Torsion zero-cycles and the Abel–Jacobi map over the real numbers

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ABSTRACT. This is a study of the torsion in the Chow group of zero-cycles on a variety over the real numbers. The first section recalls important results from the literature. The rest of the paper is devoted to the study of the Abel–Jacobi map $\alpha\colon A_0(X) \to \mathrm{Alb}(X)(\mathbf{R})$ restricted to torsion subgroups. Using Roitman's theorem over the complex numbers and a version of Bloch's cohomological Abel–Jacobi map over the real numbers, it is shown that this map can be described completely in terms of étale cohomology. For some examples (products of curves, abelian varieties, certain fibre bundles) the torsion in the kernel and cokernel of the Abel–Jacobi map α is computed explicitly.

Introduction

Consider the Abel–Jacobi map

$$\alpha\colon A_0(X) \to \mathrm{Alb}(X)(k)$$

from the group of zero-cycles of degree 0 modulo rational equivalence into the k-points of the Albanese variety of a nonsingular projective geometrically irreducible variety X over a field k. If k is algebraically closed, then α is well-known to be surjective, but it need not be injective. However, Roitman's theorem on torsion zero-cycles asserts that the restriction

$$\alpha_{\mathrm{tors}}\colon A_0(X)_{\mathrm{tors}} \to \mathrm{Alb}(X)(k)_{\mathrm{tors}}$$

to the torsion subgroups is an isomorphism when k is algebraically closed (with the p-part in characteristic p due to Milne). In particular, we get for k algebraically closed of characteristic zero that $A_0(X)_{\mathrm{tors}} \simeq (\mathbf{Q}/\mathbf{Z})^{2q}$, where $q = \dim_k H^1(X, \mathcal{O}_X)$.

For $k = \mathbf{R}$, and $X(\mathbf{R}) \neq \emptyset$, it was shown by Colliot-Thélène and Scheiderer that

$$A_0(X)_{\mathrm{tors}} \simeq (\mathbf{Q}/\mathbf{Z})^{q} \times (\mathbf{Z}/2)^{s-1},$$

with q as above and s the number of connected components of the set of real points $X(\mathbf{R})$ (for the euclidean topology; let me state here that in this paper the set of real points $X(\mathbf{R})$ and the set of complex points $X(\mathbf{C})$ will always be equipped with the euclidean topology). However, the methods of [**CTS96**] do not give precise information on the Abel–Jacobi map.

In general the map α_{tors} is not an isomorphism over $\mathbf{R}$: it is easy to construct varieties having $q = 0$, but a non-connected set of real points (for example, suitable hypersurfaces of dimension ≥ 2 in projective space). In that case $\mathrm{Alb}(X)$ is trivial, so α_{tors} is not injective. Another easy example is an abelian variety X the set of real points of which has more than two connected components (see Example 5.2). Surjectivity of α_{tors} can fail as well; examples are curves of odd genus without real points (see Section 1.2), but also certain types of 'twisted' fibre bundles that do have real points (see Example 5.1).

A standard trace argument, combined with Roitman's theorem on α_{tors} over $\mathbf{C}$, implies that the kernel and cokernel of α_{tors} over $\mathbf{R}$ are purely 2-torsion. In particular, the restriction

$$\alpha_{\mathrm{tors,div}}\colon\ A_0(X)_{\mathrm{tors,div}} \to \mathrm{Alb}(X)_{\mathrm{tors,div}}$$

to the maximal divisible torsion subgroups is surjective. Since both groups are isomorphic to $(\mathbf{Q}/\mathbf{Z})^q$ one might be led to hope that at least $\alpha_{\mathrm{tors,div}}$ is an isomorphism, but even this analogue of Roitman's theorem fails: see again Example 5.1.

In this paper we will see that over $\mathbf{R}$ the torsion Abel–Jacobi map α_{tors} is determined by the étale cohomology of X with coefficients in $\mathbf{Q}/\mathbf{Z}(j) = \varinjlim_n \mu_n^{\otimes j}$ for $j \in \mathbf{Z}$. More precisely, we have the following result, which is a combination of Theorem 3.3.i and Proposition 4.1.

THEOREM. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$ of dimension d.*

(i) *The image of α_{tors} is canonically isomorphic to the image of the natural mapping*

$$H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d)) \to H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d))^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})}.$$

(ii) *The kernel of $\alpha_{\mathrm{tors,div}}$ is isomorphic to the cokernel of the natural mapping*

$$H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d+1)) \to H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d+1))^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})}.$$

Given the image of α_{tors}, the kernel of $\alpha_{\mathrm{tors,div}}$, and the number of connected components of $X(\mathbf{R})$, it is of course a simple calculation to find the $\mathbf{Z}/2$-dimension of the kernel of α_{tors}. The formula is given in Theorem 3.3.iii.

