j-1}^{e_{j-1}} x_j^{e_j-1} x_{j+1}^{e_{j+1}}\cdots x_n^{e_n}$.

Formally we can work also with $\hat{V}(Q,\,R)^{\mathrm{ext}}$ (respectively $V(Q,\,R)^{\mathrm{ext}}$), the algebra of formal power series in the x_j's (respectively ∂_j's) with the given defining relations. In this case everything that we have described above is defined as in $\hat{V}(Q,\,R)$ and $V(Q,\,R)$.

3. The exponential map, e_q, and E_q

As in [Ma], we introduce the exponential map as a coevaluation for $\hat{V}(Q,\,R)$ and $V(Q,\,R)$. We say in a few lines how this works.
There is a braided Hopf algebra pairing $<\,,\,>: \hat{V}(Q,\,R)\otimes V(Q,\,R)\to K$ where K denotes the field where the braided Hopf algebras are defined, given by

$$< g(\partial_n,\,\ldots,\,\partial_1),\, f(x_1,\,\ldots,\,x_n) >= \varepsilon(g(\partial_n,\,\ldots,\,\partial_1) f(x_1,\,\ldots,\,x_n))$$

(see [Ma] for further details). The dual basis to the one given by $x_1^{e_1}\cdots x_n^{e_n}$ is then given by $(x_1^{e_1}\cdots x_n^{e_n})^* := \frac{\partial_n^{e_n}}{[e_n]_{q^2}!}\cdots\frac{\partial_1^{e_1}}{[e_1]_{q^2}!}$. The coevaluation map $\exp\colon K\to(\hat{V}(Q,\,R)\otimes V(Q,\,R))^{\mathrm{ext}}$ is uniquely determined by the image $\exp(x|\,\partial)$ of 1 in a formal extension to power series of $\hat{V}(Q,\,R)\otimes V(Q,\,R)$. The element $\exp(x|\,\partial)$ is defined as the canonical element $\sum_{E,F} c_{EF} x_1^{e_1}\cdots x_n^{e_n}\otimes\partial_n^{f_n}\cdots\partial_1^{f_1}$ such that for every $f(\underline{x})$ in $\hat{V}(Q,\,R)$ and every $g(\underline{\partial})$ in $V(Q,\,R)$, one has:

$$f(\underline{x}) = \sum_{E,F} c_{EF} x_1^{e_1}\cdots x_n^{e_n} < \partial_n^{f_n}\cdots\partial_1^{f_1},\, f(\underline{x}) > \qquad \text{and}$$
$$g(\underline{\partial}) = \sum_{E,F} c_{EF} < g(\underline{\partial}),\, x_1^{e_1}\cdots x_n^{e_n} > \partial_n^{f_n}\cdots\partial_1^{f_1}$$

Hence it is equal to

$$\sum_{e_1,\,\ldots,\,e_n\geq 0} x_1^{e_1}\cdots x_n^{e_n}\otimes(x_1^{e_1}\cdots x_n^{e_n})^* = \sum_{e_1,\,\ldots,\,e_n\geq 0} x_1^{e_1}\cdots x_n^{e_n}\otimes\frac{\partial_n^{e_n}}{[e_n]_{q^2}!}\cdots\frac{\partial_1^{e_1}}{[e_1]_{q^2}!}$$

By Example 10.4.16 in [Ma] this is equal to $e_{q^{-2}}((1-q^{-2})\sum_{i=1}^n x_i\otimes\partial_i)$ where $e_q(z) = \sum_{k=0}^\infty\frac{z^k}{(q;q)_k}$ (see [Koo] for further details). It follows by straightforward computation that $\exp(x\,|\,\partial)$ is also equal to $E_{q^2}((1-q^2)\sum_{i=1}^n x_i\otimes\partial_i)$, where $E_{q^2}(z) = \sum_{k=0}^\infty\frac{q^{\frac{1}{2}k(k-1)}z^k}{(q;q)_k}$. Properties analogous to those of the classical exponential map hold for exp as a consequence of the fact that it is a canonical element arising from a braided Hopf algebra pairing. In our case the properties we need can be also directly checked knowing the properties of $e_q(z)$ and $E_q(z)$. Corollary 10.4.17 in [Ma] gives us also a braided version of Taylor formula, which can be constructed also easily directly in the A_n case. This is given by

$$
\begin{aligned}
&\Delta(f(x_1, \dots, x_n)) \\
&\qquad = f(\Delta(x_1), \dots, \Delta(x_n)) = f(x_1 + y_1, \dots, x_n + y_n) \\
&\qquad = \sum_{e_1, \dots, e_n \geq 0} x_1^{e_1} \cdots x_n^{e_n} \otimes \left(\frac{\partial_n^{e_n}}{[e_n]_{q^2}!} \cdots \frac{\partial_1^{e_1}}{[e_1]_{q^2}!} f(x_1, \dots, x_n) \right) \\
&\qquad = \sum_{e_1, \dots, e_n \geq 0} y_1^{e_1} \cdots y_n^{e_n} \left(\frac{\partial_n^{e_n}}{[e_n]_{q^2}!} \cdots \frac{\partial_1^{e_1}}{[e_1]_{q^2}!} f(x_1, \dots, x_n) \right) \\
&\qquad = \exp(y \mid \partial) f(\underline{x})
\end{aligned}
$$

where $y_j = x_j \otimes 1$ and x_i stands for $1 \otimes x_i$ after the second equality sign. In the first equality we made an abuse of notation identifying the power series f with the series of its coefficients.

4. Integration on $\hat{V}(Q, R)^{\mathrm{ext}}$

In [KeMa] a theory of integration on braided covector and braided vector Hopf algebras is introduced. The authors work out a few particular cases in more detail. Integration on the braided line was futher worked out in [Koo] who proved, by means of finding suitable representations, the left invariance of the integral in an analytical way. The main tool was finding a representation for (an extension of) the algebra of "quantum coordinates" on the braided line, and a common eigenvector for the action. In his case, an eigenvector for the integral of an element of the braided line would have a Jackson integral in commuting variables as eigenvalue. Therefore, convergence and equalities were investigated on the eigenvalues. This is no longer possible if the dimension of the braided space is bigger than one, because noncommuting operators cannot have a common eigenvector. We have to work a little bit more to give a meaning to our version of the integral. In many cases we see that the concept of integrability may depend on the choice of the representation we choose. This is equivalent to the choice of a normal form in our case.
We start with the "indefinite" integral with respect to x_i. As in [KeMa], we view it as an operator on $\hat{V}(Q, R)$.

Definition. *The braided partial integral with respect to x_i acting on $f(x_1, \dots x_n) \in \hat{V}(Q, R)^{\mathrm{ext}}$ is given by:*

$$
\int_0^{x_i} f := (1 - q^2) \sum_{k=0}^{\infty} q^{2k} x_i f(q^{-2} x_1, \dots q^{-2} x_{i-1}, q^{2k} x_i, x_{i+1}, \dots x_n)
$$

It's easy to see that the operator defined above acts as a pseudo inverse for the partial differential operator ∂_i. It is indeed only a right inverse, since it acts as a left inverse for ∂_i only on series containing x_i in every monomial of its expansion (see remark in [KeMa], page 6815). Each $\int_0^{x_i}$ is a well defined operator from $\hat{V}(Q, R)^{\mathrm{ext}}$ to $\hat{V}(Q, R)^{\mathrm{ext}}$ since for every basis monomial $x_1^{e_1} \cdots x_n^{e_n}$ we can write $\int_0^{x_i} x_1^{e_1} \cdots x_n^{e_n}$ as a monomial in the x_j's with a coefficient that is a *convergent* series of complex numbers for $0 < q < 1$.

Following [KeMa] one could write this operator as a power series, namely:

$$(1-q^2)x_i\sum_{k=0}^{\infty}(1-(1-q^2)\partial_i x_i)^k$$

which makes sense because the usual basis is a basis of eigenvectors, with convergent eigenvalues.
We can read $\int_0^{x_i} f$ as a "function" of the x_j's, hence it makes sense to consider, for instance $\int_0^{a x_i}$ for a nonzero constant a. In particular we can define $\int_0^{-x_i} f$ and $\int_0^{q^l x_i} f$ as follows.

$$\int_0^{-x_i} f := -(1-q^2)\sum_{k=0}^{\infty} q^{2k} x_i f(q^{-2}x_1, \ldots q^{-2}x_{i-1}, -q^{2k}x_i, x_{i+1}, \ldots x_n)$$

$$\int_0^{q^l x_i} f := (1-q^2)\sum_{k=0}^{\infty} q^{2k+l} x_i f(q^{-2}x_1, \ldots q^{-2}x_{i-1}, q^{2k+l}x_i, x_{i+1}, \ldots x_n)$$

We define then for every $f \in \hat{V}(Q, R)^{\mathrm{ext}}$

$$\int_{-x_i}^{x_i} f := \int_0^{x_i} f - \int_0^{-x_i} f$$

and

$$\int_{-x_i\cdot\infty}^{x_i\cdot\infty} f := \lim_{r\to\infty}\int_{-q^{-2r}x_i}^{q^{-2r}x_i} f$$

The last definition is only formal so far, because the image of a power series is no longer a power series (coefficients might be infinite sums themselves), and we have no notion of convergence. Indeed we cannot find a convergence set because we cannot give values to the variables, since they do not commute. This issue can be solved in different ways, so that we can give a meaning to equalities as well. The ideas here are based on the apporaches of Kempf and Majid, and of Koornwinder. We approach the problem not for a single infinite integral, but for the $n-$dimensional integral:

$$I \cdot f := \int_{-x_n\cdot\infty}^{x_n\cdot\infty} \cdots \int_{-x_1\cdot\infty}^{x_1\cdot\infty} f =$$

formally

$$(1-q^2)^n \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \sum_{\varepsilon\in\{\pm1\}^n} q^{2|K|+\binom{n}{2}} x_1 \cdots x_n\, f(\varepsilon_1 q^{2k_1}x_1, \ldots, \varepsilon_n q^{2k_n}x_n)$$

As we said $I \cdot f$ is no longer an element of $\hat{V}(Q, R)$, since the coefficients with respect to the elements of the basis are not always definite. We fix then an action of $\hat{V}(Q, R)$ on the space of power series in the n commuting variables $z_1, \ldots, z_n$ with complex coefficients. This representation corresponds with the choice of a normal form for the monomials in $\hat{V}(Q, R)$. The representation, denoted by $\triangleright$, for monomials in the x_i's acting on monomials in the z_j's, is given by:

$$x_1^{e_1} \cdots x_n^{e_n} \triangleright z_1^{h_1} \cdots z_n^{h_n} = q^{\sum_{i=1}^n e_i H_i} z_1^{h_1+e_1} \cdots z_n^{h_n+e_n}$$

and can be extended linearly to an action of $\hat{V}(Q, R)$ on formal power series in the z_j's. We can restrict the space on which we act by taking the space V of power series which are absolutely convergent in a neighbourhood of zero. This makes sense because the z_i's commute with each other. We see that this space is invariant under the action of $\hat{V}(Q, R)$. Moreover we see that we can extend the representation of $\hat{V}(Q, R)$ on V to a representation of the class C given by the power series f in the x_i's such that $f \triangleright 1 \in V$. Indeed one can see that

(1) $\forall\, f = f(x_1, \ldots, x_n) \in C$ and $\forall\, g = g(z_1, \ldots, z_n) \in V$ it holds that $(f \triangleright g)(z_1, \ldots, z_n)$ belongs to V because the associated series of absolute values is majorized by the product in V of series of absolute values associated to $f \triangleright 1$ and g.

(2) $\forall\, f$ and $g \in C$, their product $fg \in C$ because $(fg) \triangleright 1 = f \triangleright (g \triangleright 1) \in V$ by (1)

Moreover,

(3) $\forall f \in C, \quad f \triangleright 1 = 0 \Leftrightarrow f \equiv 0 \quad$ i.e. all the coefficients of f have to be zero.

From now on we write $f_{\cdot 1}$ for $f \triangleright 1$, for any expression $f(\underline{x})$ for which the action on 1 makes sense.

We would like to extend the representation now to the formal expressions of type $I \cdot f$ for $f \in C$. This does not always make sense, hence we have to add further conditions. Let us take the subclass C' of C given by the series in C such that

(a) $f_{\cdot 1}$ can be continued analytically on $\mathbf{R}^n + iU$ for some open neighboorhood U of 0 in $\mathbf{R}^n$;

(b) $f_{\cdot 1}$ is absolutely q^2-integrable for every $\underline{z} \in \mathbf{R}^n$ for which every $z_j \neq 0$, i.e.

$$(1-q^2)^n \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \sum_{\varepsilon \in \{\pm 1\}^n} q^{2|K|} |z_1| \cdots |z_n| \, |f_{\cdot 1}(\varepsilon_1 q^{2k_1} z_1, \ldots, \varepsilon_n q^{2k_n} z_n)| < \infty$$

for $\underline{z}$ outside the standard hyperplanes.

Then we can say, for $f \in C'$ what is $(I \cdot f)_{\cdot 1}$:

$$
\begin{aligned}
&(I \cdot f)_{\cdot 1} = \\
&= \left[\int_{-x_n \cdot \infty}^{x_n \cdot \infty} \cdots \int_{-x_1 \cdot \infty}^{x_1 \cdot \infty} f \right] \triangleright 1 \\
&= \left[(1-q^2)^n \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \sum_{\varepsilon \in \{\pm 1\}^n} q^{2K + \binom{n}{2}} x_1 \cdots x_n \, f\left(\varepsilon_1 q^{2k_1} x_1, \ldots, \varepsilon_n q^{2k_n} x_n\right) \right] \triangleright 1
\end{aligned}
$$

By expressing every term in the x_i's as a sum of elements of the basis (with in principle infinite sums as coefficients) and applying the action, we see that the coefficients make sense and that

$$
\begin{aligned}
&(I \cdot f)_{\cdot 1} = \\
&= q^{\binom{n}{2}} \int_{-z_1\cdot\infty}^{z_1\cdot\infty} \cdots \int_{-z_n\cdot\infty}^{z_n\cdot\infty} f_{\cdot 1}(q^{n-1}t_1, \ldots, q^{(n-i)}t_i, \ldots, t_n) d_{q^2}t_n \cdots d_{q^2}t_1 = \\
&= \int_{-q^{n-1}z_1\cdot\infty}^{q^{n-1}z_1\cdot\infty} \cdots \int_{-q^{n-i}z_i\cdot\infty}^{q^{n-i}z_i\cdot\infty} \cdots \int_{-z_n\cdot\infty}^{z_n\cdot\infty} f_{\cdot 1}(t_1, \ldots, t_n) d_{q^2}t_n \cdots d_{q^2}t_1
\end{aligned}
$$

where the q^2-integral in the second line is the q^2-Jackson integral in n variables obtained by iterating (8.11) in [Koo]. It is clear that if $f \in C'$ then $(I \cdot f)_{\cdot 1}$ converges whenever $z_j \neq 0$ for every j.

Because of (3), two objects in C are equal if and only if they act in the same way on 1. Following this philosophy, we say that two q^2-integrals of objects in C' are equal if and only if they act on the same way on 1. This will be our tool to show equalities then.

The first purpose is to show in a less formal way than in [KeMa] where this appeared first, the invariance of the operator I under translation. As in [KeMa] and in [Koo], we make use of the Taylor formula. For this we need an extra assumption on the elements in C', since we have to use partial derivatives. We consider $f \in C'$ satisfying:

(c) For some $\eta > 0$ there exists for each $J \in (\mathbf{Z}_{\geq 0})^n$ some constant C_J such that

$$
|(D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f_{\cdot 1})(z_1, \ldots, z_n)| \leq C_J \prod_{k=1}^{n} (1 + |z_k|^2)^{-(1+\eta)}
$$

if $\underline{z} \in \mathbf{R}^n$.

For an f in C', condition (c) implies that all the Jackson derivatives of $f_{\cdot 1}$ are absolutely q^2-integrable for all $\underline{z}$ in a neighbourhood of 0 minus the intersection with the standard hyperplanes.

One sees immediately that $(\partial_i f)_{\cdot 1} = (D_{i,q^2} f_{\cdot 1})(qz_1, \ldots, qz_{i-1}, z_i \ldots, z_n)$.

We show that it makes sense to compute $I(\partial_n^{j_n} \cdots \partial_1^{j_1} f) \triangleright 1$, and that this is equal to zero whenever $(j_1, \ldots, j_n) \neq (0, \ldots, 0)$.

We write: $F_{\cdot 1}^J(z_1, \ldots, z_n) := (\partial_n^{j_n} \cdots \partial_1^{j_1} f) \triangleright 1 = (D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n}(f_{\cdot 1}))(q^{J^1} z_1, \ldots, q^{J^n} z_n)$.

