Cyclic cohomology of Hopf algebras, and a non-commutative Chern-Weil theory *

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Abstract

We give a construction of Connes-Moscovici's cyclic cohomology for any Hopf algebra equipped with a character. Furthermore, we introduce a non-commutative Weil complex, which connects the work of Gelfand and Smirnov with cyclic cohomology. We show how the Weil complex arises naturally when looking at Hopf algebra actions and invariant higher traces, to give a non-commutative version of the usual Chern-Weil theory.

Keywords: Cyclic homology, Hopf algebras, Weil complex

1 Introduction

In their computation of the cyclic cocycles involved in the non-commutative index formula in the context of the transverse index theorem, A. Connes and H. Moscovici discovered that the action of the operators involved can be organized in a Hopf algebra action, and that the computation takes place on the cyclic cohomology of their Hopf algebra ([6]). This led them to a definition of the cyclic cohomology $HC^*_\delta(\mathcal{H})$ of a Hopf algebra $\mathcal{H}$, endowed with a character satisfying certain conditions. In their context, this provides a new beautiful relation of cyclic cohomology with Gelfand-Fuchs cohomology, while, in general, it can be viewed as a non-commutative extension of the Lie algebra cohomology.

Although the definition is very well motivated, there are some quite strong restrictions on the given Hopf algebra: one requires the existence of an algebra A, endowed with an action of $\mathcal{H}$, and with a faithful (!) invariant trace $\tau : A \to \mathbf{C}$.

Our first goal is to give a simple new definition/interpretation (à la Quillen) of $HC^*_\delta(\mathcal{H})$, which shows that the only requirement which is needed is that the associated twisted antipode S_δ (see Section 2) is an involution. This is the minimal (and natural) requirement. It also shows that $HC^*_\delta(\mathcal{H})$ plays the same role as that played by the Lie algebra cohomology in the usual Chern-Weil theory in the context of flat connections, and is the starting point in answering a second problem raised by Connes and Moscovici ([6]): which is the source of the characteristic homomorphism in the case of higher traces?

Independently, in the work of Gelfand and Smirnov on universal Chern-Simons classes, there are implicit relations with cyclic cohomology ([12, 13], see also [3]). Our second goal is to make these connections explicit. So, in the second part of the paper (independent of the first part), we introduce a non-commutative Weil complex, extending the constructions from [13, 12, 19]. In this general setting, we explain some of the connections with cyclic cohomology, interpret the Chern-Simons transgression, and describe the relevant associated cohomologies.

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This non commutative Weil complex is used to define a non commutative Chern-Weil homomorphism, and explains the construction of characteristic classes associated to equivariant higher traces, already mentioned above. The approach is inspired by the construction of the usual Chern-Weil homomorphism (see [2, 11]), and of the secondary characteristic classes for foliations ([1, 14]).

Here is a short outline of the paper. In Section 2 we bring together some basic results about characters δ and the associated twisted antipodes S_{δ} on Hopf algebras. In section 3, we start by introducing a localized cyclic homology (and cohomology), denoted by $HC_{*}^{\delta}(R)$, associated to a "flat" action of $\mathcal{H}$ on the algebra R, and to a character δ on $\mathcal{H}$. The Cuntz-Quillen machinery ([9]) can be adapted to this context, but we leave this for the last section. The relevant information which is needed for the cyclic cohomology of Hopf algebras, only requires a small part of this machinery. This is captured by a localized X-complex (denoted by $X_{\delta}(R)$), which we compute in the case where $R = T(V)$ is the tensor algebra of an $\mathcal{H}$-module V.

In Section 4 we introduce $HC_{\delta}^{*}(\mathcal{H})$. If the algebra A is endowed with a flat action of $\mathcal{H}$ and a character $\tau : A \to \mathbf{C}$, there is an obvious induced (characteristic) map $HC^{*}(\mathcal{H}) \longrightarrow HC^{*}(A)$, whose source is the cyclic cohomology of $\mathcal{H}$, viewed as a coalgebra. This simple map can be interpreted as arising from a DG algebra map $T(\mathcal{H}) \longrightarrow Hom(B(A), A)$, where $T(\mathcal{H})$ is the tensor algebra of $\mathcal{H}$, and $B(A)$ is the dual construction for algebras, that is, the bar coalgebra of A (cf. [18]). Now, if the trace is invariant with respect with δ, i.e. if:

$$\tau(h \cdot a) = \delta(h)\tau(a) \ , \quad \forall h \in \mathcal{H}, \ a \in A,$$

than the previous map factors through the localization of $T(\mathcal{H})$, and induces a characteristic map $HC_{\delta}^{*}(\mathcal{H}) \to HC^{*}(A)$. Here $HC_{\delta}^{*}(\mathcal{H})$ is naturally defined by the localized X-complex of $T(\mathcal{H})$ (and this is similar with Quillen's interpretation of cyclic homology in terms of the bar construction, [18]). We easily show that the usual formulas become, after localization, the same as the ones used by Connes and Moscovici ([6]), and we conclude the section with a detailed computation of the fundamental example where $\mathcal{H} = U(\mathfrak{g})$ is the envelopping algebra of a Lie algebra $\mathfrak{g}$.

In section 5 we introduce the non-commutative Weil complex (by collecting together "forms and curvatures" in a non-commutative way), and describe its connections to cyclic cohomology. We show that there are two relevant types of cohomologies involved (which, in the case considered by Gelfand and Smirnov, correspond to Chern classes, and Chern-Simons classes, respectively), and explain that the Chern-Simons transgression is a (boundary) isomorphism between these two types of cohomologies. In connection with cyclic cohomology, we show that the non-commutativity of the Weil complex naturally gives rise to an S-operator, and to cyclic bicomplexes computing these cohomologies.

In the last section we come back to Hopf algebras actions, and higher traces. In this situation, the presence of curvatures brings the Weyl complex into the picture. We briefly discuss its localization , we describe a non-commutative Chern-Weil homomorphism, and prove its compatibility with the S-operator.

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2 Preliminaries on Hopf Algebras

In this section we review some basic properties of Hopf algebras (see [20]) and twisted antipodes.

Let $\mathcal{H}$ be a Hopf algebra. As usual, denote by S the antipode, by ϵ the counit, and by $\Delta(h) = \sum h_0 \otimes h_1$ the coproduct. Recall some of the basic relations they satisfy:

$$\sum \epsilon(h_0)h_1 = \sum \epsilon(h_1)h_0 = h, \tag{1}$$

$$\sum S(h_0)h_1 = \sum h_0 S(h_1) = \epsilon(h) \cdot 1, \tag{2}$$

$$S(1) = 1, \epsilon(S(h)) = \epsilon(h), \tag{3}$$

$$S(gh) = S(h)S(g), \tag{4}$$

$$\Delta S(h) = \sum S(h_1) \otimes S(h_0). \tag{5}$$

Throughout this paper, the notions of $\mathcal{H}$-module and $\mathcal{H}$-algebra have the usual meaning, with $\mathcal{H}$ viewed as an algebra. The tensor product $V \otimes W$ of two $\mathcal{H}$-modules is an $\mathcal{H}$-module with the diagonal action:

$$h(v \otimes w) = \sum h_0(v) \otimes h_1(w).$$

A character on $\mathcal{H}$ is any non-zero algebra map $\delta : \mathcal{H} \longrightarrow \mathbf{C}$. Characters will be used for "localizing" modules: for an $\mathcal{H}$-module V, define V_δ as the quotient of V by the space of co-invariants (linear span of elements of type $h(v) - \delta(h)v$, with $h \in \mathcal{H}, v \in V$). In other words,

$$V_\delta = \mathbf{C}_\delta \otimes_{\mathcal{H}} \mathbf{V},$$

where $\mathbf{C}_\delta = \mathbf{C}$ is viewed as an $\mathcal{H}$-module via δ. Before looking at very simple localizations (see 2.3), we need to discuss the "twisted antipode" $S_\delta := \delta * S$ associated to a character δ (recall that $*$ denotes the natural product on the space of linear maps from the coalgebra $\mathcal{H}$ to the algebra $\mathcal{H}$, [20]). Explicitly,

$$S_\delta(h) = \sum \delta(h_0)S(h_1), \quad \forall h \in \mathcal{H}.$$

Lemma 2.1 *The following identities hold:*

$$\sum S_\delta(h_0)h_1 = \delta(h) \cdot 1, \tag{6}$$

$$S_\delta(1) = 1, \epsilon(S_\delta(h)) = \delta(h), \tag{7}$$

$$\Delta S_\delta(h) = \sum S(h_1) \otimes S_\delta(h_0), \tag{8}$$

$$S_\delta(gh) = S_\delta(h)S_\delta(g), \tag{9}$$

proof: These follow easily from the previous relations. For instance, the first relation follows from the definition of S_δ, (2), and (1), respectively:

$$\sum S_\delta(h_0)h_1 = \sum \delta(h_0)S(h_1)h_2 = \sum \delta(h_0)\epsilon(h_1) \cdot 1 = \delta(\sum h_0\epsilon(h_1)) \cdot 1 = \delta(h) \cdot 1.$$

The other relations are proved in a similar way. $\square$

Lemma 2.2 . *For any two $\mathcal{H}$-modules V, W:*

$$h(v) \otimes w \equiv v \otimes S_\delta(h)(w) \quad mod\ co\text{-}invariants$$

proof: From the definition of S_δ,

$$v \otimes S_\delta(h)w = \sum \delta(h_0)v \otimes S(h_1)w,$$

so, modulo coinvariants, it is:

$$\sum h_0(v) \otimes h_1 S(h_2)(w) = \sum \epsilon(h_1)h_0(v) \otimes w = h(v) \otimes w,$$

where for the last two equalities we have used (2) and (1), respectively. $\square$

It follows easily that:

Corollary 2.3 . *For any $\mathcal{H}$-module V, there is an isomorphism:*

$$(\mathcal{H} \otimes V)_\delta \cong V, \qquad (h, v) \mapsto S_\delta(h)v.$$

There is a well known way to recognize Hopf algebras with $S^2 = Id$ (see [20], pag. 74). We extend this result to twisted antipodes:

Lemma 2.4 . *For a character δ, the following are equivalent:*

(i) $S_\delta^2 = Id$,

(ii) $\sum S_\delta(h_1)h_0 = \delta(h) \cdot 1, \ \forall h \in \mathcal{H}$

proof: The first implication follows by applying S_δ to (6), using (9), and *(i)*. Now, assume *(ii)* holds. First, remark that $S \circ S_\delta = \delta$. Indeed,

$$(S * (S \circ S_\delta))(h) = \sum S(h_0)S(S_\delta(h_1)) = \sum S(S_\delta(h_1)h_0) = \delta(h) \cdot 1,$$