My method of proving the above results is strongly inspired by Bloch's proof of Roitman's theorem as presented in [**CT93**]. There the key step is the construction of a cohomological Abel–Jacobi map

$$\lambda_i\colon\ CH^i(X)_{\mathrm{tors}} \to H^{2i-1}(X, \mathbf{Q}/\mathbf{Z}(d))$$

for the torsion of the Chow group of cycles of codimension i. This map is then shown to be an isomorphism in the case of zero-cycles (i.e., $i = d$) by inspection of the Bloch–Ogus spectral sequence.

When the ground field is not algebraically closed, the construction of λ_i usually runs into trouble. We will see in Section 3.1, however, that over $\mathbf{R}$ there are no problems, thanks to the finiteness of the absolute Galois group. Contrary to the case of an algebraically closed field, we do not get that λ_d is an isomorphism, but an analysis of the upper part of the Bloch–Ogus spectral sequence carried out in Section 2 does give us a short exact sequence

$$0 \to CH_0(X)_{\mathrm{tors}} \xrightarrow{\lambda_d} H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d)) \longrightarrow \bigoplus_{i>0} H^{d-2i}(X(\mathbf{R}), \mathbf{Z}/2) \to 0$$

(see Theorem 3.2), and also the fact that the image of λ_d maps surjectively onto the image of the base change map from $H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d))$ to $H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d))^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})}$. From these two facts it is not so hard to derive the main theorem.

STRUCTURE OF THE PAPER. In the first section I will present a few existing results on real zero-cycles and I will discuss some of the special features of étale cohomology and the Bloch–Ogus spectral sequence for varieties over $\mathbf{R}$.

Section 2 contains a new result on the Bloch–Ogus spectral sequence for varieties over **R**. Only Corollary 2.2 will be used in the rest of the paper.

The cohomological Abel–Jacobi map will be constructed and applied to the study of the real Abel–Jacobi map in Section 3 with Theorem 3.2 and Theorem 3.3 as main results. Also the filtration on $CH_0(X)_{\mathrm{tors}}$ induced by the cohomological Abel–Jacobi map and the Hochschild–Serre spectral sequence is introduced.

In Section 4 some of the criteria of Theorem 3.3 are translated into criteria that are easier to check in practice. This section is by nature quite technical; from a conceptual point of view, it does not add anything to the insight provided by Section 3. The reader might prefer to skip it at first reading (or to read the introduction only) and go directly to the examples of Section 5 where the criteria of Section 4 are actually put to work.

REMARK. Undoubtedly, all results in this paper will be valid over arbitrary real closed fields when rephrased properly, and the proofs should be relatively straightforward to adapt. However, I lacked the courage and the expertise to carry this out myself.

ACKNOWLEDGEMENTS. I would like to thank J.-L. Colliot-Thélène and C. Scheiderer for the stimulating discussions. I also would like to thank the latter for his help with the historical remarks and the references to the literature in the first two sections. Finally, I would like to thank the referees for the careful reading and detailed comments that helped me to improve the exposition, and for pointing out an error in the original version of Example 5.1.

1. Known results

This section contains an overview of some known results on zero-cycles, Albanese varieties and étale cohomology over the real numbers. Throughout the paper a *variety* will be a reduced, separated, but not necessarily irreducible scheme of finite type over a field.

SOURCES AND FURTHER READING. The theory of real abelian varieties as used in Section 1.1 is essentially due to Comessatti and put into a modern framework by Silhol; the reader is referred to [**Si89**, Chap. IV] for more information. The results on divisors on nonsingular projective curves over **R** in Section 1.2 go back to Harnack, Weichold, Klein and Witt; see [**PW91**] for a historical overview with precise references, and also for results concerning non-projective and/or singular curves. In [**CTS96**] the reader can find more about the Chow group of zero-cycles of (not necessarily complete or nonsingular) varieties of higher dimensions.