One sees that $F_{\cdot 1}^J \in V$ if f satisfies condition (c). As far as q^2-integrability is concerned: $F_{\cdot 1}^J$ is absolutely q^2- Jackson integrable if and only if for every choice of $(h_1, \ldots, h_n) \in \{\pm 1\}^n$

$$
\sum_{\underline{\varepsilon} \in \{\pm 1\}^n} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} q^{2K\cdot H} |z_1 \cdots z_n| \, |F_{\cdot 1}^J(q^{2k_1h_1}\varepsilon_1 z_1, \ldots, q^{2k_nh_n}\varepsilon_n z_n)|
$$

has a positive radius of convergence. Hence, if condition (c) holds for $f(\underline{x})$, and since the above sums are of the form:

$$
\begin{aligned}
&\sum_{\underline{\varepsilon}} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} q^{2K\cdot H} |z_1 \cdots z_n \, (D_{1,q^2}^{j_1} \cdots D_{n,q^2}^{j_n} f_{\cdot 1})(q^{2k_1h_1+J^1}\varepsilon_1 z_1, \ldots, q^{2k_nh_n+J^n}\varepsilon_n z_n)| \\
&\leq C_J |z_1 \cdots z_n| \sum_{\underline{\varepsilon}} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} q^{2K\cdot H} \prod_{r=1}^{n} (1 + |q^{2k_rh_r} z_r|^2)^{-(1+\eta)}
\end{aligned}
$$

that converges since $0 < q < 1$, one sees that the $F^J_{\cdot 1}$ are q^2-integrable.
Moreover for $\gamma = (\gamma_1, \ldots, \gamma_n)$ with γ_j in $\mathbf{R} - \{0\}$ we have

$$q^{\sum_{k=1}^n J^k} \int_{-\gamma_1\cdot\infty}^{\gamma_1\cdot\infty} \cdots \int_{-\gamma_n\cdot\infty}^{\gamma_n\cdot\infty} |F^J_{\cdot 1}(t_1, \ldots, t_n)| d_{q^2}t_n \cdots d_{q^2}t_1$$
$$= \int_{-q^{J^1}\gamma_1\cdot\infty}^{q^{J^1}\gamma_1\cdot\infty} \cdots \int_{-q^{J^n}\gamma_n\cdot\infty}^{q^{J^n}\gamma_n\cdot\infty} |D^{j_1}_{1,q^2} \cdots D^{j_n}_{n,q^2} f_{\cdot 1}(t_1, \ldots, t_n)| d_{q^2}t_n \cdots d_{q^2}t_1 = 0$$

The proof is as in the one-dimensional case (see [Koo]).
Hence we can state that

Lemma 4-1. *Let $f \in C'$ satisfy condition* (c). *Then for every $J \neq \underline{0}$ there holds: $(I(\partial_n^{j_n} \cdots \partial_1^{j_1} f)) \triangleright 1 \equiv 0$, so that we can conclude that $I(\partial_n^{j_n} \cdots \partial_1^{j_1} f) = 0$.*

Proof: One has

$$(I(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{\cdot 1} =$$
$$= \int_{-q^{n-1}z_1\cdot\infty}^{q^{n-1}z_1\cdot\infty} \cdots \int_{-q^{n-i}z_i\cdot\infty}^{q^{n-i}z_i\cdot\infty} \cdots \int_{-z_n\cdot\infty}^{z_n\cdot\infty} F^J_{\cdot 1}(t_1, \ldots, t_n) d_{q^2}t_n \cdots d_{q^2}t_1 = 0 \qquad \square$$

Proposition 4-2. *Let $f \in C'$ satisfy* (c). *Then $(id \otimes I)\Delta(f) = 1 \otimes (If)$.*

Proof: By the braided Taylor formula we have

$$(id \otimes I)(\Delta f) = \sum_{j_1, \ldots, j_n \geq 0} \frac{y_1^{j_1} \cdots y_n^{j_n}}{[j_1]_{q^2}! \cdots [j_n]_{q^2}!} I(\partial_n^{j_n} \cdots \partial_1^{j_1} f)$$

but this is by Lemma 4-1 equal to the term with all j_k's equal to 0. $\square$.

Proposition 4-3. *Let $f(\underline{x}) \in C$ such that* (c) *holds for every $\eta > 0$. Then the statement of Proposition 4-2 holds for every polynomial $p(\underline{x})$ times $f(\underline{x})$.*

Proof: It is not restrictive to assume that $p(\underline{x})$ is a monomial. By (1) it follows that also $p(\underline{x}) \cdot f(\underline{x}) \in C$. We have to check that condition (c) holds for every element of the form $x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})$. One sees immediately that

$$(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})) \triangleright 1 = f_{\cdot 1}(q^{E^1} z_1, \ldots, q^{E^n} z_n) z_1^{e_1} \cdots z_n^{e_n} \in V$$

if $f(\underline{x}) \in C$, so that $(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x}))_{\cdot 1}$ makes sense.
Condition (c) is on the q^2-Jackson partial derivatives on commuting variables, and it holds for $x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})$ as a consequence of the fact that for two functions $a(\underline{z})$ and $b(\underline{z})$ and for any $j = 1, \ldots, n$:

$$D_{j,q^2}(a(\underline{z})b(\underline{z})) = (D_{j,q^2}a(\underline{z}))(b(z_1, \ldots, z_{j-1}, q^2 z_j, z_{j+1}, \ldots, z_n)) + a(\underline{z})D_{j,q^2}(b(\underline{z}))$$

Then, by Proposition 4-2 we have the statement. $\square$
We have a description of a class of power series in the x_i's for which integration makes sense, although we are not able so far to make a complete classification of integrable functions. The same problem is treated in [OR] for the one-dimensional case. Still, what we have is enough to allow computations in the following case.

Example 1: The q^2-Gaussian $g_{q^2}(\underline{x})$ is

$$g_{q^2}(\underline{x}) := e_{q^4}(-\underline{x}\cdot\underline{x}) = e_{q^4}(-\sum_{j=1}^{n} x_j^2) = \prod_{j=1}^{n} e_{q^4}(-x_j^2)$$

where the last equality holds because of Proposition 3.1 in [Koo] (this result is due to [Sch]) and the product in the above formula is taken with increasing order on the variables. It satisfies conditions (a), (b) and (c) for every $\eta > 0$. This is a consequence of the one dimensional case (see [Koo]) and the fact that

$$(g_{q^2}(\underline{x}))_{\cdot 1}(\underline{z}) = \prod_{j=1}^{n} e_{q^4}(-z_j^2) \qquad \partial_j(g_{q^2}(\underline{x})) = -\frac{x_j}{(1-q^2)} g_{q^2}(\underline{x})$$

and that

$$D^{j_1}_{q^2,1}\cdots D^{j_n}_{q^2,n}(e_{q^4}(-\sum_{k=1}^{n} z_k^2)) = p(\underline{z})\, e_{q^4}(-\sum_{k=1}^{n} z_k^2)$$

where $p(\underline{z})$ is a polynomial in the z_j's.
It follows then that also elements of the form:

$$x_1^{a_1} g_{q^2}(x_1)\cdots x_n^{a_n} g_{q^2}(x_n) = x_1^{a_1}\cdots x_n^{a_n}\, e_{q^4}(-\sum_j (q^{-A^j} x_j)^2)$$

satisfy condition (c), so that for every $p_j(x_j) \in \hat{V}(Q,\, R)$ we can integrate every element of the form $p_1(x_1)g_{q^2}(x_1)\cdots p_n(x_n)g_{q^2}(x_n)$.
In particular, for $f(\underline{x}) = g_{q^2}(x_1)x_1^{a_1}\cdots g_{q^2}(x_n)x_n^{a_n}$ one can compute

$$\begin{aligned}
&(I\cdot f)_{\cdot 1}|_{\underline{z}=\gamma}\\
&= \int_{-q^{n-1}\gamma_1\cdot\infty}^{q^{n-1}\gamma_1\cdot\infty}\cdots\int_{-q^{n-i}\gamma_i\cdot\infty}^{q^{n-i}\gamma_i\cdot\infty}\cdots\int_{-\gamma_n\cdot\infty}^{\gamma_n\cdot\infty} f_{\cdot 1}(t_1,\,\ldots,\,t_n)d_{q^2}t_n\cdots d_{q^2}t_1\\
&= \prod_{j=1}^{n}\left(\int_{-q^{n-j}\gamma_j\cdot\infty}^{q^{n-j}\gamma_j\cdot\infty} e_{q^4}(-t_j^2)t_j^{a_j}\, d_{q^2}t_j\right)\\
&= \begin{cases} \prod_{j=1}^{n}\left(c_{q^2}(\gamma_j q^{n-j})q^{-\frac{a_j^2}{2}}(q^2;q^4)_{\frac{a_j}{2}}\right) & \text{if } a_j \text{ even } \forall j\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

where

$$c_{q^2}(\gamma) = \frac{2(1-q^2)(q^4,\, -q^2\gamma^2,\, -q^2\gamma^{-2};\, q^4)_\infty}{(-\gamma^2,\, -q^4\gamma^{-2},\, q^2;\, q^4)_\infty}$$

as in formula (8.15) in [Koo]. In particular

$$(I\cdot f)_{.1}|_{\underline{z}=\gamma} = q^{-\sum_j a_j^2/2}\prod_j (q^2;q^4)_{a_j/2}(I\cdot g_{q^2}(\underline{x}))|_{\underline{z}=\gamma}$$

if all a_j's are even, and 0 otherwise, hence we can conclude that:

$$I \cdot f = \begin{cases} q^{-\sum_j a_j^2/2} \prod_j (q^2; q^4)_{a_j/2} \big(I \cdot g_{q^2}(\underline{x})\big) & \text{if } a_j \text{ is even } \forall j \\ 0 & \text{otherwise} \end{cases}$$

The observation that the integral of the Gaussian times a monomial is equal to a constant times the integral of the Gaussian appeared in [KeMa] and in [Ma] first. In our case we have to deal with the shift and this depends on the choice of our global integral. ♠

5. Lattice integrability

In the previous Section we saw a definition of integrable series in $\hat{V}(Q,\ R)^{\text{ext}}$. Unfortunately the above method fails for another analogue of the Gaussian we would like to q^2-integrate, namely the q^2-Gaussian

$$G_{q^2}(\underline{x}) := E_{q^4}(-\underline{x} \cdot \underline{x}) = E_{q^4}(-\sum_{j=1}^{n} x_j^2) = E_{q^4}(-x_n^2) \cdots E_{q^4}(-x_1^2)$$

where the last equality holds because of the result in [Sch] to be found in [Koo], Proposition 3.1. In [Koo], section 9, it is also shown that $G_{q^2}(\underline{x})$ does not satisfy condition (c), nor condition (b) even for $n = 1$. On the other hand it is also shown there that for a given choice of a q^2-lattice of the form $\{q^{2k}\gamma \mid k \in \mathbf{Z}\}$, namely for $\gamma = 1$, $(I \cdot G_{q^2}(x_1))_{\cdot 1} \mid_{z_1=1}$ is absolutely convergent. Hence one can introduce a weaker version of integrability in $\hat{V}(Q,\ R)^{\text{ext}}$, that we will call *"lattice integrability"* requiring for an $f(\underline{x})$ such that $f_{\cdot 1}$ is entire, that there is a q^2-lattice $L(\gamma) = \{\, (\gamma_1 q^{2k_1}, \ldots, \gamma_n q^{2k_n}) \mid K \in \mathbf{Z}^n\}$ in $\mathbf{R}^n_{\neq 0}$ such that $(I \cdot f)_{\cdot 1} \mid_{\underline{z} \in L(\gamma)}$ is absolutely convergent. One can easily see that the series $f(\underline{x}) = E_{q^4}(-x_1^2) \cdots E_{q^4}(-x_n^2)$ is lattice-integrable for $\gamma = (q^{n-1}, \ldots, q^{n-j}, \ldots, 1)$. This f is unfortunately not the q^2-Gaussian $G_{q^2}(\underline{x})$ that we wanted to integrate, for $n \geq 2$. Moreover, we can show that already for $n = 2$, $G_{q^2}(\underline{x})$ is not lattice integrable although it is entire.

(Counter)example 1: Let us consider $G_{q^2}(\underline{x})$ for $n = 2$. We write for simplicity $x_1 = x$ and $x_2 = y$, and $z_1 = z$, $z_2 = w$. Then

$$G_{q^2}(\underline{x}) = \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} q^{2k^2+2l^2-2k-2l} y^{2l} x^{2k}}{(q^4; q^4)_l \, (q^4; q^4)_k} = \sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} E_{q^4}(-(q^{2l}x)^2) y^{2l}}{(q^4; q^4)_l}$$

hence

$$G_{q^2}(\underline{x})_{\cdot 1} = \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} q^{2k^2+2l^2-2k-2l} q^{4kl} z^{2k} w^{2l}}{(q^4; q^4)_l \, (q^4; q^4)_k} =$$

$$\sum_{l=0}^{\infty} \frac{(-1)^l q^{2l^2-2l} E_{q^4}(-(q^{2l}z)^2) w^{2l}}{(q^4; q^4)_l} = \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k^2-2k} E_{q^4}(-(q^{2k}w)^2) z^{2l}}{(q^4; q^4)_k}$$

which is entire since it is majorized by $E_{q^4}(|w|^2)E_{q^4}(|z|^2)$. Now we wonder whether this expression is lattice-integrable or not. In order to have that we would need that for some $\gamma=(\gamma_1,\,\gamma_2)$

$$\int_{-q\gamma_1\cdot\infty}^{q\gamma_1\cdot\infty}\int_{-\gamma_2\cdot\infty}^{\gamma_2\cdot\infty}(G_{q^2})_{\cdot 1}(t_1,\,t_2)d_{q^2}\underline{t}$$

is absolutely convergent. For this we would need that

$$\sum_{h_1=-\infty}^{\infty}\sum_{h_2=-\infty}^{\infty}q^{2|H|}\Big|\sum_{l=0}^{\infty}\frac{(-1)^l q^{2l^2-2l}E_{q^4}(-(q^{2l+2h_1}q\gamma_1)^2)(q^{2h_2}\gamma_2)^{2l}}{(q^4;q^4)_l}\Big|<\infty$$

therefore we have to look at the limit for $h_j\to-\infty$ of the summands. Clearly by the discussion in section 9 of [Koo], we see that we would need to have $\gamma_1=q$. With a similar reasoning we see that γ_2 must be equal to 1. Now, for general z and w we have:

$$(G_{q^2}(\underline{x})_{\cdot 1})(z,\,w)=\sum_{l=0}^{\infty}\frac{(-1)^l q^{2l^2-2l}w^{2l}(q^{4l}z^2;q^4)}{(q^4;q^4)_l}=$$
$$(z^2;q^4)_\infty\sum_{l=0}^{\infty}\frac{(-1)^l q^{2l^2-2l}w^{2l}}{(q^4;q^4)_l(z^2;q^4)_l}=(z^2;q^4)_\infty\,{}_1\phi_1(0;z^2;q^4,w^2)$$

which is the q^4 version of the q-Bessel function described in [KooSw]. For $(z,\,w)=(q^{2-2r},\,q^{2s})$ with $r\geq 0$ and s any integer, we have, by the extimates (2.6) and the following extimates in [KooSw], that

$$|(G_{q^2}(\underline{x})_{\cdot 1})(q^{2-2r},\,q^{2s})|=q^{2r(r-1)}q^{4rs}(q^{4r+4};q^4)_\infty|\,{}_1\phi_1(0;q^{4r+4};q^4,q^{4r+4s})|$$

For $r\to\infty$ and $s=-r$ this behaves like $q^{2r(r-1)-4r^2}\to\infty$. Hence $G_{q^2}(\underline{x})$ is not lattice integrable. ♠

Remarks: An analogue of the symmetry (2.2) for q-Bessel functions in [KooSw] holds in our case, namely:

$$E_{q^4}(-x_1^2)\,{}_1\phi_1(0;x_1^2;q^4,x_2^2)=G_{q^2}(\underline{x})=E_{q^4}(-x_2^2)E_{q^4}(-x_1^2)=\,{}_1\phi_1(0;x_2^2;q^4,x_1^2)E_{q^4}(-x_2^2)$$

once we agree that in ${}_1\phi_1$ every time we have a product of type $x_2^l/(x_1^2;q^4)_l$, the terms in x_1 have to be taken *before* the terms in x_2. Hence in general one has

$$E_q(-x_1-x_2)=E_q(-x_2)E_q(-x_1)=E_q(-x_1)\,{}_1\phi_1(0;x_1;q,x_2)=\,{}_1\phi_1(0;x_2;q,x_1)E_q(-x_2)$$

with the above meaning for ${}_1\phi_1$ in noncommuting variables.
Another equality involving a ${}_1\phi_1$ and exponentials in q-commuting variables is obtained by writing $E_q(-x_1)E_q(-x_2)$ as

$$(x_2;q)_\infty\sum_{l=0}^{\infty}\frac{(-1)^l q^{\frac{1}{2}(l^2-l)}(q^{-l}x_2;q)_l x_1^l}{(q;q)_l}$$

and using (3.12) in [Koo] with $x_1 = -y$ and $x_2 = -x$. Then one obtains

$$\sum_{l=0}^{\infty} \frac{(-1)^l q^{\frac{1}{2}(l^2-l)} x_1^l (x_2; q)_l}{(q; q)_l} = E_q(x_1 x_2) E_q(-x_1)$$

where the sum on the left hand side can be considered as a ${}_1\phi_1$ in noncommuting variables once assumed that x_1 always precedes x_2 in products. These facts were pointed to me by T. Koornwinder. ♠

Of course if a generalized function is q^2-integrable then it is lattice integrable for every choice of a lattice. One can easiliy check that for every $A = (a_1, \ldots, a_n)$ in $\mathbf{R}^n_{>0}$ and every $E = (e_1, \ldots, e_n)$ the formal power series $x_1^{e_1} \cdots x_n^{e_n} E_{q^4}(-(a_1 x_1)^2) \cdots E_{q^4}(-(a_n x_n)^2)$ is lattice-integrable in the q^2-lattice generated by $(a_1^{-1} q^{n-1+E^1}, \ldots, a_j^{-1} q^{n-j+E^j}, \ldots, a_n^{-1})$. Unfortunately, lattice integrability carries a lot of technical work with it whenever one wants to prove anything like translation invariance, for instance. This is a consequence of the fact that, in order to state that the integral of $\partial_n^{e_n} \cdots \partial_1^{e_1} f(\underline{x})$ is zero, one needs to keep track of the lattice in which this series is integrable, which in general is not the same as the lattice in which $f(\underline{x})$ is integrable, unless e_j is even for every j. Indeed, consider $n = 2$ and $f(\underline{x}) = E_{q^4}(-x_1^2) E_{q^4}(-x_2^2)$. Then, $f(\underline{x})$ is integrable for $(\gamma_1, \gamma_2) = (q, 1)$ while $\partial_2(f(\underline{x})) = -E_{q^4}(-q^2 x_1^2) \frac{x_2}{(1-q^2)} E_{q^4}(-q^4 x_2^2)$ is integrable for $(\gamma_1, \gamma_2) = (1, 1)$. However, since "morally" the integral of a function which is odd in a variable is zero, we might as well *define* the integral of every odd function to be zero by changing the definition of the integral. Namely, we define the new integral I' to be the integral of the even part of the series $f(\underline{x})$. We formalize this definition now.