(where we have used the definition of $*$, (5), and (ii), respectively.) Multiplying this relation by Id on the left, we get $S \circ S_\delta = Id * \delta$. Using the definition of S_δ, (8), and the previous relation, respectively,

$$S_\delta^2(h) = \sum \delta(S_\delta(h)_0)S(S_\delta(h)_1) = \sum \delta(S(h_1)S(S_\delta(h_0)) = \sum \delta(S(h_2))h_0\delta(h_1),$$

which is (use that δ is a character, and the basic relations again):

$$\delta(\sum h_1 S(h_2))h_0 = \sum \delta(\epsilon(h_1))h_0 = \sum \epsilon(h_1)h_0 = h. \ \square$$

3 The localized X- complex

Recall that an even (higher) trace on an algebra R is given by an extension $0 \longrightarrow I \longrightarrow L \longrightarrow R \longrightarrow 0$ and a trace on L/I^{n+1} (for some n), while an odd trace is given by an extension as before, and an I- adic trace, i.e. a linear functional on I^{n+1} vanishing on $[I^n, I]$. Via a certain equivalence, higher traces correspond exactly to cyclic cocycles on R (for the precise relations, see pp. 417- 419 in [9]). Now, if $\mathcal{H}$ acts on R, and δ is a character of $\mathcal{H}$, there is a natural notion of δ- invariance of a trace τ:

$$\tau(ha) = \delta(h)\tau(a), \qquad \forall h \in \mathcal{H}, \ a \in R.$$

One can talk about invariant higher traces on R, and then one expects an extension of the previous discussion to this situation. In particular higher traces should be in duality with a certain "localized" cyclic homology $HC_*^\delta(R)$. We expect then a localized X-complex to appear, which allows us to apply the powerful ideas of [9] to this case. We show here how this can be done.

3.1 Flat algebras: We start with a (non-unital) algebra R, an action of $\mathcal{H}$ on R, and a character δ of $\mathcal{H}$. We will usually assume that the action is flat, in the sense that:

$$h(ab) = \sum h_0(a)h_1(b), \quad \forall h \in \mathcal{H},\ a, b \in R \tag{10}$$

The motivation for our terminology is that these actions will play for us a similar role as the flat connections in geometry (see 4.3, and Proposition 5.2). Remark that, in general, the action of $\mathcal{H}$ on itself is not flat. A basic example of flat action is the usual (diagonal) action of $\mathcal{H}$ on its tensor algebra (and, in parallelism with geometry, the tensor algebra plays the same role as the the Eilenberg-Chevalley complex plays in geometry, when the connections are flat, see 4.3). And another basic example of flat actions is the one on non-commutative forms, which we describe below.

Denote by $\Omega^*(R)$ the algebra of non-commutative differential forms on R, and by d, b, k, B the usual operators acting on it (see [8], paragraph 3. of [9], and [10] for the non-unital case). Recall that:

$$\Omega^n(R) = \tilde{R} \otimes R^{\otimes n},$$

where $\tilde{R}$ is R with a unit adjoined). Extending the action of $\mathcal{H}$ to $\tilde{R}$ by $h \cdot 1 := \epsilon(h)1$, we have an action of $\mathcal{H}$ on $\Omega^*(R)$ (the diagonal action). To check the flatness condition: $h(\omega\eta) = \sum h_0(\omega)h_1(\eta)$, $\forall \omega, \eta \in \Omega(R)$, remark that one can formally reduce to the case where ω and η are degree 1 forms, in which case the computation is easy. Since the operator b does not commute with the action in general, it is not a priori clear that the operators b, k, B descend to $\Omega^*(R)_\delta$.

Assumption 3.2 *From now on we assume that δ satisfies the (equivalent) relations of Lemma 2.4. Unless otherwise specified, all the actions of $\mathcal{H}$ on algebras will be assumed to be flat.*

3.3 Example: If there is an algebra A, endowed with an action of $\mathcal{H}$, and with a δ-invariant faithful trace $\tau : A \to \mathbf{C}$, than $S_\delta^2 = Id$. In [6], the existence of such an algebra A was assumed in order to define the cyclic cohomology of the Hopf algebra $\mathcal{H}$.

Proposition 3.4 . *Given these data, the operators d, b, k, B on $\Omega^*(R)$ descend to $\Omega^*(R)_\delta$.*

proof: Since d commutes with the action of $\mathcal{H}$, and k, B (and all the other operators appearing in paragraph 3 of [9]) are made out of d, b, it suffices to prove that, modulo co-invariants,

$$b(h \cdot \eta) \equiv b(\delta(h)\eta), \quad \forall h \in \mathcal{H},\ \eta \in \Omega^*(R).$$

For $\eta = \omega da$, one has:

$$(-1)^{|\omega|} b(h \cdot \omega da) = \sum h_0(\omega)h_1(a) - \sum h_1(a)h_0(\omega).$$

Using Lemma2.2 and (6):

$$\sum h_0(\omega)h_1(a) \equiv \sum \omega \cdot S_\delta(h_0)h_1 a = \delta(h)\omega a.$$

Similarly, using Lemma 2.2, and (ii) of Lemma2.4,

$$\sum h_1(a)h_0(\omega) \equiv \sum a \cdot S_\delta(h_1)h_0\omega = \delta(h)a\omega,$$

which ends the proof. $\square$

Definition 3.5 . *Define the localized cyclic homology $HC_*^\delta(R)$ of R as the cyclic homology of the mixed complex $\Omega^*(R)_\delta$. Similarly for Hochschild and periodic cyclic homologies, and also for cohomology.*

3.6 The localized X-complex: Recall that the X-complex of R is the super-complex:

$$R \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural \quad , \tag{11}$$

where $b(xdy) = [x, y], d(x) = dx$. We define $X_\delta(R)$ as the degree one level of the Hodge tower associated to $\Omega^*(R)_\delta$, i.e. the super-complex:

$$R_\delta \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta} \quad ,$$

where:

$$\Omega^1(R)_{\natural,\delta} := \Omega^1(R)_\delta / b\Omega^2(R)_\delta = \Omega^1(R)_\natural / \natural(\textit{co-invariants}).$$

and the formulas for b, d are similar to the ones for $X(R)$. Actually $X_\delta(R)$ can be viewed as a localization of $X(R)$. There is one remark about the notation: $\Omega^1(R)_{\natural,\delta}$ is not the localization of $\Omega^1(R)_\natural$; in general, there is no natural action of $\mathcal{H}$ on it.

The construction extends to the graded case, where R is a DG algebra, provides one uses the sign convention: interchanging two elements x, y, introduces a $(-1)^{deg(x)deg(y)}$ sign. For instance, $b(xdy) = [x, y] = xy - (-1)^{deg(x)deg(y)}yx$. The resulting X-complexes are naturally super-complexes of complexes (and can be viewed as double complexes, if one takes care of the signs on the differentials).

3.7 Example. Before proceeding, let's look at a very important example: the (non-unital) tensor algebra $R = T(V)$ of an $\mathcal{H}$-module V. Adjoining a unit, one gets the unital tensor algebra $\tilde{R} = \tilde{T}(V) = \sum_{n\geq 0} V^{\otimes n}$. The computation of $X(R)$ was carried out in [18], Example 3.10. One knows then that ([9], pag. 395):

$$R = T(V), \quad \Omega^1(R)_\natural = V \oplus \tilde{T}(V) = T(V),$$

and also the description of the boundaries: $d = \sum_{i=0}^{i=n} t^i, b = (t-1)$ on $V^{\otimes(n+1)}$, where t is the backward-shift cyclic permutation. The second isomorphism is essentially due to the fact that, since V generates $T(V)$, any element in $\Omega^1(T(V))$ can be written in the form $xd(v)y$, with $x, y \in \tilde{T}(V), v \in V$ (see also the proof of the next proposition). To compute $X_\delta(R)$, one still has to compute its odd part. The final result is:

Proposition 3.8 . *For $R = T(V)$:*

$$X^0_\delta(R) = T(V)_\delta, \quad X^1_\delta(R) = T(V)_\delta,$$

where the action of $\mathcal{H}$ on $T(V)$ is the usual (diagonal), and the boundaries have the same description as the boundaries of $X(R)$: they are $(t-1)$, N (which descend to the localization). The same holds when V is a graded linear space, provided we replace the the backward-shift cyclic permutation t by its graded version.

proof: One knows ([9], pag. 395):

$$\tilde{R} \otimes V \otimes \tilde{R} \xrightarrow{\sim} \Omega^1(R), \quad x \otimes v \otimes y \mapsto x(dv)y,$$

which, passing to commutators, gives (compare to [9], pag 395):

$$R = V \otimes \tilde{R} \xrightarrow{\sim} \Omega^1(R)_\natural, \quad v \otimes y \longrightarrow \natural(dvy),$$

and the projection map $\natural : \Omega^1(R) \longrightarrow \Omega^1(R)_\natural$ identifies with:

$$\natural : \tilde{R} \otimes V \otimes \tilde{R} \longrightarrow V \otimes \tilde{R}, \quad x \otimes v \otimes y \longrightarrow v \otimes yx.$$

So $X^1_\delta(R)$ is obtained from $T(V)$, dividing out by the linear subspace generated by elements of type:

$$\natural(h \cdot x \otimes v \otimes y - \delta(h)x \otimes v \otimes y) = \sum h^1(v) \otimes h^2(y)h^0(x) - \delta(h)v \otimes yx \in T(V).$$

Now, for $y = 1$, this means exactly that we have to divide out by coinvariants (of the diagonal action of $\mathcal{H}$ on $T(V)$). But this is all, because modulo these coinvariants we have (from Lemma 2.2):

$$\sum h_1(v) \otimes h_2(y)h_0(x) \equiv \sum v \otimes S_\delta(h_1) \cdot (h_2(y)h_0(x)),$$

while, from (8), (2) and (ii) of Lemma 2.2, (1):

$$\sum S_\delta(h_1) \cdot (h_2(y)h_0(x)) = \sum S(h_2)h_3(y)S_\delta(h_1)h_0(x) = \sum \epsilon(h_1)\delta(h_0)yx = \delta(h)yx. \ \square$$

4 Cyclic Cohomology of Hopf Algebras

In this section, after recalling the definition of the cyclic cohomology of coalgebras, we give an interpretation "a la Quillen" of the characteristic map associated to a (flat) action on an algebra endowed with a trace. When the trace is invariant with respect to a given character, there is a natural "localized" cyclic cohomology of Hopf algebras, and we show that it can be introduced also using the formulas appearing in the work of Connes and Moscovici ([6]). We also treat in detail the fundamental example where $\mathcal{H} = U(\mathfrak{g})$, is the envelopping algebra of a Lie algebra $\mathfrak{g}$.