The relation between étale cohomology and the equivariant cohomology of the set of complex points of a variety over **R** discussed in Section 1.5 seems to be established first in [**Cox79**]. The relation between equivariant cohomology and the cohomology of the set of fixed points for an action of a group of prime order lies at the very origin of the equivariant cohomology theory, as developed by A. Borel (see [**Bo60**], [**Hs75**], [**AP93**]) after pioneering work of P.A. Smith. For us it will be convenient to follow Grothendieck's algebraic approach as described in [**Gr57**, Ch. V], rather than using Borel's geometric definition. In [**Sch94**] the equivariant cohomology theory is set up in the context of topos theory, allowing in particular to obtain analogues of the results in Sections 1.5 and 1.6 for varieties over arbitrary real closed fields. For the readers who wish to compare the topological and the topos-theoretic approach I have included references to analogous results in [**Sch94**] whenever I was able to locate them. A basic paper on the application of equivariant cohomology of $X(\mathbf{C})$ to real algebraic geometry is [**Kr83**]; see also [**Si89**, Ch. I]. The treatment of the

Bloch–Ogus spectral sequence over the real numbers in Section 1.7 is based on [**Sch94**, Ch. 19].

1.1. Albanese varieties over R. The Albanese variety $\mathrm{Alb}(X)$ of a nonsingular projective geometrically irreducible variety X over $\mathbf{R}$ is an abelian variety over $\mathbf{R}$ admitting a universal regular Galois-equivariant homomorphism from the group of zero-cycles of degree 0 on $X_{\mathbf{C}}$ into the set of complex points of $\mathrm{Alb}(X)$ (see for example [**Mu94**] for the definition of a regular homomorphism, and [**Ra98**] for a modern definition in terms of flat sheaves over the ground field). The universal regular homomorphism induces the Abel–Jacobi map

$$\alpha\colon A_0(X) \to \mathrm{Alb}(X)(\mathbf{R}),$$

that we considered in the introduction.

We have a canonical isomorphism

$$\mathrm{Alb}(X)(\mathbf{C}) \simeq \mathrm{Hom}(H^0(X(\mathbf{C}),\Omega^1),\mathbf{C})/(H_1(X(\mathbf{C}),\mathbf{Z})/\mathrm{tors})$$

of complex tori with an anti-holomorphic involution. Here the involution on $\mathrm{Hom}(H^0(X(\mathbf{C}),\Omega^1),\mathbf{C})$ and on $H_1(X(\mathbf{C}),\mathbf{Z})$ is induced by the anti-holomorphic involution on $X(\mathbf{C})$ and complex conjugation on $\mathbf{C}$. Taking $q = \dim_{\mathbf{C}} H^0(X(\mathbf{C}),\Omega^1)$ and $\Lambda = H_1(X(\mathbf{C}),\mathbf{Z})/\mathrm{tors}$, we get that

$$\mathrm{Alb}(X)(\mathbf{R}) \simeq (\mathbf{C}^q/\Lambda)^G.$$

From the theory of lattices with a G-action we see that Λ is of the form

$$\Lambda \simeq \mathbf{Z}^a \times \mathbf{Z}(1)^a \times \mathbf{Z}[G]^{q-a},$$

for some $0 \le a \le q$. Here $\mathbf{Z}(1)$ is $\mathbf{Z}$ with the involution $z \mapsto -z$, and $\mathbf{Z}[G]$ is the free abelian group generated by the elements of G with G acting in the obvious way. Since $(\mathbf{R}[G]/\mathbf{Z}[G])^G \simeq \mathbf{R}/\mathbf{Z}$ and $(\mathbf{R}(1)/\mathbf{Z}(1))^G \simeq \mathbf{Z}/2$, we get an isomorphism

$$\mathrm{Alb}(X)(\mathbf{R}) \simeq (\mathbf{R}/\mathbf{Z})^q \times (\mathbf{Z}/2)^a.$$

It is clear from the above construction that the torsion of $\mathrm{Alb}(X)(\mathbf{R})$ is isomorphic to $H_1(X(\mathbf{C}),\mathbf{Q}/\mathbf{Z})^G$; Poincaré duality gives an isomorphism

$$\mathrm{Alb}(X)(\mathbf{R})_{\mathrm{tors}} \simeq H^{2d-1}(X(\mathbf{C}),\mathbf{Q}/\mathbf{Z}(d))^G,$$

where $d = \dim(X)$ and $\mathbf{Q}/\mathbf{Z}(d)$ is $\mathbf{Q}/\mathbf{Z}$ with involution $x \mapsto (-1)^d x$. The d-fold twist of the coefficients comes from the fact that the fundamental class of $X(\mathbf{C})$ lies in the G-invariant part of $H_{2d}(X(\mathbf{C}),\mathbf{Z}(d))$, since complex conjugation on $X(\mathbf{C})$ preserves the orientation if and only if d is even.