Let $f(\underline{x})$ be any formal power series in the x_j's. We want to decompose it in 2^n series depending on the parity with respect to each variable. Let

$$\begin{aligned} \Pi_j^{\pm} \colon \hat{V}(Q, R)^{\text{ext}} &\to \hat{V}(Q, R)^{\text{ext}} \\ f(\underline{x}) &\mapsto \frac{1}{2}(f(\underline{x}) \pm f(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n)) \end{aligned}$$

for every j and for any choice of $\pm$. This makes sense formally, and makes sense even concretely for the series in C. Clearly those operators commute, they are projections on the space of power series that are even (resp. odd) in the j^{th} variable, so that $\Pi_j^+ \Pi_j^- = 0$ for every j. We define then for every choice of β in $\{\pm\}^n$ the operators $\Pi_\beta \colon \hat{V}(Q, R) \to \hat{V}(Q, R)$ as $(\Pi_1^{\beta_1}) \circ \cdots \circ (\Pi_n^{\beta_n})$. They are all projections on their image E_β, and clearly the decomposition of the space of power series in the x_i's descends to a decomposition of the space C in 2^n spaces that we will call C_β. We also write $V^\beta := C_\beta \triangleright 1$. We will denote $\Pi_{(+,\ldots,+)}$ by Π_0 for simplicity.

In particular $\Pi_0 f(\underline{x}) = 2^{-n} \sum_{\varepsilon \in \{\pm 1\}^n} f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$ is even in every variable, and we define the integral I' to be the composition $I \circ \Pi_0$.

Remarks: Since we work in characteristic zero, $I' \cdot f$ is also formally equal to

$$\int_0^{x_n \cdot \infty} \cdots \int_0^{x_1 \cdot \infty} \sum_{\varepsilon \in \{\pm 1\}^n} f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$$

Clearly the class of I' integrable series in bigger than the class of I integrable series, since all odd series are integrable and their integral is zero. Since I' integrability of a series $f(\underline{x})$ coincides with I integrability of $\Pi_0 f(\underline{x})$, if $f(\underline{x})$ is a series which is even in all the variables, then $f(\underline{x})$ is I integrable $\Leftrightarrow$ $f(\underline{x})$ is I' integrable since $f(\underline{x}) = \Pi_0 f(\underline{x})$. ♠

One can also introduce lattice I' integrability. Again, for series in $C_{(+,\ldots,+)}$, lattice I integrability and lattice I' integrability trivially coincide, and for a generic $f(\underline{x})$, lattice I' integrability trivially coincides with lattice I integrability of $\Pi_0 f(\underline{x})$ in the same lattice.

We can provide generalizations of Lemma 4-1, Proposition 4-2 and Proposition 4-3 by introducing condition (c') for an f such that $\Pi_0 f \in C'$:
(c') For some $\eta > 0$, there exists for each $J = (j_1, \ldots, j_n) \in \mathbf{Z}^n_{\geq 0}$ and $\beta \in \{\pm\}^n$ such that j_k is even (resp. odd) if $\beta_k = +$ (resp. $-$), some constant C_J such that

$$|(D^{j_1}_{1,q^2} \cdots D^{j_n}_{n,q^2}(\Pi_\beta f)_{\cdot 1})(z_1, \ldots, z_n)| \leq K_J \prod_{k=1}^{n} (1 + |z_k|^2)^{-(1+\eta)}$$

if $\underline{z} \in \mathbf{R}^n$.
Then we have

Lemma 5-1. *Let $f \in C'$ satisfy condition* (c'). *Then for every $J \neq \underline{0}$ there holds: $(I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f)) \triangleright 1 \equiv 0$, so that we can conclude that $I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f) = 0$.*

Proof: $I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f) = I(\partial_n^{j_n} \cdots \partial_1^{j_1} \Pi_\beta f)$ for β related to J as in condition (c'). □

Proposition 5-2. *Let $f \in C'$ satisfy* (c'). *Then $(id \otimes I')\Delta(f) = 1 \otimes (I' f)$.* □

Proposition 5-3. *Let $f(\underline{x}) \in C$ such that* (c) *holds for every $\eta > 0$. Then the statement of Proposition 4-2 holds for every polynomial $p(\underline{x})$ times $f(\underline{x})$.* □

We also have another invariance property, that is analogous of the classical property (for $n = 1$):

$$\int_{-\infty}^{\infty} \frac{1}{2}(f(x) + f(-x))dx = \int_{-\infty}^{\infty} \frac{1}{2}(f(x+y) + f(x-y))dx$$

Proposition 5-4. *Let $f(\underline{x}) \in C'$ satisfy* (c'). *Then $(id \otimes I)(\Pi_0 \otimes id)\Delta(f) = 1 \otimes (I' f)$. If $f(\underline{x})$ satisfies condition* (c) *for every $\eta > 0$ then the statement is true for every series of the form $x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})$.*

Proof: The proof uses Taylor's formula with summation only on even j_k's. □

Observe that for even j_k's

$$(I(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{\cdot 1} = (I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f))_{\cdot 1} =$$
$$= q^{-\sum_k J^k} \int_{-q^{n-1} z_1 \cdot \infty}^{q^{n-1} z_1 \cdot \infty} \cdots \int_{-z_n \cdot \infty}^{z_n \cdot \infty} D^{j_1}_{1,q^2} \cdots D^{j_n}_{n,q^2} f_{\cdot 1}(t_1, \ldots, t_n) d_{q^2} t_n \cdots d_{q^2} t_1$$

Hence the proposition above is interesting because it can be proved for lattice integrability with simple changes in the hypothesis and in the proof. This reads as follows.

Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{R}^n$. We define the following spaces:

$$C_\gamma = \{ f(\underline{x}) \in \hat{V}(Q, R)^{\mathrm{ext}} \mid f_{\cdot 1}|_{\underline{z}=\gamma} \text{ is absolutely convergent } \}$$

and $C_{q^{2K}\gamma}$ as the space of $f(\underline{x}) \in C_\gamma$ such that $f_{\cdot 1}$ can be continued analytically on a domain containing the q^2-lattice generated by γ. Clearly C_γ is closed with respect to the multiplication, hence it acts on the space V_γ of power series in commuting variables $z_1, \ldots, z_n$ that are absolutely convergent for $z = \gamma$, hence on a polydisc with polyradius $(|\gamma_1|, \ldots, |\gamma_n|)$.
Let $f(\underline{x})$ be a series in $C_{q^{2K}\gamma}$ for a given γ. Then it makes sense to investigate $I'(f(\underline{x}))_{\cdot 1}$ at $z_j = q^{n-j}\gamma_j$ and if this expression is absolutely convergent, then we say that $f(\underline{x})$ is lattice integrable. Actually we would only need $\Pi_0(f) \in C_{q^{2K}\gamma}$ but since we want to compute integrals of products, we keep the restriction on $f(\underline{x})$.
Consider now the lattice version of condition (c):
(c") Let $f(\underline{x}) \in C_{q^{2K}\gamma}$ be such that all Jackson partial derivatives $D^{j_1}_{1,q^2} \cdots D^{j_n}_{n,q^2} f_{\cdot 1}$ with all j_k's even exist on the lattice $L(\gamma)$, and are such that

$$|(D^{j_1}_{1,q^2} \cdots D^{j_n}_{n,q^2}(f_{\cdot 1}))(q^{\pm 2k_1}\gamma_1, \ldots, q^{\pm 2k_n}\gamma_n)| = O(q^{2(1+\eta)K_-})$$

for $k_i \to \infty$, for some $\eta > 0$, where K_- is the sum of the k_j's appearing with the minus sign.
We introduce the equivalence relation $\sim_\gamma$ between two expressions $f(\underline{x})$ and $g(\underline{x})$ belonging to $C_{q^{2K}\gamma}$ as follows:

$$f(\underline{x}) \sim_\gamma g(\underline{x}) \quad \Leftrightarrow \quad f_{\cdot 1}(\underline{z}) = g_{\cdot 1}(\underline{z}) \quad \forall \underline{z} \in L(\gamma)$$

Proposition 5-5. *Let $f(\underline{x})$ satisfy condition (c") for a given γ, and let γ' denote the n-tuple $(q^{n-1}\gamma_1, \ldots, q^{n-j}\gamma_j, \ldots, \gamma_n)$. Then*

(i) For every J such that every j_k is even, $I'(\partial_n^{j_n} \cdots \partial_1^{j_1} f) \sim_{\gamma'} = 0$
(ii) $(id \otimes I)(\Pi_0 \otimes id)\Delta(f) \sim_{\gamma'} 1 \otimes (I' f)$
Moreover, if $f(\underline{x})$ satisfies condition (c") for every η and for every $J \in \mathbf{Z}^n_{\geq 0}$ then for every monomial $x_1^{e_1} \cdots x_n^{e_n}$
(iii) $I'(\partial_n^{2j_n} \cdots \partial_1^{2j_1}(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x}))) \sim_{\gamma''} 0$ for every J, where $\gamma''_j = q^{n-j+E^j}\gamma_j$
(iv) $(id \otimes I)(\Pi_0 \otimes id)\Delta(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})) \sim_{\gamma''} 1 \otimes I'(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x}))$ where γ'' is as above.
Proof : Statements *(i)* and *(ii)* are clear by the remark after the proof of Proposition 4-4. In order to prove *(iii)* we recall that

$$I'(\partial_n^{2j_n} \cdots \partial_1^{2j_1}(x_1^{e_1} \cdots x_n^{e_n} f(\underline{x})))_{\cdot 1} =$$
$$q^{-4\sum_k J^k} \int_{-q^{n-1}z_1\cdot\infty}^{q^{n-1}z_1\cdot\infty} \cdots \int_{-z_n\cdot\infty}^{z_n\cdot\infty} D^{2j_1}_{1,q^2} \cdots D^{2j_n}_{n,q^2}(t_1^{e_1} \cdots t_n^{e_n} f_{\cdot 1}(q^{E^1}t_1, \ldots, q^{E^n}t_n)) d_{q^2}\underline{t}$$

hence for $\underline{z} = \gamma''$ this expression converges, and it converges to zero. By invariance under q^2 shifts of the Jackson integral we get the statement. Statement (iv) follows from statement (iii). $\square$

Remark: Observe that in the proof of *(iii)* in Proposition 4-5 the lattice in which we compute the equality depends only on the parity of the e_j's and that it would be enough to be able to keep under control the partial Jackson derivatives of $(P_\beta f)_{\cdot 1}$ with $\beta_j = +$ (resp. $-$) if e_j is even (resp. odd). ♠

One may check that $E_{q^4}(-x_1^2)\cdots E_{q^4}(-x_n^2)$ satisfies all the conditions of Proposition 4-5. Computations are left to the reader.

6. Lattice order integrability

We are still left with the problem that the q^2-Gaussian $G_{q^2}(\underline{x})$ is not lattice integrable, even with respect to I'. We have to weaken again our condition and introduce the concept *lattice order integrability*. To simplify notation, we use analogues of I' instead of I. We first give a simple example of what the procedure is, and the formalize the definition.

Example 1 Let $f_1(x_1)$ and $f_2(x_2)$ be even lattice integrable power series for γ_1 and γ_2 respectively, if we view them as power series in the one dimensional space. Let $f(\underline{x}) = f_2(x_2)f_1(x_1)$. As we have seen in the Example in the previous Section, this is not necessarily lattice integrable. However, we can consider formally the following construction. We write $f_1(x_1) = \sum_k c_k x_1^k$ and $f_2(x_2) = \sum_k b_k x_2^k$. Then, for $\gamma' = (q\gamma_1, q\gamma_2)$ we have

$$\left(\int_{-x_2\cdot\infty}^{x_2\cdot\infty}\int_{-x_1\cdot\infty}^{x_1\cdot\infty} f\right)_{\cdot 1}|_{\underline{z}=\gamma'} = \left(\int_{-x_2\cdot\infty}^{x_2\cdot\infty}\int_{-x_1\cdot\infty}^{x_1\cdot\infty}\sum_{k,l} c_k b_l q^{kl} x_1^k x_2^l\right)_{\cdot 1}|_{\underline{z}=\gamma'} =$$
$$\left(\int_{-x_2\cdot\infty}^{x_2\cdot\infty}\int_{-x_1\cdot\infty}^{x_1\cdot\infty}\sum_{l} f_1(q^l x_1) b_l x_2^l\right)_{\cdot 1}|_{\underline{z}=\gamma'}$$

If we formally interchange integration and sum, knowing that from lattice integrability of f_1 the result is convergent, we can write

$$\left(\int_{-x_2\cdot\infty}^{x_2\cdot\infty}\int_{-x_1\cdot\infty}^{x_1\cdot\infty} f\right)_{\cdot 1}|_{\underline{z}=\gamma'} = \left(\sum_l \int_{-x_2\cdot\infty}^{x_2\cdot\infty}\int_{-q^l x_1\cdot\infty}^{q^l x_1\cdot\infty} f_1(x_1) b_l (q^{-1}x_2)^l\right)_{\cdot 1}|_{\underline{z}=\gamma'}$$

and since the power series f_2 is even, the q^l-shift in the integration bound can be neglected. If we write $I_1(x_1) := \int_{-x_1\cdot\infty}^{x_1\cdot\infty} f_1(x_1)$ we have that the above integral is equal to

$$\left(\sum_l \int_{-x_2\cdot\infty}^{x_2\cdot\infty} I_1(x_1) b_l (q^{-1}x_2)^l\right)_{\cdot 1}|_{\underline{z}=\gamma'} =$$
$$\left(I_1(q^{-1}x_1)\sum_l \int_{-x_2\cdot\infty}^{x_2\cdot\infty} b_l (q^{-1}x_2)^l\right)_{\cdot 1}|_{\underline{z}=\gamma'} = \left(I_1(q^{-1}x_1)\int_{-x_2\cdot\infty}^{x_2\cdot\infty} f_2(q^{-1}x_2)\right)_{\cdot 1}|_{\underline{z}=\gamma'}$$

where we inverted again formally the sum over l and the q^2-integral. After applying the action on 1, and evaluating at γ' we get that the result is q times the product of the q^2-Jackson integrals of the two powers series evaluated at γ_1 and γ_2 respectively. ♠

We are ready now for the definition of lattice order integrability. What we will do is repeatedly applying a one dimensional integral with respect to a noncommutative variable, say x_j. If this expression "has a meaning" (i.e. this expression applied to 1 converges after evaluation at $z_j = \gamma_j$) then we will identify it with a power series in noncommuting variables, in one variable less, and repeat the procedure. Namely:

Definition 6-1. *A formal power series $f(\underline{x}) \in C$ is said to be lattice order integrable (l.o. integrable) if there is an ordening of $1, \ldots, n$, denoted by the corresponding permutation $\sigma \in S_n$, and an n-tuple $\gamma \in \mathbf{R}^n_{>0}$ such that for every $j \in \{1, \ldots, n\}$ the expression $\int_{\sigma(j)}(I_{\sigma(j-1)} \cdots I_{\sigma(1)} f)$ is entire, where $\int_{\sigma(k)} g$ and $I_{\sigma(k)} g$ are defined inductively as follows. For a formal power series f in $\{x_1, \ldots, x_n\} - \{x_{\sigma(1)}, \ldots, x_{\sigma(k-1)}\}$, $\int_{\sigma(k)} f$ is the formal expression in $\{z_1, \ldots, z_n\} - \{z_{\sigma(1)}, \ldots, z_{\sigma(k)}\}$ defined as*

$$\left(\int_{\sigma(k)} f \right) (\underline{z}) := \left(\int_{-x_{\sigma(k)} \cdot \infty}^{x_{\sigma(k)} \cdot \infty} \Pi_0 f \right)_{\cdot 1} \Bigg|_{z_{\sigma(k)} = \gamma_{\sigma(k)}}$$

and if $\int_{\sigma(k)} f$ is entire, $I_{\sigma(k)} f$ will be the unique power series in the noncommuting indeterminates $\{x_1, \ldots, x_n\} - \{x_{\sigma(1)}, \ldots, x_{\sigma(k)}\}$ such that $(\int_{\sigma(1)} f) = (I_{\sigma(1)} f)_{\cdot 1}$. If $f(\underline{x})$ is l.o. integrable we define the constant $I''_{(\sigma, \gamma)} f := \int_{\sigma(n)}(I_{\sigma(n-1)} \cdots I_{\sigma(1)} f)$ to be the lattice-order integral of $f(\underline{x})$ associated to the order σ and the lattice $L(\gamma)$.

