Axioms and (Counter)examples in Synthetic Domain Theory

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Introduction

The central idea of *Synthetic Domain Theory* (SDT) is that, if one formalizes abstract properties that a category of domains should have (relative to an intended range of applications), one might find full subcategories of the category of sets enjoying these properties. Domains, then, would just be (special) sets, maps of domains would be arbitrary set-theoretic functions, and various constructions on domains would (ideally) be set-theoretic constructions.

Unfortunately, one soon observes that this idea runs into trouble with classical set theory. For example, there are precious few sets with the property that *any* endofunction on them has a fixed point.

Remarkably however, as Dana Scott observed in [25], such inconsistencies do not arise if intuitionistic set theory is used instead. For this reason, Scott proposed that intuitionistic set theory might provide an intuitive and powerful framework for deriving domain-theoretic structure as set-theoretic structure. This proposal has since been vindicated both axiomatically in (many versions of) intuitionistic set theory, and semantically in models of intuitionistic set theory (especially toposes).

To carry out an axiomatic treatment, one first needs to fix on a version of intuitionistic set theory. In this paper we adopt the most popular choice, the internal logic of an elementary topos (with nno), also chosen, e.g., in [23, 8, 26]. The principal benefits are that models of the logic (toposes) are ubiquitous, and the methods for constructing and analysing them are very well-established. For the purposes of the axiomatic part of this paper, we believe that it would also be

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possible to use an (impredicative) intuitionistic type theory, as in [22], or even intuitionistic Zermelo-Fraenkel set theory ([27]), without changing the nature of the mathematics (only the metamathematics). An interesting challenge would be to attempt an axiomatic development in a predicative type theory.

It seems that the best way of isolating a full subcategory of sets is to identify a category of *predomains*, carriers of computational values, not necessarily including any specific "undefined" value (In classical domain-theoretic terms, predomains are cpo's without the requirement of a least element). One aims to place axioms that guarantee that the category is closed under important set-theoretic constructions (e.g. function spaces), and allows a treatment of domain-theoretic phenomena, such as recursion.

The fundamental axiom of synthetic domain theory is now well accepted. Following [23], one first identifies a set, Σ, of "termination properties" (which one can often think of as classifying an abstract notion of "semidecidable property" or "open subset"). As in [15], our main axiom, Axiom 1, asks for Σ to satisfy a certain "completeness" property. In the presence of this axiom alone, it is possible to identify a number of different notions of predomain. Amongst these, the *replete sets* [8, 30], and the *well-complete sets* [15, 22, 26] form two extreme choices. The former form the smallest full reflective subcategory of sets containing Σ, and the latter form what appears to be the largest full subcategory of sets supporting an adequate treatment of recursion. Although it is not known if well-complete sets form a reflective subcategory in general, their restriction to well-complete Σ-*posets* do [29]. These form a category intermediate between the replete and well-complete sets. Each of these three notions provides a complete full subcategory of sets closed under important domain-theoretic constructions, especially: internal limits (including exponentials) and the derived "lifting" functor which classifies Σ-partial functions. In general (always?), the containments between the categories are proper.

From examining the topos models of Axiom 1 that have been investigated to date [20, 6, 15, 3, 4] one may extrapolate another axiom: Σ is a $\neg\neg$-separated set (our Axiom 2). Although hard to motivate conceptually, by permitting classical forms of reasoning about Σ-properties, the axiom has powerful and useful consequences. It also allows yet another category of predomains to be identified. In the presence of Axiom 2, a strengthened notion of *regular* Σ-poset (corresponding to the *extensional* objects in [6, 8, 15]) is useful. Such objects arise very naturally in certain models. For example, in Johnstone's topological topos [12], the regular Σ-posets are exactly the sequential T_0 topological spaces (Matías Menni, private communication). In general, the well-complete regular Σ-posets form a full reflective category of sets [29] and provide yet another respectable notion of predomain. Well-complete regular Σ-posets also arise very naturally in models. For example, in the effective topos, [7], the well-complete regular Σ-posets are (equivalent to) the *complete extensional PERs* of [6]. Even more strikingly, in Fiore and Rosolini's topos $\mathcal{H}$ [3, 4], the category of well-complete regular Σ-posets is equivalent to the familiar category of ω-complete partial orders from classical domain theory.

We have already mentioned a proliferation of candidate categories of predomains. We believe it would be wrong to advocate one notion as being preferable to the others in all instances. Instead, the most suitable category is likely to depend upon any intended application. However, in order to appreciate the choices available, it is important to have a thorough understanding of the properties of each, as well as of the general consequences of the axioms.

The goal of this paper is to contribute to the development of such a thorough understanding, by filling in some of the most prominent gaps in the existing literature. On the one hand, we shall demonstrate new consequences of the two axioms above (and also of two additional axioms). On the other hand, we shall also demonstrate some non-consequences, many of which had been previously conjectured. As well as providing proofs of these conjectures, our techniques are of independent interest, because they involve the analysis of new models of SDT with interesting and hitherto unobserved properties.

The paper begins, in Section 1, with our axiomatic development. In order to make the paper self contained, we give a full treatment of: the dominance $\Sigma \subseteq \Omega$, the construction of the associated lifting functor L, its final coalgebra F, and its initial algebra I (including a new proof that the latter is initial). We then introduce the notion of *completeness*, fundamental to the development of SDT. Our first main results are a conceptual breakdown of the completeness axiom, Axiom 1, valid in the presence of Axiom 2 (Propositions 1.16–1.20), and a proof that the complete regular Σ-posets are closed under lifting (Theorem 1.22). This shows that the notions of completeness and well-completeness coincide for regular Σ-posets. Finally, we analyse the consequences of assuming that Σ possesses various kinds of join under the implication order. Axiom 3 states that it contains $\perp$ (the least element in Ω). We show that our Axiom 4, *Phoa's Principle* ([30]), is equivalent to Σ possessing either binary joins (Theorem 1.27), or equivalently N-indexed joins (Corollary 1.29). Further, if Σ is closed under N-indexed joins in Ω (existential quantification over N) then every complete object is well-complete (Proposition 1.31).

In Section 2, we consider the Modified Realizability Topos ([18]), as our first new model of SDT. We show that, under an appropriate choice of dominance, Axioms 1–4 are satisfied. This model allows us to obtain a number of independence results. Firstly, the *Scott Principle* ([30]) fails (Proposition 2.8), although its weak version is a consequence of the axioms (Proposition 1.24). Also, the decidable subobject classifier, 2, is complete but not well-complete (Proposition 2.9). This shows both that completeness and well-completeness do not coincide in general, even for Σ-posets, and also that well-complete objects are not necessarily closed under finite coproducts in the topos. These results were conjectured (in a more restrictive setting) in [15]. They justify the necessity, in general, of considering the somewhat clumsy notion of well-completeness rather than the cleaner notion of completeness.

In the brief Section 3 we revisit the best known model of SDT, that given by the Effective Topos ([7]). Our purpose here is to point out some unexpected ways in which the Effective Topos is less well-behaved than one might expect. In particular, we show that the initial lift-algebra is not an internal colimit of its standard chain of approximating iterates (it is trivially not an external colimit of this chain). This corrects a claim made in [8]. Further, we establish the surprising property that the internal colimit of the chain is (well-)complete (Theorem 3.3). A consequence of this is that an internal version of the limit-colimit coincidence of ordinary domain theory fails for (well-)complete objects in the Effective Topos.

Finally, in Section 4, we consider a Grothendieck topos, constructed specifically to obtain one further independence result. Again we find a dominance such that Axioms 1–4 are satisfied. This time, although well-complete objects are closed under finite coproducts in the topos, the natural numbers object is not well-complete (although it is complete). This result shows that a situation which cannot arise in ordinary realizability toposes (see [15, Theorem 7.5]) can nonetheless arise in the context of SDT in an arbitrary elementary topos.

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1 Axiomatics

In this section we develop basic Synthetic Domain Theory on the basis of 4 axioms, which are (in a way familiar from [8]) introduced just where the treatment needs them.

1.1 Basic Notions

Throughout this section, we assume that we are working in a topos $\mathcal{E}$ with natural numbers object N. The subobject classifier of $\mathcal{E}$ is denoted Ω, with generic mono $\top : 1 \to \Omega$.

The reader should be aware that from now on in this section, *all* our reasoning will be in the internal logic of $\mathcal{E}$ (whenever this makes sense. There are minor deviations from this viewpoint, as in subsection 1.2). Statements that a diagram is a pullback, an arrow an isomorphism, epi or monic, etcetera, should be rigorously interpreted as their equivalents in the internal logic. Even statements about, e.g., a "functor $L : \mathcal{E} \to \mathcal{E}$ which has a monad structure", can be done in an (innocuous) extension of the internal language by an extra symbol for definable type formation.

The basic theory of dominances and lifting, as laid out below, is due to Rosolini ([23]).

Definition 1.1 A *dominance* is a subobject Σ of Ω, satisfying the axioms

$(\Sigma 1)$	$\top \in \Sigma$
$(\Sigma 2)$	$\forall p, q{:}\Omega.\, p \in \Sigma \wedge (p \Rightarrow (q \in \Sigma)) \Rightarrow ((p \wedge q) \in \Sigma)$

Given a dominance Σ, for each object X the notion of Σ-subobject of X is given: $A \subseteq_\Sigma X$ iff $\forall x{:}X.[x \in A] \in \Sigma$.

We have then an endofunctor L on $\mathcal{E}$, the "lift functor", together with a natural transformation $\zeta : \mathrm{id} \Rightarrow L$, which structure classifies Σ-partial maps. This means: given a Σ-subobject A of X and a map: $A \to Y$, there is a unique function: $X \to L(Y)$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\zeta_Y"'] & L(Y) \end{tikzcd}$$

is a pullback.

Definition 1.2 Define

$$\begin{array}{rcl} L(X) & = & \{\alpha \in \Omega^X \mid \forall xy{:}X.(x \in \alpha \wedge y \in \alpha \Rightarrow x = y) \wedge \\ & & \qquad [\exists x{:}X.x \in \alpha] \in \Sigma\} \end{array}$$

If $f : X \to Y$ is a map, $L(f) : L(X) \to L(Y)$ is given by

$$L(f)(\alpha) = \{f(x) \mid x \in \alpha\}$$

The natural transformation $\zeta : \mathrm{id} \Rightarrow L$ is given by

$$\zeta_X(x) = \{x\}$$

It is a consequence of definition 1.1 that there is a natural transformation $\mu : L^2 \Rightarrow L$ giving (L, ζ, μ) the structure of a monad on $\mathcal{E}$:

$$\mu_X(\mathcal{A}) = \bigcup \mathcal{A} = \{x \in X \mid \exists \alpha \in \mathcal{A}.x \in \alpha\}$$

To see that $\mu_X(\mathcal{A}) \in L(X)$, note that $\exists x{:}X.x \in \mu_X(\mathcal{A})$ is equivalent to the conjunction

$$(\exists \alpha{:}L(X).\alpha \in \mathcal{A}) \wedge \forall \alpha{:}L(X).(\alpha \in \mathcal{A} \Rightarrow \exists x{:}X.x \in \alpha)$$

The first conjunct is in Σ since $\mathcal{A} \in L^2(X)$, and it implies that the second conjunct is in Σ (since, for $\beta \in \mathcal{A}$, the second conjunct is equivalent to $\exists x{:}X.x \in \beta$); hence by ($\Sigma$2) of definition 1.1, the conjunction is in Σ.

The monad equations for μ and ζ are easily verified.

In the following, we shall write $\exists x.x \in \alpha$ or even $\exists x \in \alpha$ for $\exists x{:}X.x \in \alpha$. We note that $L(1) \cong \Sigma$ and that under this correspondence, $\mu_1 : L(\Sigma) \to \Sigma$ is given by

$$\mu_1(\alpha) = \exists \sigma \in \alpha.\sigma$$

We have at once:

Proposition 1.3 *For all $\alpha \in L(\Sigma)$,*

$$\alpha = \{\mu_1(\alpha) \mid \exists\sigma \in \alpha\}$$

Proof. Let $\sigma \in \alpha$. Then $\exists\sigma \in \alpha$ and $\sigma \Rightarrow \mu_1(\alpha)$; conversely, $(\tau \in \alpha) \wedge \tau \Rightarrow \sigma$ since $(\tau \in \alpha) \wedge \tau \Rightarrow (\tau = \sigma) \wedge \tau$, by $\alpha \in L(\Sigma)$. So $\alpha \subseteq \{\mu_1(\alpha) \mid \exists\sigma \in \alpha\}$. Conversely, if $\tau \in \{\mu_1(\alpha) \mid \exists\sigma \in \alpha\}$ then $\exists\sigma \in \alpha$ and since $\sigma \in \alpha \Rightarrow \sigma = \mu_1(\alpha)$ as we have just seen, $\sigma = \tau$. ∎

We shall consider both algebras for the *functor* and the *monad* L: here, L-algebra means algebra for the functor L (that is, a diagram $L(X) \xrightarrow{a} X$); a *strict* L-algebra is an algebra for the monad (L, ζ, μ). An L-coalgebra is a diagram $X \xrightarrow{b} L(X)$. The definition of morphisms is as usual; the strict algebras are a full subcategory of the algebras.

Note that if $(L(X) \xrightarrow{a} X)$ is an L-algebra and X is $\neg\neg$-separated (the equality relation on X is a $\neg\neg$-closed subobject of $X \times X$), a is strict if and only if $a{\cdot}\zeta_X = \mathrm{id}_X$.

Using the natural numbers object N, we define the following L-coalgebra F:

$$F = \{\psi \in \Sigma^N \mid \forall n{:}N.\psi(n+1) \Rightarrow \psi(n)\}$$

with coalgebra structure $\tau : F \to L(F)$ given by

$$\tau(\psi) = \{\lambda n{:}N.\psi(n+1) \mid \psi(0)\}$$

Note, that τ is an isomorphism, with inverse $\sigma : L(F) \to F$:

$$\sigma(\alpha) = \begin{cases} \exists\psi{:}F.\psi \in \alpha & \text{if } n = 0 \\ \exists\psi{:}F.\psi \in \alpha \wedge \psi(n-1) & \text{if } n > 0 \end{cases}$$

Given any L-coalgebra $(X \xrightarrow{a} L(X))$ there is a unique coalgebra morphism $f : X \to F$ given by

$$f(x) = \lambda n{:}N. \begin{cases} \exists y{:}X.y \in a(x) & \text{if } n = 0 \\ \exists y{:}X.y \in a(x) \wedge f(y)(n-1) & \text{if } n > 0 \end{cases}$$

Hence, $(F \xrightarrow{\tau} L(F))$ is a *terminal L-coalgebra.*

It is a useful comment that in fact, F is a retract of Σ^N, by the map $\varphi \mapsto \lambda n{:}N.\bigwedge_{k \le n} \varphi(k)$.

L has also an initial algebra $(L(I) \xrightarrow{\sigma} I)$. I can be constructed as the least L-subalgebra of $(L(F) \xrightarrow{\sigma} F)$. Mamuka Jibladze ([11]) has given a beautiful formula for I:

$$I = \{\psi \in F \mid \forall\phi{:}\Omega.(\forall n{:}N.((\psi(n) \Rightarrow \phi) \Rightarrow \phi)) \Rightarrow \phi\}$$

Note that $\sigma : L(F) \to F$ restricts to $\sigma : L(I) \to I$ which is the L-algebra structure on I.

There are several ways of proving that I is in fact the initial L-algebra ([11],[27]). The proof below is new and highlights the role of an *induction principle* that Jibladze's formula plays.

Theorem 1.4 *$L(I) \xrightarrow{\sigma} I$ is the initial L-algebra.*

Proof. Let $(L(X) \xrightarrow{g} X)$ be any L-algebra. First we prove that there is at most one algebra map from I to X. Any such $h : I \to X$ with $h{\cdot}\sigma = g{\cdot}L(h)$ must satisfy:

$$h(\varphi) = g(\{h(\lambda n{:}N.\varphi(n+1) \mid \varphi(0)\})$$

(since $\varphi = \sigma(\tau(\varphi))$). We write $H(h, \varphi)$ for this relation.

Suppose $\forall\varphi{:}I.H(h_1, \varphi)$ and $\forall\varphi{:}I.H(h_2, \varphi)$. Let $\phi \in \Omega$ be the proposition

$$\forall k{:}N.h_1(\lambda n{:}N.\varphi(n+k)) = h_2(\lambda n{:}N.\varphi(n+k))$$

Suppose ("induction hypothesis") $\varphi(n) \Rightarrow \phi$. Then for all $k \in N$,

$$\{h_1(\lambda n{:}N.\varphi(n+k)) \mid \varphi(n)\} = \{h_2(\lambda n{:}N.\varphi(n+k)) \mid \varphi(n)\}$$

Hence for all $k \in N$, since $H(h_1, \lambda n{:}N.\varphi(n+k))$, $h_1(\lambda n{:}N.\varphi(n+k))$ is equal to

$$g(\{g(\{\ldots g(\{h_1(\lambda n{:}N.\varphi(n+k+w+1) \mid \varphi(k+w)\} \ldots \mid \varphi(k)\})$$

(where w is such that $k + w \geq n$); which is

$$g(\{g(\{\ldots g(\{h_2(\lambda n{:}N.\varphi(n+k+w+1) \mid \varphi(k+w)\} \ldots \mid \varphi(k)\})$$

by induction hypothesis; which is $h_2(\lambda n{:}N.\varphi(n+k))$ by $H(h_2, \lambda n{:}N.\varphi(n+k))$.

We conclude that for all $n \in N$, $(\varphi(n) \Rightarrow \phi) \Rightarrow \phi$. Since $\varphi \in I$ we have ϕ, so in particular,

$$h_1(\varphi) = h_2(\varphi)$$

which shows uniqueness of h.