4.1 Cyclic cohomology of coalgebras: Let $\mathcal{H}, \delta$ be as in the previous section. Looking first just at the coalgebra structure of $\mathcal{H}$, one defines the cyclic cohomology of $\mathcal{H}$ by duality with the case of algebras. As in [6], we define the Λ-module ([4]), denoted $\mathcal{H}^\sharp$, which is $\mathcal{H}^{\otimes(n+1)}$ in degree n, whose co-degeneracies are:

$$d^i(h^0, \ldots, h^n) = \begin{cases} (h^0, \ldots, h^{i-1}, \Delta h^i, h^{i+1}, \ldots, h^n) & \text{if } 0 \le i \le n \\ \sum (h^0_{(1)}, h^1, \ldots, h^n, h^0_{(0)}) & \text{if } i = n+1 \end{cases} .$$

and whose cyclic action is:

$$t(h^0, \ldots, h^n) = (h^1, h^2, \ldots, h^n, h^0).$$

Denote by $HC^*(\mathcal{H})$ the corresponding cyclic cohomology, by $C^*_\lambda(\mathcal{H})$ the cyclic complex, and by $CC^*(\mathcal{H})$ the cyclic (upper plane) bicomplex (Quillen-Loday-Tsygan's) computing it. We dualize (in a straightforward manner) Quillen's interpretations of cyclic cocycles on an algebra, in terms of its bar (DG) coalgebra ([18]). We then introduce the DG tensor algebra of $\mathcal{H}$, $T(\mathcal{H})$, which is $\mathcal{H}^{\otimes n}$ in degrees $n \ge 1$ and 0 otherwise, and with the differential $b' = \sum_0^n (-1)^i d^i$. We then have (see [18], or use our Lemma 3.8):

Proposition 4.2 *(Quillen). Up to a shift on degrees, the cyclic bicomplex of $\mathcal{H}$, $CC^*(\mathcal{H})$ coincides with the X-complex of the DG algebra $T(\mathcal{H})$, and the cyclic complex $C^*_\lambda(\mathcal{H})$ is isomorphic to $T(\mathcal{H})_\natural$. This is true for any coalgebra.*

Let us be more precise about the shifts. In a precise way, the proposition identifies $CC^*(\mathcal{H})$ with the super-complex of complexes:

$$\ldots \longrightarrow X^1(T\mathcal{H})[-1] \longrightarrow X^0(T\mathcal{H})[-1] \longrightarrow X^1(T\mathcal{H})[-1] \longrightarrow \ldots \quad ,$$

and gives an isomorphism:

$$C^*_\lambda(\mathcal{H}) \cong T(\mathcal{H})_\natural[-1] \ .$$

We emphasize that, when considering $CC^*(\mathcal{H})$ as a bicomplex with anticommuting differentials, one has to change the sign in the differential of $X^0(\mathcal{H})$, i.e. use $-b'$ instead of b'.

4.3 The (localized) characteristic map: Let A be a $\mathcal{H}$-algebra, and let $\tau : A \longrightarrow \mathbf{C}$ be a trace on A. There is an obvious map induced in cyclic cohomology (which uses just the coalgebra structure of $\mathcal{H}$):

$$\gamma^\tau : HC^*(\mathcal{H}) \longrightarrow HC^*(A),\ (h^0, \ldots, h^n) \longrightarrow \gamma(h^0, \ldots, h^n), \tag{12}$$
$$\gamma(h^0, \ldots, h^n)(a_0, \ldots, a_n) = \tau(h^0(a_0) \cdot \ldots \cdot h^n(a_n)).$$

In order to find the relevant complexes in the case of invariant traces, we give a different interpretation of this simple map. We can view the action of $\mathcal{H}$ on A, as a linear map:

$$\gamma_0 : \mathcal{H} \longrightarrow Hom(B(A), A)^1 = Hom_{lin}(A, A)$$

where $B(A)$ is the (DG) bar coalgebra of A. Recall that $B(A)$ is $A^{\otimes n}$ in degrees $n \geq 1$ and 0 otherwise, with the coproduct:

$$\Delta(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1} (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n),$$

and with the usual b' boundary as differential. Then $Hom(B(A), A)$ is naturally a DG algebra (see [18]), with the product: $\phi * \psi := m \circ (\phi \otimes \psi) \circ \Delta$ (m stands for the multiplication on A). Explicitly, for $\phi, \psi \in Hom(B(A), A)$ of degrees p and q, respectively,

$$(\phi * \psi)(a_1, \ldots, a_{p+q}) = (-1)^{pq} \phi(a_1, \ldots, a_p) \psi(a_{p+1}, \ldots, a_{p+q}) \ ,$$

The map γ_0 uniquely extends to a DG algebra map:

$$\tilde{\gamma} : T(\mathcal{H}) \longrightarrow Hom(B(A), A). \tag{13}$$

This can be viewed as a characteristic map for the flat action (see Proposition 5.2). Recall also ([18]) that the norm map N can be viewed as a closed cotrace $N : C^\lambda_*(A)[1] \to B(A)$ on the DG coalgebra $B(A)$, that is, N is a chain map with the property that $\Delta \circ N = \sigma \circ \Delta \circ N$, where σ is the graded twist $x \otimes y \mapsto (-1)^{deg(x) deg(y)} y \otimes x$. A formal property of this is that, composing with N and τ, we have an induced trace:

$$\tau_\natural : Hom(B(A), A) \longrightarrow C^*_\lambda(A)[1], \quad \tau_\natural(\phi) = \tau \circ \phi \circ N. \tag{14}$$

Composing with $\tilde{\gamma}$, we get a trace on the tensor algebra:

$$\tilde{\gamma}^\tau : T(\mathcal{H}) \longrightarrow C^*_\lambda(A)[1], \tag{15}$$

and then a chain map:

$$\gamma^\tau : T(\mathcal{H})_\natural \longrightarrow C^*_\lambda(A)[1]. \tag{16}$$

Via the previous proposition, it induces (12) in cohomology.

Let's now start to use the Hopf algebra structure of $\mathcal{H}$, and the character δ. First of all remark that the map $\tilde{\gamma}$ is $\mathcal{H}$-invariant, where the action of $\mathcal{H}$ on the right hand side of (13) comes from the action on A: $(h \cdot \phi)(a) = h\phi(a),\ \forall\ a \in B(A)$. To check the invariance condition: $\tilde{\gamma}(hx) =$

$h\tilde{\gamma}(x)$, $\forall\ x \in T(\mathcal{H})$, remark that the flatness of the action reduces the checking to the case where $x \in \mathcal{H} = T(\mathcal{H})^1$, and that is obvious. Secondly, remark that if the trace τ is δ-invariant, then so is (14). In conclusion, $\tilde{\gamma}^\tau$ in (15) is an invariant trace on the tensor algebra, so our map (16) descends to a chain map:

$$\gamma_\delta^\tau : T(\mathcal{H})_{\natural,\delta} \longrightarrow C_\lambda^*(A)[1],$$

and we have an induced (characteristic) map:

$$\gamma_\delta^\tau : HC_\delta^*(\mathcal{H}) \longrightarrow HC^*(A),$$

where the definition of the first cohomology is the natural one:

Definition 4.4 *Define the localized (by δ) cyclic bicomplex of $\mathcal{H}$, as the localized X-complex of its DG tensor algebra:*

$$CC_\delta^*(\mathcal{H}) := X_\delta(T(\mathcal{H})), \quad \text{shifted by one},$$

and the localized cyclic complex of $\mathcal{H}$:

$$C_{\lambda,\delta}^*(\mathcal{H}) := T(\mathcal{H})_{\natural,\delta}, \quad \text{shifted by one}.$$

The corresponding cyclic (Hochschild, etc.) cohomologies are denoted by $HC_\delta^*(\mathcal{H})$, etc. Here, by the shift, we mean the same shift as in Proposition 4.2.

4.5 The associated cyclic module: We identify now our definition with Connes and Moscovici's. Recall that, given $\mathcal{H}$, δ as before, one defines a Λ- module $\mathcal{H}_\delta^\natural$. It is $\mathcal{H}^{\otimes n}$ on degree n, and (cf. [6], formulas (37), (38)):

$$d_\delta^i(h^1,\dots,h^n) = \begin{cases} (1,h^1,\dots,h^n) & \text{if } i=0 \\ (h^1,\dots,h^{i-1},\Delta h^i,h^{i+1},\dots,h^n) & \text{if } 1\le i\le n \\ (h^1,\dots,h^n,1) & \text{if } i=n+1 \end{cases}.$$

$$s_\delta^i(h^1,\dots,h^n) = (h^1,\dots,\epsilon(h^{i+1}),\dots,h^n), \quad 0\le i\le n-1,$$

$$t_\delta(h^1,\dots,h^n) = S_\delta(h_1)\cdot(h^2,\dots,h^n,1)$$

(where $''S_\delta(h_1)\cdot''$ stands for the diagonal action, cf. Section 2).

As pointed out by Connes and Moscovici, checking directly the cyclic relation $t_\delta^{n+1} = Id$ is a difficult task. They have proved this relation in [6], under the assumption mentioned in Example 3.3.

Theorem 4.6 . *For any Hopf algebra $\mathcal{H}$ and any character δ as before (i.e. with $S_\delta^2 = Id$), the previous formulas define a cyclic module whose cyclic complex is isomorphic to $C_{\lambda,\delta}^*(\mathcal{H})$, whose cyclic bicomplex is $CC_\delta^*(\mathcal{H})$, and whose cohomology is $HC_\delta^*(\mathcal{H})$.*

proof: We have seen in Proposition 3.8:

$$\Omega^1(T\mathcal{H})_\natural \cong T\mathcal{H}, \quad \Omega^1(T\mathcal{H})_{\natural,\delta} \cong (T\mathcal{H})_\delta.$$

The first isomorphism is the one which gives the identification $X(T\mathcal{H}) \cong CC^*(\mathcal{H})$ of Proposition 4.2. The second isomorphism, combined with the isomorphism (cf. Lemma 2.3):

$$(T\mathcal{H})_\delta^{n+1} \cong \mathcal{H}^{\otimes n},\ [h_0\otimes h_1\otimes\dots\otimes h_n] \mapsto S_\delta(h_0)\cdot(h_1\otimes\dots\otimes h_n),$$

(with the inverse $h_1\otimes\dots\otimes h_n \mapsto [1\otimes h_1\otimes\dots\otimes h_n]$), gives the identification $X_\delta(T\mathcal{H}) \cong CC_\delta^*(\mathcal{H})$. The cyclic relation for $\mathcal{H}_\delta^\natural$ follows from the similar property for $CC_\delta^*(\mathcal{H})$ (namely: $(1-t)\circ N = 0$), and the identification $(T\mathcal{H})_{\natural,\delta} \cong C_{\lambda,\delta}^*(\mathcal{H})$ follows from the fact that these complexes are coaugmentations of the first quadrant part of $X_\delta(T\mathcal{H})$ and $CC_\delta^*(\mathcal{H})$, respectively. $\square$

Corollary 4.7 . *For any (flat) action of $\mathcal{H}$ on A and any δ-invariant trace τ on A, there are induced maps:*

$$\tau_\delta^\tau : HC_\delta^*(\mathcal{H}) \longrightarrow HC^*(A),$$

which are compatible with the S-operation.