Clearly there is quite a difference between I and $I''_{(\sigma, \gamma)}$ since I maps formal power series to formal expressions in $x_1, \ldots, x_n$ while $I''_{(\sigma, \gamma)}$ maps l.o. integrable power series to constants. We will see later what is the relation between the two maps, on the space where they are both defined. We will also see in the examples that even if a power series in l.o. integrable for every order, it could still be not lattice integrable. In this case, we will show the relation between $I''_{(\sigma, \gamma)} f$ and $I''_{(\sigma', \gamma')} f$.
Observe that by definition of $\int_l$, power series that are odd in some variables are automatically defined to be l.o. integrable and that the integral will be zero for every choice of σ. For this reason, we will only investigate lattice order integrability for even power series. We can state a few results about lattice order integrability.

Proposition 6-2. *Let $f(\underline{x})$ be an even element of $\hat{V}(Q, R)^{\mathrm{ext}}$ such that, for some $\tau \in S_n$, and for some power series in one indeterminate $f_1, \ldots, f_n$ we can write $f(\underline{x}) = f_{\rho(1)}(x_{\rho(1)}) \cdots f_{\rho(n)}(x_{\rho(n)})$ where $\rho = \tau^{-1}$. Then $f(\underline{x})$ is l.o. integrable if and only if each f_j, viewed as a power series in commuting variables) is entire and lattice integrable. In this case $f(\underline{x})$ is lattice order integrable for every order σ and a suitable lattice depending on σ. Moreover, one has*

$$I''_{(\sigma, \gamma)}(f(\underline{x})) = q^{\mathrm{l}(\sigma) + \mathrm{l}(\tau)} \prod_{j=1}^{n} \int_{-\gamma_j \cdot \infty}^{\gamma_j \cdot \infty} (f_j)_{\cdot 1}(t_j) d_{q^2} t_j$$

where l denotes the usual length of a permutation.

Proof : ($\Rightarrow$) Suppose that $f(\underline{x})$ is as in the hypothesis, and that each f_j is entire, and lattice-integrable for a given $\tilde{\gamma}_j$. We write $f_j(x_j) = \sum_k c_{jk} x_j^k$ for every j. For an $n-$tuple K and for $p \in \{1, \ldots, n\}$, we will also write

$$K_{\tau,p} := \sum_{\substack{j > p \\ \tau(j) < \tau(p)}} k_j \quad \text{and} \quad K^{\tau,p} := \sum_{\substack{j < p \\ \tau(j) > \tau(p)}} k_j$$

We fix a σ. Then for $\sigma(1)=l$ and $\gamma_l=\tilde{\gamma}_l$ one has

$$\left(\int_l f\right)(z_1,\ldots,z_{l-1},z_{l+1},\ldots,z_n)$$
$$=2(1-q^2)\sum_{h=-\infty}^{\infty} q^{2h}\gamma_l\sum_K c_{1k_1}\cdots c_{nk_n}q^{-K_l}z_1^{k_1}\cdots z_{l-1}^{k_{l-1}}z_{l+1}^{k_{l+1}}\cdots z_n^{k_n}q^{\sum_j k_jK_{\tau,j}}(q^{2h}\gamma_l)^{k_l}$$
$$=2(1-q^2)\sum_{h=-\infty}^{\infty} q^{2h}\gamma_l\sum_{K'} c_{1k_1}\cdots \hat{c}_{lk_n}\cdots c_{nk_n}(q^{-1}z_1)^{k_1}\cdots(q^{-1}z_{l-1})^{k_{l-1}}z_{l+1}^{k_{l+1}}\cdots z_n^{k_n}$$
$$\times q^{\sum_{j\neq l}k_jK'_{\tau,j}}f_l(q^{2h+K'_{\tau,p}+K'^{\,\tau,p}}\gamma_l)$$

where K' is the $(n-1)$-tuple obtained by K by deleting $\sigma(1)=l$, and

$$K'_{\tau,p}:=\sum_{\substack{l\neq j>p\\ \tau(j)<\tau(p)}}k_j \quad\text{and}\quad K'^{\,\tau,p}:=\sum_{\substack{l\neq j<p\\ \tau(j)>\tau(p)}}k_j$$

The last equality holds because

$$\sum_p k_pK_{\tau,p}=k_lK_{\tau,l}+\sum_{p>l}k_pK_{\tau,p}+\sum_{p<l}k_pK_{\tau,p}$$
$$=k_lK_{\tau,l}+\sum_{p>l}k_pK'_{\tau,p}+\sum_{p<l}k_pK'_{\tau,p}+\sum_{\substack{p<l\\ \tau(l)<\tau(p)}}k_pk_l=k_l(K_{\tau,l}+K^{\tau,l})+\sum_{p\neq l}k_pK'_{\tau,p}$$

Then one can use the convergence of the q^2-Jackson integral of $f_{l\,\cdot 1}$, together with the fact that the other f_k's are entire and the fact that $K'_{\tau,p}+K'^{\,\tau,p}$ is an even number because the f_j's are even to show that one can invert the order of summation in the above sum, using dominated convergence. One gets:

$$\left(\int_l f\right)(z_1,\ldots,z_{l-1},z_{l+1},\ldots,z_n)$$
$$=\sum_{K'}q^{\sum_{j\neq l}k_jK'_{\tau,j}}c_{1k_1}\cdots c_{nk_n}(q^{-1}z_1)^{k_1}\cdots(q^{-1}z_{l-1})^{k_{l-1}}z_{l+1}^{k_{l+1}}\cdots z_n^{k_n}$$
$$\times 2(1-q^2)\sum_{h=-\infty}^{\infty}q^{2h}\gamma_lf_l(q^{2h+K'_{\tau,p}+K'^{\,\tau,p}}\gamma_l)$$

but this is nothing but

$$=\sum_{K'}q^{\sum_{j\neq l}k_jK'_{\tau,j}}c_{1k_1}\cdots c_{nk_n}(q^{-1}z_1)^{k_1}\cdots(q^{-1}z_{l-1})^{k_{l-1}}z_{l+1}^{k_{l+1}}\cdots z_n^{k_n}q^{-K'_{\tau,p}-K'^{\,\tau,p}}$$
$$\times\int_{-q^{K'_{\tau,p}+K'^{\,\tau,p}}\gamma_l\cdot\infty}^{q^{K'_{\tau,p}+K'^{\,\tau,p}}\gamma_l\cdot\infty}(f_l)_{\cdot 1}(t_l)d_{q^2}t_l$$

The above power series is entire since all the f_j's are, and one finds that

$$I_{\sigma(1)}f = \left(\int_{-\gamma_l\cdot\infty}^{\gamma_l\cdot\infty}(f_l)_{\cdot 1}(t_l)d_{q^2}t_l\right)\prod_{\substack{j\\ \rho(j)\neq l}} f_{\rho(j)}(q^{\eta_{\rho(j)}+\theta_{\rho(j)}}x_{\rho(j)})$$

with

$$\eta_k = \begin{cases} -1 & \text{if } k<\sigma(1)\\ 0 & \text{if } k>\sigma(1)\end{cases} \qquad \theta_k = \begin{cases} -1 & \text{if } k<l \text{ and } \tau(k)>\tau(l)\\ 1 & \text{if } k>l \text{ and } \tau(k)<\tau(l)\\ 0 & \text{otherwise}\end{cases}$$

Therefore we are again in the hypothesis of the Proposition, but then in case $(n-1)$. Since the statement in one dimension is obvious, lattice order integrability is proved, considering the following shifted lattice: for every new step we make, the argument of the f_j that still has to be integrated will be shifted by powers of q. If one goes through computations one finds that the exponential of q in the shift of the argument of f_r with $r=\sigma(s)$ is

$$-\Sigma_s(\sigma,\tau) = -\left[\#\{j<s \mid \sigma(s)<\sigma(j)\} + \#\{j<s \mid (\sigma(j)-\sigma(s))(\tau\sigma(j)-\tau\sigma(s))<0\}\right]$$

hence the right lattice to integrate is the one defined by $\gamma_{\sigma(s)} = \tilde{\gamma}_{\sigma(s)}q^{\Sigma_s(\sigma,\tau)}$. In this setting, the integral will be the product of the q^2-Jackson integrals of the $f_{j\,\cdot 1}$'s multiplied by a power of q with exponent

$$\begin{aligned}\sum_{s=1}^{n}\Sigma_s(\sigma,\tau) &= \mathrm{l}(\sigma) + \sum_{s=1}^{n}\#\{j<s \mid (\sigma(j)-\sigma(s))(\tau\sigma(j)-\tau\sigma(s))<0\}\\ &= \mathrm{l}(\sigma)+\mathrm{l}(\tau)\end{aligned}$$

since the second term in the sum is equal to the cardinality of

$$\{j,\,s \mid j<s\}\cap\left(\{j,\,s\mid \sigma(j)<\sigma(s),\ \tau\sigma(j)>\tau\sigma(s)\}\cup\{j,\,s\mid\sigma(s)<\sigma(j),\ \tau\sigma(s)>\tau\sigma(j)\}\right)$$

For the converse of the statement: one sees that if $f(\underline{x})$ can be written as a product of one dimensional power series, those series have to be entire, and if there is a σ such that $f(\underline{x})$ is lattice order integrable, this means that $f_{\sigma(1)}$ is lattice integrable on $q^{2k_1}\gamma_{\sigma(1)}$, and so on, for the following f_j's, with shifted argument. By the $\Rightarrow$ part, we see that lattice order integrability has to hold for every σ'. $\square$.

Example 2: By the above Proposition, the formal power series $f(\underline{x}) = x_n^{e_n}E_{q^4}(-a_n^2x_n^2)\cdots x_1^{e_1}E_{q^4}(-a_1^2x_1^2)$ is l.o. integrable for every σ and $\gamma_{\sigma(k)} = a_{\sigma(k)}^{-1}q^{(k-1)+\#\{j<k\,\mid\,\sigma(k)<\sigma(j)\}}$, since in this case $\tau(k)=n-k+1$. In particular, $G_{q^2}(q^2\underline{x})$ and all products of type $G_{q^2}(q^2\underline{x})x_1^{e_1}\cdots x_n^{e_n}$ are l.o. integrable for every choice of the order σ. One has

$$I''_{(\sigma,\gamma)}f = \begin{cases} b_{q^2}^n\left[\prod_j(q^2;q^4)_{f_j}\right]q^{2n+2|E|}q^{\binom{n}{2}}q^{\mathrm{l}(\sigma)}\prod_{j=1}^n a_j^{-1-e_j} & \text{if } e_j=2f_j \text{ for every } j\\ 0 & \text{otherwise}\end{cases}$$

where $b_{q^2} = (1-q^2)(q^2,-q^2,-1;\,q^2)_\infty$ and the result follows by [Koo]. In particular we observe that the result depends on the choice of σ only in a straightforward way and that

$L(\gamma)$ does not depend on E but only on σ and the a_j's. Therefore it makes sense to consider the relation between $I''_{(\sigma,\,\gamma)}f$ and $I''_{(\sigma,\,\gamma)}(E_{q^4}(-\sum_k a_k^2 x_k^2))$. One immediately sees that if all the e_j's are even

$$I''_{(\sigma,\,\gamma)}(f) = \frac{(\prod_j (q^2;q^4)_{f_j})q^{2|E|}}{(\prod_j a_j^{e_j})} I''_{(\sigma,\,\gamma)}(E_{q^4}(-\sum_k a_k^2 x_k^2))$$

We say in this case (and whenever an equivalence of integrals $I''_{(\sigma,\gamma)}$ holds, with the same σ and γ on both sides) that the integral $I(x_n^{e_n}E_{q^4}(-a_n^2x_n^2)\cdots x_1^{e_1}E_{q^4}(-a_1^2x_1^2))$ is *"weakly equivalent"* to $I(E_{q^4}(-\sum_k a_k^2 x_k^2))$. In particular $I(x_n^{e_n}E_{q^4}(-q^2x_n^2)\cdots x_1^{e_1}E_{q^4}(-q^2x_1^2))$ is weakly equivalent to $\prod_j (q^2;q^4)_{f_j} I(G_{q^2}(q^2\underline{x}))$. We also want to point out that the above $f(\underline{x})$ is an example of the fact that one can have l.o. integrability for every order and still not have lattice-integrability. ♠

Example 3: If we changed the definition of $\int_l$ for l.o. integrability removing Π_0 we would no longer have the fact that all constants are l.o. integrable, but then l.o. integrability would strongly depend on the choice of σ, even in the simplest cases. Take for instance $n=2$, and $f(\underline{x}) = f_1(x_1)f_2(x_2)$ where $f_1(x_1)$ is a series that is not lattice integrable, and $f_2(x_2)$ is lattice-integrable, or even $I-$integrable as a series in one variable, with integral equal to zero. For instance, take $f_1(x_1) = x_1$ and $f_2(x_2) = \partial_2^2 e_{q^4}(-x_2^2)$. Then for $\sigma = \mathrm{id}$ the power series f would not be l.o. integrable for any choice of γ, while for $\sigma = (2\ 1)$ we would have $I''_{(\sigma,\,\gamma)}f = 0$ for every $\gamma \in (\mathbf{R}_+)^2$. ♠

Properties and Remarks:
(a) It is easy to check that if $f(\underline{x})$ is l.o. integrable for the order σ and the lattice $L(\gamma)$, then, for every n-tuple of nonzero real numbers $(a_1, \ldots, a_n)$, the power series $f_A(\underline{x}) = f(a_1x_1, \ldots, a_nx_n)$ is also l.o. integrable for the same order σ and for γ replaced by $\tilde{\gamma}$ where $\tilde{\gamma}_j = a_j^{-1}\gamma_j$ for every j. Then one has equivalence of the numbers $I''_{(\sigma,\,\gamma)}(f) = (a_1\cdots a_n)I''_{(\sigma,\,\tilde{\gamma})}(f_A)$.

(b) It is also obvious that if $f(\underline{x})$ is l.o. integrable for σ and γ then the resulting $I''_{(\sigma,\,\gamma)}(f)$ is invariant under shifts of each γ_j by even powers of q.
(c) If f is as in Proposition 6-1, with $\tau = \mathrm{id}$, and if it is l.o. integrable, then it is lattice integrable.
(d) In the definition of lattice order integrability the requirement on the $\int_{\sigma(k)} f$'s to be entire for every k can be weakened to analyticity. The weaker version of the definition is left to the reader. ♠

By the discussion above, one can conclude that for well behaved series (by this we mean series satisfying condition (c), (c') etcetera) also the integral $I''_{(\sigma,\,\gamma)}$ is invariant under translation.

Remarks: The whole construction of lattice order integrability may look artificial, and it may seem to be a definition that is useful only in a noncommutative setting. However, this is not the case. One can define a similar concept of integrability also for power series in commutative variables. In this case, the definition can be made much simpler.

Definition 6-3 (T. *Koornwinder). Let* $f(z_1, \ldots, z_n) = \sum_{e_1, \ldots, e_n} C_E z_1^{e_1} \cdots, z_n^{e_n}$ *be a formal power series in commuting variables with coefficients in* $\mathbf{R}$. *The order integral of* f *with respect to the order id and the lattice* $L(\gamma)$, *for* γ *an* n*-tuple of real positive numbers,*

$$\int_{-\gamma_n\cdot\infty}^{\gamma_n\cdot\infty}\left(\cdots\left(\int_{-\gamma_1\cdot\infty}^{\gamma_1\cdot\infty} f(t_1, \ldots, t_n) d_{q^2} t_1\right)\cdots\right) d_{q^2} t_n$$

is defined recursively as follows, provided the assumptions in each recursive step are satisfied: $I_E^0 := c_E$ *and for* $j = 1, \ldots, n$ $f^j_{e_{j+1},\ldots,e_n}(z_j) := \sum_{e_j} I^{j-1}_{e_j,\ldots,e_n} z_j^{e_j}$ *is analytic on* $\mathbf{R} + iU$ *for some open neighbourhood of 0 in* $\mathbf{R}$, *and* $I^j_{e_{j+1},\ldots,e_n} := \int_{-\gamma_j\cdot\infty}^{\gamma_j\cdot\infty} f^j_{e_{j+1},\ldots,e_n}(t_j) d_{q^2} t_j$ *is absolutely convergent. Then* I^n *is the required integral. Similarly, if* σ *is a permutation of* $\{1, \ldots, n\}$ *we define*

$$\int_{-\gamma_{\sigma(n)}\cdot\infty}^{\gamma_{\sigma(n)}\cdot\infty}\left(\cdots\left(\int_{-\gamma_{\sigma(1)}\cdot\infty}^{\gamma_{\sigma(1)}\cdot\infty} f(t_1, \ldots, t_n) d_{q^2} t_{\sigma(1)}\right)\cdots\right) d_{q^2} t_{\sigma(n)}$$

The definition of order integrability with respect to a lattice in the commutative case is due to T. Koornwinder.