To show existence of h: let, for $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots) \in L(X)^N$, $G(\vec{\alpha}, \varphi)$ be the statement

$$\forall k{:}N.\alpha_k = \{g(\alpha_{k+1}) \mid \varphi(k)\}$$

Quite in the same way as in the uniqueness part of the proof, one shows that there can be at most one $\vec{\alpha} \in L(X)^N$ with $G(\vec{\alpha}, \varphi)$. Now, let $\phi \in \Omega$ be the proposition $\exists\vec{\gamma} \in L(X)^N.G(\vec{\gamma}, \varphi)$. Suppose for induction hypothesis, $\varphi(m) \Rightarrow \phi$. For $m' \geq m$ put

$$\beta_{m'} = \{g(\alpha_{m'+1}) \mid G(\vec{\alpha}, \varphi) \wedge \varphi(m')\}$$

Then $\beta_{m'}$ has at most one element, and, by induction hypothesis, $\exists x{:}X.x \in \beta_{m'}$ is equivalent to $\varphi(m')$, so $\beta_{m'} \in L(X)$ for $m' \geq m$. We extend the definition of $\vec{\beta}$ by putting

$$\beta_{m''} = \{g(\beta_{m''+1}) \mid \varphi(m'')\}$$

for $m'' < m$.

Now suppose $G(\vec{\gamma}, \varphi)$. Then for $m' \geq m$ we have

$$\begin{array}{rcl} \beta_{m'} & = & \{g(\alpha_{m'+1}) \mid \varphi(m') \wedge G(\vec{\alpha}, \varphi)\} \\ & = & \{g(\gamma_{m'+1}) \mid \varphi(m')\} \\ & = & \gamma_{m'} \end{array}$$

and therefore, we have $\beta_k = \gamma_k$ for all $k \in N$. So, $G(\vec{\gamma}, \varphi)$ implies $\vec{\beta} = \vec{\gamma}$ whence $G(\vec{\beta}, \varphi)$. Since $\varphi(m) \Rightarrow \exists\vec{\gamma}.G(\vec{\gamma}, \varphi)$ we have for all $m' \geq m$,

$$\begin{array}{rcl} \beta_{m'} & = & \{g(\alpha_{m'+1}) \mid \varphi(m') \wedge G(\vec{\alpha}, \varphi)\} \\ & = & \{g(\beta_{m'+1} \mid \varphi(m')\} \end{array}$$

But for $m' < m$ this holds by definition. Therefore, $G(\vec{\beta}, \varphi)$ holds, and hence ϕ. We have proved $(\varphi(m) \Rightarrow \phi) \Rightarrow \phi$; since m was arbitrary and $\varphi \in I$, ϕ. We may conclude $\forall\varphi{:}I.\exists!\vec{\alpha}.G(\vec{\alpha}, \varphi)$. Put $h(\varphi) = g(\alpha_0)$ for the unique $\vec{\alpha}$ satisfying $G(\vec{\alpha}, \varphi)$. It follows that $h(\lambda n{:}N.\varphi(n+1)) = g(\alpha_1)$, so

$$g(\{h(\lambda n{:}N.\varphi(n+1)) \mid \varphi(0)\}) = g(\{g(\alpha_1) \mid \varphi(0)\}) = g(\alpha_0) = h(\varphi)$$

whence h is the desired algebra map. ∎

From theorem 1.4, another initiality property of I can be derived. This is a general fact about initial algebras for functors which have a monad structure, discovered by Bénabou and Jibladze, and proved in [13]. Let (L, ζ, μ) be a monad on a category. A *strict L-algebra with successor* is a structure $(X, L(X) \xrightarrow{a} X, X \xrightarrow{g} X)$ such that a is a strict L-algebra structure, and g is arbitrary. A *morphism* of strict L-algebras with successor: $(X, a, g) \to (Y, b, h)$ is a morphism $f : X \to Y$ of L-algebras such that $f{\cdot}g = h{\cdot}f$.

The theorem is, that if (I, σ) is the initial L-algebra, then $(I, \sigma{\cdot}\mu_I{\cdot}L(\sigma^{-1}), \sigma{\cdot}\zeta_I)$ is the initial strict L-algebra with successor (note, that σ is always an isomorphism by Lambek's Lemma).

In our case, putting $\rho = \sigma{\cdot}\mu_I{\cdot}L(\sigma^{-1})$ and $s = \sigma{\cdot}\zeta_I$, we have:

$$\begin{array}{rcl} \rho(\alpha) & = & \lambda n{:}N.\exists\varphi \in I.\varphi \in \alpha \wedge \varphi(n) \\ s(\varphi) & = & \lambda n{:}N.\left\{ \begin{array}{cl} \top & \text{if } n = 0 \\ \varphi(n-1) & \text{if } n > 0 \end{array} \right. \end{array}$$

For the record:

Theorem 1.5 *(I, ρ, s) is the initial strict L-algebra with successor.*

It is not hard to see that I can be approximated by two first-order definitions:

Proposition 1.6

$$\forall\psi \in F.[(\exists n{:}N.\neg\psi(n)) \;\Rightarrow\; \psi \in I \;\Rightarrow\; (\neg\neg\exists n{:}N.\neg\psi(n))]$$

In general, none of the implications in proposition 1.6 can be reversed; the converse to the first entailment fails in the Effective topos, and that of the second entailment fails in the Modified realizability topos (for certain dominances), as we shall see in sections 3 and 2, respectively (more directly, if Σ consists of the two elements $\perp$ and $\top$ of Ω, then $I \cong N$ and the first implication can be reversed, so the converse to the second implication is what is generally known as Markov's Principle, about which we shall see more in this paper).

It is immediate from proposition 1.6 that for $\psi \in F \subseteq \Sigma^N$, one has the implication $\psi \in I \Rightarrow \neg\neg(\perp \in \Sigma)$, where $\perp = \neg(\top) \in \Omega$. It is a nice application of theorem 1.4 to see that in fact:

Proposition 1.7 *I is inhabited iff $\bot \in \Sigma$.*

Proof. If $\bot \in \Sigma$ then $(\lambda n{:}N.\bot) \in I$ so I is inhabited; for the converse, observe that $\Sigma \cong L(1)$ has the L-algebra structure $\mu_1 : L(\Sigma) \to \Sigma$. Let $f : I \to \Sigma$ be the unique morphism of L-algebras. If $\bot \in \Sigma$ then it is easily seen that $f = \lambda i{:}I.\bot$; hence $\forall i{:}I.(\bot \in \Sigma \Rightarrow \neg f(i))$, so also $\forall i{:}I.(\neg\neg(\bot \in \Sigma) \Rightarrow \neg f(i))$. Since $\forall i{:}I.\neg\neg(\bot \in \Sigma)$ we have $\forall i{:}I.f(i) = \bot$, so $\forall i{:}I.\bot \in \Sigma$. ∎

To finish this subsection we record the easy fact that I is a downset of F: for $\psi, \varphi \in F$, if $\psi \in I$ and $\forall n{:}N.\varphi(n) \Rightarrow \psi(n)$, then $\varphi \in I$, which is immediate from Jibladze's formula.

1.2 I as an internal colimit?

This seems to be an appropriate point to take up the issue of whether I, the internal L-algebra, is "essentially" the colimit of an internal N-indexed diagram

$$0 \xrightarrow{!} L(0) \xrightarrow{L(!)} L^2(0) \xrightarrow{L^2(!)} \dots$$

as claimed in [8]. For further reference, let us call this "the initial L-chain".

In order to perform a precise calculation (and to know exactly what one is saying) one has to be a little bit delicate here; clearly, it will not do to exhibit a "chain" of objects X_n and to prove "$X_n \cong L^n(0)$", since one doesn't know where n is living in this argument. So, let us be excused for being pedantic for a while.

Let $L_N : \mathcal{E}/N \to \mathcal{E}/N$ be the functor with the same internal definition as L, of course relative to the dominance $\Sigma \times N \to N$ in $\mathcal{E}/N$. On objects, $L_N(X \xrightarrow{\gamma} N)$ can be rendered as the subobject of $L(X) \times N$ classified by

$$L(X) \times N \xrightarrow{L(\gamma)\times\zeta_N} L(N) \times L(N) \xrightarrow{\subseteq} \Omega$$

together with the projection to N.

What we are looking for is an *internal representation* of the initial L-chain as a diagram in $\mathcal{E}/N$. That is, one wants an object $X \xrightarrow{\gamma} N$, and an N-indexed family of functions $a_n : X_n \to X_{n+1}$ such that, writing $X_L \xrightarrow{\gamma_L} N$ and $b_n : (X_L)_n \to (X_L)_{n+1}$ for $L_N(X \xrightarrow{\gamma} N)$ and the induced structure on this object, one has $X_0 = \emptyset$ and there is (internally) an N-indexed family of bijections $c_n : X_{n+1} \to (X_L)_n$ such that the squares

$$\begin{tikzcd} X_{n+1} \arrow[r, "c_n"] \arrow[d, "a_{n+1}"'] & (X_L)_n \arrow[d, "b_n"] \\ X_{n+2} \arrow[r, "c_{n+1}"'] & (X_L)_{n+1} \end{tikzcd}$$

commute.

In the case of our lifting functor L, we may use the terminal L-coalgebra F to construct such an object. We have a function

$$\Omega^F \to \Omega^{L(F)}$$

which represents the action of L on subobjects of F. Formally, it is the transpose of the composite

$$\Omega^F \times L(F) \xrightarrow{s} L(F \times \Omega^F) \xrightarrow{L(\in)} L(\Omega) \to \Omega^\Omega \xrightarrow{\bigwedge} \Omega$$

where s is the strength of the monad L: $s(\psi, \alpha) = \{(\psi, \varphi) | \varphi \in \alpha\}$.

Composing with $\tau^{-1} : \Omega^{L(F)} \to \Omega^F$ (which is an isomorphism) we have a map $f_L : \Omega^F \to \Omega^F$ which represents the action of L on subobjects of F and preserves inclusions. We have for $X \subseteq F$:

$$f_L(X) = \{\psi \in F \mid \tau(\psi) \in L(X)\} = \{\psi \in F \mid \tau(\psi) \subseteq X\}$$

Now define the following sequence of subobjects of F: $X_0 = \emptyset$, and $X_{n+1} = \{\psi \in F \mid \neg\psi(n)\}$.

Proposition 1.8 *For each $n \in N$, $f_L(X_n) = X_{n+1}$.*

Proof. Let $\psi \in f_L(X_n)$ so for some $\alpha \in L(X_n)$,

$$\psi = \tau^{-1}(\alpha) = \lambda k{:}N. \begin{cases} \exists\varphi \in \alpha & k = 0 \\ \exists\varphi \in \alpha.\varphi(k-1) & k > 0 \end{cases}$$

Since $\alpha \subseteq X_n = \{\varphi \in F \mid \neg\varphi(n)\}$ we have $\psi(n+1) = \exists\varphi \in \alpha.\varphi(n) = \bot$, so $\psi \in X_{n+1}$. Conversely, if $\psi \in X_{n+1}$ then $\tau(\psi) = \{\lambda k{:}N.\psi(k+1) \mid \psi(0)\} \subseteq X_n$, so $\tau(\psi) \in L(X_n)$ and $\psi \in f_L(X_n)$. ∎

Corollary 1.9 *The subobject X of $F \times N$, defined by*

$$X = \{(\varphi, n) \mid \varphi \in X_n\}$$

together with its projection to N, can be equipped with the structure necessary to form an internal representation of the initial L-chain.

Corollary 1.10 *The colimit of the internal representation of the initial L-chain has, up to isomorphism, the object*

$$\{\varphi \in F \mid \exists n{:}N.\neg\varphi(n)\}$$

as underlying object.

Therefore, if the first entailment of proposition 1.6 cannot be reversed, the functor L does not generally preserve colimits of internal N-chains. Again, this happens in the case of the standard model in the Effective Topos (section 3).

1.3 Completeness, Σ-order and (regular) Σ-posets

We use $\iota : I \to F$ to denote the inclusion map.

Definition 1.11 Call an object X *complete* if the map

$$X^\iota : X^F \to X^I$$

is an isomorphism, i.e. internally every map $f : I \to X$ has a unique extension to $\bar{f} : F \to X$.

The following lemma gives a characterization of complete objects. We introduce the following notation for F: for $\varphi, \psi \in F$ we write $\varphi \wedge \psi$ for $\lambda n{:}N.\varphi(n) \wedge \psi(n)$. Moreover, we shall write ∞ for the element $\lambda n{:}N.\top$ of F.

Lemma 1.12 *An object X is complete if and only if there exists a (necessarily unique) function $\bigsqcup : X^I \to X$ satisfying:*

$$\forall g{:}X^F,\ \psi{:}F.\ \bigsqcup(\lambda\varphi{:}I.\, g(\varphi \wedge \psi)) = g(\psi). \tag{1}$$

Proof. For the $\Rightarrow$ direction, suppose X is complete. Define $\bigsqcup(f) = \overline{f}(\infty)$. Then, for any g and ψ, we have that $\bigsqcup(\lambda\varphi{:}I.\, g(\varphi \wedge \psi)) = g((\infty) \wedge \psi) = g(\psi)$ because $\lambda\varphi : F.\, g(\varphi \wedge \psi)$ is the unique extension of $\lambda\psi : I.\, f(\psi \wedge \psi)$.

For the $\Leftarrow$ implication, suppose that such a function $\bigsqcup$ exists. Given $f : I \to X$ define $\overline{f} : F \to X$ by $\overline{f}(\psi) = \bigsqcup(\lambda\varphi : I.\, f(\varphi \wedge \psi))$. To see that $\overline{f}$ extends f, suppose $\varphi' : I$. Define $f_{\varphi'} : F \to X$ by $f_{\varphi'}(\psi) = f(\psi \wedge \varphi')$. Then $\overline{f}(\varphi') = \bigsqcup(\lambda\varphi{:}I.\, f(\varphi{\wedge}\varphi')) = \bigsqcup(\lambda\varphi{:}I.\, f_{\varphi'}(\varphi{\wedge}\varphi')) = f_{\varphi'}(\varphi') = f(\varphi')$, where the penultimate equality is by (1). For uniqueness, suppose $g : F \to X$ is another extension. Then $g(\psi) = \bigsqcup(\lambda\varphi{:}I.\, g(\varphi \wedge \psi)) = \bigsqcup(\lambda\varphi{:}I.\, f(\varphi \wedge \psi)) = \overline{f}(\psi)$.

For the uniqueness of $\bigsqcup$, it is clear that if X is complete then $\bigsqcup(f)$ is determined to be $\overline{f}(\infty)$. ∎

Theorem 1.5 gives a nice way of proving the *fixed point property* for complete strict L-algebras:

Theorem 1.13 *Let $(X, a : L(X) \to X)$ be a strict L-algebra. If X is complete, then every function $g : X \to X$ has a fixed point.*

Proof. Let $h : I \to X$ the unique map of strict L-algebras with successor, from (I, ρ, s) to (X, a, g). Then $h{\cdot}s = g{\cdot}h$. By completeness of X, h has a unique extension to $h' : F \to X$. The map $s : I \to I$ extends to $s : F \to F$ (by the same formula), and also $h'{\cdot}s = g{\cdot}h'$ since both maps extend $h{\cdot}s$. Since, in F, the map s has a fixed point ∞, its h'-image is a fixed point of g. ∎

Note that from the proof of theorem 1.13 we have for the function $\bigsqcup : X^I \to X$ of lemma 1.12, that $\bigsqcup(f) = \bigsqcup(f{\cdot}s)$.

Since the notion of completeness is defined by an orthogonality property, it follows as usual that the complete objects are closed under all internal limits in $\mathcal{E}$; in particular, they form an exponential ideal, and they are closed under retracts.

Let us immediately give an example of an object which is not complete:

Proposition 1.14 *I is not complete.*

Proof. Suppose I complete. Then the inclusion $\iota : I \to F$ has a retraction $j : F \to I$, hence I is inhabited, as F is; so $\bot \in \Sigma$ by proposition 1.7. Then evaluation at 0: $I \to \Sigma$ has a section

$$\sigma \mapsto \lambda n{:}N. \begin{cases} \sigma & \text{if } n = 0 \\ \bot & \text{else} \end{cases}$$

So Σ is complete, and therefore F, being a retract of a power of Σ. Since both the identity on F and the composition ιj extend $\iota : I \to F$, $\iota j = \mathrm{id}_F$ and ι is surjective, which contradicts Jibladze's formula (by 1.6). ∎

At this point we introduce our first two axioms, which will be in force for the rest of this section.

Axiom 1 Σ is complete.

This axiom is called the *Completeness Axiom* in [15]. We shall shwo in section 2 that this axiom does not imply that the complete objects are closed under lifting. However, under Axiom 1, it makes sense to introduce the following

Definition 1.15 Call an object X *well-complete* if $L(X)$ is complete.

For, in [27] and [15] it is shown that (under Axiom 1) well-complete implies complete, and that the well-complete objects are closed under all internal limits in $\mathcal{E}$ as well as under L.

It is then an easy consequence that the well-complete objects form the *largest* full subcategory of the complete objects which is closed under L.

Our next axiom is, intuitively, far from obvious, although it holds in every model of SDT so far investigated. Our reason for including it, is the number of useful consequences of it and the simplicity of the resulting theory.

Axiom 2 Σ is $\neg\neg$-separated.

Note that Axiom 2 can be written in two equivalent ways: $\forall p \in \Sigma(\neg\neg p \Rightarrow p)$ or $\forall pq \in \Sigma(\neg\neg(p \Rightarrow q) \Rightarrow (p \Rightarrow q))$. It follows from Axiom 2 that if $\bot \in \Sigma$ and $g, h : \Sigma \to \Sigma$ have the same values on $\bot$ and $\top$, then $g = h$.