4.8 Example: Although we look at $HC_\delta^*(\mathcal{H})$ as the analogue of the Lie algebra homology (see also Theorem 4.11) , it is still instructive to look at the case where $\mathcal{H} = C(G)$, is the algebra of functions on a finite group G. As a linear space, it is generated by the functions ϵ_a, concentrated on $a \in G$, with the product $\epsilon_a \epsilon_b = \epsilon_a$ if $a = b$ and 0 otherwise, with the coproduct $\Delta(\epsilon_a) = \sum_{bc=a} \epsilon_b \epsilon_c$, the counit $\epsilon(\epsilon_a) = 1$ if $a = 1$ and 0 otherwise, and the antipode $S(\epsilon_a) = \epsilon_{a^{-1}}$. We choose $\delta = \epsilon$. The corresponding cyclic module $C_\delta^*(\mathcal{H})$ computing $HC_\delta^*(\mathcal{H})$ is easily seen to be the dual of the (contravariant) cyclic module $C_*(G)$ with $C_n(G) = \mathbf{C}[G^n]$,

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \leq i \leq n-1 \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \end{cases},$$

$$t(g_1, \dots, g_n) = ((g_1 \dots g_n)^{-1}, g_1, \dots, g_{n-1}).$$

Although the identification $C_\delta^*(\mathcal{H}) = Hom(C_*(G), \mathbf{C})$ (and even the definition of $\mathcal{H}$) requires G to be finite, $C_*(G)$ is defined in general, and is a well known object (see e.g. 3.13 in [7], and the references therein) which computes the "localization at units" of the cyclic homologies of the group ring $\mathbf{C[G]}$. In particular we mention that:

$$HP^*(C_\delta^*(\mathcal{H})) = \bigoplus_k H^{*+2k}(G, \mathbf{C}),$$

which is $\mathbf{C}$ in degree 0, and 0 in degree 1, since our G was assumed to be finite.

We present now a detailed computation for the case where $\mathcal{H} = U(\mathfrak{g})$ is the envelopping algebra of a Lie algebra $\mathfrak{g}$, following [6] (Theorem 6.(i)). Although, roughly speaking, the computation is similar to the computation of the cyclic cohomology of the algebra of functions on a smooth manifold, technically there are some differences. To start with, remark that the Hochschild boundary of the cyclic module $\mathcal{H}_\delta^\natural$ depends just on the coalgebra structure of $\mathcal{H}$ and on the unit of $\mathcal{H}$. More precisely:

Corollary 4.9 *We have isomorphisms:*

$$HH_\delta^*(\mathcal{H}) \cong Cotor_{\mathcal{H}}^*(\mathbf{C}_\eta, \mathbf{C}_\eta) ,$$

where $\mathbf{C}_\eta$ is the (right/left) $\mathcal{H}$ comodule $\mathbf{C}$, with the coaction induced by the unit map $\eta : \mathbf{C} \longrightarrow \mathcal{H}$.

proof: For the reference, recall the standard (bar) resolution $\mathbf{C}_\eta$ by (free) left $\mathcal{H}$ comodules, denoted by $B(\mathcal{H}, \mathbf{C}_\eta)$:

$$0 \longrightarrow \mathbf{C}_\eta \xrightarrow{\eta} \mathcal{H} \xrightarrow{\mathbf{d}} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\mathbf{d}} \dots$$

$$d = \sum_{i=1}^{n+1} (-1)^{i+1} d_\delta^i : \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes (n+1)}.$$

Then $Cotor_{\mathcal{H}}(\mathbf{C}_\eta, \mathbf{C}_\eta)$ is computed by the chain complex $\mathbf{C}_\eta \Box_{\mathcal{H}} \mathbf{B}(\mathcal{H}, \mathbf{C}_\eta)$, that is, by the Hochschild complex of $\mathcal{H}_\delta^\natural$. $\Box$

As a first application of this, let's look at the symmetric (Hopf) algebra $S(V)$ on a vector space V. Recall that the coproduct is defined on generators by $\Delta(v) = v \otimes 1 + 1 \otimes v, \ \forall v \in V$.

Proposition 4.10 *For any vector space V, the maps $A : \Lambda^n(V) \longrightarrow S(V)^{\otimes n}$, $v_1 \wedge \ldots \wedge v_n \mapsto (\sum_\sigma sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!$ induce isomorphisms:*

$$HH^*_\delta(S(V)) \cong \Lambda^*(V).$$

proof: We will use a Koszul type resolution for the left $S(V)$ comodule $\mathbf{C}_\delta$. Let $e_1, ..., e_k$ be a basis of V, and $\pi^i \in V^*$ the dual basis. The linear maps π^i extend uniquely to algebra homomorphisms $\pi^i : S(V) \longrightarrow S(V)$ with the property $\pi^i(1) = 0$. Remark that each of the π^i's are maps of left $S(V)$ comodules. Indeed, to check that $(1 \otimes \pi_i) \circ \Delta = \Delta \circ \pi^i$, since the maps involved are algebra homomorphisms, it is enough to check it on the generators $e_i \in S(V)$, and that is easy. Consider now the coaugmented complex of lefty $S(V)$ comodules:

$$0 \longrightarrow \mathbf{C}_\eta \xrightarrow{\eta} \mathbf{S(V)} \otimes \mathbf{\Lambda^0(V)} \xrightarrow{\mathbf{d}} \mathbf{S(V)} \otimes \mathbf{\Lambda^1(V)} \xrightarrow{\mathbf{d}} \ldots,$$

with the boundary $d = \sum \pi^i \otimes e_i$, that is:

$$d(x \otimes v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{k} \pi^i(x) \otimes e_i \wedge v_1 \wedge \ldots \wedge v_n.$$

Point out that the definition does not depend on the choice of the basis, and it is dual to the Cartan boundary on the Weil complex of V, viewed as a commutative Lie algebra. This also explains the exactness of the sequence. Alternatively, one can use a standard "Koszul argument", or, even simpler, remark that $(S(V) \otimes \Lambda^*(V)) \otimes (S(W) \otimes \Lambda^*(W)) \cong (S(V \oplus W) \otimes \Lambda^*(V \oplus W))$ as chain complexes (for any two vector spaces V and W), which reduces the assertion to the case where $dim(V) = 1$. So we get a resolution $\mathbf{C}_\eta \longrightarrow \mathbf{S(V)} \otimes \mathbf{\Lambda^*(V)}$ by free (hence injective) left $S(V)$ comodules. Then 4.9 implies that $HH^*_\delta(S(V))$ is computed by $\mathbf{C}_\eta \Box_{\mathbf{S(V)}} (\mathbf{S(V)} \otimes \mathbf{\Lambda^*(V)})$, that is, by $\Lambda^*(V)$ with the zero differential. This proves the second part of the theorem.

To show that the isomorphism is induced by A, we have to compare the previous resolution with the standard bar resolution $B(S(V), \mathbf{C}_\eta)$ (see the proof of 4.9). We define a chain map of left $S(V)$ comodules:

$$P : B(S(V), \mathbf{C}_\eta) \longrightarrow \mathbf{S(V)} \otimes \mathbf{\Lambda^*(V)} ,$$

$$P(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = x_0 \otimes pr(x_1) \wedge \ldots \wedge pr(x_n),$$

where $pr : S(V) \longrightarrow V$ is the obvious projection map. We check now that it is a chain map, i.e.:

$$dP(x_0 \otimes x_1 \otimes ... \otimes x_n) = Pd(x_0 \otimes x_1 \otimes \ldots \otimes x_n).$$

First of all, we may assume $x_1, \ldots, x_n \in V$ (otherwise, both terms are zero). The left hand side is then:

$$\sum_{i=1}^{k} \pi^i(x_0) \otimes e_i \wedge x_1 \wedge \ldots \wedge x_n,$$

while the right hand side is:

$$P(\Delta(x_0) \otimes x_1 \otimes \ldots \otimes x_n) = (id \otimes pr)(\Delta(x_0)) \wedge x_1 \wedge \ldots \wedge x_n \ .$$

So we are left with proving that:

$$(id \otimes pr)\Delta(x) = \sum_{i=1}^{k} \pi^i(x) \otimes e_i, \quad \forall\ x \in S(V),$$

and this can be checked directly on the linear basis $x = e_{i_1} \ldots e_{i_n} \in S(V)$. In conclusion, P is a chain map between our free resolutions of $\mathbf{C}_\eta$ (in the category of left $S(V)$ comodules). By the usual homological algebra, the induced map $\bar{P}$ obtained after applying the functor $\mathbf{C}_\eta \Box_{\mathbf{S(V)}}$- , induces isomorphism in cohomology. From the explicit formula:

$$\bar{P}(x_1 \otimes \ldots \otimes x_n) = pr(x_1) \wedge \ldots \wedge pr(x_n),$$

we see that $\bar{P} \circ A = Id$, so our isomorphism is induced by both $\bar{P}$ and A. $\Box$

We are ready now to present one of the main examples, which relates cyclic cohmology with Lie algebra cohomology. Let $\mathfrak{g}$ be a Lie algebra. Any $\delta \in \mathfrak{g}^*$ (i.e. $\delta : \mathfrak{g} \to \mathbf{C}$ linear, with $\delta|_{[\mathfrak{g},\mathfrak{g}]} = 0$), extends uniquely to an algebra homomorphism $\delta : U(\mathfrak{g}) \longrightarrow \mathbf{C}$ and serves as a character of the Hopf (envelopping) algebra $U(\mathfrak{g})$. One has $S_\delta^2 = Id$ because, on generators,

$$S_\delta(v) = -v + \delta(v) \quad \forall \ v \in \mathfrak{g} \ .$$

Denote by $\mathbf{C}_\delta$ the $\mathfrak{g}$ module $\mathbf{C}$ with the action induced by δ.