We have an easy result:

Proposition 6-4. *Let* $f(\underline{x}) = \sum_E c_E x_1^{e_1} \cdots x_n^{e_n} \in C$ *be even and lattice order integrable with respect to the order* σ *and the lattice* $L(\gamma)$. *Then* $f_{\cdot 1}(\underline{z})$ *is integrable with respect to the order* σ *and the lattice* $L(\tilde{\gamma})$ *with* $\tilde{\gamma}_{\sigma(l)} = q^{\#\{j<l \,|\, \sigma(l)<\sigma(j)\}}\gamma_{\sigma(l)}$. *Moreover* $q^{\mathrm{l}(\sigma)} I^n = I''_{(\sigma,\gamma)} f$.

Proof: By definition $f_{\cdot 1}(\underline{z}) = \sum_E c_E z_1^{e_1} \cdots z_n^{e_n}$. Since by definition of lattice order integrability for $f(\underline{x})$,

$$\left(\int_{\sigma(1)} f\right)(z_{\sigma(2)}, \ldots, z_{\sigma(n)}) =$$

$$2(1-q^2)\sum_{k=-\infty}^{\infty} q^{2k}\gamma_{\sigma(1)} \sum_E c_E (q^{-1}z_1)^{e_1} \cdots (q^{-1}z_{\sigma(1)-1})^{e_{\sigma(1)-1}} (q^{2k}\gamma_{\sigma(1)})^{e_{\sigma(1)}} \cdots z_n^{e_n}$$

can be written as an *entire power series* in $z_{\sigma(2)}, \cdots, z_{\sigma(n)}$, for every $e_{\sigma(2)}, \ldots, e_{\sigma(n)}$ $I^{\sigma(1)}_{e_{\sigma(2)},\ldots,e_{\sigma(n)}}$ is a finite constant and $\sum_{e_{\sigma(2)}} I^{\sigma(1)}_{e_{\sigma(2)},\ldots,e_{\sigma(n)}} z_{\sigma(2)}^{e_{\sigma(2)}}$ converges everywhere. By the fact that $\int_{\sigma(2)} f$ can be expressed as an entire power series, $\sum_{e_{\sigma(2)}} I^{\sigma(1)}_{e_{\sigma(2)},\ldots,e_{\sigma(n)}} z_{\sigma(2)}^{e_{\sigma(2)}}$ is also integrable with respect to $z_{\sigma(2)}$ in the lattice $L(\tilde{\gamma})$ where

$$\tilde{\gamma}_{\sigma(2)} = \begin{cases} \gamma_{\sigma(2)} & \text{if } \sigma(2) > \sigma(1) \\ q\gamma_{\sigma(2)} & \text{if } \sigma(2) < \sigma(1) \end{cases}$$

By repeating the same arguments one obtains the statement. □

Remark: The reader may wonder whether we could have chosen another realization of $\hat{V}(Q, R)^{\mathrm{ext}}$ and of the integrals of elements of $\hat{V}(Q, R)^{\mathrm{ext}}$ other than $f \triangleright 1$. Of course one

might consider a different representation, or a different choice of the normal form. The advantages of a representation associated to the choice of a normal form is the fact that it is enough to test operators on 1 to state an equivalence in $\hat{V}(Q,\, R)$. The advantage of the particular normal form that we have chosen is based on the fact that C and C_γ are closed under product, hence we have a map from formal expressions in $x_1\, \ldots,\, x_n$ to formal expressions in the $z_1,\, \ldots,\, z_n$ such that on rather big subspaces it comes exactly from an algebra homomorphism. If we had chosen another normal form, we could no longer extend the representation π of $\hat{V}(Q,\, R)$ on V to a representation of the subset S of $\hat{V}(Q,\, R)^{\rm ext}$ such that $\pi(S)(1) \subset V$.
Take for instance $n = 2$, and the representation of $\hat{V}(Q,\, R)$ on $\mathbf{R}[[z,\, w]]$ given by: $\pi(x_1)(f(z,\, w)) = zf(z,\, q^{-1}w)$ and $\pi(x_2)(f(z,\, w)w) = f(z,\, w)w$. This is the representation associated to the choice of the normal form with x_2 preceding x_1.
Then $a = \sum_{k=0}^\infty x_1^k$ and $b = \sum_{l=0}^\infty x_2^l$ belong to S but ab does not belong to S since $\pi(\sum_{k=0}^\infty x_1^k)(\pi(\sum_{l=0}^\infty x_2^l))(1)) = \sum_{k,l=0}^\infty q^{-kl} z^k w^l \notin V$.
On the other hand it is true that other realizations, even though they cannot be associated to a representation on C may be more useful to give a meaning to $I \cdot G_{q^2}(\underline{x})$ ♠

7. The braided Fourier transform

Now we have all ingredients for the introduction of braided Fourier transforms on a subspace of $\hat{V}(Q,\, R)^{\rm ext}$. We introduce two transforms, related to each other by a shift in the arguments and the application of the antipode to one of them. As we already said, the first time that a Fourier transform for this kind of algebras appeared was in [KeMa], from which we took inspiration. A quantum Fourier transform has been studied in the one dimensional case in [Koo] and in [OR] where also a theory of distributions is developed. The transforms that we present are based on [KeMa], [Koo] and [Ch]. One of the goals of this section is to provide an n-dimensional analogues to formulae (8.19), (8.20) and (8.21); hence to Theorem 8.1 in [Koo]. The difference with [Koo] is that in our version, the algebra $\hat{V}(Q,\, R)^{\rm ext} \otimes V(Q,\, R)^{\rm ext}$ has the braided product $(m \otimes m)(\mathrm{id} \otimes \Phi \otimes \mathrm{id})$ instead of the ordinary one, although in normal form his formulae and ours for $n = 1$ coincide. The difference with [KeMa] lies mainly in the fact that our integral is not bosonic (i.e. it does not have trivial braiding with elements of the algebras $\hat{V}(Q,\, R)^{\rm ext}$ and $V(Q,\, R)^{\rm ext}$). We also apply some minor changes like shifting the argument of the exponential and using the antipode in the definition$^{(*)}$. The use of the antipode appears also in [Ch] where the case of finite dimensional braided Hopf algebras is treated. These transforms behave nicely with respect to a convolution product and with respect to the action of $V(Q,\, R)$ on $\hat{V}(Q,\, R)^{\rm ext}$. They also respect various classical properties of the Fourier transform. These facts are developed in [KeMa] and in [Ca].
We say that an element $f(\underline{x})$ of $\hat{V}(Q,\, R)^{\rm ext}$ is *of class* $\mathcal{R}$ if $f(\underline{x})\, x_1^{e_1} \cdots x_n^{e_n}$ is I'-integrable for every monomial $x_1^{e_1} \cdots x_n^{e_n}$. We say that it is *of class* $\mathcal{R}_{(\sigma,\, \gamma)}$ if for every monomial $x_1^{e_1} \cdots x_n^{e_n}$, the power series $f(\underline{x})\, x_1^{e_1} \cdots x_n^{e_n}$ is lattice order integrable for σ and γ.
Again, we do not provide a complete classification of $\mathcal{R}$ but we give a class for which this makes sense, which is big enough to reach our goal. Indeed, power series satisfying

$^{(*)}$ We do this because the dual of a braided Hopf algebra is the opposite of the dual of a genuine Hopf algebra.

condition (c) of Section 4 belong to $\mathcal{R}$, hence product of $e_{q^2}(-x_j^2)$ and polynomials belong to $\mathcal{R}$ provided every $e_{q^2}(-x_j^2)$ appears in the product.

Definition 7-1. *The braided Fourier transforms F and F_S are defined on the class $\mathcal{R}$ and they have images in $\hat{V}(Q,\, R)^{\mathrm{ext}} \otimes V(Q,\, R)^{\mathrm{ext}}$. They are given by*

$$F := (I' \otimes \mathrm{id})(m \otimes \mathrm{id})\left(\mathrm{id} \otimes \exp\left(x \mid \frac{i}{(1-q^2)}(\partial_1,\, \ldots,\, q^{-j+1}\partial_j, \ldots,\, q^{-n+1}\partial_n)\right)\right)$$

$$F_S := (I' \otimes S)(m \otimes \mathrm{id})\left(\mathrm{id} \otimes \exp\left(x \mid \frac{iq^2}{(1-q^2)}(q^{n-1}\partial_1,\, \ldots,\, q^{n-j}\partial_j,\, \ldots,\, \partial_n)\right)\right)$$

For an $f(\underline{x}) \in \mathcal{R} \cap C_\beta$ one has that: $F(f(\underline{x})) := (I' \otimes id)\Big(f(\underline{x})\, E_{q^2}\big(i \sum_{j=1}^n x_j \otimes q^{-(j-1)}\partial_j\big)\Big)$

$$= \sum_{\substack{e_1,\, \ldots,\, e_n \\ A(e_j)=\beta_j}} I\big(f(\underline{x})\, x_1^{e_1} \cdots\, x_n^{e_n}\big) \otimes \frac{i^{|E|} q^{-\sum_j E^j}}{\prod_{j=1}^n (q^2; q^2)_{e_j}} \partial_n^{e_n} \cdots\, \partial_1^{e_1}$$

and $F_S(f(\underline{x})) := (I' \otimes S)\Big(f(\underline{x}) E_{q^2}\big(iq^2 \sum_{j=1}^n x_j \otimes q^{(n-j)}\partial_j\big)\Big)$

$$= \sum_{\substack{e_1,\, \ldots,\, e_n \\ A(e_j)=\beta_j}} I\big(f(\underline{x})\, x_1^{e_1} \cdots\, x_n^{e_n}\big) \otimes \frac{(-i)^{|E|} q^{|E|^2 - \sum_j e_j E_j + |E| + \sum_j E_j}}{\prod_{j=1}^n (q^2; q^2)_{e_j}} \partial_1^{e_1} \cdots,\, \partial_n^{e_n}$$

where $A(e_j) = +$ (resp. $-$) if e_j is even (resp. odd). Here we used that $S(\partial_n^{e_n} \cdots\, \partial_1^{e_1}) = (-1)^{|E|} q^{|E|^2 - |E| - \sum_j e_j E_j} \partial_1^{e_1} \cdots \partial_n^{e_n}$ Therefore it is clear that the second components in the tensor product of $F(f(\underline{x}))$ and of $F_S(f(\underline{x}))$ will also have parity β. In order to provide formulae analogous to (8.21) and (8.19) in [Koo], we need to compute F_S for
$M(\underline{x},\, A) = e_{q^4}(-x_1^2)x_1^{a_1} \cdots\, e_{q^4}(-x_n^2)x_n^{a_n} = x_1^{a_1} \cdots x_n^{a_n} e_{q^4}\big(-\sum_j (q^{-A_j} x_j)^2\big)$ and for
$H(\underline{x},\, A) = e_{q^4}(-x_1^2)\tilde{h}_{a_1}(x_1; q^2) \cdots e_{q^4}(-x_n^2)\tilde{h}_{a_n}(x_n; q^2)$ where the $\tilde{h}_{a_j}$'s are the *discrete $q-$Hermite II polynomials* (see [KoSw] and references in there), that are defined by:

$$\tilde{h}_l(z;\, q) := z^l\, {}_2\phi_1(q^{-n},\, q^{-n+1};\, 0;\, q^2,\, -q^2 z^{-2}) = (q;\, q)_l \sum_{k=0}^{[\frac{l}{2}]} \frac{(-1)^k q^{-2kl+k(2k+1)} z^{l-2k}}{(q^2;\, q^2)_k (q;\, q)_{l-2k}}$$

Both $M(\underline{x},\, A)$ and $H(\underline{x},\, A)$ satisfy condition (c) of section 4, so that the transform in defined on both series. We first compute the transform F_S on a generic $f(\underline{x})$. In order to give a meaning to the transform we apply the realization map π_γ sending $g(\underline{x})$ to $g_{\cdot 1}(\underline{z})$, followed by evaluation at $\underline{z} = \gamma$, to the first component of $F_S(f(\underline{x}))$. By the assumption that $f(\underline{x}) \in \mathcal{R}$ we know that this is well defined so that $(\pi_\gamma \otimes \mathrm{id})F_S(f(\underline{x}))$ is a genuine power series in the noncommuting ∂_j's. By the computations in the previous section, one obtains, for an $f(\underline{x}) \in C_\beta \cap \mathcal{R}$:

$$(\pi_\gamma \otimes \mathrm{id})\big(F_S(f(\underline{x}))\big) = \sum_{\substack{e_1,\,\dots,\,e_n \\ A(e_j)=\beta_j}} \frac{(-i)^{|E|} q^{\sum_j (e_j^2+e_j)}}{\prod_{k=1}^n (q^2;\, q^2)_{e_k}} \times$$

$$\times \left(\int_{-\gamma_1 q^{n-1+E_1}\cdot\infty}^{\gamma_1 q^{n-1+E_1}\cdot\infty} \cdots \int_{-\gamma_n q^{E_n}\cdot\infty}^{\gamma_n q^{E_n}\cdot\infty} f_{\cdot 1}(\underline{t})\, t_1^{e_1} \cdots t_n^{e_n}\, d_{q^2}\underline{t} \right) \times \partial_1^{e_1} \cdots \partial_n^{e_n}$$

By invariance of the q^2-integral we see that the integration bounds do not depend on E but only on the parity of its components, hence they only depend on β. In particular, for $f(\underline{x}) = f_1(x_1) \cdots f_n(x_n) \in \mathcal{R} \cap C_\beta$ one has

$$(\pi_\gamma \otimes \mathrm{id}) F_S(f(\underline{x})) = \prod_{k=1}^n \Bigg[\sum_{A(e_k)=\beta_k} \frac{(-i)^{e_k} q^{e_k^2+e_k} \partial_k^{e_k}}{(q^2;q^2)_{e_k}} \left(\int_{-q^{B(\beta)_k+n-k}\gamma_k\cdot\infty}^{q^{B(\beta)_k+n-k}\gamma_k\cdot\infty} [(f_k)_{\cdot 1}(t_k)]\, t_k^{e_k}\, d_{q^2} t_k \right) \Bigg]$$

where the product is taken in *increasing* order and $B(\beta) = (b(\beta)_1, \dots, b(\beta)_n)$ is the n-tuple $\{0,\, 1\}^n$ such that the k^{th} entry is 0 (resp. 1) if β_k is even (resp. odd) and $B(\beta)_k = \sum_{j=1}^{k-1} b(\beta)_j$ as usual.
Hence we come to an $n-$dimensional version of formula (8.21) in [Koo]. Let $M(\underline{x},\, A)$ as above. We remind that in this case $\beta_j = +$ (resp. $-$) if a_j is even (resp. odd). Then we use the one-dimensional case in [Koo] to obtain our result. We can do so because when one chooses the normal forms we have chosen for $\hat{V}(Q,\, R)^{\text{ext}}$ and $V(Q,\, R)^{\text{ext}}$ computations for each factor look exactly the same as in (8.21) and (9.8) in [Koo]. Indeed, expanding in power series the left hand side of (8.21) and using q^2 instead of q one has

$$c_{q^2}(\gamma) q^{-a^2-a} i^a h_a(t; q^2) E_{q^4}(-q^4 t^2) = \sum_{\substack{k\geq 0 \\ k+a \text{ even}}} \frac{i^k q^{k^2+k}}{(q^2;q^2)_k} \left(\int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} x^{a+k} e_{q^4}(-x^2) d_{q^2} x \right) t^k =$$

$$\sum_{\substack{k\geq 0 \\ k+a \text{ even}}} \frac{i^k q^{k^2+k}}{(q^2;q^2)_k} c_{q^2}(\gamma) q^{\frac{-(a+k)^2}{2}} (q^2;q^4)_{\frac{a+k}{2}} t^k$$

Using the above formula we obtain:

$$(\pi_\gamma \otimes \mathrm{id}) F_S(M(\underline{x},\, A)) =$$

$$\prod_{j=1}^n \left[c_{q^2}\left(q^{n-j+B(\beta)_j}\gamma_j\right) \sum_{\substack{e_j\geq 0 \\ e_j+a_j \text{ even}}} \left(\frac{(-i)^{e_j} q^{e_j^2+e_j}}{(q^2;\, q^2)_{e_j}} q^{\frac{-(e_j+a_j)^2}{2}} (q^2;\, q^4)_{\frac{e_j+a_j}{2}} \right) \partial_j^{e_j} \right] \tag{7.1}$$

$$= (-i)^{|A|} \left[\prod_{j=1}^n c_{q^2}(q^{n-j+B(\beta)_j}\gamma_j) \right] q^{\sum_j a_j(1-a_j)} \prod_{k=1}^n \left[E_{q^4}(-q^4 \partial_k^2) h_{a_k}(\partial_k;\, q^2) \right]$$

where
(i) The product is taken in *increasing* order.
(ii) $E_{q^4}(-z^2) =: G_{q^2}(z)$ is the so-called big q^2-Gaussian
(iii) $h_l(z;\, q^2)$ is the *discrete q^2-Hermite I polynomial* of degree l (see [KoSw] and references in there) and is defined as

$$h_l(z;\, q^2) := z^l\,{}_2\phi_0(q^{-2l}, q^{2-2l};\, ;\, q^4,\, q^{4l-2}z^{-2}) = (q^2;\, q^2)_l \sum_{k=0}^{[\frac{l}{2}]} \frac{(-1)^k q^{2k(k-1)} z^{l-2k}}{(q^4;\, q^4)_k (q^2;\, q^2)_{l-2k}}$$