For the rest of this subsection, Axioms 1 and 2 are the only ones that we assume.

Let us derive an interesting corollary of Theorem 1.13 and Axioms 1 and 2:

Proposition 1.16 *If $\bot \in \Sigma$, then for any function $g : \Sigma \to \Sigma$ we have $g(\bot) \Rightarrow g(\top)$.*

Proof. As in the proof of proposition 1.7, $(\Sigma, \zeta_\Sigma, \mu_1)$ is a strict L-algebra so by Axiom 1 and Theorem 1.13, g has a fixed point x. Now assume $g(\bot)$; then since $\neg x = [x = \bot]$ and $x = g(x)$, $\neg x \Rightarrow \neg g(\bot)$; hence $\neg\neg x$, so x by Axiom 2. Therefore $x = \top$, so $g(\top)$. The implication $g(\bot) \Rightarrow g(\top)$ is proved. ∎

The Σ-*order* on an object X is defined by:

$$x \sqsubseteq y \quad \text{iff} \quad \forall P{:}\Sigma^X . P(x) \Rightarrow P(y)$$

Henceforth we reserve $\sqsubseteq$ for the Σ-order. It is, in general, only a preorder. Note that proposition 1.16 can be reformulated as: if $\bot \in \Sigma$, then $\bot \sqsubseteq \top$. The next proposition gives a strengthening of this:

Proposition 1.17

i) For $\sigma, \tau \in \Sigma$ we have:

$$(\sigma \sqsubseteq \tau) \Leftrightarrow (\sigma \Rightarrow \tau)$$

ii) for $\varphi, \psi \in I$ we have:

$$\varphi \sqsubseteq \psi \Leftrightarrow (\forall n{:}N.\varphi(n) \Rightarrow \psi(n))$$

Proof. i): Since the implication $(\sigma \sqsubseteq \tau) \Rightarrow (\sigma \Rightarrow \tau)$ is trivial (consider the identity on Σ), we prove the converse. So let $g : \Sigma \to \Sigma$, $x \Rightarrow y \in \Sigma$ and assume $g(x)$. Suppose $\neg g(y)$. Then $\neg x$ since $x \Rightarrow (x = y = \top)$ hence $g(y)$ by the assumption $g(x)$. So, $x = \bot$ and we have $\bot \in \Sigma$, whence we may apply proposition 1.16 to the function $h : \Sigma \to \Sigma$ defined by $h(p) = g(p \wedge y)$. We see that $h(\bot) = g(x) = \top$ and $h(\top) = g(y) = \bot$ whereas 1.16 gives $h(\bot) \Rightarrow h(\top)$. This contradiction establishes $\neg\neg g(y)$, so we have $g(y)$ by Axiom 2.

ii): Again, $\Rightarrow$ is evident since for each $n \in N$, evaluation at n gives a map: $I \to \Sigma$. For $\Leftarrow$: suppose $\varphi, \psi \in I$ satisfy the RHS. Then by proposition 1.7, $\bot \in \Sigma$ and we also have

$$\neg\neg\exists n{:}N.(\neg\varphi(n) \wedge \forall k < n.\varphi(k))$$

If $\neg\varphi(n) \wedge \forall k < n.\varphi(k)$, we can define $w : \Sigma \to I$ by

$$w(p) = \lambda k{:}N.\begin{cases} \top & \text{if } k < n \\ p \wedge \psi(k) & \text{if } k \geq n \end{cases}$$

Then $\varphi = w(\bot)$, $\psi = w(\top)$ so by proposition 1.16:

$$P(\varphi) = P(w(\bot)) \Rightarrow P(w(\top)) = P(\psi)$$

for all $P : I \to \Sigma$.

Therefore we have $\neg\neg(P(\varphi) \Rightarrow P(\psi))$, which gives $P(\varphi) \Rightarrow P(\psi)$ by Axiom 2. So $\varphi \sqsubseteq \psi$. ∎

Corollary 1.18

i) for $\varphi, \psi \in F$ we have:

$$\varphi \sqsubseteq \psi \Leftrightarrow \forall n{:}N.\varphi(n) \Rightarrow \psi(n)$$

ii) If $\bot \in \Sigma$ then for all $P \in \Sigma^F$:

$$P(\infty) \Rightarrow \neg\neg\exists\varphi{:}I.P(\varphi)$$

iii) If $\bot \in \Sigma$ then for all $P \in \Sigma^I$,

$$[\neg\neg\exists\varphi{:}I.P(\varphi)] \in \Sigma$$

Proof. For i): as usual, that $\sqsubseteq$ implies pointwise $\Rightarrow$, is clear. Conversely suppose $\forall n{:}N.\varphi(n) \Rightarrow \psi(n)$ and let $R : F \to \Sigma$ satisfy $R(\varphi), \neg R(\psi)$. Then $\neg(\varphi \in I)$ because $\varphi \in I$ would allow the same argument as in 1.17. But $\neg(\varphi \in I)$ implies $\forall n{:}N.\varphi(n)$ by Axiom 2, hence $\varphi = \psi$ by assumption. The contradiction gives $\neg\neg R(\psi)$, so $R(\psi)$ by again Axiom 2.

For ii), if $\neg\exists\varphi{:}I.P(\varphi)$ then $P{\cdot}\iota = \lambda\varphi{:}I.\bot$, hence $P = \lambda\psi{:}F.\bot$ by completeness of Σ;

For iii), note that in fact, $\neg\neg\exists\varphi{:}I.P(\varphi)$ is equivalent to $\tilde{P}(\infty)$, where $\tilde{P} = (\Sigma^\iota)^{-1}(P)$. ∎

Definition 1.19 Suppose $\bot \in \Sigma$. Then we define the map step : $N \to I$ by

$$\mathrm{step}(n) = \lambda k{:}N. \begin{cases} \top & \text{if } k < n \\ \bot & \text{else} \end{cases}$$

Note that, if $\bot \in \Sigma$, step is a $\neg\neg$-dense inclusion: $N \rightarrowtail I$ (use Jibladze's formula).

Let us use the map step to prove the following converse to corollary 1.18:

Proposition 1.20 *Assume only Axiom 2. If $\bot \in \Sigma$ and the three (conclusions of) statements of corollary 1.18 hold, then Axiom 1 holds, i.e. Σ is complete.*

Proof. To show that $\Sigma^\iota : \Sigma^F \to \Sigma^I$ monic, suppose $R_1, R_2 \in \Sigma^F$ satisfy $\forall\varphi{:}I.R_1(\varphi) \Leftrightarrow R_2(\varphi)$. For a given $\psi \in F$, apply ii) of 1.18 to $\lambda\varphi{:}I.R_1(\psi \wedge \varphi)$ to obtain $R_1(\psi) \Rightarrow \neg\neg\exists n{:}N.R_1(\psi \wedge \mathrm{step}(n))$ (using the $\neg\neg$-density of step). Hence, $R_1(\psi) \Rightarrow \neg\neg\exists n{:}N.R_2(\psi \wedge \mathrm{step}(n))$ by assumption, and applying now i) of 1.18 we get $R_1(\psi) \Rightarrow \neg\neg R_2(\psi)$, so $R_1(\psi) \Rightarrow R_2(\psi)$ by Axiom 2. By symmetry, we have $R_1 = R_2$.

To show that Σ^ι is surjective, let $R \in \Sigma^I$. Define $R' \in \Sigma^F$ by $R'(\psi) \equiv \neg\neg\exists n{:}N.R(\psi \wedge \mathrm{step}(n))$. This is well-defined by iii) of 1.18, applied to $\lambda\varphi{:}I.R(\psi \wedge \varphi)$, and again density of step. For $\varphi \in I$, since $\neg\neg\exists n{:}N.\varphi = \mathrm{step}(n)$, $R(\varphi)$ implies $R'(\varphi)$; and conversely if $R'(\varphi)$ so $\neg\neg\exists n{:}N.R(\varphi \wedge \mathrm{step}(n))$, an application of i) of 1.18 yields $R(\varphi)$. ∎

Remark. Looking back at the proofs of 1.17 and part i) of 1.18, we see that we have been proving these facts from proposition 1.16 directly, without invoking Axiom 1. Therefore, we might have stated proposition 1.20 also in the following way:
Suppose Axiom 2 and $\bot \in \Sigma$. If the conclusions of proposition 1.16 and statements ii) and iii) of 1.18 hold, then Axiom 1 holds.
In this way, proposition 1.20 is an internalization and generalization of Proposition 6.1 of [15].

We let $\eta_X : X \to \Sigma^{\Sigma^X}$ be the function $\lambda x{:}X.\lambda P{:}\Sigma^X.P(x)$. We call X a *Σ-poset* if η_X is a monomorphism, and we say that X is a *regular Σ-poset* if η_X is a $\neg\neg$-closed monomorphism.
[The terminology "Σ-poset" is quite clear, since this is equivalent to the property that the Σ-order on X is antisymmetric. What we call "regular Σ-poset" has been called "extensional object" in the literature (e.g., [8]). We consider this terminology less fortunate, since only in the case of one particular dominance in the Effective Topos the notion has anything to do with extensionality (of properties of indices of partial recursive functions)]
Since every $f : X \to \Sigma^Y$ factors as

$$X \stackrel{\eta_X}{\to} \Sigma^{\Sigma^X} \stackrel{\tilde{f}}{\to} \Sigma^Y$$

(with $\tilde{f}(Q) = \lambda y{:}Y.Q(\lambda x{:}X.f(x)(y))$) and both the classes of monomorphisms and $\neg\neg$-closed monomorphisms are the $\mathcal{M}$-parts of factorization systems on $\mathcal{E}$ (which, among other things, means that $f{\cdot}g \in \mathcal{M}$ implies $g \in \mathcal{M}$), we have that X is a (regular) Σ-poset iff there is a ($\neg\neg$-closed) mono $X \to \Sigma^Y$ for some object Y.

Proposition 1.21 *Given $\eta_X : X \to \Sigma^{\Sigma^X}$ define $l_X : L(X) \to \Sigma^{\Sigma^X}$ by*

$$l_X(\alpha) = \lambda P{:}\Sigma^X.\exists x{:}X.(x \in \alpha \wedge P(x))$$

Then if η_X is a ($\neg\neg$-closed) mono, so is l_X, hence if X is a (regular) Σ-poset, so is $L(X)$.

Proof. Suppose η_X is mono, and $l_X(\alpha) = l_X(\beta)$, which means $\forall P{:}\Sigma^X.(\exists x \in \alpha \wedge P(x) \Leftrightarrow \exists y \in \beta \wedge P(y))$. Let $x \in \alpha$. Then $\forall P{:}\Sigma^X.(x \in P \Rightarrow \exists y \in \beta \wedge P(y))$ so $\exists y \in \beta$ and hence, since $\beta \in L(X)$, $\exists ! y \forall P.(x \in P \Rightarrow y \in \beta \wedge P(y))$.

For such y then, $\forall P\,(x \in P \Leftrightarrow y \in P)$ must hold, i.e. $\eta_X(x) = \eta_X(y)$ so $x = y$; therefore $x \in \beta$. We have proved $\alpha \subseteq \beta$ and by symmetry of the argument, $\alpha = \beta$.
Now suppose η_X is a $\neg\neg$-closed mono, and $\mathcal{A} \in \Sigma^{\Sigma^X}$ such that

$$\neg\neg\exists\alpha{:}L(X).\mathcal{A} = l_X(\alpha)$$

Now $\mathcal{A} = l_X(\alpha)$ implies the equivalence

$$\mathcal{A}(\lambda x{:}X.\top) \Leftrightarrow \exists x{:}X.\mathcal{A} = \eta_X(x)$$

so $\neg\neg\exists\alpha{:}L(X).\mathcal{A} = l_X(\alpha)$ implies the same equivalence, since both sides of it are $\neg\neg$-stable. Therefore we have this equivalence. Let $\beta = \{x \in X \mid \mathcal{A} = \eta_X(x)\}$. Then β is at most a singleton since η_X is mono, and $\exists x{:}X.x \in \beta$ is equivalent to $\mathcal{A}(\lambda x{:}X.\top)$, hence in Σ. So $\beta \in L(X)$ and we have $\mathcal{A}(P) \Leftrightarrow \exists x{:}X.x \in \beta \wedge P(x)$ whence $\mathcal{A} = l_X(\beta)$. ∎

We prove now the main theorem of this subsection. In itself, the theorem is not new (it was also proved in [21]); what is new is that our proof requires nothing more than Axioms 1 and 2. In fact, we find it rather surprising that in this generality, the property of being a regular Σ-poset suffices to restore the implication: if X complete then $L(X)$ complete.

Theorem 1.22 *If X is complete and regular, so is $L(X)$.*

Proof. By proposition 1.21 we know that $L(X)$ is regular. Moreover, since $L(X) \subseteq \Sigma^{\Sigma^X}$ and Σ^{Σ^X} is complete, we know that any $f : I \to L(X)$ can have at most one extension to an $\bar{f} : F \to L(X)$.

To prove existence, suppose $f : I \to L(X)$ given and let $g : F \to \Sigma^{\Sigma^X}$ be the unique extension of $l_X \cdot f$. We aim to show that $\forall \psi{:}F.\neg\neg\exists \alpha{:}L(X).g(\psi) = l_X(\alpha)$ and then use the fact that l_X is a $\neg\neg$-closed embedding. Since the desired conclusion is $\neg\neg$-stable we may distinguish cases as to $\bot \in \Sigma$ or $\neg(\bot \in \Sigma)$.

If $\neg(\bot \in \Sigma)$ then, since Σ is $\neg\neg$-separated, $\Sigma = \{\top\}$, $L(X) \cong X$ and we have the conclusion by completeness of X.

If $\bot \in \Sigma$ we know by lemma 1.12 and Corollary 1.18ii) that

$$g(\psi) = \lambda P{:}\Sigma^X.\neg\neg\exists\varphi{:}I.\exists x{:}X.(P(x) \wedge x \in f(\psi \wedge \varphi))$$

Define $h(\psi) = \{x \in X \mid g(\psi) = \eta_X(x)\}$. Then since η_X is a $\neg\neg$-closed mono, $h(\psi)$ contains at most one element, and $\exists x{:}X.x \in h(\psi)$ is $\neg\neg$-stable.

Moreover, for $\psi \in I$ we have

$$\begin{aligned} h(\psi) &= \{x \in X \mid l_X(f(\psi)) = \eta_X(x)\} \\ &= \{x \in X \mid \forall P{:}\Sigma^X.(P(x) \Leftrightarrow \exists y{:}X.P(y) \wedge y \in f(\psi))\} \\ &= f(\psi) \end{aligned}$$

again using that η_X is monic. So if we can prove that always $h(\psi) \in L(X)$, we have found the extension of f. We claim:

$$\exists x{:}X.x \in h(\psi) \Leftrightarrow \neg\neg\exists\varphi{:}I.\exists y{:}X.y \in f(\varphi \wedge \psi)$$

so that $[\exists x{:}X.x \in h(\psi)] \in \Sigma$. To prove the claim,
$\Rightarrow$ is easy: if $x \in h(\psi)$ then

$$\forall P{:}\Sigma^X.(P(x) \Leftrightarrow \neg\neg\exists\varphi{:}I.\exists y{:}X.P(y) \wedge y \in f(\psi \wedge \varphi))$$

which by specializing to $\lambda x{:}X.\top \in \Sigma^X$ gives

$$\neg\neg\exists\varphi{:}I.\exists y{:}X.y \in f(\psi \wedge \varphi)$$

For $\Leftarrow$, suppose $\exists y{:}X.y \in f(\psi \wedge \varphi)$ for $\varphi \in I$. We have $\neg\neg\exists n{:}N.\varphi = \mathrm{step}(n)$ so assume $\varphi = \mathrm{step}(n)$. Then for all $\chi \in I$, since $\varphi \sqsubseteq s^n(\chi)$ by 1.17, $\exists y{:}X.y \in f(\psi \wedge s^n(\chi))$; let $k : I \to X$ be the unique function satisfying

$$f(\psi \wedge s^n(\chi)) = \{k(\chi)\}$$

If, using completeness of X, $\bar{k} : F \to X$ is its unique extension and $\bar{x} = \bar{k}(\infty)$, then it is readily verified (using 1.17 and 1.18) that $g(\psi) = \eta_X(\bar{x})$, hence $\bar{x} \in h(\psi)$. By $\neg\neg$-stability of $\exists x{:}X.x \in h(\psi)$ as noted before, we are done. ∎

1.4 Chain Completeness and the Phoa Principle

In the preceding pages, the reader has seen many statements depending on the assumption that $\perp \in \Sigma$. Our reason for not adopting this as an axiom yet, was mainly to emphasize that proposition 1.21 and theorem 1.22 do not need it.

That having been accomplished, we introduce

Axiom 3 $\perp \in \Sigma$

Apart from the equivalence already noticed (that I is inhabited), axiom 3 is equivalent to the statement that for every object X, every decidable subobject of X is a Σ-subobject. Another equivalent is that $1 = L(0)$ (or, 0 is well-complete. The statement that 0 is complete is weaker, and equivalent to the condition $\neg\neg(\perp \in \Sigma)$).

It also follows, that for decidable objects X, all maps $I \to X$ are constant so that these objects are complete; and that the complete objects are closed under internal sums indexed by a decidable object: examples are 2 and N. We shall see that this does not hold for well-complete objects. In fact, we have (under Axiom 1 alone) the following implications:

Proposition 1.23

$$N \textit{ well-complete} \Rightarrow 2 \textit{ well-complete} \Rightarrow \textit{ Axiom 3}$$

This is proved in [27]. Another fact which is proved there, is that under Axioms 1 and 2 together with the well-completeness of 2, the well-completeness of N is equivalent to *Markov's Principle*, which is the statement

$$\forall P{:}2^N.(\neg\neg\exists n{:}N.P(n)) \Rightarrow \exists n{:}N.P(n)$$

We shall see in section 2 an example where 2 is not well-complete, and in section 4 an example where 2 is well-complete, but N is not.