1.2. Divisors on curves. Zero-cycles on nonsingular curves are divisors, so the Albanese variety of a nonsingular projective geometrically irreducible curve X coincides with the Picard variety; it is better known as the Jacobian variety $\mathrm{Jac}(X)$ of X. From general theory (valid over arbitrary fields) it follows that the mapping

$$\alpha\colon A_0(X) \to \mathrm{Jac}(X)(\mathbf{R})$$

is injective for X as above and surjective if $X(\mathbf{R}) \neq \emptyset$.

If X has genus g and $X(\mathbf{R})$ has $s > 0$ connected components, then

$$A_0(X) \simeq \mathrm{Jac}(X)(\mathbf{R}) \simeq (\mathbf{R}/\mathbf{Z})^g \times (\mathbf{Z}/2)^{s-1}.$$

More precisely, we have that the class of a divisor Z of degree zero lies in the maximal divisible subgroup of $A_0(X)$ if and only if Z is an integral linear combination of

(i) real points with multiplicity 2,

(ii) closed points with complex residue field,
(iii) divisors of the form $[P_1]-[P_2]$ with P_1 and P_2 real points in the same connected component of $X(\mathbf{R})$.

This also holds when $X(\mathbf{R})=\emptyset$, so then

$$A_0(X)\simeq(\mathbf{R}/\mathbf{Z})^g.$$

However, in this case we actually have

$$\mathrm{Jac}(X)(\mathbf{R})\simeq\begin{cases}(\mathbf{R}/\mathbf{Z})^g & \text{if } g \text{ is even,}\\ (\mathbf{R}/\mathbf{Z})^g\times\mathbf{Z}/2 & \text{if } g \text{ is odd,}\end{cases}$$

so when g is odd and X has no real points, then the mapping α has cokernel $\mathbf{Z}/2$. Otherwise α is surjective.

1.3. Zero-cycles on higher dimensional varieties. For a nonsingular projective geometrically irreducible variety X over $\mathbf{R}$ of arbitrary dimension it was shown in [**CTI81**], using the calculation for curves given above, that $A_0(X)$ modulo the maximal divisible subgroup $A_0(X)_{\mathrm{div}}$ is a $\mathbf{Z}/2$-vector space of dimension $s-1$ if $X(\mathbf{R})$ has $s>0$ connected components, and that $A_0(X)$ is divisible if X has no real points. In fact, the criteria for a zero-cycle to be divisible modulo rational equivalence are precisely the same as the criteria for a divisor on a curve given above.

In [**CTS96**] it was shown using Roitman's theorem over $\mathbf{C}$ and a trace argument, that the torsion of $A_0(X)_{\mathrm{div}}$ is isomorphic to $(\mathbf{Q}/\mathbf{Z})^q$ with $q=\dim_{\mathbf{R}}H^1(X,\mathcal{O}_X)$. Together with the above expression for $A_0(X)/A_0(X)_{\mathrm{div}}$, this gives the formula of the introduction:

$$A_0(X)_{\mathrm{tors}}\simeq(\mathbf{Q}/\mathbf{Z})^q\times(\mathbf{Z}/2)^{s-1}$$

when $X(\mathbf{R})\neq\emptyset$ (see [**CTS96**, Th. 1.6.b]). The group $A_0(X)/A_0(X)_{\mathrm{tors}}$ is uniquely divisible, which follows from the analogous statement for varieties over $\mathbf{C}$ and a trace argument (see [**CTS96**, Th. 1.3]).

1.4. Étale and Galois cohomology. The group $G=\mathrm{Gal}(\mathbf{C}/\mathbf{R})=\mathbf{Z}/2$ does not have finite cohomological dimension for 2-torsion coefficients, so $\operatorname{Spec}\mathbf{R}$ does not have finite étale cohomological dimension, nor does any variety X over $\mathbf{R}$ with $X(\mathbf{R})\neq\emptyset$.

As a graded ring, the cohomology ring $H^*(G,\mathbf{Z})$ is isomorphic to the (commutative!) ring $\mathbf{Z}[\eta^2]/(2\eta^2)$, where η^2 has degree 2. Also, $H^*(G,\mathbf{Z}/2)\simeq\mathbf{Z}/2[\eta]$, with η of degree 1. The notation already indicates what the natural map $H^*(G,\mathbf{Z})\to H^*(G,\mathbf{Z}/2)$ looks like. The cohomology of G is *periodic*: cup product with η^2 induces for any $q\geq 0$ and any G-module M a surjection

$$H^q(G,M)\to H^{q+2}(G,M),$$

which is an isomorphism if $q>0$. If M is a $\mathbf{Z}/2$-module, cup product with η induces for any $q\geq 0$ a surjection $H^q(G,M)\to H^{q+1}(G,M)$ which is an isomorphism if $q>0$.