Theorem 4.11 *For any Lie algebra $\mathfrak{g}$, and any $\delta \in \mathfrak{g}^*$:*

$$HP_\delta^*(U(\mathfrak{g})) \cong \bigoplus_{i \equiv * \mathrm{mod}\ 2} H_i(\mathfrak{g}; \mathbf{C}_\delta).$$

proof: Consider the mixed complex:

$$\Lambda : \ \Lambda^0(\mathfrak{g}) \underset{0}{\overset{d_{Lie}}{\leftrightarrows}} \Lambda^1(\mathfrak{g}) \underset{0}{\overset{d_{Lie}}{\leftrightarrows}} \Lambda^2(\mathfrak{g}) \underset{0}{\overset{d_{Lie}}{\leftrightarrows}} \cdots \ ,$$

where d_{Lie} stands for the usual boundary in the Chevalley-Eilenberg complex computing $H_*(\mathfrak{g})$. Denote by $\mathcal{B}$ the mixed complex associated to the cyclic module $\mathcal{H}^\natural$, and by $\mathcal{B}_\delta$ its localization, i.e. the mixed complex associated to the cyclic module $\mathcal{H}_\delta^\natural$ (so they are the mixed complexes computing $HC^*(\mathcal{H})$, and $HC_\delta^*(\mathcal{H})$), respectively). Here $\mathcal{H} = U(\mathfrak{g})$. Let $\pi : \mathcal{B} \to \mathcal{B}_\delta$ be the projection map, which, after our identifications (Theorem 4.6), is degreewise given by:

$$\pi : \mathcal{H}^{\otimes(n+1)} \longrightarrow \mathcal{H}^{\otimes n}, \ \pi(h_0 \otimes \ldots \otimes h_n) = S_\delta(h_0) \cdot (h_1 \otimes \ldots \otimes h_n).$$

Denote by B and B_δ the usual (degree (-1)) "B- boundaries" of the two mixed complexes $\mathcal{B}, \mathcal{B}_\delta$. Recall that $B = N\sigma_{-1}\tau$, where:

$$\sigma_{-1}(h_0, \ldots, h_n) = \epsilon(h_0)(h_1, \ldots, h_n), \ \tau(h_0, \ldots, h_n) = (-1)^n(h_1, \ldots, h_n, h_0) \ ,$$

and $N = 1 + \tau + \ldots + \tau^n$ on $\mathcal{H}^{\otimes(n+1)}$.

We will show that Λ and $\mathcal{B}_\delta$ are quasi-isomorphic mixed complexes (which easily implies the theorem), but for the computation we have to use the mixed complex $\mathcal{B}$, where explicit formulas are easier to write. We define the map:

$$A : \Lambda^n(\mathfrak{g}) \longrightarrow \mathcal{H}^{\otimes n}, \ A(v_1 \wedge \ldots \wedge v_n) = (\sum_\sigma sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!.$$

The fact that the (localized) Hochschild boundary depends just on the coalgebra structure of $U(\mathfrak{g})$ and on the unit, which are preserved by the Poincare-Birkhoff-Witt Theorem (see e.g. [21]), together with the previous proposition, shows that A is a quasi-isomorphism of mixed complexes, once we prove its compatibility with the degree (-1) boundaries, that is:

$$B_\delta(A(x)) = A(d_{Lie}(x)), \quad \forall\ x = v_1 \wedge \ldots \wedge v_n \in \Lambda^n(\mathfrak{g}). \tag{17}$$

Using that $A(x) = \pi(y)$, where $y = (\sum sign(\sigma) 1 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/n!$, we have

$$\begin{aligned} B_\delta A(x) &= \pi(B(y)) = \\ &= \pi N \sigma_{-1}(\sum sign(\sigma)\,(1 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} - (-1)^n v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \otimes 1))/n! \\ &= \pi(N(\sum sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}))/n! \\ &= \pi(\sum sign(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})/(n-1)!\ . \end{aligned}$$

But

$$\pi(v \otimes v_1 \otimes \ldots \otimes v_n) = \delta(v) v_1 \otimes \ldots \otimes v_n - \sum_{i=1}^{n} v_1 \otimes \ldots \otimes v v_i \otimes \ldots v_n\ ,$$

and, with these, it is straightforward to see that $B_\delta A(x)$ equals to:

$$A(\sum_{i=1}^{n} (-1)^{i+1} \delta(v_1) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_n + \sum_{i<j} (-1)^{i+j} [v_i, v_j] \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_n),$$

i.e. with $A(d_{Lie}(x))$. $\square$

5 A non-commutative Weil complex

Let $\mathcal{H}$ be a (not necessarily counital) coalgebra. Define its Weil algebra $W(\mathcal{H})$ as the (non-commutative, non-unital) DG algebra freely generated by the symbols h of degree 1, $\omega(h)$ of degree 2, linear on $h \in \mathcal{H}$. The differential of $W(\mathcal{H})$ is denoted by b', and is the unique derivation which acts on generators by:

$$b'(h) = \omega_h - \sum h_0 h_1,$$

$$b'(\omega_h) = \sum \omega_{h_0} h_1 - \sum h_0 \omega_{h_1}.$$

Example 5.1 This algebra is intended to be a non-commutative analogue of the usual Weil complex of a Lie algebra ([2], see also [11]). Particular cases have been used in the study of universal Chern-Simons forms. When $\mathcal{H} = \mathbf{C}\rho$ (i.e. $\mathbf{C}$, with 1 denoted by ρ), with $\Delta(\rho) = \rho \otimes \rho$, it is the complex introduced in [19]; for $\mathcal{H} = \mathbf{C}\rho_1 \bigoplus \ldots \bigoplus \mathbf{C}\rho_n$ with $\Delta(\rho_i) = \rho_i \otimes \rho_i$, we obtain one of the complexes studied on [12, 13].

We discuss now its "universal property". Given a DG algebra Ω^*, and a linear map:

$$\phi : \mathcal{H} \longrightarrow \Omega^1,$$

define its curvature:

$$\omega_\phi : \mathcal{H} \longrightarrow \Omega^2, \omega_\phi(h) = d\phi(h) + \sum \phi(h_0)\phi(h_1).$$

Alternatively, using the natural DG algebra structure of $Hom(\mathcal{H}, \Omega^*)$,

$$\omega_\phi := d(\phi) - 1/2[\phi, \phi] \in Hom(\mathcal{H}, \Omega)^2.$$

There is a unique algebra homomorphism (the characteristic map of ϕ):

$$k(\phi) : W(\mathcal{H}) \longrightarrow \Omega^*,$$

sending h to $\phi(h)$ and ω_h to $\omega_{\phi}(h)$.

One can easily see that (compare with the usual Weil complex of a Lie algebra):

Proposition 5.2 . *The previous construction induces a $1-1$ correspondence between linear maps $\phi : \mathcal{H} \longrightarrow \Omega^1$ and DG algebra maps $k : W(\mathcal{H}) \longrightarrow \Omega^*$. In particular, there is a $1-1$ correspondence between flat linear maps $\phi : \mathcal{H} \longrightarrow \Omega^1$ (i. e. with the property that $\omega_{\phi} = 0$), and DG algebra maps $k : T(\mathcal{H}) \longrightarrow \Omega^*$.*

An immediate consequence is that $W(\mathcal{H})$ does not depend on the co-algebra structure of $\mathcal{H}$. Actually one can see directly that $(W(\mathcal{H}), b') \cong (W(\mathcal{H}), d_1)$, where d_1 is the derivation on $W(\mathcal{H})$ defined on generators by:

$$d_1(h) = \omega_h, \quad d_1(\omega_h) = 0,$$

(i.e. the differential corresponding to $\mathcal{H}$ with the trivial co-product). An explicit isomorphism sends h to h and ω_h to $\omega_h + \sum h_0 h_1$.

Corollary 5.3 . *The Weil algebra $W(\mathcal{H})$, and the complex $W(\mathcal{H})_\natural$ are acyclic.*

Corollary 5.4 . *The Weil algebra $W(\mathcal{H})$ is free in the category of DG algebras.*

5.5 Extra-structure on $W(\mathcal{H})$: Now we look at the extra-structure of $W(\mathcal{H})$. First of all, denote by $I(\mathcal{H})$ the ideal generated by the curvatures ω_h. The powers of $I(\mathcal{H})$, and the induced truncations are denoted by:

$$I_n(\mathcal{H}) := I(\mathcal{H})^{n+1}, \quad W_n(\mathcal{H}) := W(\mathcal{H})/I(\mathcal{H})^{n+1}.$$

Remark that $W_0(\mathcal{H}) = T(\mathcal{H})$ is the tensor (DG) algebra of $\mathcal{H}$ (up to a minus sign in the boundary, which is irrelevant, and will be ignored). In particular, one has the free resolution (in the sense of cyclic homology, [9]) of the DG algebra $T(\mathcal{H})$:

$$0 \longrightarrow I(\mathcal{H}) \longrightarrow W(\mathcal{H}) \longrightarrow T(\mathcal{H}) \longrightarrow 0.$$

Dual to even higher traces, we introduce the complex:

$$W_n(\mathcal{H})_\natural := W_n(\mathcal{H})/[W_n(\mathcal{H}), W_n(\mathcal{H})]$$

obtained dividing out the (graded commutators). In the terminology of [13] (pag 103), it is the space of "cyclic words". Dual to odd higher traces:

$$I_n(\mathcal{H})_\natural := I_n(\mathcal{H})/[I(\mathcal{H}), I_{n-1}(\mathcal{H})].$$

The Chern-Simons contraction (see Theorem 5.8) leads us to a slight modification of the last complex:

$$\tilde{I}_n(\mathcal{H})_\natural := I_n(\mathcal{H})/I_n(\mathcal{H}) \cap [W(\mathcal{H}), W(\mathcal{H})].$$

Secondly, we point out a bi-grading on $W(\mathcal{H})$: defining d_2 such that $b' = d_1 + d_2$, then $W(\mathcal{H})$ has a structure of bi-DG algebra, with $deg(h) = (1,0), deg(\omega_h) = (1,1)$. Actually $W(\mathcal{H})$ can be viewed as the tensor algebra of $\mathcal{H}^{(1,0)} \bigoplus \mathcal{H}^{(1,1)}$ (two copies of $\mathcal{H}$ on the indicated xi-degrees). With this bi-grading, q in $W^{p,q}$ counts the number of curvatures. The first boundary d_1 increases q, while d_2 increases p.

5.6 Example: Let's have a closer look at $\mathcal{H} = \mathbf{C}\rho$ with $\Delta(\rho) = h\rho \otimes \rho$, for which the computations were carried out by D. Quillen [19], recalling the main features of our complexes:

1. ω^n are cocycles of $W(\mathcal{H})_\natural$ (where $\omega = \omega_{h\rho}$). They are trivial in cohomology (cf. Corollary 5.3).
2. the place where ω^n give non-trivial cohomology classes is $I_{m,\natural}$, with m sufficiently large.
3. the cocycles ω^n (trivial in $W(\mathcal{H})_\natural$) transgress to certain (Chern-Simons) classes. The natural complex in which these classes are non-trivial (in cohomology) is $W_m(\mathcal{H})_\natural$.
4. there are striking "suspensions" (by degree 2 up) in the cohomology of all the complexes $W_n(\mathcal{H})_\natural$, $I_n(\mathcal{H})_\natural$, $\tilde{I}_n(\mathcal{H})_\natural$.