Let us observe that up to the introduction of Axiom 3, nothing has brought us in conflict with classical set theory: classically, we'd have been forced to the conclusion $\Sigma = \{\top\}$, but not to any contradiction. Axiom 3, however, marks our departure from the realm of classical sets.

Let us also note the following consequence of Axiom 3, which is a weakening of what [30] calls the "Scott Principle":

Proposition 1.24 (Weak Scott Principle)

$$\forall P{:}\Sigma^{\Sigma^N}.P(\lambda n{:}N.\top) \Rightarrow \neg\neg\exists n{:}N.P(\mathrm{step}(n))$$

Proof. Immediate from Axiom 3, corollary 1.18ii) and lemma 1.12). ∎

The main topic of this subsection is the study of chain completeness. In the whole set-up of synthetic domain theory, the guiding intuition has been that I, the initial L-algebra, is the "generic chain": even without any reference to an order on X, the object X^I is seen as the "object of chains in X", and sometimes (as in [8]) the desire was expressed to do away with the Σ-order altogether (this, by the way, in contrast with [21, 19, 20], where the Σ-order is taken as basic and the notion of completeness is defined as: having lubs of N-chains for the Σ-order. One should however note that, in order to prove the desirable property that every function preserves them, [21] has a non-standard definition of "lub of N-chain", which only in specific cases (cf. corollary 2.12) is equivalent to the natural one. From an axiomatic point of view, this is a drawback).

Whatever one's point of view, it seems wise to acknowledge that the Σ-order is there, whether one loves it or not. Here we investigate the *axiomatic content* of the two notions of completeness: what is the relation between them, and what do we need for them to coincide? What is the relation between the object of N-chains in X (for the Σ-order) and the object X^I?

First, a formal definition:

Definition 1.25 An *N-chain* in X is a function $f : N \to X$ satisfying

$$\forall n{:}N.f(n) \sqsubseteq f(n+1)$$

We use $\mathrm{Ch}(X)$ to denote the object of N-chains in X.

X is *N-complete* if for every $f \in \mathrm{Ch}(X)$ there is $\sup_n f(n) \in X$ satisfying:

$$\forall x{:}X.((\forall n{:}N.f(n) \sqsubseteq x) \Leftrightarrow \sup_n f(n) \sqsubseteq x))$$

Clearly, if X is an N-complete Σ-poset then the assignment $f \mapsto \sup_n f(n)$ is a function $\mathrm{Ch}(X) \to X$, since sups are unique.

One simple relation between completeness and N-completeness, in an important case:

Proposition 1.26 *If X is a Σ-poset, then X N-complete implies X complete.*

Proof. Suppose $f : I \to X$. Again, since X is a Σ-poset, f can have at most one extension to F. Now define $g : F \to X$ by

$$g(\psi) = \sup_n f(\psi \wedge \mathrm{step}(n))$$

By Axiom 3, $\mathrm{step}(n) \in I$ hence $\psi \wedge \mathrm{step}(n) \in I$, so g is well-defined. If $\varphi \in I$ then $g(\varphi) \sqsubseteq f(\varphi)$ because $\forall n{:}N.f(\varphi \wedge \mathrm{step}(n)) \sqsubseteq f(\varphi)$; conversely, if $\varphi = \mathrm{step}(m)$ then $f(\varphi) \sqsubseteq g(\varphi)$. Since $\neg\neg\exists n{:}N.\varphi = \mathrm{step}(n)$, $f(\varphi) \sqsubseteq g(\varphi)$ by Axiom 2. Since X is a Σ-poset then, $f(\varphi) = g(\varphi)$, so g extends f, and X is complete. ∎

The fourth, and last, axiom that we introduce in this section, appears to be just a bit stronger than we need. It has a nice equivalent (given in theorem 1.27 below) but in general it might just be a bit too strong. We adopt it because of its useful consequences and because it holds in many models. On the other hand, it is exactly theorem 1.27 which makes Axiom 4, unlike our three other axioms, look rather special. The existence of a "parallel termination test" on Σ rules out models of SDT based on "sequential" partial combinatory algebras (such as, e.g., the ones considered in [17], [14] and [15]; [4] gives a non-realizability model where nonetheless Axiom 4 fails); and inasmuch one is interested in models of sequential computation, Axiom 4 cannot be recommended.

Axiom 4 $\forall\sigma,\tau{:}\Sigma.((\sigma \Rightarrow \tau) \Rightarrow \exists h{:}\Sigma^{\Sigma}.(h(\bot) = \sigma \wedge h(\top) = \tau))$

Note that by Axiom 2 and proposition 1.16, this is equivalent to:

$$\forall\sigma,\tau{:}\Sigma.((\sigma \Rightarrow \tau) \Leftrightarrow \exists! h{:}\Sigma^{\Sigma}.(h(\bot) = \sigma \wedge h(\top) = \tau))$$

Axiom 4 is called the *Phoa Principle* in [30].

It is straightforward that Axiom 4 is equivalent to the statement that the map $\psi : \Sigma^{\Sigma} \to L(\Sigma)$ defined by $\psi(h) = \{h(\bot) \mid h(\top)\}$, is an isomorphism (use proposition 1.3).

Another equivalent form of Axiom 4 is given in the following theorem:

Theorem 1.27 *Under Axioms 1,2 and 3, Axiom 4 is equivalent to the statement that Σ has binary joins for the implication order.*

Proof. If Σ has binary joins $\sqcup$ for the implication order, Axiom 4 follows, for given $\sigma \Rightarrow \tau$ define $h : \Sigma \to \Sigma$ by

$$h(x) = (\sigma \sqcup x) \wedge \tau$$

Conversely, assume Axiom 4.
For $\sigma \in \Sigma$ let $h_\sigma : \Sigma \to \Sigma$ be the unique map with $h_\sigma(\bot) = \sigma$ and $h_\sigma(\top) = \top$. Put $\sigma \otimes \tau = h_\sigma(\tau)$.

It is easy to see that $h_\bot$ is the identity on Σ and $h_\top$ is the constant function with value $\top$, so $\bot \otimes \sigma = \sigma = \sigma \otimes \bot$ and $\top \otimes \sigma = \top = \sigma \otimes \top$. Since Σ is $\neg\neg$-separated, it follows that $\otimes$ is commutative.

Since each h_σ preserves the $\Rightarrow$-order by proposition 1.17, we have

$$\sigma = h_\sigma(\bot) \Rightarrow h_\sigma(\tau) = \sigma \otimes \tau$$

and hence also $\tau \Rightarrow \tau \otimes \sigma = \sigma \otimes \tau$. Moreover, the two implications

$$\begin{array}{lcl} (\sigma \Rightarrow \bot) & \Rightarrow & \sigma \otimes \bot = \bot \\ (\sigma \Rightarrow \top) & \Rightarrow & \sigma \otimes \top = \top \end{array}$$

give, by $\neg\neg$-separatedness of Σ,

$$\forall x{:}\Sigma.((\sigma \Rightarrow x) \Rightarrow \sigma \otimes x = x)$$

Therefore, $\sigma \Rightarrow x$ and $\tau \Rightarrow x$ together imply

$$\sigma \otimes \tau \Rightarrow \sigma \otimes x = x$$

So, $\sigma \otimes \tau$ is the join of σ and τ for the $\Rightarrow$-order. ∎

Corollary 1.28 *Every N-chain in Σ has a unique extension to a function $I \to \Sigma$ via* step $: N \to I$.

Proof. Let $f \in \mathrm{Ch}(\Sigma)$ and consider $\uparrow(f) = \{g \in \mathrm{Ch}(\Sigma) \mid \forall n{:}N. f(n) \Rightarrow g(n)\}$. Then Axiom 4 (via 1.27) implies that $\uparrow(f)$ has the structure of a strict L-algebra: define $a : L(\uparrow(f)) \to \uparrow(f)$ by

$$a(\alpha) = \lambda n{:}N.(\exists g \in \alpha. g(n)) \sqcup f(n)$$

Let $\gamma : \uparrow(f) \to \uparrow(f)$ be given by

$$\gamma(g) = \lambda n{:}N. g(n+1)$$

By theorem 1.5 there is a unique $h : I \to \uparrow(f)$ such that the diagrams

$$\begin{tikzcd} L(I) \arrow[r, "L(h)"] \arrow[d, "\beta"'] & L(\uparrow(f)) \arrow[d, "a"] \\ I \arrow[r, "h"'] & \uparrow(f) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "h"] \arrow[d, "s"'] & \uparrow(f) \arrow[d, "\gamma"] \\ I \arrow[r, "h"'] & \uparrow(f) \end{tikzcd}$$

commute. From the first one, one obtains that

$$h(\mathrm{step}(0)) = h(\lambda n{:}N.\perp) = h(\beta(\emptyset)) = a(\emptyset) = f$$

and the second one gives that

$$h(\mathrm{step}(n+1)) = h(s(\mathrm{step}(n))) = \gamma(h(\mathrm{step}(n))) = \lambda m{:}N. h(\mathrm{step}(n))(m+1)$$

Therefore by induction $h(\mathrm{step}(n)) = \lambda m{:}N. f(n+m)$. Now let $\bar{f} : I \to \Sigma$ be defined by $\bar{f}(\varphi) = h(\varphi)(0)$. Then $\bar{f}$ is the required extension. It is unique because of the $\neg\neg$-density of $\mathrm{step} : N \to I$. ∎

Corollary 1.29 *Σ is N-complete.*

Proof. By the preceding corollary and the completeness of Σ, every N-chain f in Σ extends uniquely to a map $\bar{f} : F \to \Sigma$; it is readily checked (using corollary 1.18) that $\bar{f}(\infty) = \sup_n f(n)$. ∎

We leave the proof of the following generalization to the reader.

Corollary 1.30 *For regular Σ-posets X: X complete implies X N-complete.*

The reader will have noted that, as a consequence of Axiom 4, every function $f : N \to \Sigma$ has a supremum $\bigsqcup_n f(n)$ in Σ. In general, however, these suprema are different from $\exists n{:}N. f(n)$. In fact, we have the following:

Proposition 1.31 *Assume axioms 1,2,3. If for every function $f : N \to \Sigma$, $[\exists n{:}N. f(n)] \in \Sigma$, then X complete implies X well-complete, for every object X.*

Proof. We use the characterization of complete objects given in Lemma 1.12. Consider any $f : I \to LX$. For any $n : N$ such that $\exists x : X. x \in f(\mathrm{step}(n))$, it follows from Proposition 1.17 that $\forall \varphi : I. \exists! x : X. x \in f(s^n(\varphi))$, determining an evident function $f_n : I \to X$. As X is complete, this determines a value $\bigsqcup(f_n) \in X$ by Lemma 1.12. Define:

$$\bigsqcup(f) \quad = \quad \{\bigsqcup(f_n) \mid n : N,\ \exists x : X.\, x \in f(\mathrm{step}(n))\}$$

If $\exists x : X.\, x \in f(\mathrm{step}(n))$ then $\bigsqcup(f_n) = \overline{f_n}(\lambda n : N.\, \top) = \overline{f_{n+k}}(\lambda n : N.\, \top) = \bigsqcup(f_{n+k})$, because $f_{n+k} = f_n \cdot s^k$. So $x, x' \in \bigsqcup(f)$ implies $x = x'$. Also $\exists x : X.\, x \in \bigsqcup(f)$ iff $\exists n : N.\, \exists x : X.\, x \in f(\mathrm{step}(n))$, so $[\exists x : X.\, x \in \bigsqcup(f)] \in \Sigma$ by the closure of Σ under existensial quantification over N. Therefore $\bigsqcup(f) \in L(X)$.

We have defined a function $\bigsqcup : (LX)^I \to LX$. It suffices to show that this satisfies the condition of Lemma 1.12. Accordingly, take any $g : F \to LX$ and $\psi : F$. Define $f_\psi : I \to LX$ by $f_\psi(\varphi) = g(\varphi \wedge \psi)$. We must show that $\bigsqcup(f_\psi) = g(\psi)$.

We first show that $\exists n : N.\, \exists x : X.\, x \in f_\psi(\mathrm{step}(n))$ implies $\bigsqcup(f_\psi) = g(\psi)$. Suppose $\exists n : N.\, \exists x : X.\, x \in f_\psi(\mathrm{step}(n))$. Then, as above, we have $f_{\psi n} : I \to X$ defined by $f_{\psi n}(\varphi) =$ the unique $x \in g(s^n(\varphi) \wedge \psi)$. By the completeness of X, we have $\bigsqcup f_{\psi n} =$ the (necessarily existing) unique $x \in g(\psi)$. Thus indeed $\bigsqcup(f_\psi) = \{\bigsqcup(f_{\psi n})\} = g(\psi)$.

Now suppose $\exists x : X.\, x \in \bigsqcup(f_\psi)$. By the definition of $\bigsqcup(f_\psi)$, we have that $\exists n : N.\, \exists x : X.\, x \in f_\psi(\mathrm{step}(n))$. So, by the above, $\bigsqcup(f_\psi) = g(\psi)$. On the other hand, suppose $\exists x : X.\, x \in g(\psi)$. Then, by Axiom 1 and the $\neg\neg$-density of step, $\neg\neg(\exists n : N.\, \exists x : X.\, x \in f_\psi(\mathrm{step}(n)))$. Hence, by Axiom 2, $\exists n : N.\, \exists x : X.\, x \in f_\psi(\mathrm{step}(n))$, because $[\exists n : N.\, \exists x : X.\, x \in f_\psi(\mathrm{step}(n))] \in \Sigma$. Thus again $\bigsqcup(f_\psi) = g(\psi)$, concluding the proof. ∎

Note, that the hypotheses of proposition 1.31 imply Markov's Principle. But in fact, if Axiom 3 holds (so 2 and N are complete) and complete implies well-complete, then Markov's Principle follows, by the reasoning from [26].

2 SDT in Modified Realizability

2.1 The category of Modified Assemblies

The purpose of this section is to give an exposition of a particular model of SDT in the Modified Realizability Topos $\mathcal{M}od$, by which we mean the one investigated in [18]. In fact, most of the treatment takes place inside the subcategory of $\neg\neg$-separated objects of $\mathcal{M}od$ which is therefore of prime importance; but we also use the internal logic of the full topos.

The precise definition of $\mathcal{M}od$ can be found in [18] and does not need to be repeated here. Suffice it to say that the non-standard truth values (in the tripos representing $\mathcal{M}od$) are inclusions $A \subseteq B$ of subsets of the set $\mathbb{N}$ of natural numbers, such that always $0 \in B$ where 0 is such that $0 \cdot x = 0$ (we write partial recursive function application with a dot $\cdot$) and $\langle 0, 0\rangle = 0$ ($\langle -, -\rangle$ is a recursive coding of pairs, which we assume to be bijective, with recursive projections $(-)_0$ and $(-)_1$). We shall see that so long as we restrict to $\neg\neg$-separated objects, we don't have to bother about 0.

Definition 2.1 The category **ModAss** of *modified assemblies* (not to be confused with those of Thomas Streicher, in [28]!) is the following:

Objects are triples $(X, |\cdot|^X, P_X)$ where X is a set, $|\cdot|^X : X \to \mathcal{P}(\mathbb{N})$ a function assigning to each $x \in X$ a nonempty set $|x|^X$ of *realizers* of x, and $P_X \subseteq \mathbb{N}$ a nonempty set such that $|x|^X \subseteq P_X$ for all $x \in X$, the set of *global realizers* of X. We often denote the object by its underlying set.

Morphisms $(X, |\cdot|^X, P_X) \to (Y, |\cdot|^Y, P_Y)$ are functions $f : X \to Y$ such that there is a partial recursive function ϕ which is defined on P_X, maps P_X into P_Y and every $|x|^X$ into $|f(x)|^Y$. We say that ϕ tracks f.

Proposition 2.2 **ModAss** *is equivalent to the category of* $\neg\neg$*-separated objects of* $\mathcal{M}od$*.*

Proof. Let $(\mathcal{M}od)_{\mathrm{sep}}$ denote the category of $\neg\neg$-separateds in $\mathcal{M}od$. From [18], proposition 3.1 and beyond, $(\mathcal{M}od)_{\mathrm{sep}}$ looks, up to equivalence, like **ModAss** except for the requirement that always $0 \in P_X$. But, writing A^+ for the set $\{a{+}1 \mid a \in A\}$, it is evident that every object $(X, |\cdot|^X, P_X)$ of **ModAss** is isomorphic in **ModAss** to $(X, (|\cdot|^X)^+, (P_X)^+ \cup \{0\})$ and therefore the full embedding $(\mathcal{M}od)_{\mathrm{sep}} \to$ **ModAss** is essentially surjective on objects. ■

2.2 Some structure of ModAss

Limits, colimits and ccc-structure of **ModAss** are simple calculations; we omit proofs. The product of $(X, |\cdot|^X, P_X)$ and $(Y, |\cdot|^Y, P_Y)$ can be rendered as $(X \times Y, |(x, y)| = |x|^X \times |y|^Y, P_X \times P_Y)$. The products $|x| \times |y|$, $P_X \times P_Y$, etc. should be read as sets of coded pairs.

Regular subobjects of $(X, |\cdot|^X, P_X)$ are, up to isomorphism, of the form $(X', |\cdot|^X, P_X)$ for a subset X' of X.