We observe that the only part of $(\pi_\gamma \otimes \mathrm{id})F_S(M(\underline{x},\, A))$ involving the γ_j's is the coefficient $\left[\prod_{j=1}^n c_{q^2}(q^{n-j+B(\beta)_j}\gamma_j)\right]$ that is also equal to

$$\left(q^{\sum_j B(\beta)_j} I\big(e_{q^4}(-\sum_j (q^{B(\beta)_j}\gamma_j)^2)\big)_{\cdot 1}\right)|_{\underline{z}=\gamma} = \left[\int_{-q^{B(\beta)_n}x_n\cdot\infty}^{q^{B(\beta)_n}x_n\cdot\infty} \cdots \int_{-q^{B(\beta)_1}x_1\cdot\infty}^{q^{B(\beta)_1}x_1\cdot\infty} g_{q^2}\right]_{\cdot 1} |_{\underline{z}=\gamma}$$

hence the coefficient is a shifted integral of the Gaussian $g_{q^2}(\underline{x})$ where the shift only depends on the parity of the function $M(\underline{x},\, A)$, i.e. only on the parity of the a_j's. So we conclude that

$$F_S(e_{q^4}(-x_1^2)x_1^{a_1} \cdots e_{q^4}(-x_n^2)x_n^{a_n}) =$$
$$\left[\int_{-q^{B(\beta)_n}x_n\cdot\infty}^{q^{B(\beta)_n}x_n\cdot\infty} \cdots \int_{-q^{B(\beta)_1}x_1\cdot\infty}^{q^{B(\beta)_1}x_1\cdot\infty} g_{q^2}\right] \otimes (-i)^{|A|} q^{\sum_j a_j(1-a_j)} \prod_{k=1}^{n} \left[E_{q^4}(-q^4\partial_k^2) h_{a_k}(\partial_k;\, q^2)\right] \tag{7.2}$$

The above result gives the analogue of the classical reciprocity between { Gaussian times a monomial } and { rescaled Gaussian times a Hermite polynomial } under the Fourier transform in $\mathbf{R}^n$. From the above result we derive an analogue of formula (8.19) in [Koo], for $H(\underline{x},\, A)$ defined above. $H(\underline{x},\, A)$ is also contained in one of the subspaces C_β since each $\tilde{h}_a(x_j; q^2)$ has constant parity. We obtain:

$$\begin{aligned} F_S(e_{q^4}(-x_1^2)\tilde{h}_{a_1}(x_1; q^2) \cdots e_{q^4}(-x_n^2)\tilde{h}_{a_n}(x_n; q^2)) &= \left[\int_{-q^{B(\beta)_n}x_n\cdot\infty}^{q^{B(\beta)_n}x_n\cdot\infty} \cdots \int_{-q^{B(\beta)_1}x_1\cdot\infty}^{q^{B(\beta)_1}x_1\cdot\infty} g_{q^2}\right] \otimes \\ \otimes \prod_{k=1}^{n} \left[(q^2;\, q^2)_{a_k} \sum_{s_k=0}^{[\frac{1}{2}a_k]} \frac{(-i)^{a_k} q^{-a_k^2+a_k}}{(q^4;\, q^4)_{s_k}(q^2;\, q^2)_{a_k-2s_k}} h_{a_k-2s_k}(\partial_k;\, q^2) E_{q^4}(-q^4\partial_k^2)\right] \\ = \left[\int_{-q^{B(\beta)_n}x_n\cdot\infty}^{q^{B(\beta)_n}x_n\cdot\infty} \cdots \int_{-q^{B(\beta)_1}x_1\cdot\infty}^{q^{B(\beta)_1}x_1\cdot\infty} g_{q^2}\right] (-i)^{|A|} q^{\sum_k (a_k - a_k^2)} \otimes \prod_{j=1}^{n} (\partial_j^{a_j} E_{q^4}(-q^4\partial_j^2)) \end{aligned} \tag{7.3}$$

where the last equality follows from (8.5) in [Koo] and the product is taken in increasing order. After applying the realization map one would get:

$$\begin{aligned} &(\pi_\gamma \otimes \mathrm{id})F_S(e_{q^4}(-x_1^2)\tilde{h}_{a_1}(x_1; q^2) \cdots e_{q^4}(-x_n^2)\tilde{h}_{a_n}(x_n; q^2)) \\ &= \left[\prod_{j=1}^{n} c_{q^2}(q^{n-j+B(\beta)_j}\gamma_j)\right] (-i)^{|A|} q^{\sum_k (a_k - a_k^2)} \otimes \prod_{j=1}^{n} (\partial_j^{a_j} E_{q^4}(-q^4\partial_j^2)) \end{aligned} \tag{7.4}$$

We have another analogue of (8.21), which is less helpful, though. It can be obtained with the same techniques as formulae (5.1) and (5.2). It reads as follows:

$$(\pi_\gamma\otimes\mathrm{id})F_S\big(e_{q^4}(-\sum_j x_j^2)x_1^{a_1}\cdots x_n^{a_n}\big)=q^{-\sum_j a_jA_j}(\pi_\gamma\otimes\mathrm{id})F_S(M(q^{A_1}x_1,\ldots,q^{A_n}x_n,A))$$
$$=q^{-\sum_j(a_j+1)A_j}\sum_{\substack{e_1,\ldots,e_n\\ A(e_j)=\beta_j}}\frac{(-i)^{|E|}q^{|E|^2+|E|}}{\prod_j(q^2;q^2)_{e_j}}\times$$
$$\times\left[(I(M(\underline{x},\,A))x_1^{e_1}\cdots\,x_n^{e_n}).{}_1(q^{A_1}\gamma_1,\,\ldots,\,q^{A_n}\gamma_n)\right](q^{-A_n}\partial_n)^{e_n}\cdots(q^{-A_1}\partial_1)^{e_n}$$
$$=q^{-\sum A_j(a_j+1)}\left(\prod_{j=1}^n c_{q^2}(q^{n-j}\gamma_j)\right)(-i)^{|A|}q^{\sum(a_j^2-a_j)}\times$$
$$\prod_{k=1}^n\left[E_{q^4}(-q^{4-2A_k}\partial_k^2)\,h_{a_k}(q^{-A_k}\partial_k;\,q^2)\right]\tag{7.5}$$

where again the product is taken in *increasing* order. Hence by the same reasoning as before we can conclude that

$$\begin{aligned}F_S\Big(e_{q^4}(-\sum_j x_j^2)\,x_1^{a_1}\cdots\,x_n^{a_n}\Big)&=q^{-\sum_j(a_j+1)A_j}\left[\int_{-x_n\cdot\infty}^{x_n\cdot\infty}\cdots\int_{-x_1\cdot\infty}^{x_1\cdot\infty}g_{q^2}\right]\otimes\\ \otimes(-i)^{|A|}q^{\sum_j a_j(1-a_j)}&\prod_{k=1}^n\left[E_{q^4}(-q^{4-2A_k}\partial_k^2)h_{a_k}(q^{-A_k}\partial_k;\,q^2)\right]\end{aligned}\tag{7.6}$$

This result is less satisfactory because although the coefficients of the x_j's do not depend on the a_k's, the shift in the ∂_k's depends on the a_k's and not only on the position, hence it cannot go through to polynomials. Hence one cannot use this result in order to obtain another analogue of (8.19).

We observe that for well behaved functions, and for an n-tuple $A=(a_1,\,\ldots,\,a_n)$ of nonzero real numbers, the braided Fourier transform of $f(a_1x_1,\,\ldots,\,a_nx_n)$ can be obtained by the braided Fourier transform of $f(\underline{x})$. More precisely, $(\pi_\gamma\otimes\mathrm{id})F_S(f(a_1x_1,\,\ldots,\,a_nx_n))=\big(\prod_j a_j\big)^{-1}(\pi_{\tilde\gamma}\otimes\mathrm{id})\big(F_S(f(\underline{x}))\big)(a_1^{-1}\partial_1,\,\ldots,\,a_n^{-1}\partial_n)$ where $\tilde\gamma$ denotes the $n-$tuple obtained by γ multiplying each component γ_j by a_j. Clearly, similar results holds for $(\pi_\gamma\otimes\mathrm{id})F$, hence they hold for F_S and for F. This was used to derive formula (7.5).

Now we want to compute the braided Fourier transform for monomials or polynomials times a q^2-Gaussian of type $G_{q^2}(\underline{x})$. We cannot use the same definition since $G_{q^2}(\underline{x})$ is not even lattice integrable. Therefore, we introduce a weaker notion of braided Fourier transform.

Definition 7-2. *The "weak" braided Fourier transforms $F''(\sigma,\gamma)$ and $F''_S(\sigma,\gamma)$ are defined* *on the class* $\mathcal{R}_{(\sigma,\,\gamma)}$. *They map this class to* $V(Q,\,R)^{\text{ext}}$ *and they are defined as*

$$F''(\sigma,\gamma) := (I''_{(\sigma,\,\gamma)}\otimes \mathrm{id})(m\otimes \mathrm{id})\left(\mathrm{id}\otimes \exp\left(x\mid \frac{i}{(1-q^2)}(\partial_1,\,\ldots,\,q^{1-j}\partial_j,\,\ldots,\,q^{1-n}\partial_n)\right)\right)$$

$$F''_S(\sigma,\gamma) := (I''_{(\sigma,\,\gamma)}\otimes S)(m\otimes \mathrm{id})\left(\mathrm{id}\otimes \exp\left(x\mid \frac{iq^2}{(1-q^2)}(q^{n-1}\partial_1,\,\ldots,\,q^{n-j}\partial_j,\,\ldots,\,\partial_n)\right)\right)$$

We will use this new notion in order to derive an n-dimensional version of (8.20) in [Koo]. Namely, we will derive the transform $F''(\sigma,\,\gamma)$ of the formal power series $N(\underline{x},\,A) = E_{q^4}(-q^4x_n^2)\,x_n^{a_n}\cdots E_{q^4}(-q^4x_1^2)\,x_1^{a_1}$ for given positive integers $a_1,\,\ldots,\,a_n$. One has to compute $I''_{(\sigma,\,\gamma)}(N(\underline{x},\,A)\,x_1^{e_1}\cdots\, x_n^{e_n})$ for every E with each $e_j\equiv a_j \pmod 2$, for some fixed σ and γ. This makes sense by the computations in Section 4 because

$$N(\underline{x},\,A)\,x_1^{e_1}\cdots\, x_n^{e_n} = q^{-\sum_j E^j(e_j+a_j)}E_{q^4}(-q^{4-2E^n}x_n^2)x_n^{e_j+a_j}\cdots E_{q^4}(-q^{4-2E^1}x_1^2)x_1^{e_1+a_1}$$

is l.o. integrable for every choice of σ, with $\gamma_{\sigma(k)} = q^{(k-1)+A^{\sigma(k)}+\#\{j<k\,\mid\,\sigma(j)>\sigma(k)\}}$. In particular, since we showed that the resulting integrals differ only by a factor $q^{l(\sigma)}$, we compute it only for $\sigma=\mathrm{id}$ and $\gamma_k = q^{k-1+A^k}$. Then, denoting by $N'(\underline{x},\,A)$ the power series obtained by $N(\underline{x},\,A)$ by multiplying the argument of the $E_{q^4}(-x_j^2)$ by q^{-E^j} for every j, one has:

$$\begin{aligned}&F''(\sigma,\,\gamma)(N(\underline{x},\,A))\\ &= \sum_{\substack{e_1,\,\ldots,\,e_n\\ e_j+a_j=2h_j}} \frac{i^{|E|}q^{-\sum_j E_j-\sum_j E^j(e_j+a_j)}}{\prod_j (q^2;q^2)_{e_j}}\big(I''_{(\mathrm{id},\gamma)}(N'(\underline{x},\,A+E))\big)\partial_n^{e_n}\cdots\partial_1^{e_1}\end{aligned}$$

So that

$$\begin{aligned}&F''(\sigma,\,\gamma)(N(\underline{x},\,A))\\ &= \sum_{\substack{e_1,\,\ldots,\,e_n\\ e_j+a_j=2h_j}} \frac{i^{|E|}q^{-\sum_j E^j-\sum_j E^j(e_j+a_j)+2|E+A|}}{\prod_j (q^2;q^2)_{e_j}\,q^{2|E|+|A|}}q^{\sum_j E^j(a_j+e_j)}\times\\ &\times\big(\prod_{j=1}^{n}(q^2;q^4)_{h_j}\big)I''_{(\mathrm{id},\,\gamma)}\big(E_{q^4}(-\sum_j q^{4-2E^j}x_j^2)\big)\partial_n^{e_n}\cdots\partial_1^{e_1}\\ &= (-1)^{|A|}I''_{(\mathrm{id},\,\tilde\gamma)}(E_{q^4}(-\sum_j q^4x_j^2))\prod_{j=n}^{1}\left[\sum_{e_j+a_j=2h_j}\frac{(-i)^{e_j}(q^2;q^4)_{h_j}}{\prod_j(q^2;q^2)_{e_j}}\partial_j^{e_j}\right]\\ &= q^{\sum_j(a_j^2-a_j)}i^{|A|}b^n_{q^2}q^{\binom{n}{2}}\left(\prod_{j=n}^{1}\tilde h_{a_j}(\partial_j;q^2)e_{q^4}(-\partial_j^2)\right)\end{aligned}\tag{7.7}$$

Here $\tilde\gamma$ denotes the $n-$tuple such that $\tilde\gamma_k = q^{A^k}\gamma_k$ and the product is taken in decreasing order. For the last equality we used (9.15) and (8.20) in [Koo].

Using the definition of the h_{a_j}'s, formula (7.7) above and formula (8.17) in [Koo] one also gets the following result:

$$
\begin{aligned}
&F''(\sigma,\gamma)(E_{q^4}(-q^4x_n^2)\,h_{a_n}(x_n;q^2)\cdots E_{q^4}(-q^4x_1^2)\,h_{a_1}(x_1;q^2))\\
&= i^{|A|}q^{\sum_j(a_j^2-a_j)}I''_{(\sigma,\tilde{\gamma})}\big(E_{q^4}(-q^4\sum_j x_j^2)\big)\prod_{j=n}^{1}\partial_j^{a_j}\,e_{q^4}(-\partial_j^2)
\end{aligned}
\tag{7.8}
$$

where the product is taken in *decreasing* order and $\tilde{\gamma}$ is given by $\tilde{\gamma}_k = q^{A^k}\gamma_k$ for every k as before.

Remark: One could avoid the definition of the weak transform by finding other ways to give a meaning to the integral of power series $f(\underline{x})$ which is not lattice-integrable. One of the ways could be checking whether, for another choice of the normal form on $\hat{V}(Q,R)^{\text{ext}}$, we could provide another realization π' for which both $\pi'(f)$ and $\pi'(I'(f(\underline{x})\,x_1^{e_1}\cdots x_n^{e_n}))$ for every $x_1^{e_1}\cdots x_n^{e_n}$ are convergent, or at least converge on a lattice. For instance if we choose the reverse normal form for monomials in $\hat{V}(Q,R)$, then $\pi' N(\underline{x},A)$ would be integrable on a lattice, being the product of $E_{q^4}(-b_j^2z_j^2)z_j^{a_j}$ up to a constant. One could define an equivalence relation on $V(Q,R)^{\text{ext}}$ by saying that two elements are the same if for every realization they either both diverge, or they both converge to the same element (at least on a q^2-lattice), and there is at least one realization for which they coincide. This is a subject that has to be further developed. These ideas are due to T. Koornwinder. ♠

8. Integral on $V(Q,R)$ and Inverse Transform

We want now to build an inverse for the braided Fourier transforms, at least on a subspace of $\mathcal{R}$ and $\mathcal{R}_{(\sigma,\gamma)}$. In order to do this one needs an integral on $V(Q,R)^{\text{ext}}$, and to develop the theory in this algebra. Since there is a symmetry between $\hat{V}(Q,R)^{\text{ext}}$ and $V(Q,R)^{\text{ext}}$, one can simply repeat the definitions and computations keeping in mind that whenever we had a left action involving $\hat{V}(Q,R)^{\text{ext}}$, we will need a right action in the case of $V(Q,R)^{\text{ext}}$. We will only provide the necessary formulae, while the properties and the proofs of similar statements as those of Section 4, 5 and 6 are left to the reader. We observe that all the results in this Section can be achieved both by direct computation or by using the symmetry $\psi\colon \hat{V}(Q,R)^{\text{ext}}\to V(Q,R)^{\text{ext}}$ defined in Section 2.