Our intention is also to explain these fenomena (in our general setting).

5.7 "Chern-Simons contractions". Starting with two linear maps:

$$\rho_0, \rho_1 : \mathcal{H} \longrightarrow \Omega^1,$$

we form:

$$t\rho_0 + (1-t)\rho_1 := \rho_0 \otimes t + \rho_1 \otimes (1-t) : \mathcal{H} \longrightarrow (\Omega^* \otimes \Omega(1))^1,$$

where $\Omega(1)$ is the algebraic DeRham complex of the line: $\mathbf{C[t]}$ in degree 0, and $\mathbf{C[t]dt}$ in degree 1, with the usual differential. Composing its characteristic map $W(\mathcal{H}) \longrightarrow \Omega \otimes \Omega(1)$, with the degree -1 map $\Omega \otimes \Omega(1) \longrightarrow \Omega$ coming from the integration map $\int_0^1 : \Omega(1) \longrightarrow \mathbf{C}$ (emphasize that we use the graded tensor product, and the integration map has degree -1), we get a degree -1 chain map:

$$k(\rho_0, \rho_1) : W(\mathcal{H}) \longrightarrow \Omega.$$

As usual,

$$dk(\rho_0, \rho_1) + k(\rho_0, \rho_1)b' = k(\rho_1) - k(\rho_0).$$

The particular case where $\Omega = W(\mathcal{H}), \rho_0 = 0, \rho_1 = Id_{\mathcal{H}}$ gives a contraction of $W(\mathcal{H})$:

$$H := k(Id_{\mathcal{H}}, 0) : W(\mathcal{H}) \longrightarrow W(\mathcal{H}).$$

We point out that H preserves commutators. From this, one can easily see that there is a well defined map:

$$CS : H^*(\tilde{I}_n(\mathcal{H})_\natural) \longrightarrow H^{*-1}(W_n(\mathcal{H})_\natural),\ [x] \mapsto [H(x)],$$

to which we will refer as the Chern Simons map. It also exists at the level of complexes, as a chain map $\tilde{I}_n(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural[1]$.

The formulas for the contraction H resemble the usual ones ([12, 13, 18, 19]). For instance, at the level of $W(\mathcal{H})_\natural$, one has:

$$H(\frac{\omega_h^{n+1}}{(n+1)!}) = \int_0^1 \frac{1}{n!} h(t\omega_h + (t^2 - t)\sum_{(h)} h_0 h_1)^n dt \tag{18}$$

Theorem 5.8 *The Chern-Simons map is an isomorphism* $H^*(\tilde{I}_n(\mathcal{H})_\natural) \xrightarrow{\sim} H^{*-1}(W_n(\mathcal{H})_\natural)$.

We will also prove, after discussing the S-operator that:

Theorem 5.9 *The projection* $I_n(\mathcal{H})_\natural \twoheadrightarrow \tilde{I}_n(\mathcal{H})_\natural$ *induces isomorphism in cohomology.*

proof of Theorem 5.8: It is almost obvious. The boundary in the long exact sequence induced by the short exact sequence:

$$0 \longrightarrow \tilde{I}_n(\mathcal{H})_\natural \longrightarrow W(\mathcal{H})_\natural \longrightarrow W_n(\mathcal{H})_\natural \longrightarrow 0$$

is an isomorphism by Corollary 5.3. One can easily check that composed with the Chern-Simons map CS we get the identity. □

5.10 The S-operator: We will explain now the "suspensions" (by degree 2 up) in the various cohomologies we deal with, by describing a degree 2 operator. For this we introduce certain bi-complexes computing these cohomologies, in which S can be described as a shift. Although we can define these bicomplexes directly and check all the formulas (in a very similar way to cyclic cohomology), we prefer to show first how these complexes show up naturally using the ideas from cyclic cohomology. Recall (see Section 3, or [18, 19]) that for any DG algebra R, one has a sequence:

$$0 \longrightarrow R_\natural \xrightarrow{d} \Omega^1(R)_\natural \xrightarrow{b} R \xrightarrow{\natural} R_\natural \longrightarrow 0, \tag{19}$$

which is exact on the right. Here b and d come from the boundaries of the X- complex of R (see (11)), and $R_\natural = R/[R, R]$. When the sequence is exact, it can be viewed as an Ext^2 class, and induces a degree 2 operator $S : H^*(R_\natural) \longrightarrow H^{*+2}(R_\natural)$, explicitly described by the following diagram chasing ([19], pag. 120). Given $\alpha \in H^k(R_\natural)$, we represent it by a cocycle c, and use the exactness to solve successively the equations:

$$\begin{cases} c &= \natural(u) \\ \partial(u) &= b(v) \\ \partial(v) &= d(w) \end{cases}$$

where ∂ stands for the vertical boundary. Then $S(\alpha) = [\natural(w)] \in H^{k+2}(R_\natural)$.

Equivalently, pasting together (19), we get a resolution:

$$0 \longrightarrow R_\natural \xrightarrow{d} \Omega^1(R)_\natural \xrightarrow{b} R \xrightarrow{d} \Omega^1(R)_\natural \xrightarrow{b} R \longrightarrow \dots$$

of $R_\natural$ by a bicomplex (which is usually denoted by $X^+(R)$, and is $X(R)$ in positive degrees). We emphasize that when working with bicomplexes with anti-commuting differentials, one has to introduce a "$-$" sign for the even vertical boundaries (i.e. for those of R). So, one can use this bicomplex to compute the cohomology of $R_\natural$, and then S is simply the shift operator. The computation for $R = W(\mathcal{H})$, can be carried out as in the case of the tensor algebra (Lemma 3.8), and this is done in the proof of Proposition 6.8. We end up with the following exact sequence of complexes (which can be taken as a definition):

$$\dots \longrightarrow W^b(\mathcal{H}) \xrightarrow{t-1} W(\mathcal{H}) \xrightarrow{N} W^b(\mathcal{H}) \xrightarrow{t-1} W(\mathcal{H}) \longrightarrow \dots,$$

where we have to explain the new objects. First of all, $W^b(\mathcal{H})$ is the same as $W(\mathcal{H})$ but with a new boundary $b = b' + b_t$ with b_t described below. The t operator is the backward cyclic permutation:

$$t(ax) = (-1)^{|a||x|} xa,$$

for $a \in H$ or of type ω_h. This operator has finite order in each degree of $W(\mathcal{H})$: we have $t^p = 1$ on elements of bi-degree (p, q), so $t^{k!} = 1$ on elements of total degree k. The norm operator N is $N := 1 + t + t^2 + \dots + t^{p-1}$ on elements of bi-degree (p, q). The boundary b of $W^b(\mathcal{H})$ is $b = b' + b_t$,

$$b_t(ax) = t(d_2(a)x),$$

for $a \in \mathcal{H}$ or of type ω_h. Recall (5.5) that d_2 is the differential on $W(\mathcal{H})$ which, on generators:

$$d_2(h) = -\sum h_0 h_1, \quad d_2(\omega_h) = \sum(\omega_{h_0} h_1 - h_0 \omega_{h_1}).$$

Of course, one can check directly that b is a boundary and that $(t-1)$ and N are chain maps.

Obviously, the powers $I(\mathcal{H})^{n+1}$ are invariant by $b, t-1, N$, so we get similar sequences for $I_n(\mathcal{H})$, $W_n(\mathcal{H})$. For reference, we conclude:

Corollary 5.11 *There are exact sequences of complexes:*

$$\ldots \longrightarrow W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow \ldots \tag{20}$$

$$0 \longrightarrow W_n(\mathcal{H})_\natural \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow \ldots \tag{21}$$

$$0 \longrightarrow I_n(\mathcal{H})_\natural \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \longrightarrow \ldots \tag{22}$$

Corollary 5.12 *There are short exact sequences of complexes:*

$$0 \longrightarrow W_n(\mathcal{H})_\natural \xrightarrow{N} W_n^b(\mathcal{H}) \xrightarrow{t-1} W_n(\mathcal{H}) \longrightarrow W_n(\mathcal{H})_\natural \longrightarrow 0 \tag{23}$$

$$0 \longrightarrow I_n(\mathcal{H})_\natural \xrightarrow{N} I_n^b(\mathcal{H}) \xrightarrow{t-1} I_n(\mathcal{H}) \longrightarrow I_n(\mathcal{H})_\natural \longrightarrow 0 \tag{24}$$

In particular, (21), (22), give two bicomplexes which compute the cohomologies of $W_n(\mathcal{H})_\natural$, $I_n(\mathcal{H})_\natural$. These bicomplexes are similar to the (first quadrant) cyclic bicomplexes appearing in cyclic cohomology, and are denoted by $\underline{CC}^*(I_n(\mathcal{H}))$ and $\underline{CC}^*(W_n(\mathcal{H}))$. There are obvious shifts maps in these bicomplexes which induce our S operator:

$$S : H^*(W_n(\mathcal{H})_\natural) \longrightarrow H^{*+2}(W_n(\mathcal{H})_\natural),$$

(and similarly for $I_n(\mathcal{H})_\natural$). Alternatively, one can obtain S as cup-product by the Ext^2 classes arising from Corollary 5.12.

proof of theorem 5.9: Denote for simplicity by $\underline{CC}^*(I_n), \underline{CC}^*(W), \underline{CC}^*(W_n)$ the (first quadrant) cyclic bicomplexes (or their total complexes) of I_n, W, and W_n, respectively. We have a map of short exact sequences of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{I}_n(\mathcal{H})_\natural \arrow[r] \arrow[d, "N"] & W(\mathcal{H})_\natural \arrow[r] \arrow[d, "N"] & W_n(\mathcal{H})_\natural \arrow[r] \arrow[d, "N"] & 0 \\
0 \arrow[r] & \underline{CC}^*(I_n) \arrow[r] & \underline{CC}^*(W) \arrow[r] & \underline{CC}^*(W_n) \arrow[r] & 0
\end{tikzcd}$$

where we have used the fact that $N : I_n(\mathcal{H})_\natural \longrightarrow I_n(\mathcal{H})$ factors through the projection $I_n(\mathcal{H})_\natural \twoheadrightarrow \tilde{I}_n(\mathcal{H})_\natural$ (being defined on the entire $W(\mathcal{H})_\natural$). Applying the five lemma to the exact sequences induced in cohomology by the previous two short exact sequences, the statement follows. $\square$