For $A, B \subseteq \mathbb{N}$ write $A + B = (\{0\} \times A) \cup (\{1\} \times B)$. The coproduct of $(X, |\cdot|^X, P_X)$ and $(Y, |\cdot|^Y, P_Y)$ can be rendered as

$$(X \sqcup Y, |x| = \{0\} \times |x|^X, |y| = \{1\} \times |y|^Y, P_X + P_Y)$$

A morphism $f : (X, |\cdot|^X, P_X) \to (Y, |\cdot|^Y, P_Y)$ is regular epi if and only if f is a surjective function and, up to isomorphism, $|y|^Y = \bigcup_{f(x)=y} |x|^X$ and $P_Y = P_X$. A diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "h"'] & W \end{tikzcd}$$

is a pushout if and only if the underlying diagram of sets is a pushout in Set and moreover, the induced map $Z + Y \to W$ is a regular epimorphism.

The function space $(X, |\cdot|^X, P_X)^{(Y, |\cdot|^Y, P_Y)}$ has a underlying set the set of morphisms $f : (Y, |\cdot|^Y, P_Y) \to (X, |\cdot|^X, P_X)$, a realizer of f is an index of a partial recursive function which tracks f, and a global realizer is an index for a partial recursive function that is defined on P_Y and maps P_Y into P_X.

This is a good place to comment on the notions "discrete" and "modest" for modified assemblies. The natural numbers object of $\mathcal{M}od$ is represented in **ModAss** as the object $N = (\mathbb{N}, |\cdot|^N, \mathbb{N})$ with $|n|^N = \{n\}$. The functor ∇ : Set $\to \mathcal{M}od$ which inserts Set as $\neg\neg$-sheaves in $\mathcal{M}od$, factors through **ModAss** via: $\nabla(X) = (X, |x| = \mathbb{N}, \mathbb{N})$.

Mimicking known terminology for the Effective Topos, we say that a modified assembly X is *modest* if it is a regular image of a regular subobject of N. X is *discrete*, if X is internally orthogonal to $\nabla(2)$, that is if the diagonal: $X \to X^{\nabla(2)}$ is an isomorphism. It is a result of [10] that in the effective topos, for separated objects, these notions coincide (even fiberwise for families of separated objects indexed by a separated object; obviously these notions can and really should be defined for families).

In **ModAss**, an object $(X, |\cdot|^X, P_X)$ is modest if and only if, up to isomorphism, $P_X = \mathbb{N}$ and $x \neq y$ implies $|x|^X \cap |y|^X = \emptyset$; it is discrete if again, $x \neq y$ implies $|x|^X \cap |y|^X = \emptyset$ but no condition on P_X. So in **ModAss** there is a difference, even in the fiber over 1:

Proposition 2.3 *The object N^N is discrete, but not modest.*

Proof. N^N is the object $(\mathcal{R}, |\cdot|, \mathrm{Tot})$ where $\mathcal{R}$ is the set of all total recursive functions, $|\varphi|$ is the set of codes for φ and Tot is the set of all codes for total recursive functions. Were N^N modest, there would be an isomorphism

$$(Y, |\cdot|, \mathbb{N}) \to (\mathcal{R}, |\cdot|, \mathrm{Tot})$$

tracked by some recursive function ψ, but then ψ would enumerate all total recursive functions; which runs into a familiar diagonal argument. ∎

The logic of $\mathcal{M}od$ and the tripos underlying it is described in [18] (see also [9]). Salient features are the following principles:

IP	$(\neg A \to \exists n{:}N.B) \to \exists n{:}N.(\neg A \to B)$
CT	$\forall f{:}N^N \exists e{:}N \forall x{:}N \exists y{:}N.(T(e,x,y) \wedge U(y) = f(x))$
AC-N	$\forall n{:}N \exists x{:}X.R(n,x) \to \exists f{:}X^N \forall n{:}N.R(n,f(n))$

which are true in $\mathcal{M}od$. On the negative side, we have the failure of Markov's Principle in $\mathcal{M}od$. In fact, in the presence of IP, CT and AC-N, Markov's Principle is *inconsistent*, see [31].

Since we shall work in the category **ModAss** of $\neg\neg$-separated objects of $\mathcal{M}od$, a few remarks about the internal logic of this category, related to the one of $\mathcal{M}od$:

1) There is a functor $\Gamma{:}\mathcal{M}od \to$ Set, left adjoint to ∇.

2) The regular subobjects in **ModAss** are precisely those which are $\neg\neg$-closed in $\mathcal{M}od$. If φ is a $\neg\neg$-stable formula with free variable x of type

$X = (X, |\cdot|^X, P_X)$ then the regular subobject $\{x \in X \mid \varphi(x)\}$ is represented by the object $(X_\varphi, |\cdot|^X, P_X)$ where

$$X_\varphi = \{x \in X \mid \varphi \text{ has an actual realizer}\} = \Gamma(\llbracket \varphi \rrbracket)$$

(taking realizers in the tripos underlying $\mathcal{M}od$)

3) For an object $X = (X, |\cdot|^X, P_X)$, if R is a $\neg\neg$-closed equivalence relation on X (hence, by the preceding remark, represented by an ordinary equivalence relation on the *set* X), the quotient X/R is represented by the object $(X/R, |\cdot|, P_X)$ where

$$|[x]| = \bigcup_{(x,y)\in R} |y|^X$$

4) The inclusion functor from **ModAss** into $\mathcal{M}od$ does not preserve epimorphisms, only regular epimorphisms. Therefore, if R is a relation from X to Y, the statement $\forall y{:}Y \exists x{:}X.R(x, y)$ is true in $\mathcal{M}od$, if and only if the composite

$$R \to X \times Y \to Y$$

is a regular epimorphism in **ModAss**.

2.3 A Model of SDT in $\mathcal{M}od$

The dominance in $\mathcal{M}od$ that we study in this paper, is

$$\Sigma = \{p \in \Omega \mid \exists n : N.p \leftrightarrow \neg\neg(n \in K)\}$$

Here N is the natural numbers object, and K the halting set. In order to see that this is a dominance, we use the principle IP. If $p \leftrightarrow \neg\neg(n \in K)$ and $p \to \exists m : N.(q \leftrightarrow \neg\neg(m \in K))$ then by IP, $\exists m : N.(p \to (q \leftrightarrow \neg\neg(m \in K)))$ from which one obtains $p \wedge q \in \Sigma$.

The following closure properties hold for Σ: if $p, q \in \Sigma$ then $p \wedge q \in \Sigma$ and $\neg\neg(p \vee q) \in \Sigma$; if $f : N \to \Sigma$ is a morphism, then $[\neg\neg\exists n{:}N.f(n)] \in \Sigma$. Verifications are left to the reader. It follows at once that our Σ satisfies Axioms 2 and 3, and since it has binary sups, by theorem 1.27 it will satisfy Axiom 4 if Axiom 1 holds.

Modulo our identification of $(\mathcal{M}od)_{\text{sep}}$ with **ModAss**, the object Σ can be represented as

$$\Sigma = (\{\top, \bot\}, |\cdot|, \mathbb{N})$$

with $|\top| = K$ and $|\bot| = \bar{K}$, the complement of K. Note that Σ is the quotient of N by the equivalence relation:

$$n \sim m \text{ iff } \neg\neg(n \in K) \leftrightarrow \neg\neg(m \in K)$$

Since this equivalence relation is $\neg\neg$-closed, the representation of Σ follows from our remarks on the internal logic of **ModAss**.

[The reader should notice the double use of the symbol K: both for the usual halting set and for the "internal halting set", i.e. the subobject of N defined by the formula $\exists y{:}N.T(x,x,y)$. We trust that the reader will be able to tell these two K's apart]

Next, we calculate the lift functor L and the objects I and F.

In **ModAss**, the Σ-subsets of $(X, |\cdot|^X, P_X)$ are in 1-1 correspondence with subsets X' of X such that for some r.e. set A,
$\bigcup_{x\in X'} |x|^X \subseteq A$ and $(\bigcup_{x\notin X'} |x|^X) \cap A = \emptyset$. The object corresponding to this subobject is then $(X', |\cdot|^X, P_X)$. Define now:

$$L(X, |\cdot|^X, P_X) = (Y, |\cdot|^Y, P_Y)$$

where $Y = X_\perp = X \sqcup \{\perp\}$, $|x|^Y = |x|^X \times K$, $|\perp|^Y = P_X \times \bar{K}$ and $P_Y = P_X \times \mathbb{N}$.

Using the above description of Σ-subsets, one sees that $L(X, |\cdot|^X, P_X)$ classifies Σ-partial maps out of $(X, |\cdot|^X, P_X)$ and hence is indeed object part of the lift functor; its morphism part sends $f : X \to Y$ to $f_\perp = f \cup \{(\perp,\perp)\} : X_\perp \to Y_\perp$. The natural transformation ζ embeds X in $X_\perp$.

Proposition 2.4 *The functor L preserves regular epimorphisms and pushouts.*

Proof. An easy verification using the explicit descriptions of the notions involved. ∎

Incidentally, that $L(X) + L(Y) \to L(X + Y)$ is regular epi follows from Axiom 2 and IP: suppose $\alpha \in L(X + Y)$. Then

$$\exists w.w \in \alpha \Rightarrow (\exists x \in X.x \in \alpha) \vee (\exists y \in Y.y \in \alpha)$$

so by IP, since $\exists w.w \in \alpha$ is $\neg\neg$-stable, $\alpha \in L(X) \vee \alpha \in L(Y)$.

The object F, underlying object of the terminal L-coalgebra, is, by its internal definition, a regular subobject of Σ^N. Working out this internal definition, one sees that F is represented by the object

$$F = (\omega + 1, |\cdot|^F, \mathbb{N})$$

(using ω for the least infinite ordinal), where

$$|n|^F = \{e \mid W_e = \{x|x < n\}\} \text{ and } |\omega|^F = \{e \mid W_e = \mathbb{N}\}$$

As usual, W_e denotes the domain of the e-th partial recursive function. The coalgebra structure $\tau : F \to L(F)$ sends 0 to $\perp$, $n + 1$ to n and ω to ω. It is tracked by the recursive function which, given $e \in \mathbb{N}$, returns the pair $\langle \Lambda x.e{\cdot}(x + 1), \mu(e)\rangle$, where μ is total recursive such that $\mu(e) \in K$ if and only if $e{\cdot}0$ is defined; the notation $\Lambda x.\varphi(x)$ means: a standard index for the indicated partial recursive function.

As to the initial L-algebra I, we have the following general theorem:

Theorem 2.5 *For any* $\neg\neg$*-separated dominance* Σ *in a topos* $\mathcal{E}$ *satisfying* IP*, with associated lift functor* L *and terminal* L*-coalgebra* F*, the initial* L*-algebra is given by:*

$$I = \{p \in F \mid \exists n : N.\neg p(n)\}$$

Proof. Indeed, using Jibladze's formula for I, take $\exists n : N.\neg p(n)$ for ϕ. If $p(n) \to \exists m : N.\neg p(m)$ then $\exists m : N.p(n) \to \neg p(m)$ by IP (since Σ is $\neg\neg$-separated), so $\exists m : N.\neg p(\max(n, m))$ since $p \in F$, so $\exists n : N.\neg p(n)$. ∎

Therefore, by the interpretation of the internal logic in $\mathcal{M}od$, I is represented by the object

$$I = (\omega, |\cdot|^I, \mathbb{N})$$

where

$$|n|^I = \{\langle e, m\rangle \mid W_e = \{x \in \mathbb{N} \mid x < n\} \text{ and } m \geq n\}$$

The algebra structure $\sigma : L(I) \to I$ is the function sending $\perp$ to 0 and n to $n+1$; it is tracked by the recursive function which, when given a pair $\langle\langle e, m\rangle, k\rangle$, returns a pair $\langle e', m + 1\rangle$ where e' is an index for the partial recursive function

$$x \mapsto \begin{cases} 0 & \text{if } x = 0 \wedge k \in K \\ e\cdot(x-1) & \text{if } x > 0 \wedge k \in K \\ \text{undefined} & \text{else} \end{cases}$$

We shall also consider another L-algebra: let

$$I' = (\omega, |\cdot|^{I'}, \mathbb{N})$$

with $|n|^{I'} = \{e \mid W_e = \{x | x < n\}\}$. I' is $\{p \in F \mid \neg\neg\exists n : N.\neg p(n)\}$, the $\neg\neg$-closure of I in F. The algebra structure on I' is the same as for I and also tracked by (virtually) the same recursive function. Note, that I' is the $\neg\neg$-closure of I in F.

As before, $\iota : I \to F$ is the inclusion, as are $\iota_0 : I \to I'$ and $\iota_1 : I' \to F$. We have:

Proposition 2.6 *For separated objects* X *in* $\mathcal{M}od$*:* X *is complete if and only if both* X^{ι_0} *and* X^{ι_1} *are isomorphisms.*

Proof. One direction is immediate; for the other, if X is complete then X^{ι_0} is regular epi and X^{ι_1} is monic. But X^{ι_0} is monic since X is separated and ι_0 is a dense inclusion, and X^{ι_1} is regular epi for if $f : I' \to X$ let $g : F \to X$ the unique extension of the restriction of f to I. Again by density and separation, g extends f. ∎

2.4 The Completeness Axiom in $\mathcal{M}od$

Now, we verify that Σ, as we have defined it, is indeed complete. It turns out that the proof can be given entirely in the internal logic, using the internal descriptions of Σ, I and F, and the axiom schemes IP, CT and AC-N. As in the case of the Effective topos ([19]), the mathematical content of the proof is virtually the same as that of the Rice-Shapiro theorem in recursion theory. We make use of proposition 1.20, so we have to check the conditions of that proposition.

Theorem 2.7 *Σ is complete.*

Proof. First we check i) of 1.18: for $R \in \Sigma^F$, if $\varphi \in R$ and $\forall n{:}N.\varphi(n) \to \psi(n)$, then $\psi \in R$.

Since $\varphi \in F \subset \Sigma^N$ we have $\forall n{:}N\exists m{:}N.\varphi(n) \leftrightarrow \neg\neg(m \in K)$; applying AC-N and CT we get

$$\forall \varphi \in F \exists a{:}N \forall n{:}N. a{\cdot}n{\downarrow} \wedge (\varphi(n) \leftrightarrow \neg\neg(a{\cdot}n \in K)) \tag{2}$$

Define an operation $S : N \to F$ by

$$S(e)(n) \equiv \forall m \leq n \neg\neg(e{\cdot}m{\downarrow})$$

This is clearly well-defined.

Since R is a Σ-subset of F we have $\forall \varphi{:}F\exists m.(\varphi \in R \leftrightarrow \neg\neg(m \in K))$ so again applying AC-N and CT we obtain a total recursive function G such that

$$\forall e{:}N.S(e) \in R \leftrightarrow \neg\neg(G(e) \in K) \tag{3}$$

For the proof of our first claim, suppose $\varphi, \psi \in F$ and a_1, a_2 satisfy (2) for φ, ψ respectively. By the recursion theorem, find a code e such that

$$e{\cdot}x \simeq \mu z.T(a_1{\cdot}x, a_1{\cdot}x, z) \vee (T(a_2{\cdot}x, a_2{\cdot}x, (z)_0) \wedge T(G(e), G(e), (z)_1))$$

Then $\neg(G(e) \in K)$ implies $e{\cdot}x{\downarrow} \leftrightarrow a_1{\cdot}x \in K$, hence

$$\forall n(\neg\neg\forall m \leq n.e{\cdot}m{\downarrow} \leftrightarrow \forall m \leq n.\neg\neg a_1{\cdot}m \in K \leftrightarrow \forall m \leq n\varphi(m) \leftrightarrow \varphi(n))$$

so $S(e) = \varphi$. Therefore if $\varphi \in R$, we obtain, by (3), $\neg\neg(G(e) \in K)$ and $S(e) \in R$. Now suppose $\forall n.\varphi(n) \to \psi(n)$. Then for all n,

$$\begin{array}{rcl} \neg\neg\forall m \leq n.e{\cdot}m{\downarrow} & \leftrightarrow & \forall m \leq n.\neg\neg(a_1{\cdot}m \in K \vee a_2{\cdot}m \in K) \\ & \leftrightarrow & \forall m \leq n.\neg\neg(a_2{\cdot}m \in K) \\ & \leftrightarrow & \forall m \leq n.\psi(m) \\ & \leftrightarrow & \psi(n) \end{array}$$

Hence $S(e) = \psi$, so, since $S(e) \in R$, $\psi \in R$. The first claim is proved.

Next we check ii) of 1.18, which is (equivalent to): if $\varphi \in R$ then $\neg\neg\exists n.\varphi|n \in R$, where $\varphi|n$ abbreviates $\varphi \wedge \mathrm{step}(n)$.

In order to prove it, let again a satisfy (2) for φ. By the recursion theorem, find e such that

$$e{\cdot}x \simeq \begin{cases} \mu z.T(a{\cdot}x, a{\cdot}x, z) & \text{if } \forall y \leq x.\neg T(G(e), G(e), y) \\ \uparrow & \text{else} \end{cases}$$

Again, $\neg(G(e) \in K)$ implies $S(e) = \varphi$, so $\varphi \in R$ gives $\neg\neg(G(e) \in K)$ and $S(e) \in R$.
Now, by decidability of the T-predicate, if n is minimal with $T(G(e), G(e), n)$, then clearly for all k:

$$\begin{array}{rcl} \neg\neg\forall m \leq k.e{\cdot}m{\downarrow} & \leftrightarrow & k < n \wedge \forall m \leq k.a{\cdot}m \in K \\ & \leftrightarrow & k < n \wedge \varphi(k) \\ & \leftrightarrow & (\varphi|n)(k) \end{array}$$

so $S(e) = \varphi|n$ and $\varphi|n \in R$. Since $\neg\neg G(e) \in K$, we have $\neg\neg\exists n.\varphi|n \in R$, as required.