Just as for $\hat{V}(Q,R)^{\text{ext}}$, there are partial derivatives on $V(Q,R)^{\text{ext}}$. Those are given on a series $g(\underline{\partial})$ by the generalized coefficient of $1\otimes\partial_i$ in $\Delta(g(\underline{\partial}))$, i.e. for a monomial $\partial_n^{e_n}\cdots\partial_1^{e_1}$, and for $j\in\{1,\ldots,n\}$

$$(\partial_n^{e_n}\cdots\partial_1^{e_1})\leftharpoondown D_i = [e_i]_{q^2}\partial_n^{e_n}\cdots\partial_{i+1}^{e_{i+1}}\partial_i^{e_i-1}(q\partial_{i-1})^{e_{i-1}}\cdots(q\partial_1)^{e_1}$$

i.e. for $g(\underline{\partial})\in V(Q,R)^{\text{ext}}$:

$$g(\underline{\partial})\leftharpoondown D_i = \frac{g(\partial_n,\ldots,q^2\partial_i,\ldots,q^2\partial_1)-g(\partial_n,\ldots,\partial_i,q^2\partial_{i-1},\ldots,q^2\partial_1)}{(q^2-1)}\partial_i^{-1}$$

The operators D_i define a right action of $\hat{V}(Q,\,R)$ on $V(Q,\,R)$ such that x_j acts as D_j. There holds a right version of Taylor's formula, namely:

$$\Delta(g(\underline{\partial})) = g(\Delta(\underline{\partial})) = g(\underline{\partial}) \leftharpoondown \left(\sum_{e_1,\ldots,\,e_n \geq 0} \frac{D_1^{e_1}\cdots\, D_n^{e_n} \otimes \partial_n^{e_n}\cdots\,\partial_1^{e_1}}{[e_1]_{q^2}!\cdots\,[e_n]_{q^2}!} \right)$$

A partial inverse for D_i is then given by the (indefinite) q^2-integral acting from the right:

$$g(\underline{\partial}) \leftharpoondown \int_0^{\partial_i} := (1-q^2)\sum_{k=0}^{\infty} g(\partial_n,\ldots,\,q^{2k}\partial_i,\,q^{-2}\partial_{i-1},\ldots,\,q^{-2}\partial_1)q^{2k}\partial_i$$

and again as in the case of $\hat{V}(Q,\,R)$, one can define $\int_0^{-\partial_i}$, $\int_0^{q^p a}$. The global integral is formally obtained as the limit for all $r_j \to \infty$ of:

$$g(\underline{\partial}) \leftharpoondown \int_{-q^{-2r_1}\partial_1}^{q^{-2r_1}\partial_1} \cdots \int_{-q^{-2r_n}\partial_n}^{q^{-2r_n}\partial_n}$$

where

$$g(\underline{\partial}) \leftharpoondown \int_{-\partial_j}^{\partial_j} := \left(g(\underline{\partial}) \leftharpoondown \int_0^{\partial_j}\right) - \left(g(\underline{\partial}) \leftharpoondown \int_0^{-\partial_j}\right)$$

Hence we have formally

$$\begin{aligned} g(\underline{\partial}) \leftharpoondown \int^{\underline{\partial}} &:= g(\underline{\partial}) \leftharpoondown \int_{-\partial_1\cdot\infty}^{\partial_1\cdot\infty} \cdots \int_{-\partial_n\cdot\infty}^{\partial_n\cdot\infty} \\ &= (1-q^2)^n \sum_{\underline{\varepsilon}\in\{\pm 1\}^n} \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} g(\varepsilon_n q^{2k_n}\partial_n,\,\ldots,\,\varepsilon_1 q^{2k_1}\partial_1)q^{2|K|}\partial_1\cdots\partial_n \end{aligned}$$

As in Section 4, one can define an action of $V(Q,\,R)$ on the power series in the n commuting indeterminates $z_1,\,\ldots,\,z_n$, in order to give a meaning to the integral. One can use this action to define integrability, lattice integrability and lattice order integrability as in Section 5 and 6, and we leave this to the reader. The action will be the right regular action after the choice of a normal form. It is denoted by $\triangleleft$ and it is defined on monomials as:

$$(z_1^{k_1}\cdots z_n^{k_n}) \triangleleft \partial_n^{e_n}\cdots\,\partial_1^{e_1} := q^{\sum_j K_j e_j} z^{e_1+k_1}\cdots z_n^{e_n+k_n} = z_1^{k_1}(q^{K_1}z_1)^{e_1}\cdots z_n^{k_n}(q^{K_n}z_n)^{e_n}$$

For brevity we will denote $1 \triangleleft g(\underline{\partial}) :=_1 . g(\underline{z})$ for every expression $g(\underline{\partial})$ for which the action on 1 makes sense.
As in Section 5 we construct projections $P_j^{\pm}$ defined on $V(Q,\,R)$ and $V(Q,\,R)^{\text{ext}}$ for every $j = 1,\ldots,\,n$ and every choice of $+$ or $-$ as follows:

$$\begin{aligned} P_j^{\pm}\colon V(Q,\,R)^{\text{ext}} &\to V(Q,\,R)^{\text{ext}} \\ g(\underline{\partial}) &\mapsto \frac{1}{2}[g(\underline{\partial}) \pm g(\ldots,\,-\partial_j,\,\ldots)] \end{aligned}$$

Again, the $P_j^{\pm}$'s commute with each other, they are projections on the subspaces of $V(Q,\,R)^{\mathrm{ext}}$ consisting of even (resp. odd) elements with respect to the j^{th} variable, and $P_j^+P_j^- = 0$ for every j. Then, for every $\beta \in \{-,\,+\}^n$ we define $P_\beta := P_1^{\beta_1}\circ\cdots\circ P_n^{\beta_n}$. The P_β's are all projections on their image G_β and clearly the decomposition $V(Q,\,R)^{\mathrm{ext}} = \bigoplus_\beta G_\beta$ corresponds to the decomposition of $\mathbf{C}[[\underline{z}]]$ in series that are either odd or even in each variable after applying the action on 1. We write P_0 for $P_{(+,\dots,+)}$. We can define again the integral J' defined as applying $\int^{\underline{\partial}}$ only to the even part of the power series.
In particular one can check that for a $g(\underline{\partial})$ for which this makes sense one has:

$$_{1\cdot}\!\big(P_0 g(\underline{\partial}) \leftharpoondown J'\big) =_{1\cdot} \left(\Big(P_0\big(g(\underline{\partial})\big)\Big) \leftharpoondown \int^{\underline{\partial}}\right) = \int_{-z_1q^{n-1}\cdot\infty}^{z_1q^{n-1}\cdot\infty}\cdots\int_{-z_nq^{n-n}\cdot\infty}^{z_nq^{n-n}\cdot\infty} {}_{1\cdot}\!(P_0(g))(\underline{t})d_{q^2}\underline{t}$$

and

$$_{1\cdot}\!(g \leftharpoondown D_1^{j_1}\cdots D_n^{j_n}) = (D_{q^2}^{J}({}_{1\cdot}g))(q^{J^1}z_1,\dots,\,q^{J^n}z_n)$$

We also observe that under the "symmetry" $\psi\colon \hat{V}(Q,\,R)^{\mathrm{ext}} \to V(Q,\,R)^{\mathrm{ext}}$ mapping x_j to $q^{-j+\frac{1}{2}(n-1)}\partial_{n-j+1}$ one has

$$\psi(I\cdot f(x_1,\dots,x_n)) = q^{-n}\left[(\psi(f(\underline{x}))(\underline{\partial})) \leftharpoondown \int_{-q^{-n+1}\partial_1\cdot\infty}^{q^{-n+1}\partial_1\cdot\infty}\cdots\int_{-q^{-n+2j-1}\partial_j\cdot\infty}^{q^{-n+2j-1}\partial_j\cdot\infty}\cdots\int_{-q^{n-1}\partial_n\cdot\infty}^{q^{n-1}\partial_n\cdot\infty}\right] \tag{8.1}$$

One defines the integral $J''_{(\sigma,\,\gamma)}$ for l.o. integrable power series as the right handed version of $I''_{(\sigma,\,\gamma)}$. We will need $J''_{(\sigma,\,\gamma)}(E_{q^4}(-q^4\partial_1^2)\partial_1^{a_1}\cdots,\,E_{q^4}(-q^4\partial_n^2)\partial_n^{a_n})$. One checks as for the case of $\hat{V}(Q,\,R)^{\mathrm{ext}}$ that the integrand is actually is l.o. integrable for every σ for a suitable choice of γ, and that the results differ only by a factor $q^{\mathrm{l}(\sigma)}$. In particular, for $\sigma = \mathrm{id}$ one needs $\gamma_k = q^{(k-1)+A^k}$ and one gets

$$J''_{(\sigma,\,\gamma)}(E_{q^4}(-q^4\partial_1^2)\partial_1^{a_1}\cdots,\,E_{q^4}(-q^4\partial_n^2)\partial_n^{a_n}) = \begin{cases} q^{\binom{n}{2}}b_{q^2}^n\prod_j(q^2;q^4)_{b_j} & \text{if } a_j = 2b_j\ \forall j\\ 0 & \text{otherwise}\end{cases}$$

hence

$$J''_{(\sigma,\,\gamma)}(E_{q^4}(-q^4\partial_1^2)\partial_1^{a_1}\cdots,\,E_{q^4}(-q^4\partial_n^2)\partial_n^{a_n}) = \big(J''_{(\sigma,\,\gamma)}(E_{q^4}(-q^4\sum_j\partial_j^2))\prod_j(q^2;q^2)_{b_j}$$

if $a_j = 2b_j$ for every j.
Properties like right invariance of the integral, nullity of the integral of the partial derivative of a power series and so on can be proved as in Section 4.

Now we are ready to introduce an inversion formula for F_S and F and their weak analogues using the symmetry between $\hat{V}(Q,\,R)^{\mathrm{ext}}$ and $V(Q,\,R)^{\mathrm{ext}}$ and the results at the end of the previous Section. This will provide an analogue of Theorem 8.1 in [Koo]. The first inversion formula for the braided Fourier transform was due to [KeMa], but we will use a different one and the braiding will not appear in our formulation. As we said our direct transforms are also different from Kempf and Majid's one. The reason why we looked for a different formula is that the element *vol* in [KeMa] is not necessarily convergent. Moreover, their formula does not necessarily work in our case because our integral is not "bosonic", i.e. it has a nontrivial braiding with $\hat{V}(Q,\, R)$, or $V(Q,\, R)$ for $n \geq 2$, as the reader can easily check (see also [Ch] for a few remarks about this property of the integral). Another inversion formula, for the case $n=1$ appears in [OR], where the transforms are shown to be isomorphisms of topological spaces. We could not extend their results, because we have not yet defined a topology on the noncommutative spaces we are working with, while for $n=1$ the spaces of functions in x or in ∂ can be identified with the classical ones. This is material for future research.

We say that a power series $g(\underline{\partial})$ in $V(Q,\, R)^{\mathrm{ext}}$ is *of class* $\mathcal{S}$ if every monomial times $g(\underline{\partial})$ is $J'-$integrable. This is possible for instance if $g(\underline{\partial})$ satisfies conditions similar to condition (c) of Section 4. We say that $g(\underline{\partial})$ is *of class* $\mathcal{S}_{(\sigma,\,\gamma)}$ if there is an order σ and a lattice $L(\gamma)$ such that every monomial times $g(\underline{\partial})$ are lattice order integrable for σ and γ.

Definition 8-1. *We define the linear maps* $G,\, G_S : \mathcal{S} \to \hat{V}(Q,\, R)^{\mathrm{ext}} \otimes V(Q,\, R)^{\mathrm{ext}}$ *by*

$$G := \left[(\mathrm{id} \otimes m)\left(\exp(x \,|\, \frac{i}{(1-q^2)}(\partial_1,\, \ldots,\, q^{-j+1}\partial_j,\, \ldots,\, q^{-n+1}\partial_n)) \otimes \mathrm{id}\right)\right] \leftharpoonup (\mathrm{id} \otimes J')$$

$$G_S := \left[(S \otimes m)\left(\exp(x \,|\, \frac{iq^2}{(1-q^2)}(q^{n-1}\partial_1,\, \ldots,\, q^{n-j}\partial_j,\, \ldots,\, \partial_n)) \otimes \mathrm{id}\right)\right] \leftharpoonup (id \otimes J')$$

For instance for G_S, the transform of a given $g(\underline{\partial}) \in \mathcal{S}$ of fixed parity β will be

$$G_S(g(\underline{\partial})) = \left((S \otimes \mathrm{id})(E_{q^2}(iq^2 \sum_j q^{(n-j)} x_j \otimes \partial_j))g(\underline{\partial})\right) \leftharpoonup (\mathrm{id} \otimes J')$$

$$= \sum_{\substack{e_1,\, \cdots,\, e_n \\ A(e_j)=\beta_j}} \frac{(-i)^{|E|} q^{|E|^2+|E|} q^{\sum_j (n-j)e_j}}{\prod_j (q^2;q^2)_{e_j}} x_1^{e_1} \cdots x_n^{e_n} \otimes (\partial_n^{e_n} \cdots \partial_1^{e_1} g(\underline{\partial}) \leftharpoonup (\mathrm{id} \otimes J'))$$

If $\tau : V(Q,\, R)^{\mathrm{ext}} \otimes \hat{V}(Q,\, R)^{\mathrm{ext}} \to \hat{V}(Q,\, R)^{\mathrm{ext}} \otimes V(Q,\, R)^{\mathrm{ext}}$ denotes the usual flip operator, putting $c_j = q^{-j+\frac{1}{2}(n-1)}$ for every $j = 1, \ldots,\, n$ we observe that for $f(\underline{x}) = f_1(x_1) \cdots f_n(x_n)$ with f_j of parity β_j, there holds:

$$\tau(\psi \otimes \psi^{-1})F_S(f(\underline{x})) =$$

$$= \sum_{\substack{e_1, \ldots,\, e_n \\ A(e_j)=\beta_j}} \frac{(-i)^{|E|} q^{|E|^2+|E|+\sum_j E_j}}{\prod_j (q^2;q^2)_{e_j}} (c_1^{-1}x_1)^{e_n} \cdots (c_j^{-1}x_j)^{e_{n-j+1}} \cdots (c_n^{-1}x_n)^{e_1} \otimes$$

$$q^{-n}\left[f_1(c_1 q^{E_1}\partial_n)(c_1\partial_n)^{e_1} \cdots f_n(c_n q^{E_n}\partial_1)(c_n\partial_1)^{e_n}\right] \leftharpoonup \int_{-q^{-n+1}\partial_1\cdot\infty}^{q^{-n+1}\partial_1\cdot\infty} \cdots \int_{-q^{n-1}\partial_n\cdot\infty}^{q^{n-1}\partial_n\cdot\infty}$$

and this is equal to

$$= \sum_{\substack{e_1,\ldots,e_n\\ A(e_j)=\beta_j}} \frac{(-i)^{|E|}q^{|E|^2+|E|-\sum_j e_j E_j}}{\prod_j (q^2;q^2)_{e_j}} (c_1^{-1}x_1)^{e_n}\cdots(c_j^{-1}x_j)^{e_{n-j+1}}\cdots(c_n^{-1}x_n)^{e_1}\otimes$$

$$\left[\partial_n^{e_1}\cdots\partial_1^{e_n} f_1(q^{-E^1}\partial_n)\cdots f_n(q^{-E^n}\partial_1)\right] \leftharpoonup \int_{-q^{-n+1+E_n}c_n\partial_1\cdot\infty}^{q^{-n+1+E_n}c_n\partial_1\cdot\infty}\cdots\int_{-c_1q^{n-1+E_1}\partial_n\cdot\infty}^{c_1q^{n-1+E_1}\partial_n\cdot\infty} =$$

$$= \sum_{\substack{e_1,\ldots,e_n\\ A(e_j)=\beta_j}} \frac{(-i)^{|E|}q^{|E|^2+|E|+\sum_j E^j}}{\prod_j (q^2;q^2)_{e_j}} (c_1^{-1}x_1)^{e_n}\cdots(c_j^{-1}x_j)^{e_{n-j+1}}\cdots(c_n^{-1}x_n)^{e_1}\otimes$$

$$\left[\partial_n^{e_1}\cdots\partial_1^{e_n} f_1(\partial_n)\cdots f_n(\partial_1)\right] \leftharpoonup \int_{-q^{-n+1+E_n-E_1}c_n\partial_1\cdot\infty}^{q^{-n+1+E_n-E^1}c_n\partial_1\cdot\infty}\cdots\int_{-c_1q^{n-1+E_1-E^n}\partial_n\cdot\infty}^{c_1q^{n-1+E_1-E^n}\partial_n\cdot\infty}$$

Hence we see that the formal expression of $\tau(\psi\otimes\psi^{-1})F_S(f(\underline{x}))$ coincides with the formal expression of

$$G_S(f_1(\partial_n)\cdots f_n(\partial_1))((c_1^{-1}x_1,\ldots,c_n^{-1}x_n)\otimes(q^{n-1+E_1-E^1}c_1\partial_n,\ldots,q^{E_n-E^n+1-n}c_n\partial_1))$$