Corollary 5.13 *The Chern Simons map induces isomorphisms:*

$$CS : H^*(I_n(\mathcal{H})_\natural) \longrightarrow H^{*-1}(W_n(\mathcal{H})_\natural),$$

compatible with the S *operation.*

5.14 Example: There are canonical Chern and Chern-Simons classes induced by any group-like element $\rho \in \mathcal{H}$ (i.e. with the property $\Delta(\rho) = \rho \otimes \rho$). Denote by ω its curvature. Since $b'(\omega^n) = [\omega^n, \rho]$ is a commutator, ω^n define cohomology classes:

$$ch_{2n}(\rho) := [\natural(\frac{1}{n!}\omega^n)] \in H^{2n}(I_m(\mathcal{H})_\natural)\ , \tag{25}$$

for any $n \geq m$. The associated Chern-Simons class $cs_{2n-1}(\rho) := CS(ch_{2n}(\rho))$ is given by the formula (see (18)):

$$cs_{2n-1}(\rho) = [\natural\{\frac{1}{(n-1)!}\int_0^1 \rho(tb'(\rho) + t^2\rho^2)^{n-1}dt\}] \in H^{2n-1}(W_m(\mathcal{H})_\natural).$$

To compute $S(ch_{2n}(\rho))$, we have to solve succesively the equations:

$$\begin{cases} b'(\frac{1}{n!}\omega^n) &= (t-1)(v) \\ b(v) &= N(w) \end{cases}$$

and then $S(ch_{2n}(\rho)) = [\natural(w)]$. The first equation has the obvious solution $v = \frac{1}{n!}\rho\omega^n$, whose $b(v) = \frac{1}{n!}\omega^{n+1}$, so the second equation has the solution $w = \frac{1}{(n+1)!}\omega^{n+1}$. In conclusion,

$$S(ch_{2n}(\rho)) = ch_{2(n+1)}(\rho), \quad S(cs_{2n-1}(\rho)) = cs_{2n+1}(\rho). \tag{26}$$

(where the second relation follows from the first one and Corrolary 5.13.) In other words, at the level of the periodic cohomology (in the sense of cyclic cohomology, [5]), there is precisely one cohomology class induced by the group-like element ρ.

6 A non-commutative Chern-Weil characteristic map

We explain now how the Weil complex introduced in the previous paragraph appears naturally in the case of higher traces, and Hopf algebra actions. We will obtain in particular the case of usual traces discussed in Section 4. Also, for $\mathcal{H} = \mathbf{C}\rho$ (example 5.1), we reobtain the results , and interpretations of some of the computations of [18] (see Example 6.9 below).

Although the same discussion applies to coalgebras (with some simplifications), we present here the case of Hopf algebras. So, let $\mathcal{H}$ be a Hopf algebra, and δ a character on $\mathcal{H}$ with the property $S_\delta^2 = Id$. Through this paragraph, A is an algebra (endowed with an action of $\mathcal{H}$), and:

$$0 \longrightarrow I \longrightarrow R \xrightarrow{u} A \longrightarrow 0 \tag{27}$$

is an equivariant extension of A. These are the data which give "equivariant characteristic classes" in the (cyclic) cohomology of (the base space) A, once we have certain invariant traces.

6.1 Localizing $W(\mathcal{H})$: First of all remark that the Weil complex $W(\mathcal{H})$ is naturally an $\mathcal{H}$ DG algebra. By this we mean a DG algebra, endowed with a (flat) action, compatible with the grading and with the differentials. The action is defined on generators by:

$$g \cdot i(h) := i(gh), \quad g \cdot \omega_h := \omega_{gh}, \quad \forall\ g, h \in \mathcal{H}.$$

and extended by $h(xy) = \sum h_0(x)h_1(y)$. Here, to avoid confusions, we have denoted by $i : \mathcal{H} \longrightarrow W(\mathcal{H})$ the inclusion. Remark that the action does preserve the bi-degree (see 5.5), so $W(\mathcal{H})_\delta$ has an induced bi-grading. We briefly explain how to get the localized version for the constructions and the properties of the previous section. First of all one can localize with respect to δ as in Section 3, and (with the same proof as of Proposition 3.4), all the operators descend to the localized spaces. The notation $I_n(\mathcal{H})_{\natural,\delta}$ stands for $I_n(\mathcal{H})$ divided out by commutators and co-invariants. For Theorem 5.8, remark that the contraction used there is compatible with the action. To get the exact sequences from Corollary 5.11 and 5.12, we may look at them as a property for the cohomology of finite cyclic groups, acting (on each fixed bi-degree) in our spaces. Or we can use the explicit map $\alpha : W(\mathcal{H}) \rightarrow W(\mathcal{H})$ defined by $\alpha := (t + 2t^2 + \dots + (p-1)t^{p-1})$ on elements of bi-degree (p, q), which has the properties: $(t-1)\alpha + N = pId$, $\alpha(I(\mathcal{H})^{n+1}) \subseteq I(\mathcal{H})^{n+1}$, and α descends (because t does). So, also the analogue of Theorem 5.9 follows. In particular $H^*_\delta(W_n(\mathcal{H})_\natural)$ is computed either by the complex $W_n(\mathcal{H})_{\natural,\delta}$, or by the (localized) cyclic bicomplex $CC^*_\delta(W_n(\mathcal{H}))$. Similarly, we consider the S operator, and the periodic versions of these cohomologies.

6.2 The cyclic cohomologies involved: We define $HC^*_\delta(\mathcal{H}, n) := H^{*+1}(W_n(\mathcal{H})_{\natural,\delta})$. Remark that for $n = 0$ we obtain Connes-Moscovici's cyclic cohomology, while, in general, there are obvious maps:

$$\dots \longrightarrow HC^*_\delta(\mathcal{H}, 2) \longrightarrow HC^*_\delta(\mathcal{H}, 1) \longrightarrow HC^*_\delta(\mathcal{H}, 0) \cong HC^*_\delta(\mathcal{H}). \tag{28}$$

Denote by $CC^*_\delta(\mathcal{H}, n)$ the cyclic bicomplex computing $HC^*_\delta(\mathcal{H}, n)$, that is, the localized version of the cyclic bicomplex (20) (shifted by one in the vertical direction, and with the boundary b' replaced by $-b'$). Its first-quadrant part is denoted by $\underline{CC}^*_\delta(\mathcal{H}, n)$. There are similar complexes $CC^*(\mathcal{H}, n)$, $\underline{CC}^*(\mathcal{H}, n)$ living before the localization. Remark that for $n = 0$, these are the complexes used in Section 4:

$$CC^*(\mathcal{H}, 0) = CC^*(\mathcal{H}) \ , \quad CC^*_\delta(\mathcal{H}, 0) = CC^*_\delta(\mathcal{H}).$$

6.3 The case of even equivariant traces: Let now τ be an even equivariant trace over A, i.e. an extension (27) and an (n-dimensional) trace $\tau : R \longrightarrow \mathbf{C}$ vanishing on I^{n+1}. To describe the induced characteristic map, we choose a linear splitting $\rho : A \longrightarrow R$ of (27). As in the case of the usual Weil complex, there is a unique equivariant map of DG algebras:

$$\tilde{k} : W(\mathcal{H}) \longrightarrow Hom(B(A), R),$$

sending $1 \in \mathcal{H}$ to ρ. This follows from Proposition 5.2 and from the equivariance condition (with the same arguments as in 4.3). Here, the action of $\mathcal{H}$ on $Hom(B(A), R)$ is induced by the action on R. Since ρ is a homomorphism modulo I, $\tilde{k}$ sends $I(\mathcal{H})$ to $Hom(B(A), I)$, so induces a map $W_n(\mathcal{H}) \longrightarrow Hom(B(A), R/I^{n+1})$. As in 4.3, composing with the δ-invariant trace:

$$\tau_\natural : Hom(B(A), R/I^{n+1}) \longrightarrow C^*_\lambda(A)[1], \quad \phi \mapsto \tau \circ \phi \circ N \ ,$$

we get a δ-invariant trace on $W_n(\mathcal{H})$, so also a chain map:

$$k^{\tau,\rho} : W_n(\mathcal{H})_{\natural,\delta} \longrightarrow C^*_\lambda(A)[1] \ . \tag{29}$$

Denote by the same letter the map induced in cohomology:

$$k^{\tau,\rho} : HC^*_\delta(\mathcal{H}, n) \longrightarrow HC^*(A) \tag{30}$$

Theorem 6.4 *The characteristic map (30) of an even higher trace τ does not depend on the choice of the splitting ρ and is compatible with the S-operator.*

proof:(compare to [18]) We proceed as in 5.7. If ρ_0, ρ_1 are two liftings, form $\rho = t\rho_0 + (1-t)\rho_1 \in Hom(A, R[t])$, viewed in the degree one part of the DG algebra $Hom(B(A), R \otimes \Omega(1))$. It induces a unique map of $\mathcal{H}$DG algebras $\tilde{k}_\rho : W(\mathcal{H}) \longrightarrow Hom(B(A), R \otimes \Omega(1))$, sending 1 to ρ, which maps $I(\mathcal{H})$ to the DG ideal $Hom(B(A), I \otimes \Omega(1))$ (since ω_ρ belongs to the former). Using the trace $\tau \otimes \int : R/I^{n+1} \otimes \Omega(1) \longrightarrow \mathbf{C}$, and the universal cotrace on $B(A)$, it induces a chain map:

$$k_{\rho_0,\rho_1} : W_n(\mathcal{H}) \longrightarrow C^*_\lambda(A)[1],$$

which kills the coinvariants and the commutators. The induced map on $W_n(\mathcal{H})_{\natural,\delta}$ is a homotopy between k^{τ,ρ_0} and k^{τ,ρ_1}. The compatibility with S follows from the fact that the characteristic map (29) can be extended to a map between the cyclic bicomplexes $CC^*_\delta(\mathcal{H}, n)$ and $CC^*(A)$. We will prove this after shortly discussing the case of odd higher traces. $\square$

6.5 The case of odd equivariant traces: A similar discussion applies to the case of odd equivariant traces on A, i.e. extensions (27) endowed with a linear map $\tau : I^{n+1} \longrightarrow \mathbf{C}$, vanishing on $[I^n, I]$ (i.e. I-adic trace, cf. [9]). The resulting map $HC^*_\delta(I_n(\mathcal{H})) \longrightarrow HC^{*-1}(A)$, combined with Corollary 5.13 (and the comments in 6.1), give the characteristic map:

$$k^{\tau,\rho} : HC^*_\delta(\mathcal{H}, n) \longrightarrow HC^{*+1}(A),$$

which has the same properties as in the even case:

Theorem 6.6 *The characteristic map (30) of an odd higher trace τ does not depend on the choice of the splitting ρ and is compatible with the S-operator.*

6.7 The localized tower $\mathcal{X}_\delta(R, I)$: Recall that given an ideal I in the algebra R, one has a tower of supercomplexes $\mathcal{X}_\delta(R, I)$ given by ([9], pag. 396):

$$\mathcal{X}^{2n+1}(R, I) : R/I^{n+1} \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural/\natural(I^{n+1}dR + I^n dI) \quad ,$$

$$\mathcal{X}^{2n}(R, I) : R/(I^{n+1} + [I^n, R]) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_\natural/\natural(I^n dR) \; ,$$

where $\natural : \Omega^1(R) \to \Omega^1(R)_\natural$ is the projection. The structure maps $\mathcal{X}^n(R, I) \to \mathcal{X}^{n+1}(R, I)$ of the tower are the obvious projections. We have a localized version of this, denoted by $\mathcal{X}_\delta(R, I)$, and which is defined by:

$$\mathcal{X}^{2n+1}_\delta(R, I) : R/(I^{n+1} + \mathrm{coinv}) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}/\natural(I^{n+1}dR + I^n dI) \quad ,$$

$$\mathcal{X}^{2n}_\delta(R, I) : R/(I^{n+1} + [I^n, R] + \mathrm{coinv}) \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1(R)_{\natural,\delta}/\natural(I^n dR) \; ,$$

where this time, $\natural$ denotes the projection $\Omega^1(R) \to \Omega^1(R)_{\natural,\delta}$.