Finally, we check iii) of 1.18. Let $R \in \Sigma^I$. We want to show that the formula

$$\neg\neg\exists n.\varphi|n \in R$$

defines a Σ-subset R' of F, which is equivalent to that statement.
Since R is a Σ-subset of I we have

$$\forall\varphi \in F(\varphi \in I \rightarrow \exists m{:}N.(\varphi \in R \leftrightarrow \neg\neg m \in K))$$

Since $\varphi|n \in I$ for all n, by AC-N and CT

$$\forall\varphi \in F\exists a{:}N\forall n{:}N.(a{\cdot}n{\downarrow} \wedge (\varphi|n \in R \leftrightarrow \neg\neg a{\cdot}n \in K)$$

Suppose $\varphi \in F$, $a \in N$ satisfy this; then

$$\neg\neg\exists n.\varphi|n \in R \leftrightarrow \neg\neg\exists n.\neg\neg a{\cdot}n \in K \leftrightarrow \neg\neg\exists n.a{\cdot}n \in K$$

(Since $\neg\neg\exists x\neg\neg\exists y \leftrightarrow \neg\neg\exists xy$ intuitionistically) We see that R' is indeed a Σ-subset of F. ∎

2.5 Counterexamples in ModAss

Our first counterexample is the failure of the strong form of the Scott Principle (cf. proposition 1.24), which is the Weak Scott Priciple without $\neg\neg$.

Proposition 2.8 *In $\mathcal{M}od$, the Scott Principle fails.*

Proof. Σ^{Σ^N} is isomorphic to the object whose underlying set is the set of all r.e. sets which are extensional in codes for r.e. sets, $|A|$ is the set of codes for A, and the set of global realizers is $\mathbb{N}$. We have $\{\Phi \in \Sigma^{\Sigma^N} \,|\, N \in \Phi\}$ and $\{(n, \Phi) \in N \times \Sigma^{\Sigma^N} \,|\, \{m \,|\, m < n\} \in \Phi\}$ as $\neg\neg$-closed subobjects of Σ^{Σ^N} and $N \times \Sigma^{\Sigma^N}$ respectively, and validity of the Scott Principle means that the projection from the latter to the former is a regular epimorphism. This boils down to: there is a *total* recursive function ψ such that for all e, if W_e is an extensional r.e. set containing (all codes for) $\mathbb{N}$, then W_e contains (all codes for) $\{m \mid m < \psi(e)\}$. This, of course cannot be, for then define by the recursion theorem:

$$e{\cdot}x \simeq \begin{cases} 0 & \text{if } \forall m < \psi(e)+1.x{\cdot}m\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Clearly, W_e would be an extensional r.e. set containing $\mathbb{N}$ but not $\{m \mid m < \psi(e)\}$. ∎

Our next counterexample shows that the implication

$$2 \text{ well-complete} \Rightarrow \text{ Axiom } 3$$

of proposition 1.23 cannot be reversed, and that the well-complete objects are not closed under coproducts. It also shows that the complete objects are not closed under L in $\mathcal{M}od$; that 2 is not a regular Σ-poset, and therefore that the regular Σ-posets are not closed under coproducts.

Proposition 2.9 *$L(2)$ is not complete.*

Proof. $L(2)$ is the object with underlying set $\{\bot, 0, 1\}$, with $|\bot| = \{0,1\} \times \bar{K}$, $|0| = \{0\} \times K$, $|1| = \{1\} \times K$ and $\{0,1\} \times \mathbb{N}$ is the set of global realizers. Definitely, $L(2)^\iota : L(2)^F \to L(2)^I$ is a bijective function, but we show that its inverse cannot have a realizer. Suppose the contrary; let ψ track $(L(2)^\iota)^{-1}$.

Let c be a fixed element of $\bar{K}$, $d \in |\omega|^F$. Using the recursion theorem, let e be an index such that:

$$e{\cdot}\langle f, m\rangle \simeq \begin{cases} \langle 0, c\rangle & \text{if there is no computation} \\ & \text{in } \leq \langle f, m\rangle \text{ steps of } \psi(e){\cdot}d \\ \langle 1-i, \mu(f,h)\rangle & \text{otherwise, if } h \text{ is the} \\ & \text{least computation of } \psi(e){\cdot}d \text{ with outcome} \\ & \langle i, y\rangle, \text{ and } \mu \text{ is total, such that} \\ & \mu(f,h) \in K \text{ if and only if } f{\cdot}x\downarrow \\ & \text{for all } x \leq \max\{m \mid \exists g.\langle g, m\rangle \leq h\}+1 \end{cases}$$

Clearly, $e \in \mathbb{N} \to \{0,1\} \times \mathbb{N}$. Also, $\psi(e){\cdot}d$ must be defined since otherwise e realizes the function $\lambda i : I.\bot$. But then, e is a realizer of an element φ of $L(2)^I$, whereas $\psi(e)$ cannot realize an extension of φ to F. ∎

Remark An alternative proof of 2.9 can be given using the fact that $L(2)^N$ represents only the *recursively separable* disjoint pairs of r.e. sets.

Referring back to proposition 2.6, one would like to see where the obstruction to the completeness of $L(2)$ is located. In fact, we have (because L preserves pushouts) a regular epimorphism $\Sigma + \Sigma \to L(2)$, and since F has a top element, also the map

$$\Sigma^F + \Sigma^F \to L(2)^F$$

is a regular epimorphism.

But in fact it is not so much the top element which is essential here, since also $\Sigma^{I'} + \Sigma^{I'} \to L(2)^{I'}$ is regular epi, as follows from the proof of the following proposition.

Proposition 2.10 *$L(2)^{\iota_1} : L(2)^F \to L(2)^{I'}$ is an isomorphism.*

Proof. Since the composite $L(2)^F \to L(2)^{I'} \to L(2)^I$ is monic (we've seen that the function is bijective; monics in **ModAss** are simply 1-1 functions), definitely $L(2)^{\iota_1}$ is monic; it suffices to show that it is a regular epimorphism.

Let $f : I' \to L(2)$ be realized by ψ. Then ψ is total recursive, maps $\mathbb{N}$ into $\{0,1\} \times \mathbb{N}$ and for $e \in |n|^{I'}$ we have $f(n) = i$ ($i \in \{0,1\}$) if and only if $\psi(e)_0 = i$ and $\psi(e)_1 \in K$, where $\psi(e)$ is the coded pair $\langle \psi(e)_0, \psi(e)_1 \rangle$.

Let, for $n \in \mathbb{N}$, $[0]^n$ be a standard code for the function $\lambda x. \begin{cases} 0 & \text{if } x < n \\ \uparrow & \text{else} \end{cases}$

By the recursion theorem, let e be such that:

$$e{\cdot}x \simeq \begin{cases} & \text{Search for the minimal pair } \langle n, w \rangle \text{ such} \\ & \text{that } \psi([0]^n)_0 \neq \psi(e)_0 \text{ and } w \text{ testifies} \\ & \text{that } \psi([0]^n)_1 \in K; \\ \uparrow & \text{if such } \langle n, w \rangle \text{ does not exist; otherwise,} \\ 0 & \text{if } x < n \\ \uparrow & \text{else} \end{cases}$$

Then e is always a realizer of an element of I'. Moreover, if there is an n with $f(n) = i$ ($i \in \{0,1\}$), then $\psi(e)_0 = i$.

Let μ be total recursive such that:

$$\mu(a) \in K \Leftrightarrow \exists n\, (\psi([0]^n)_1 \in K \wedge \forall y < n. a{\cdot}y{\downarrow})$$

Then the total recursive function

$$\chi(a) = \langle \psi(e)_0, \mu(a) \rangle$$

realizes an extension of f to a function $F \to L(2)$ (Clearly, a code for χ is obtained recursively in a code for ψ) ∎

On the positive side, we have the following lemma, which is due to Rosolini ([24]):

Lemma 2.11 (Rosolini) *Let $(X, \leq)$ be an internal preorder and $f : N \to X$ a chain with supremum x. Then for any Σ-subset U of X:*

$$x \in U \Rightarrow \neg\neg\exists n{:}N. f(n) \in U$$

Proof. The proof is the same as in [24] except for the final appeal to Markov's Principle, which explains the $\neg\neg$ in the statement. ∎

Corollary 2.12 *Any morphism $f : X \to Y$ preserves suprema of chains w.r.t. the Σ-preorder.*

Proof. Let $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots$ have supremum x. Then $f(x)$ is an upper bound for $\{f(a_i) \mid i \in N\}$; if y is another such, and U a Σ-subset of Y such that $f(x) \in U$, then by 2.11, since $f^{-1}(U)$ is a Σ-subset of X, $\neg\neg\exists n \in N. f(a_n) \in U$; so $\neg\neg y \in U$, so $y \in U$. Hence $f(x) \sqsubseteq y$. ∎

Corollary 2.13 *If $f : N \to X$ is a chain for the Σ-order, then $x = \sup_n f(n)$ if and only if*

$$\forall P{:}\Sigma^X.(P(x) \Leftrightarrow \neg\neg\exists n{:}N.P(f(n)))$$

The proof is left to the reader. Note, that Reus takes this as a *definition* of the supremum of a chain.

We finish with a few remarks about regular Σ-posets in the model in **ModAss**. From proposition 2.9, theorem 1.22 and the validity of Axiom 3 (which implies that 2 is complete) we see that 2 is not regular; nor is N. Examples of regular Σ-posets are: Σ and all its powers, 1 (by Axiom 2, 1 is a $\neg\neg$-closed subobject of Σ), F and I' as $\neg\neg$-closed subobjects of Σ^N, and of course all L-iterates of these.

I is *not* regular; this follows from the following proposition, whose proof is left to the reader, and the remark that (in **ModAss**) the inclusion $\iota_0 : I \to I'$ is not $\neg\neg$-closed.

Proposition 2.14 *For X a regular Σ-poset, $X^{\iota_0} : X^{I'} \to X^I$ is an isomorphism.*

Example. This is an example of an object X for which the notions of completeness and chain completeness do *not* coincide. Consider, as in [19], the object Z_A: $Z_A = (\{\top, \bot\}, |\cdot|, \mathbb{N})$ where $|\top| = A$, $|\bot| = \bar{A}$, for some nonrecursive r.e. set A which is not m-equivalent to K.

Z_A is complete, but not chain complete. To see that Z_A is complete, note that every morphism from Σ to Z_A must be constant (otherwise we had a reduction of K to either A or $\bar{A}$); therefore Z_A is orthogonal to both I and F, and for that reason complete.

To see that Z_A is not chain complete we employ a trick due to Rosolini ([24]). Define the following sequence of chains c_{nm} in Z_A:

$$c_{nm} = \begin{cases} \top & \text{if } \exists m' \leq m.T(n,n,m') \\ \bot & \text{otherwise} \end{cases}$$

If Z_A were chain complete, there would be a function $f : N \to Z_A$ such that for all n, $f(n) = \sup_m c_{nm}$. But we can see that any such morphism would reduce K to A, since $f(n) = \top$ iff $n \in K$.

Remark Rather embarrassingly, we do not know whether Z_A is well-complete (and we do not yet have examples of well-complete objects which are not regular Σ-posets; clearly, Z_A cannot be regular in view of proposition 1.30).

3 The Standard Model in the Effective Topos

As the model of SDT in the Effective Topos that we deal with in this section, is the best investigated model in existence, we don't have many new results. The main theorems are 3.2 and 3.3 below. In a separate subsection, we discuss a relationship between the models in $\mathcal{E}ff$ and $\mathcal{M}od$.

The effective topos $\mathcal{E}ff$ is described at length in [7]. Its full subcategory of $\neg\neg$-separated objects is also presented there, as well as in many other papers. It is, up to equivalence, the category **Ass** of *Assemblies*:

Definition 3.1 An *assembly* is a pair $(X, |\cdot|)$ where X is a set and $|x|$ is a nonempty subset of $\mathbb{N}$, for every $x \in X$. A *morphism* of assemblies $(X, |\cdot|) \to (Y, |\cdot|)$ is a function $f : X \to Y$ such that there is a partial recursive function φ which tracks f, i.e. $\forall x \in X \forall n \in |x| (\varphi(n){\downarrow} \,\&\, \varphi(n) \in |f(x)|)$.

The structure of **Ass** is well known, so we omit details here. We do list, however, some principles from the internal logic of $\mathcal{E}ff$ that we shall need:

$$\begin{array}{rl}
\mathrm{ECT} & \forall n{:}N\,(\neg A(n) \to \exists m{:}N\, B(n,m)) \to \\
 & \exists f{:}N \forall n{:}N\,(\neg A(n) \to f{\cdot}n{\downarrow} \wedge B(n, f{\cdot}n)) \\
\mathrm{AC-N} & \forall n{:}N \exists x{:}X\, B(n,x) \to \exists f{:}X^N \forall n{:}N\, B(n, f(n)) \\
\Omega\mathrm{-cov} & \forall \phi{:}\Omega\, \exists A{:}\mathcal{P}_{\neg\neg}(N)\,(\phi \leftrightarrow \exists n{:}N\, n \in A) \\
\mathrm{MP} & \forall f{:}2^N\,(\neg\neg\exists n{:}N\, f(n) = 0 \to \exists n{:}N\, f(n) = 0)
\end{array}$$

In Ω-cov, $\mathcal{P}_{\neg\neg}(N)$ is the object of $\neg\neg$-closed subsets of N. Throughout, N is the natural numbers object of $\mathcal{E}ff$.

The dominance under discussion here is

$$\Sigma = \{p \in \Omega \mid \exists n{:}N\,(p \leftrightarrow n \in K)\}$$

By MP, Σ is $\neg\neg$-separated. Σ is a dominance for which Axioms 1–4 hold, as follows from [23, 19] and was explicitly shown in [15]. Moreover one sees that Σ is closed under existential quantification over N, so that by proposition 1.31, the notions complete and well-complete coincide in $\mathcal{E}ff$ for this dominance.

Σ, being $\neg\neg$-separated, is represented by the assembly

$$(\{\bot, \top\}, |\bot| = \bar{K}, |\top| = K)$$

In analogy with Modified Assemblies, a Σ-subobject of $(X, |\cdot|)$ is a subassembly $(X', |\cdot|)$ where X' is a subset of X such that for some r.e. set $A \subseteq \mathbb{N}$,

$$X' = \{x \in X \mid |x| \subseteq A\} = \{x \in X \mid |x| \cap A \neq \emptyset\}$$

It is then an easy matter to verify that the lift functor L on **Ass** is represented as follows:

$$L(X, |\cdot|) = (X \sqcup \{\bot\}, |\cdot|^{LX})$$

where $|\bot|^{LX} = \bar{K}$ and $|x|^{LX} = \{n \in \mathbb{N} \mid n{\cdot}n \in |x|\}$ (Note the difference between the lift functor here, and in **ModAss**!). One checks that the functor thus described (together with the natural transformation ζ which embeds X in $X \sqcup \{\bot\}$), classifies Σ-partial maps.

The terminal L-coalgebra F is represented by the assembly $(\omega + 1, |\cdot|)$ where

$$\begin{array}{rcl}
|n| & = & \{e \mid W_e = \{m | m < n\}\} \\
|\omega| & = & \{e \mid W_e = \mathbb{N}\}
\end{array}$$

as follows easily from the logical definition of F as regular subobject of Σ^N.

Regarding the initial L-algebra we have the following general theorem:

Theorem 3.2 *For every $\neg\neg$-separated dominance Σ in $\mathcal{E}ff$ with associated lift functor L and terminal L-coalgebra F, the initial L-algebra is given by*

$$I = \{\varphi \in F \mid \neg\neg\exists n{:}N\, \neg\varphi(n)\}$$

Proof. We show that $\neg\neg\exists n{:}N\, \neg\varphi(n)$ implies Jibladze's formula in the internal logic of $\mathcal{E}ff$. This suffices by proposition 1.6. So let $\phi{:}\Omega$ and suppose

$$\forall n{:}N\, ((\varphi(n) \to \phi) \to \phi)$$

By Ω-cov, let $A \in \mathcal{P}_{\neg\neg}(N)$ be such that $\phi \leftrightarrow \exists y \in A$. By ECT, since Σ is $\neg\neg$-separated, $\varphi(n) \to \phi$ is equivalent to $\exists u(\varphi(n) \to u{\cdot}0{\downarrow} \wedge u{\cdot}0 \in A)$. Since also $\varphi(n) \to u{\cdot}0{\downarrow} \wedge u{\cdot}0 \in A$ is $\neg\neg$-stable, another application of ECT yields:

$$\exists g \forall n, u((\varphi(n) \to u{\cdot}0{\downarrow} \wedge u{\cdot}0 \in A) \to g{\cdot}(n,u){\downarrow} \wedge g{\cdot}(n,u) \in A) \quad (1)$$

Let g as in (1); by a parametrized version of the recursion theorem, let u_n be such that

$$u_n{\cdot}x \simeq g{\cdot}(n+1, u_{n+1}) \qquad (2)$$

Then for all $n{:}N$ we have:

$$\begin{array}{rcl} (u_{n+1}{\cdot}0{\downarrow} \wedge u_{n+1}{\cdot}0 \in A) & \to & \\ (\varphi(n+1) \to (u_{n+1}{\cdot}0{\downarrow} \wedge u_{n+1}{\cdot}0 \in A)) & \to & \text{by (1)} \\ g{\cdot}(n+1, u_{n+1}){\downarrow} \wedge g{\cdot}(n+1, u_{n+1}) \in A & \to & \text{by (2)} \\ u_n{\cdot}0{\downarrow} \wedge u_n{\cdot}0 \in A & & \end{array}$$

Moreover we have:

$$\begin{array}{rc} \neg(u_n{\cdot}0{\downarrow} \wedge u_n{\cdot}0 \in A) & \to \\ \neg(\varphi(n+1) \to (u_{n+1}{\cdot}0{\downarrow} \wedge u_{n+1}{\cdot}0 \in A)) & \to \\ \varphi(n+1) & \end{array}$$

because $\varphi(n+1)$ is $\neg\neg$-stable. Combining we get

$$\begin{array}{rc} \neg(u_0{\cdot}0{\downarrow} \wedge u_0{\cdot}0 \in A) & \to \\ \forall n \neg(u_n{\cdot}0{\downarrow} \wedge u_n{\cdot}0 \in A) & \to \\ \forall n\, \varphi(n+1) & \to \\ \forall n\, \varphi(n) & \end{array}$$

By the assumption $\neg\neg\exists n{:}N\, \neg\varphi(n)$ we get

$$\neg\neg(u_0{\cdot}0{\downarrow} \wedge u_0{\cdot}0 \in A)$$

hence $u_0{\cdot}0{\downarrow} \wedge u_0{\cdot}0 \in A$ since this is $\neg\neg$-stable; i.e., $\exists y \in A$, i.e. ϕ, as required.■

From Theorem 3.2 we deduce that in the case we are discussing, I is represented by the assembly $(\omega, |\cdot|)$ with $|n| = \{e \mid W_e = \{m | m < n\}\}$, as it is a regular subobject of F.