In the same way one shows that the formal expression of $\tau(\psi\otimes\psi^{-1})F(f(\underline{x}))$ coincides with

$$G(f_1(\partial_n)\cdots f_n(\partial_1))((c_1^{-1}x_1,\ldots,c_n^{-1}x_n)\otimes(q^{n-1+E_1-E^1}c_1\partial_n,\ldots,q^{E_n-E^n+1-n}c_n\partial_1))$$

We can use the symmetry between F_S and G_S, together with formula (7.2) in order to compute $G_S(\partial_n^{a_n}e_{q^4}(-\partial_n^2)\cdots\,\partial_1^{a_1}e_{q^4}(-\partial_1^2))$ for given positive integers $a_1,\ldots,a_n$ of parity respectively $\beta_1,\ldots,\beta_n$. Indeed the symmetry tells that:

$$\begin{aligned}&G_S\big(e_{q^4}(-\partial_n^2)\partial_n^{a_n}\cdots e_{q^4}(-\partial_1^2)\partial_1^{a_1}\big) =\\ &(L_\psi\otimes L'_{\psi,\beta})\big[\tau(\psi\otimes\psi^{-1})\tau F_S(e_{q^4}(-x_1^2)x_1^{a_n}\cdots e_{q^4}(-x_n^2)x_n^{a_1})\big]\end{aligned}$$

where L_ψ is the shift operator mapping x_j to c_jx_j and $L'_{\psi,\beta}$ is the shift operator mapping ∂_j to $q^{n-2j+1+B(\beta)^j-B(\beta)_j}c_{n-j+1}^{-1}\partial_j$. Then the above expression is equal to:

$$(-i)^{|A|}q^{\sum_j(a_j-a_j^2)}\prod_{j=n}^{1}\left[E_{q^4}(-q^4x_j^2)h_{a_j}(x_j;q^2)\right]\otimes L'_{\psi,\beta}\psi\int_{-q^{B(\beta)^1}x_n\cdot\infty}^{q^{B(\beta)^1}x_n\cdot\infty}\cdots\int_{-q^{B(\beta)^n}x_1\cdot\infty}^{q^{B(\beta)^n}x_1\cdot\infty}g_{q^2}$$

$$= (-i)^{|A|}q^{\sum_j(a_j-a_j^2)}\prod_{j=n}^{1}\left[E_{q^4}(-q^4x_j^2)h_{a_j}(x_j;q^2)\right]\otimes$$

$$q^{-n}L'_{\psi,\beta}\left(g_{q^2}(c_1\partial_n,\ldots,c_n\partial_1)\leftharpoonup\int_{-q^{B(\beta)^1-n+1}\partial_1\cdot\infty}^{q^{B(\beta)^1-n+1}\partial_1\cdot\infty}\cdots\int_{-q^{B(\beta)^n+n-1}\partial_n\cdot\infty}^{q^{B(\beta)^n+n-1}\partial_n\cdot\infty}\right)$$

So we can conclude that

$$G_S\big(e_{q^4}(-\partial_n^2)\partial_n^{a_n}\cdots e_{q^4}(-\partial_1^2)\partial_1^{a_1}\big) =$$
$$(-i)^{|A|}q^{\sum_j(a_j-a_j^2)}\prod_{j=n}^{1}\left[E_{q^4}(-q^4x_j^2)h_{a_j}(x_j;q^2)\right]\otimes\left(g_{q^2}\leftharpoonup\int_{-q^{B(\beta)_1}\partial_1\cdot\infty}^{q^{B(\beta)_1}\partial_1\cdot\infty}\cdots\int_{-q^{B(\beta)_n}\partial_n\cdot\infty}^{q^{B(\beta)_n}\partial_n\cdot\infty}\right) \tag{8.2}$$

where the second component of the tensor product clearly depends only on the parity of the a_j's. Of course formula (8.2) can also be easily obtained by direct computation. Using the definition of the $\tilde{h}_m(z;q^2)$, with the same relation as before between the a_j's and the β_j's one obtains:

$$G_S\big(\tilde{h}_{a_n}(\partial_n;q^2)e_{q^4}(-\partial_n^2)\cdots\tilde{h}_{a_1}(\partial_1;q^2)e_{q^4}(-\partial_1^2)\big)$$
$$=(-i)^{|A|}q^{\sum_j(a_j-a_j^2)}\prod_{j=n}^{1}\left[E_{q^4}(-q^4x_j^2)x_j^{a_j}\right]\otimes\left(g_{q^2}\leftharpoonup\int_{-q^{B(\beta)_1}\partial_1\cdot\infty}^{q^{B(\beta)_1}\partial_1\cdot\infty}\cdots\int_{-q^{B(\beta)_n}\partial_n\cdot\infty}^{q^{B(\beta)_n}\partial_n\cdot\infty}\right) \tag{8.3}$$

which is the $V(Q,\,R)^{\mathrm{ext}}$-version of (7.3). By these results we can conlcude that:

Proposition 8-2. *Let $\beta=(\beta_1,\,\dots,\,\beta_n)\in\{\pm1\}^n$, σ be any order and γ be the n-tuple with components given by $\gamma_{\sigma(k)}=q^{B(\beta)^{\sigma(k)}+k-1+\#\{j<k\,|\,\sigma(j)>\sigma(k)\}}$. Then on power series in $\hat{V}(Q,\,R)^{\mathrm{ext}}$ of type $E_{q^4}(-q^4x_n^2)p_n(x_n)\cdots E_{q^4}(-q^4x_1^2)p_1(x_1)$ where the p_i's are polynomials of fixed parity $\beta_1,\,\dots,\,\beta_n$, there holds:*

$$G_S\circ F''(\sigma,\,\gamma)=\mathrm{id}\otimes\big(I''_{(\sigma,\,\tilde\gamma)}G_{q^2}(q^2\underline{x})\big)\left(g_{q^2}\leftharpoonup\int_{-q^{B(\beta)_1}\partial_1\cdot\infty}^{q^{B(\beta)_1}\partial_1\cdot\infty}\cdots\int_{-q^{B(\beta)_n}\partial_n\cdot\infty}^{q^{B(\beta)_n}\partial_n\cdot\infty}\right)$$

where $\tilde\gamma$ is such that $\tilde\gamma_k=\gamma_kq^{B(\beta)_k}$ for every k. Therefore for power series in $V(Q,\,R)^{\mathrm{ext}}$ of the form $q_n(\partial_n)e_{q^4}(-\partial_n^2)\cdots q_1(\partial_1)e_{q^4}(-\partial_1^2)$ where the q_j's are polynomials of fixed parity $\beta_1,\,\dots,\,\beta_n$, one has

$$(F''(\sigma,\,\gamma)\otimes\mathrm{id})G_S=\mathrm{id}\otimes\big(I''_{(\sigma,\,\tilde\gamma)}G_{q^2}(q^2\underline{x})\big)\left(g_{q^2}\leftharpoonup\int_{-q^{B(\beta)_1}\partial_1\cdot\infty}^{q^{B(\beta)_1}\partial_1\cdot\infty}\cdots\int_{-q^{B(\beta)_n}\partial_n\cdot\infty}^{q^{B(\beta)_n}\partial_n\cdot\infty}\right)$$

with $\tilde\gamma$ as before.

Proof: It follows by (7.8) and (8.2). □

A slightly more general version of this proposition holds by considering a proper γ and q^2-Gaussians where the argument is multiplied by a nonzero constant.

We also want to consider another inverse transform, the weak transform inverting (7.2) and (7.3).

Definition 8-3. *The "weak" transforms $G''(\sigma,\,\gamma)$ and $G''_S(\sigma,\,\gamma)$ map $\mathcal{S}_{(\sigma,\,\gamma)}$ to $\hat{V}(Q,\,R)^{\mathrm{ext}}$ and are defined as:*

$$G''(\sigma,\,\gamma) :=$$
$$\left[(\mathrm{id}\otimes m)\left(\exp(x\,|\frac{i}{(1-q^2)}(\partial_1,\ldots,q^{-j+1}\partial_j,\ldots,q^{-n+1}\partial_n))\otimes\mathrm{id}\right)\right]\leftharpoonup\left(\mathrm{id}\otimes J''_{(\sigma,\,\gamma)}\right)$$
$$G''_S(\sigma,\,\gamma) :=$$
$$\left[(S\otimes m)\left(\exp(x\,|\frac{iq^2}{(1-q^2)}(q^{n-1}\partial_1,\ldots,q^{n-j}\partial_j,\ldots,\partial_n))\otimes\mathrm{id}\right)\right]\leftharpoonup\left(\mathrm{id}\otimes J''_{(\sigma,\,\gamma)}\right)$$

As in formulae (7.7) and (7.8) one finds, for $\sigma=\mathrm{id}$ and $\gamma_k=q^{A^k+k-1}$ (and similarly for different σ's)

$$\begin{aligned}&G''_{(\mathrm{id},\,\gamma)}\big(E_{q^4}(-q^4\partial_1^2)\partial_1^{a_1}\cdots E_{q^4}(-q^4\partial_n^2)\partial_n^{a_n}\big)\\&=q^{\sum_j(a_j^2-a_j)}i^{|A|}\big(\prod_{j=1}^{n}\tilde{h}_{a_j}(x_j;q^2)e_{q^4}(-x_j^2)\big)\,[J''_{(\mathrm{id},\,\tilde{\gamma})}\big(G_{q^2}(q^2\underline{\partial})\big)]\end{aligned}\tag{8.4}$$

and

$$\begin{aligned}&G''_{(\mathrm{id},\,\gamma)}\big(E_{q^4}(-q^4\partial_1^2)h_{a_1}(\partial_1)\cdots E_{q^4}(-q^4\partial_n^2)h_{a_n}(\partial_n)\big)\\&=q^{\sum_j(a_j^2-a_j)}i^{|A|}\big(\prod_{j=1}^{n}x_j^{a_j}\,e_{q^4}(-x_j^2)\big)\,[J''_{(\mathrm{id},\,\tilde{\gamma})}\big(G_{q^2}(q^2\underline{\partial})\big)]\end{aligned}\tag{8.5}$$

where in both formulae $\tilde{\gamma}_k=q^{k-1}$ for every k and the product is taken in increasing order. These formulae can be obtained by using the symmetry or by direct computation. One has the second inversion property:

Proposition 8-4. *Let $\beta=(\beta_1,\,\ldots,\,\beta_n)\in\{\pm1\}^n$, σ be any order and γ be the n-tuple with components given by $\gamma_{\sigma(k)}=q^{B(\beta)^{\sigma(k)}+k-1+\#\{j<k\,|\,\sigma(j)>\sigma(k)\}}$. Then on power series in $V(Q,\,R)^{\mathrm{ext}}$ of type $E_{q^4}(-q^4\partial_1^2)p_1(\partial_1)\cdots E_{q^4}(-q^4\partial_n^2)p_n(\partial_n)$ where the p_j's are polynomials of fixed parity $\beta_1,\,\ldots,\,\beta_n$, there holds:*

$$F_S\circ G''(\sigma,\,\gamma)=\left[\int_{-q^{B(\beta)_n}x_n\cdot\infty}^{q^{B(\beta)_n}x_n\cdot\infty}\cdots\int_{-q^{B(\beta)_1}x_1\cdot\infty}^{q^{B(\beta)_1}x_1\cdot\infty}g_{q^2}\right]\left(J''_{(\sigma,\,\tilde{\gamma})}\big(G_{q^2}(q^2\underline{\partial})\big)\right)\otimes\mathrm{id}$$

where $\tilde{\gamma}_{\sigma(k)}=\gamma_{\sigma(k)}q^{B(\beta)^{\sigma(k)}}$. Therefore for power series in $\hat{V}(Q,\,R)^{\mathrm{ext}}$ having the form $q_1(x_1)e_{q^4}(-x_1^2)\cdots q_n(x_n)e_{q^4}(-x_n^2)$ where the q_j's are polynomials of fixed parity $\beta_1,\ldots,\beta_n$, one has

$$(\mathrm{id}\otimes G''(\sigma,\,\gamma))F_S=\left[\int_{-q^{B(\beta)_n}x_n\cdot\infty}^{q^{B(\beta)_n}x_n\cdot\infty}\cdots\int_{-q^{B(\beta)_1}x_1\cdot\infty}^{q^{B(\beta)_1}x_1\cdot\infty}g_{q^2}\right]\left(J''_{(\sigma,\,\tilde{\gamma})}\big(G_{q^2}(q^2\underline{\partial})\big)\right)\otimes\mathrm{id}$$

with $\tilde{\gamma}$ as before.

Proof: It follows by (7.2), (7.3), (8.4) and (8.5). $\square$

One observes that in this case the Plancherel measure is always a product of integrals of q^2-Gaussians, but the integration bounds depend on the parity of the power series. So these transforms could also be seen as sine and cosine transforms (see for this also [KooSw] where the q-sine and q-cosine transforms in commuting variables are defined).

Remark The break in symmetry appearing in q^2-integration is a phenomenon that has recently been observed by [CHMW]. Sometimes this lack of symmetry can be avoided, for instance if the generalized function $f(\underline{x})$ that we want to integrate (and transform) is lattice integrable in $L(\gamma)$ and $L(q\gamma)$. In this case we could replace q^2-integration by q-integration since $\int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} f_{\cdot 1}(t)d_qt = \int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} f_{\cdot 1}(t)d_{q^2}t + \int_{-q\gamma\cdot\infty}^{q\gamma\cdot\infty} f_{\cdot 1}(t)d_{q^2}t$. The new defined integral will be again invariant under translation. The q-integral of a q^2-Gaussians $g_{q^2}(\underline{x})$ times a monomial will be similar to the q^2-integral of the same expression. Using the q-integral in the definition of $(\pi_\gamma \otimes \mathrm{id})F_S$ will provide results similar to formula (7.1) but with $\prod_j c_{q^2}(q^{n-j+B(\beta)_j}\gamma_j)$ replaced by $\prod_j (c_{q^2}(\gamma_j) + c_{q^2}(q\gamma_j))$. The result will be therefore independent of the parity of the a_j's. However, this approach cannot be used for $G_{q^2}(\underline{x})$, since we have seen that there is only one q^2-lattice for which $G_{q^2}(\underline{x})$ is lattice order integrable. ♠

Acknowledgments

The author would like to thank Professor T. H. Koornwinder for many discussions, and for his useful comments and remarks.

References

[AST] M. Artin, W. Schelter, and J. Tate *"Quantum deformation of GL_n"* Comm. Pure Appl. Math. **44** (1991) 879-895

[Ca] G. Carnovale *"On the braided Fourier transform and the convolution product in the A_n case"* to appear

[Ch] C. Chryssomalakos *"Remarks on Quantum Integration"* Comm. Math. Phys. **184** (1997) 1-25

[CHMW] B. L. Cerchiai, R. Hinterding, J. Madore and J. Wess *"A Calculus Based on a q-deformed Heisenberg Algebra"*, QA/9809160

[ChZu] C. Chryssomalakos and B. Zumino *"Translations, Integrals and Fourier Transforms in the quantum plane"* in A. Ali, J. Ellis and S. Randjbar-Daemi, eds. "Salamfestschrift", proceedings of the "Conference on highlights of particles and condensed matter physics" ICTP, Trieste, Italy (1993)

[Du] M. Durdevic *"On braided quantum groups"* q-alg 9412003

[FRT] N. Yu. Reshetikhin, L.A. Takhtadjian and L.D. Faddeev *"Quantization of Lie groups and Lie algebras"* English Transl. Leningrad Math. J.**1** (1990) 193-225

[HH] M. Hashimoto and T. Hayashi *"Quantum multilinear algebra"* Tôhoku Math. J., **44** (1992), 471-521

[HW] R. Hinterding and J. Wess *"q-deformed Hemite Polynomials in q-Quantum Mechanics"*, QA/9803050

[KeMa] A. Kempf and S. Majid *"Algebraic q-integration and Fourier theory on quantum and braided spaces"* J. Math. Phys. **35**(12) (1994) 6802-6837

[Koo] T. H. Koornwinder *"Special Functions and q-commuting variables"* Special Functions, q-series and related topics, M.E.H Ismail. D.R. Masson, M. Rahman (eds.), Fields Institute Communications 14, AMS, (1997) 131-166

[KooSw] T. H. Koornwinder and R. F. Swarttouw *"On q-analogues of the Fourier and Hankel transforms"* Trans. Am. Math. Soc. 333 (1992) 445-461

[KoSw] R. Koekoek and R. Swarttouw *"The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue"* Revised Version of Report 94-05 Delft University of Technology, Faculty TWI, (1996)

[LyuMa] V. V. Lyubashenko and S. Majid *"Braided groups and quantum Fourier transform"* J. Alg. 166 506-528 (1994)

[Ma] S. Majid *"Foundations of quantum group theory"* Cambridge University Press, (1995)

[MMNNU] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno *"Representations of the Quantum Group $SU_q(2)$ and the Little q-Jacobi Polynomials"* J. Funct. Anaysis 99, (1991), 357-387

[OR] M. Olshanetsky and V. Rogov *"The q-Fourier transform of q-distributions"* IHES Preprint q-alg 9712055

[Ro] L. Rowen *"Ring Theory"* Vol. I, II Academic press (1988)

[Sch] M. P. Schützenberger *"Une interpétation de certaines solutions de l'équation fonctionelle: $F(x+y) = F(x)F(y)$"* C. R. Acad. Sci. Paris 236 (1953) 352-353