Remark that the construction extends to the graded case. The relevant fact for us in this setting is that each $\mathcal{X}^n(R, I)$ is naturally a super-complex of complexes. When refering to it as a bicomplex, we change the sign of the vertical boundaries of $\mathcal{X}^n(R, I)_+$ (so that the differentials become anticommuting).

Proposition 6.8 *The cyclic bicomplex $CC^*_\delta(\mathcal{H}, n)$ is isomorphic to the bicomplex $\mathcal{X}^{2n+1}_\delta(W(\mathcal{H}), I(\mathcal{H}))$.*

proof: The computation is similar to the one of $X(T\mathcal{H})$ (see Example 3.7 and Proposition 3.8). Denote $W = W(\mathcal{H})$, $I = I(\mathcal{H})$, and let $V \subset W(\mathcal{H})$ be the linear subspace spanned by h's and ω_h's. Remark that W, as a graded algebra, is freely generated by V. This allows us to use exactly the same arguments as in 3.7, 3.8 to conclude that $\Omega^1 W \cong \tilde{W} \otimes V \otimes \tilde{W}$, $\Omega^1(W)_\natural \cong V \otimes \tilde{W} = W$, $\Omega^1(W)_{\natural,\delta} \cong W_\delta$. Also the projection $\natural : \Omega^1(W) \to \Omega^1(W)_\natural$ identifies with:

$$\natural : \Omega^1(W) \longrightarrow V \otimes \tilde{W} = W, \quad x\partial(v)y \mapsto (-1)^\mu vyx \;, \tag{31}$$

for $x, y \in \tilde{W}, v \in V$. Here $\mu = deg(x)(deg(v) + deg(y))$ introduces a sign, due to our graded setting, and $\partial : W \to \Omega^1(W)$ stands for the universal derivation of W. Using this, we can compute the new boundary of W, coming from the isomorphism $W \cong \Omega^1(W)_\natural$, and we end up we the b-boundary of W, defined in Section 5. For instance, if $x = hx_0 \in W$ with $h \in \mathcal{H}$, since $\natural(\partial(h)x_0) = x$ by (31), its boundary is:

$$\begin{aligned}
&\natural(\partial(b'h)x_0) - \partial(h)b'(x_0)) = \\
&\quad = \natural(\partial(\omega_h - \sum h_0h_1)x_0 - \partial(h)b'(x_0)) = \\
&\quad = \natural(\partial(\omega_h)x_0 - \sum \partial(h_0)h_1x_0 - \sum h_0\partial(h_1)x_0 - \partial(h)b'(x_0)) = \\
&\quad = \omega_h x_0 - \sum h_0h_1x_0 - \sum(-1)^{deg(x)}h_1x_0h_0 - hb'(x_0) = \\
&\quad = b'(h)x_0 - \sum t(h_0h_1x_0) - hb'(x_0) = \\
&\quad = b'(hx_0) + t(d_2(h)x_0) = b(hx_0)
\end{aligned}$$

Remark also that our map (31) has the property:

$$\natural(I^n\partial I + I^{n+1}\partial W) = I^{n+1}. \tag{32}$$

These give the identification $\mathcal{X}^{2n+1}(W, I) \cong CC^*(\mathcal{H}, n)$. The localized version of this is just a matter of checking that the isomorphism $\Omega^1(W)_{\natural,\delta} \cong W_\delta$ already mentioned, induces $\Omega^1(W)_{\natural,\delta}/\natural(I^n\partial I + I^{n+1}\partial W) \cong (W/I^{n+1})_\delta$, which follows from (32). $\square$

proof of the S-relation: We freely use the dual constructions for (DG) coalgebras B, such as the universal coderivation $\Omega_1(B) \to B$, the space of co-commutators $B^\natural = Ker(\Delta - \sigma\circ\Delta : B \to B \otimes B)$, and the X-complex $X(B)$ (see [18]). Denote $B = B(A)$, $L = Hom(B, R)$, $J = Hom(B, I)$. Our goal is to prove that the characteristic map (29) can be defined at the level of the cyclic bicomplexes. Consider first the case of even traces τ. Since the $\mathcal{H}$ DG algebra map $\tilde{k} : W(\mathcal{H}) \to L$ maps $I(\mathcal{H})$ inside J, there is an induced map $\mathcal{X}^{2n+1}_\delta(W(\mathcal{H}), I(\mathcal{H})) \to \mathcal{X}^{2n+1}_\delta(L, J)$, extending $W_n(\mathcal{H})_{\natural,\delta} \to (L/J^{n+1})_{\natural,\delta}$. So, it suffices to show that the map $(L/J^{n+1})_{\natural,\delta} \to Hom(B^\natural, (R/I^{n+1})_{\natural,\delta})$ (constructed as (29)), lifts to a map of super-complexes (of complexes)

$$\mathcal{X}^{2n+1}_\delta(L, J) \to Hom(X(B), (R/I^{n+1})_{\natural,\delta}) \tag{33}$$

Indeed, using Proposition 6.8, the (similar) computation of $X(B)$ (as the cyclic bicomplex of A), the interpretation of the norm map N as the universal cotrace of B (see [18]), and the fact that any τ as above factors through $(R/I^{n+1})_{\natural,\delta} \to \mathbf{C}$, the map (33) is "universal" for our problem. The construction of (33) is quite simple. The composition with the universal coderivation of B is a derivation $L \to Hom(\Omega_1(B), R)$ on L, so it induces a map $\chi : \Omega^1(L) \to Hom(\Omega_1(B), R)$. Since χ is a L-bimodule map, and it is compatible with the action of $\mathcal{H}$, it induces a map $\Omega^1(L)_\natural \to Hom(\Omega_1(B)^\natural, (R/I^{n+1})_{\natural,\delta})$, which kills $\natural(J^n dJ + J^{n+1}dL + coinv)$. This, together with with the obvious $(L/J^{n+1})_\delta \to Hom(B, (R/I^{n+1})_{\natural,\delta})$, give (33). For the case of odd higher traces we proceed similarly. Remark that (33) was a priori defined at the level of $L, \Omega^1(L)$, so, in this case, one has to restrict to the ideals (instead of dividing out by them). $\square$

6.9 Example: Choosing $\rho = 1 \in \mathcal{H}$ (the unit of $\mathcal{H}$) in Example 5.14, and applying the characteristic map to the resulting classes, we get the Chern/Chern-Simons classes (in the cyclic cohomology of A), described in [18]. Remark that our proof of the compatibility with the S operator consists on two steps: the first one proves the universal formulas (26) at the level of the Weil complex, while the second one shows, in a formal way, that the characteristic map can be defined at the level of the cyclic bicomplexes. This allows us to avoid the explicit cochain computations.

For the completness, we include the following theorem which is analogous to one of the main results of [9] (Theorem 6.2). The notation $T(R)$ stands for the (non-unital) tensor algebra of R, and $I(R)$ is the kernel of the multiplication map $T(R) \to R$. Recall also that if M is a mixed complex, θM denotes the associated Hodge tower of M, which represents the cyclic homology type of the mixed complex (for details see [9]).

Theorem 6.10 *There is a homotopy equivalence of towers of supercomplexes:*

$$\mathcal{X}_\delta(TR, IR) \simeq \theta(\Omega^*(R)_\delta).$$

proof: The proof from [9] can be adapted. For this, one uses the fact that the projection $\Omega^*(R) \longrightarrow \Omega^*(R)_\delta$ is compatible with all the structures (with the operators, with the mixed complex structure). All the formulas we get for free, from [9]. The only thing we have to do is to take care of the action. For instance, in the computation of $\Omega^1(TR)_\natural$ (pag. $399 - 401$ in [9]), the isomorphism $\Omega^1(TR)_\natural \cong \Omega^-(R)$ is not compatible with the action of $\mathcal{H}$, but, using the same technique as in 2.2, it descends to localizations (which means that we can use the natural (diagonal) action we have on $\Omega^-(R)$). With this in mind, the analogous of Lemma 5.4 in [9] holds, that is, $\mathcal{X}_\delta(TR, IR)$ can be identified (without regarding the differentials) with the tower $\theta(\Omega^*(R)_\delta)$. Denote by k_δ the localization of k. The spectral decomposition with respect to k_δ is again a consequence of the corresponding property of k ([9], pp $389 - 391$ and pp. $402 - 403$), and the two towers are homotopically concentrated on the nillspaces of k_δ, corresponding to the eigenvalue 1. Lemma 6.1 of [9] identifies the two boundaries corresponding to this eigenvalues, which concludes the theorem. $\square$

6.11 Remark: Analogous to the usual case of algebras (see [9, 10]) there is a bivariant theory $HP^*_\delta(A, B)$ for any algebra A and any $\mathcal{H}$- algebra B, defined as the cohomology of the super-complex $Hom(\mathcal{X}(TA, IA), \mathcal{X}_\delta(TB, IB))$ (see [9]). According to the previous theorem, $HP^*_\delta(\mathbf{C}, B) = HP^\delta_*(B)$ (discussed in Section 3), while $HP^*_\delta(A, \mathbf{C}) = HP^*(A)$. There are also composition products:

$$\circ : HP^*_\delta(A, A) \otimes HP^*_\delta(A) \longrightarrow HP^*(A) \ . \tag{34}$$

Inspired by the bivariant theory, one can also consider the periodic version of (28), and also its stabilized version, which uses the supercomplex $\lim_{\underset{n}{\longleftarrow}} \widehat{CC}_\delta(\mathcal{H}, n)$ (here $\widehat{CC}_\delta(\mathcal{H}, n)$ is the $\mathbf{Z}/2\mathbf{Z}$ graded version of $CC_\delta(\mathcal{H}, n)$. One can define then a stabilized characteristic map, whose target is $HP^*_\delta(A, A)$, and which is compatible with the caracteristic maps defined in this section. These use techniques of [15], [17]. Details will be given elsewhere.

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