How does this compare with the object $I_0 = \{\varphi \in F \mid \exists n{:}N \neg\varphi(n)\}$ which is, by corollary 1.10, the colimit of the initial L-chain? The following theorem is a strong way of saying that I_0 and I are not isomorphic, in view of proposition 1.14:

Theorem 3.3 *The object I_0 is complete.*

Proof. Since I_0 is a Σ-poset, the map $(I_0)^F \to (I_0)^I$ is certainly monic; it suffices to see that it is regular epi.

I_0 is represented by the assembly $(\omega, |\cdot|_0)$ where

$$|n|_0 = \{\langle e, m\rangle \mid W_e = \{x|x < n\} \wedge n \leq m\}$$

Let $f : I \to I_0$ be a morphism, tracked by a partial recursive function φ. So if $W_e = \{x|x < n\}$ then $\varphi(e) = \langle \varphi(e)_0, \varphi(e)_1\rangle$ with $f(n) \leq \varphi(e)_1$ and $W_{\varphi(e)_0} = \{x|x < f(n)\}$

By a standard argument one shows that $f : \omega \to \omega$ must be order-preserving (the type of argument used in the Rice-Shapiro theorem).

Let B be recursive, such that for all $n \in \mathbb{N}$, $W_{B(n)} = \{x|x < n\}$. Now use the recursion theorem to find an index u of a partial recursive function, satisfying

$$u{\cdot}x \simeq \begin{cases} \uparrow & \text{if } \varphi(u)\uparrow \\ 0 & \text{if for no } v, w < x,\ w \text{ is a computation of } \varphi(B(v))_0{\cdot}\varphi(u)_1 \\ \uparrow & \text{else} \end{cases}$$

If for some $v \in \mathbb{N}$, $\varphi(B(v))_0{\cdot}\varphi(u)_1$ is defined, then there is a least $x \in \mathbb{N}$ such that there are $v, w < x$ with w a computation of $\varphi(B(v))_0{\cdot}\varphi(u)_1$. It follows that $W_u = W_{B(x)}$ for this x, hence $u \in |x|$ (w.r.t. I), hence $\varphi(B(u))_0{\cdot}\varphi(u)_1$ is undefined; this contradicts the monotonicity of f. Hence, for no $v \in \mathbb{N}$, $\varphi(B(v))_0{\cdot}\varphi(u)_1$ is defined.

Define $f' : F \to I_0$ as the unique function which is tracked by $\Lambda e.\langle\psi(e), \varphi(u)_1\rangle$, where

$$\psi(e) = \Lambda x \begin{cases} 0 & \text{if } x < \varphi(u)_1 \text{ and} \\ & \exists y\, (e{\cdot}y\downarrow \wedge \varphi(B(y+1))_0{\cdot}x\downarrow) \\ \uparrow & \text{else} \end{cases}$$

Then f' extends f and a code for a tracking of f' is found recursively in codes for trackings of f, which gives the desired operation: $(I_0)^I \to (I_0)^F$. ∎

There is a point about theorem 3.3 which deserves to be made, in particular in connection with the research in [2]. Let **WC** denote the category of (well-)complete objects of $\mathcal{E}ff$. In **WC**, the object F carries both the initial algebra and final coalgebra structures for L, and they are each other's inverse (F is a *fixed-point object* in the sense of [1]).

Now, in $\mathcal{E}ff$, F is the internal limit of

$$1 \leftarrow L(1) \leftarrow L^2(1) \leftarrow \ldots$$

whereas I_0 is the internal colimit of

$$0 \to L(0) \to L^2(0) \to \ldots$$

This shows, that F, although a fixed-point object, is not "inductive" in the sense of [2].

Moreover, for abstract reasons (see the final section of [15]), the category of (well-)complete strict lift algebras is internally algebraically compact ([5]). Here, F is still the above limit; but the colimit of the other chain will be $L(I_0)$, which is clearly not isomorphic to F.

Therefore, we have algebraic compactness, without the simplest instance of the limit-colimit coincidence holding. Thus one loses generality if one predicates algebraic compactness on the limit-colimit coincidence, as done in [2].

3.1 Relating the Models in $\mathcal{E}ff$ and $\mathcal{M}od$

The category **Ass** is a full, coreflective subcategory of **ModAss**. Define M : **Ass** $\to$ **ModAss** by

$$M(X, |\cdot|) = (X, |\cdot|, \bigcup_{x \in X} |x|)$$

if X is nonempty, and put $M(\emptyset, \emptyset) = (\emptyset, \emptyset, \mathbb{N})$.

M has a right adjoint C : **ModAss** $\to$ **Ass** given by forgetting the global realizers. The actions of M and C on morphisms are self-evident.

The following theorem relates the models of SDT in **Ass** and **ModAss** that we have been discussing. In both categories, Σ, L, F and I have their standard meaning and we use the same symbols (relying on context to make clear in which category we are); moreover in **Ass** we have the object I_0, colimit of the internal initial L-chain, and in **ModAss** we have the object I', the $\neg\neg$-closure of I in F.

Theorem 3.4

i) *M is full and faithful, and C is faithful;*

ii) *M preserves products and C preserves finite colimits;*

iii) *For objects X, Y of* **Ass**, *$X^Y \cong C(M(X)^{M(Y)})$, dinatural in X and Y;*

iv) *$M(I_0) \cong I$; $M(F) \cong F$; $M(I)$ lies strictly between I and I' as subobjects of F; hence M does not preserve equalizers;*

v) *Let L_0 :* **Ass** $\to$ **Ass** *be defined by $L_0 = CLM$. Then $ML_0 \cong LM$ naturally, and I_0 is the initial L_0-algebra;*

vi) *M reflects completeness.*

Proof. i) and ii) are easy verifications. iii) is a standard Yoneda argument, using $M \dashv C$ and i)–ii).
iv): We have $M(I_0) \cong I$ in **ModAss** since the identity function $I \to M(I_0)$ is tracked by $\Lambda n.\langle \psi(n), (n)_1 \rangle$, where

$$\psi(n) = \Lambda x. \begin{cases} 0 & \text{if } x < (n)_1 \text{ and } \forall y \leq x.(n)_0 \cdot y\downarrow \\ \uparrow & \text{else} \end{cases}$$

Similarly, $F \to M(F)$ is tracked by

$$\Lambda n.\Lambda x.\begin{cases} 0 & \text{if } \forall y \leq x.n{\cdot}y\downarrow \\ \uparrow & \text{else} \end{cases}$$

The non-isomorphism of I' and $M(I)$ is a standard exercise in recursion theory. Note that $M(I)$ is not a regular subobject of F, so M does not preserve equalizers.
v) $ML_0 \cong LM$ is easy to see. Now by this isomorphism, any L_0-algebra $L_0(X) \to X$ in **Ass** is carried by M to an L-algebra $LM(X) \to M(X)$ in **ModAss**, so there is a unique L-algebra morphism $I \xrightarrow{f} M(X)$. Since M is full, $f = M(\bar{f})$ for some $\bar{f} : I_0 \to X$. By fully faithfulness of M, there is a unique L_0-algebra structure on I_0 which is carried by M to the initial algebra structure on I. Then since $M(\bar{f})$ is an L-algebra map, by faithfulness of M $\bar{f}$ is an L_0-algebra map, and the unique one.
vi) Suppose $M(X)$ is complete, i.e. $M(X)^{\iota} : M(X)^F \to M(X)^I$ is an isomorphism. Now $I \xrightarrow{\iota} F$ is $M(I_0 \xrightarrow{\iota'} F)$. Applying the functor C and iii), we find that

$$X^{\iota'} : X^F \to X^{I_0}$$

is an isomorphism. The completeness of X follows in the same way as in 2.6, noting that X is separated and $I_0 \subseteq I$ a dense inclusion. ∎

4 A Grothendieck topos

In this section we consider an example of an entirely different nature. We analyse a Grothendieck topos in which: Axioms 1–4 hold, 2 is well-complete, but N is not well-complete. This provides a counterexample to the converse of the first implication of 1.23.

We begin by introducing notation for sites and sheaves over them. Full definitions can be found in [16]. Let $\mathbf{C}$ be any small category. We use letters $A, B, \ldots$ for objects of $\mathbf{C}$, and Greek letters $\varphi, \vartheta, \ldots$ for morphisms. We write $\widehat{\mathbf{C}}$ for the category of presheaves on $\mathbf{C}$. Given a presheaf F, an element $x \in F(B)$ and a morphism $\varphi : A \to B$ we write $x \cdot \varphi$ for the element $F(\varphi)(x) \in F(A)$.

Let $\mathbf{J}$ be a Grothendieck topology on $\mathbf{C}$. We write $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{J})$ for the full subcategory of $\widehat{\mathbf{C}}$ consisting of sheaves for $\mathbf{J}$. Given a formula ϕ of the Mitchell-Bénabou language, we write $A \Vdash \phi$ to mean that A forces ϕ according to the Kripke-Joyal semantics for $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{J})$ [16, VI.7]. Our sole application of Kripke-Joyal smantics is to derive a general characterisation of the $\neg\neg$-separated objects of $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{J})$, valid under fairly weak conditions on the site $(\mathbf{C}, \mathbf{J})$.

Proposition 4.1 *Suppose that* $\mathbf{C}$ *contains an object* I *such that:*

- I *is not covered by the empty family in* $\mathbf{J}$*;*
- $\mathbf{C}(I, A)$ *empty implies* A *is covered by the empty family in* $\mathbf{J}$*; and*

- $\mathbf{C}(I, A)$ *nonempty implies every morphism in* $\mathbf{C}(A, I)$ *is split epi.*

Then the following are equivalent for a sheaf F.

1. F *is* $\neg\neg$*-separated in* $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{J})$.

2. *For all* A *and* $x, y \in F(A)$, $x = y$ *if and only if, for all* $\varphi : I \to A$, $x \cdot \varphi = y \cdot \varphi$.

Proof. Suppose F is $\neg\neg$-separated. Assume that, for all $\varphi : I \to A$, $x \cdot \varphi = y \cdot \varphi$. We must show that $x = y$. As F is $\neg\neg$-separated, it suffices to show that $A \Vdash \neg\neg(x = y)$. Consider any $\varphi : B \to A$ such that $B \Vdash \neg(x \cdot \varphi = y \cdot \varphi)$. If there existed $\vartheta : I \to B$, then we would have both $I \Vdash x \cdot \varphi\vartheta = y \cdot \varphi\vartheta$ (by the assumption), and $I \Vdash \neg(x \cdot \varphi\vartheta = y \cdot \varphi\vartheta)$ (by the monotonicity of forcing), which is a contradiction as I is not covered by the empty family. Therefore $\mathbf{C}(I, B)$ is empty, and so B is covered by the empty family as required.

Conversely, suppose 2 holds. Given any $x, y \in F(A)$, suppose that $A \Vdash \neg\neg(x = y)$. We must show that $x = y$. By 2, it suffices to show that $x \cdot \varphi = y \cdot \varphi$, for all $\varphi : I \to A$. Consider any such φ. By monotonicity, $I \Vdash \neg\neg(x \cdot \varphi = y \cdot \varphi)$. As I is not covered by the empty family, $I \nVdash \neg(x \cdot \varphi = y \cdot \varphi)$. So there exists $\vartheta : B \to I$, where B is not covered by the empty family, such that $B \Vdash x \cdot \varphi\vartheta = y \cdot \varphi\vartheta$. However, ϑ is split epi, so there exists $\vartheta' : I \to B$ with $\theta \circ \vartheta' = \mathsf{id}_I$. By monotonicity, $B \Vdash x \cdot \varphi\vartheta\vartheta' = y \cdot \varphi\vartheta\vartheta'$, i.e. $I \Vdash x \cdot \varphi = y \cdot \varphi$ as required. ∎

We write $\omega\mathbf{cpo}$ for the category of ω-complete partial orders (i.e. partial orders for which every ascending chain has a least-upper bound) and ω-continuous functions between them (i.e. monotone functions that preserve lubs of ascending chains). Following [3, 4], we shall construct a topos, into which $\omega\mathbf{cpo}$ embeds, from a site based on a small full subcategory of $\omega\mathbf{cpo}$. However, whereas their site was chosen to ensure that ω-complete partial orders embed as nicely as possible, our site is defined specifically to prevent the natural numbers object from being well-complete.

Henceforth, let $\mathbf{C}$ be any small full subcategory of $\omega\mathbf{cpo}$ satisfying the three conditions listed below. (The conditions referring to topological properties arise by considering $\omega\mathbf{cpo}$ as a full subcategory of $\mathbf{Top}$, the category of topological spaces and continuous functions, under the standard ω-Scott topology on ωcpos.)

1. $\mathbf{C}$ contains: the initial object, 0; the terminal object, 1; Sierpinski space, $\mathbb{O}$; the object $\overline{\omega} = \{0, 1, \ldots, \omega\}$ (with the ascending order); and the discrete natural numbers, $\mathbb{N}$.

2. $\mathbf{C}$ is closed under finite products.

3. If X is an open subset of an object A of $\mathbf{C}$ then X is itself an object of $\mathbf{C}$.

The essential conditions are: that $\mathbf{C}$ contains $\overline{\omega}$, which implies that $\mathbf{C}$ is a dense subcategory of $\omega\mathbf{cpo}$; that $\mathbf{C}$ is closed under open subobjects, which allows the

Grothendieck topology below to be defined on $\mathbf{C}$; and that $\mathbf{C}$ contains $\mathbb{N}$, which will be crucial in our proof of proposition 4.8.

Definition 4.2 (Finite open cover topology) The *finite open cover topology*, $\mathbf{K}$, on $\mathbf{C}$ is the Grothendieck topology generated by basic covers consisting of finite families of inclusions $\{A_i \subseteq A\}_{1 \leq i \leq n}$ (for $n \geq 0$) where $\{A_i\}_{1 \leq i \leq n}$ is an open cover of A.

Observe that the empty ωcpo is the only object covered by the empty family.

We write $\mathbf{y} : \mathbf{C} \to \widehat{\mathbf{C}}$ for the Yoneda functor $\mathbf{y}(A) = \mathbf{C}((-), A)$. There is also an extended Yoneda functor $\mathcal{Y} : \omega\mathbf{cpo} \to \widehat{\mathbf{C}}$ defined by $\mathcal{Y}(D) = \omega\mathbf{cpo}(I(-), D)$, where we write $I : \mathbf{C} \to \omega\mathbf{cpo}$ for the full inclusion functor. As $\mathbf{C}$ is a dense subcategory of $\omega\mathbf{cpo}$, the functor $\mathcal{Y}$ is full and faithful. The next two propositions establish that $\mathcal{Y}$ also behaves well with respect to the category $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$ of sheaves, which is the category in which we are primarily interested.

Proposition 4.3 *For every ωcpo D, $\mathcal{Y}(D)$ is a $\neg\neg$-separated $\mathbf{K}$-sheaf.*

Proof. That $\mathcal{Y}(D)$ is a sheaf is easily verified (cf. the standard verification that the open cover topology is subcanonical [16, pp. 124–5]). Its $\neg\neg$-separation follows from Proposition 4.1. Setting $I = 1$, the site $(\mathbf{C}, \mathbf{K})$ clearly has the required properties. Then statement 2 of Proposition 4.1 holds for $\mathcal{Y}(D)$ because 1 is a generator in $\omega\mathbf{cpo}$. ∎

Proposition 4.4 *$\mathcal{Y}$ exhibits $\omega\mathbf{cpo}$ as a full reflective exponential ideal of* $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$.

Proof. We have already seen that $\mathcal{Y}$ gives a functor from $\omega\mathbf{cpo}$ to $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$, which is itself an exponential ideal of $\widehat{\mathbf{C}}$. By its definition, $\mathcal{Y}$ preserves all limits. It is well known that $\omega\mathbf{cpo}$ is cartesian closed. The familiar Yoneda-based argument that $\mathbf{y}$ preserves exponentials when $\mathbf{C}$ is a cartesian closed category, [25], extends (even though $\mathbf{C}$ need not be cartesian closed) to show that, for any object A of $\mathbf{C}$ and ωcpo D, we have $\mathcal{Y}(D^A) \cong \mathcal{Y}(D)^{\mathbf{y}(A)}$ (the argument uses the closure of $\mathbf{C}$ under finite products). To show that $\omega\mathbf{cpo}$ is an exponential ideal, consider any sheaf F. Then, for some diagram of representables $\{\mathbf{y}(A_i)\}$ we have that $F \cong \varinjlim \mathbf{y}(A_i)$. Therefore $\mathcal{Y}(D)^F \cong \varprojlim \mathcal{Y}(D)^{\mathbf{y}(A_i)} \cong \varprojlim \mathcal{Y}(D^{A_i}) \cong \mathcal{Y}(\varprojlim(D^{A_i}))$. Thus $\mathcal{Y}(D)^F$ is indeed in the image of $\mathcal{Y}$. Finally, $\mathcal{Y}$ has a left adjoint by the Special Adjoint Functor Theorem because it preserves all limits and Sierpinski space, $\mathbb{O}$, is a cogenerator in $\omega\mathbf{cpo}$. ∎

Next we identify a dominance in $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$, as required for the development of synthetic domain theory. Let Σ be the object $\mathcal{Y}(\mathbb{O})$. As $\mathbb{O}$ classifies open subobjects in $\mathbf{C}$, it follows that Σ is a dominance in $\widehat{\mathbf{C}}$ (see Theorem 3.1.9 of [23]). The induced lifting functor maps a presheaf F to the presheaf

$$L(F)(A) \quad = \quad \{(U, x) \mid U \text{ is an open subset of } A \text{ and } x \in F(U)\}$$

where the action on morphisms of $\mathbf{C}$ is defined by taking inverse-images. Moreover, as Σ is a $\mathbf{K}$-sheaf, it is easily seen that Σ is also a dominance on $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$

(cf. Proposition 1.6 of [2]), and hence the lifting functor above cuts down to $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$. Further, it is clear from the explicit description of the lifting functor that, for every ωcpo D, it holds that $\mathcal{Y}(D_\perp) \cong L(\mathcal{Y}D)$ (where $D_\perp$ is the usual lifting of D in $\omega\mathbf{cpo}$).

In order to understand the notions of completeness and well-completeness in $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$, we need to construct the initial algebra and final coalgebra for L. As we saw in Section 1, the final coalgebra F is a retract of Σ^N and hence, because $\omega\mathbf{cpo}$ is an exponential ideal in which idempotents split, lies in the image of $\mathcal{Y}$. Therefore $F \cong \mathcal{Y}(\overline{\omega})$, because, as is well-known, $\overline{\omega}$ is the final coalgebra for the lifting functor $(\cdot)_\perp$ on $\omega\mathbf{cpo}$.

It is instructive to give an explicit description of the initial lift algebra. For any ωcpo D, we write $[D,\omega]$ for the set of ω-continuous functions from D to the linearly-ordered *poset*, $\omega = \{0,1,2,\ldots\}$. Define I to be the presheaf:

$$I(A) \quad = \quad \{g \in [A,\omega] \mid g \text{ has finite image}\}.$$

It is routine to verify that I is, in fact, a $\mathbf{K}$-sheaf (because all basic covers are finite). Now, consider the familiar diagram in $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$:

$$0 \;\to\; L0 \;\to\; L^2 0 \;\to\; \ldots$$

obtained by iterating L over the unique morphism $0 \to L0$, where 0 is the initial object in $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$ (n.b. it is not initial in $\widehat{\mathbf{C}}$). By the direct pointwise construction of colimits in $\widehat{\mathbf{C}}$, one sees that I is the colimit of the above diagram in $\widehat{\mathbf{C}}$ and hence also in $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$. Morover, from the explicit description of L, one sees that the functor L preserves the colimit (indeed this is true for general reasons [2, Theorem 1.7]), and hence the inverse to the universal $I \to LI$ is the initial algebra for L in $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$ (it is not initial in $\widehat{\mathbf{C}}$). The canonical map $\iota : I \to F$ is the evident family of inclusions $\iota_A : I(A) \subseteq [A,\omega] \subseteq \omega\mathbf{cpo}(A,\overline{\omega})$.

Proposition 4.5 *For every ωcpo D, $\mathcal{Y}(D)$ is complete in $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$.*

Proof. Take any ωcpo, D. We must show that the canonical map $\mathcal{Y}(D)^F \to \mathcal{Y}(D)^I$ is an isomorphism. By the colimiting property of I, we have

$$\begin{aligned}
\mathcal{Y}(D)^I &\cong \mathcal{Y}(D)^{\varinjlim L^i(0)} \\
&\cong \varprojlim \mathcal{Y}(D)^{L^i(0)} \\
&\cong \varprojlim \mathcal{Y}(D)^{\mathcal{Y}(\mathbb{O}_i)}
\end{aligned}$$

where, for $i \geq 0$, we write $\mathbb{O}_i$ for the ωcpo $\{0,\ldots,i-1\}$ under the usual linear order (thus $\mathbb{O} = \mathbb{O}_2$). (Although we haven't yet proved that $\mathcal{Y}$ preserves initial objects, we do know that $L^i(0) \cong \mathbb{O}_i$ for $i \geq 1$, because $\mathbb{O}_1 \cong 1$.) As $\omega\mathbf{cpo}$ is an exponential ideal of $\underline{\mathrm{Sh}}(\mathbf{C},\mathbf{K})$, the functor $\mathcal{Y}$ preserves exponentials, so

$$\begin{aligned}
\varprojlim \mathcal{Y}(D)^{\mathcal{Y}(\mathbb{O}_i)} &\cong \varprojlim \mathcal{Y}(D^{\mathbb{O}_i}) \\
&\cong \mathcal{Y}(\varprojlim D^{\mathbb{O}_i}) \\
&\cong \mathcal{Y}(D^{\varinjlim \mathbb{O}_i})
\end{aligned}$$

But, in ω**cpo** we have that $\overline{\omega}$ is the colimit of the derived diagram of inclusions

$$\mathbb{O}_1 \subseteq \mathbb{O}_2 \subseteq \mathbb{O}_3 \subseteq \ldots$$

so it follows that

$$\begin{array}{rcl} \mathcal{Y}(D^{\varinjlim \mathbb{O}_i}) & \cong & \mathcal{Y}(D^{\overline{\omega}}) \\ & \cong & \mathcal{Y}(D)^{\mathcal{Y}(\overline{\omega})} \\ & \cong & \mathcal{Y}(D)^F \end{array}$$

By following the isomorphisms through, one sees that the isomorphism constructed is indeed the inverse to the canonical map. ∎

Let us consider the axioms of Section 1 in the light of the above results. Axiom 1 is a consequence of Proposition 4.5, because $\Sigma = \mathcal{Y}\mathbb{O}$. For the same reason, Axiom 2 is a consequence of Proposition 4.3. Axiom 3 holds because $L(0) \cong 1$ in $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$, so 0 is well-complete. Finally, Axiom 4 holds because $\mathbb{O}^{\mathbb{O}} \cong \mathbb{O}_\perp$ in ω**cpo**, hence, applying $\mathcal{Y}$, we obtain $\Sigma^\Sigma \cong L(\Sigma)$ in $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$ (the isomorphism is indeed given by the required map). It is also worth observing that, for any ωcpo D, we have that $\mathcal{Y}(D)$ is well-complete (because $L(\mathcal{Y}(D)) \cong \mathcal{Y}(D_\perp)$).

We now proceed to our main application of the chosen site, demonstrating that the first implication of 1.23 cannot be reversed. We must show that 2 is well-complete but that N is not.

Proposition 4.6 *The functor $\mathcal{Y} : \omega\mathbf{cpo} \to \underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$ preserves finite coproducts.*

Proof. It is convenient to work with the full subcategory $\mathbf{C}'$ of $\mathbf{C}$ obtained by omitting the empty ωcpo, and with the induced Grothendieck topology $\mathbf{K}'$. It is easily seen that the induced functor $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K}) \to \underline{\mathrm{Sh}}(\mathbf{C}', \mathbf{K}')$ is an equivalence of categories. Morover the evident $\mathcal{Y}' : \omega\mathbf{cpo} \to \underline{\mathrm{Sh}}(\mathbf{C}', \mathbf{K}')$ commutes with $\mathcal{Y}$ along the equivalence. Thus it suffices to show that $\mathcal{Y}'$ preserves finite coproducts.

The preservation of the initial object is trivial because all objects of $\mathbf{C}'$ are nonempty. For binary coproducts, given two ωcpos D, E, write $\mathcal{Y}'(D) + \mathcal{Y}'(E)$ for the (pointwise) coproduct of $\mathcal{Y}'(D)$ and $\mathcal{Y}'(E)$ in $\widehat{\mathbf{C}'}$. We shall exhibit $\mathcal{Y}'(D) + \mathcal{Y}'(E)$ as a $\mathbf{K}'$-dense subobject of $\mathcal{Y}'(D + E)$, showing that $\mathcal{Y}'(D + E)$ (as it is a sheaf) is the sheafification of $\mathcal{Y}'(D) + \mathcal{Y}'(E)$, and hence the coproduct.

We define a mono $\nu : \mathcal{Y}'(D) + \mathcal{Y}'(E) \to \mathcal{Y}'(D + E)$. For any $d \in (\mathcal{Y}'(D) + \mathcal{Y}'(E))(A)$ either $d = in1(f)$ for some $f \in \mathcal{Y}'(D)(A) = \omega\mathbf{cpo}(A, D)$ or $d = in2(g)$ for some $g \in \omega\mathbf{cpo}(A, E)$. In the first case define $\nu_A(d) = (in1 \circ f) \in \mathcal{Y}'(D + E)(A)$. In the second define $\nu_A(d) = (in2 \circ g) \in \mathcal{Y}'(D + E)(A)$. It is readily checked that the ν so defined is natural. It is monic because coproducts in ω**cpo** are disjoint, and all objects A of $\mathbf{C}'$ are nonempty.

It remains to show that ν is $\mathbf{K}'$-dense. Take any $h \in \mathcal{Y}'(D + E)(A) = \omega\mathbf{cpo}(A, D+E)$. We must show that there is a cover $\{A_i\}_i$ of A with a matching family $\{d_i \in (\mathcal{Y}'(D) + \mathcal{Y}'(E))(A_i)\}_i$ such that h is the unique amalgamation of

$\{\nu_{A_i}(d_i)\}_i$ in $\mathcal{Y}'(D+E)$. However, by the stability of coproducts in $\omega\mathbf{cpo}$, we have that $A \cong A_1 + A_2$, where A_1 and A_2 are disjoint open subsets of A, such that $h : A \to D+E$ is (isomorphic to) $f+g : A_1 + A_2 \to D+E$. Therefore either one of A_1, A_2 is empty or $\{A_1, A_2\}$ is a cover for A in $\mathbf{K}'$. In the latter case $\{in1(f), in2(g)\}$ is the desired matching family in $\mathcal{Y}'(D) + \mathcal{Y}'(E)$. In the former case it is either $\{in1(f)\}$ or $\{in2(g)\}$ as appropriate. ∎

It follows from Proposition 4.6 that the object 2 in $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$ lies in the image of $\mathcal{Y}$ and is hence well-complete. It remains to show that the natural numbers object, N, is not well-complete. Define the presheaf N by

$$N(A) \;=\; \{f \in \omega\mathbf{cpo}(A, \mathbb{N}) \mid f \text{ has finite image}\}$$

whose action on morphisms $\varphi : A \to B$ is defined by composition. (Thus N is a subpresheaf of $\mathcal{Y}(\mathbb{N})$.)

Proposition 4.7 *N is the natural numbers object in* $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$.

Proof. It is routine to verify that N is a sheaf (using that all covers in $\mathbf{K}$ are finite). To show it is the nno, it is convenient to work with its restriction N' to a sheaf in $\underline{\mathrm{Sh}}(\mathbf{C}', \mathbf{K}')$. Consider the nno, $\widehat{\mathbb{N}}$, in $\widehat{\mathbf{C}'}$ (defined by $\widehat{\mathbb{N}}(A) = \mathbb{N}$). We exhibit $\widehat{\mathbb{N}}$ as a $\mathbf{K}'$-dense subobject of N', hence N' is its sheafification, the nno in $\underline{\mathrm{Sh}}(\mathbf{C}', \mathbf{K}')$.

The required mono $\nu : \widehat{\mathbb{N}} \to N'$, is defined by mapping any $n \in \widehat{\mathbb{N}}(A) = \mathbb{N}$ to the constantly n function in $N'(A)$. The ν defined is clearly natural, and is monic because every A is nonempty. To show that ν is $\mathbf{K}'$-dense, consider any $f \in N'(A)$. We must find a cover $\{A_i\}_i$ of A and matching family $\{n_i \in \widehat{\mathbb{N}}(A_i)\}_i$ such that f is the unique amalgamation of $\{\nu_{A_i}(n_i)\}$ in $N'(A)$. However, as f has finite image $\{n_1, \ldots, n_k\}$, and is continuous, A splits as a finite disjoint union $A_1 \cup \cdots \cup A_k$ of nonempty open subsets such that $a \in A_i$ implies $f(a) = n_i$. Then $\{A_i\}_{1 \le i \le k}$ is the desired cover for A and $\{n_i\}_{1 \le i \le k}$ is its matching family. ∎

Proposition 4.8 *N is not well-complete in* $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$.

Proof. By the explicit description of lifting, one sees that the lift of N is isomorphic to the sheaf (which we henceforth call $L(N)$):

$$L(N)(A) \;=\; \{f \in \omega\mathbf{cpo}(A, \mathbb{N}_\perp) \mid f \text{ has finite image}\}.$$

Consider the morphism $\rho : \mathcal{Y}(\mathbb{N}) \times I \to L(N)$ in $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$ defined at A by:

$$\rho_A(f,g)(a) \;=\; \begin{cases} f(a) & \text{if } f(a) < g(a) \\ \perp & \text{otherwise} \end{cases}$$

(for $f \in \omega\mathbf{cpo}(A, \mathbb{N})$, $g \in I(A)$ and $a \in A$). Note that $\rho_A(f,g) \in \omega\mathbf{cpo}(A, \mathbb{N}_\perp)$ because f and g are continuous, and its image is finite by the definition of I. Thus indeed $\rho_A(f,g) \in L(N)(A)$. (The naturality of ρ is obvious.)

We show that ρ has no extension (along $\mathsf{id}_{\mathcal{Y}(\mathbb{N})} \times \iota$) to a morphism $\overline{\rho} : \mathcal{Y}(\mathbb{N}) \times F \to N$, and therefore $L(N)$ is not complete. Suppose, for contradiction, that $\overline{\rho}$ does exist. For $i \in \overline{\omega}$, consider the constantly i function $k_i : \mathbb{N} \to \overline{\omega}$. Clearly $k_i \in F(\mathbb{N})$, and if $i \in \omega$ then also $k_i \in I(\mathbb{N})$. Because $\overline{\rho}$ extends ρ, we have, for $i \in \omega$,

$$\overline{\rho}_{\mathbb{N}}(\mathsf{id}_{\mathbb{N}}, k_i)(n) \quad = \quad \begin{cases} n & \text{if } n < i, \\ \perp & \text{otherwise.} \end{cases}$$

For any $i \in \omega$, consider the map $\phi_i : \mathbb{N} \times \mathbb{O} \to \mathbb{N} \times \overline{\omega}$ defined by $\phi_i(n, \perp) = (n, i)$ and $\phi_i(n, \top) = (n, \omega)$. By the naturality of $\overline{\rho}$ along $n \mapsto (n, \perp)$ and $n \mapsto (n, \top) : \mathbb{N} \to \mathbb{N} \times \mathbb{O}$, we have that $\overline{\rho}_{\mathbb{N}\times\mathbb{O}}(\phi_i)(n, \perp) = \overline{\rho}_{\mathbb{N}}(\mathsf{id}_{\mathbb{N}}, k_i)(n)$ and $\overline{\rho}_{\mathbb{N}\times\mathbb{O}}(\phi_i)(n, \top) = \overline{\rho}_{\mathbb{N}}(\mathsf{id}_{\mathbb{N}}, k_\omega)(n)$. Therefore, for all $i \in \mathbb{N}$, we have $\overline{\rho}_{\mathbb{N}}(\mathsf{id}_{\mathbb{N}}, k_i)(n) \leq \overline{\rho}_{\mathbb{N}}(\mathsf{id}_{\mathbb{N}}, k_\omega)(n)$ (in the partial order on $\mathbb{N}_\perp$). It follows that $\overline{\rho}_{\mathbb{N}}(\mathsf{id}_{\mathbb{N}}, k_\omega)$ is the "identity" function from $\mathbb{N}$ to $\mathbb{N}_\perp$, which does not have finite image. Thus indeed $\overline{\rho}$ does not exist. ∎

By [26, Theorem 1], Proposition 4.8 is equivalent to the failure of Markov's Principle in $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{K})$ (see the discussion in Section 1, following proposition 1.23). In fact, our original proof that N is not well-complete was by establishing the failure of Markov's Principle directly. Here, we presented the proof above in order to keep the paper self contained.

Finally, we remark on the extent to which the results in this section hold for a more general choice of site. The basic results, Propositions 4.3, 4.4 and 4.5 go through for any category $\underline{\mathrm{Sh}}(\mathbf{C}, \mathbf{J})$ where $\mathbf{C}$ is a dense full subcategory of $\omega\mathbf{cpo}$, and $\mathbf{J}$ is any subcanonical topology. The proofs are essentially the same, using the analogous result for the canonical topology from [3, 4] to obtain the first part of Proposition 4.3. In order to obtain the preservation of finite coproducts, it is helpful to assume that $\mathbf{C}$ is sufficiently well behaved that (a fragment of) the finite coproduct topology can be defined on it. Then Proposition 4.6 generalises to any subcanonical topology that contains the finite coproduct topology. Finally, for the Proposition 4.8 to go through, it is also necessary to have $\mathbb{N}$ in $\mathbf{C}$, and to ensure that $\mathbf{J}$ is generated by (sufficiently many) finite basic covers.

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