For the étale cohomology of a variety X over $\mathbf{R}$ this implies that when $q>2\dim(X)$ we have for any torsion sheaf $\mathcal{F}$ on X that $H^q_{\text{ét}}(X,\mathcal{F})\simeq H^{q+2}_{\text{ét}}(X,\mathcal{F})$, and even $H^q_{\text{ét}}(X,\mathcal{F})\simeq H^{q+1}_{\text{ét}}(X,\mathcal{F})$ if $\mathcal{F}$ is a 2-torsion sheaf, as can be seen from the *Hochschild–Serre spectral sequence*

$$E_2^{p,q}=H^p(G,H^q_{\text{ét}}(X_{\mathbf{C}},\mathcal{F}))\Rightarrow H^{p+q}_{\text{ét}}(X,\mathcal{F})$$

(see Section 1.6 for a similar result with more details of the proof). In other words, the étale cohomology of varieties over $\mathbf{R}$ is stable in degree $>2\dim(X)$. This stable part has a very natural interpretation in terms of the cohomology of $X(\mathbf{R})$. For example, in the case $\mathcal{F} = \mathbf{Z}/2$ we have for every $q > 2\dim(X)$ an isomorphism

$$H^q_{\text{ét}}(X, \mathbf{Z}/2) \simeq \bigoplus_{i=0}^{\dim X} H^i(X(\mathbf{R}), \mathbf{Z}/2),$$

which was first constructed by D. Cox in [**Cox79**]. In the next sections we will see how this isomorphism can be obtained from a comparison theorem between étale cohomology over $\mathbf{R}$ and equivariant cohomology, combined with a localization theorem in equivariant cohomology.

1.5. Comparison between étale and equivariant cohomology. To any abelian torsion group M with an action of $G = \mathbf{Z}/2$ we can associate on the one hand a locally constant sheaf on the étale site of a variety X over $\mathbf{R}$, and on the other hand a locally constant G-sheaf on the space $X(\mathbf{C})$, equipped with the euclidean topology and the canonical $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$-action. It was first proved in [**Cox79**] using étale homotopy theory that for every $q \geq 0$ there is an isomorphism

$$H^q_{\text{ét}}(X, M) \simeq H^q(X(\mathbf{C}); G, M). \tag{1}$$

Here the right hand side denotes equivariant cohomology in the sense of either Grothendieck (see [**Gr57**, Ch. V]) or Borel (see [**Bo60**]) — both theories are equivalent since G is finite. In fact, if we take Grothendieck's definition of $H^q(X(\mathbf{C}); G, -)$ as the qth right derived functor of taking G-invariant global sections, then the isomorphism (1) follows directly from the usual comparison between the étale cohomology of $X_{\mathbf{C}}$ and the cohomology of $X(\mathbf{C})$ as given in [**SGA4**, Exp. XVI] and the fact that étale cohomology of $\operatorname{Spec}\mathbf{R}$ corresponds to Galois cohomology (see also [**Sch94**, (15.3)], [**Ni94**], and the comments in [**CTS96**, §2.3]).

Observe that the G-sheaf associated to the G-module $\mathbf{Q}/\mathbf{Z}(j)$ defined in Section 1.1 corresponds to the étale sheaf $\mathbf{Q}/\mathbf{Z}(j) = \varinjlim_n \mu_n^{\otimes j}$.

1.6. Equivariant cohomology. We will consider equivariant sheaf cohomology, as defined in [**Gr57**, Ch. V], for $G = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acting on $V = X(\mathbf{C})$ or $X(\mathbf{R})$, where X is a variety over $\mathbf{R}$. Coefficients will always be taken in (the locally constant G-sheaf on V associated to) a torsion G-module M, and we will mainly concentrate on the cases $M = \mathbf{Z}/2$ or $\mathbf{Q}/\mathbf{Z}(j)$. We denote by $i\colon V^G \hookrightarrow V$ the inclusion of the set of fixed points (note that here $V^G = X(\mathbf{R})$). By $\pi\colon V \to V/G$ we denote the quotient map.

By [**Gr57**, Th. 5.2.1] we have two spectral sequences converging to equivariant cohomology: the *Hochschild–Serre spectral sequence*

$$E_2^{p,q} = H^p(G, H^q(V, M)) \Rightarrow H^{p+q}(V; G, M), \tag{2}$$

(in this context also known as the *Borel–Serre spectral sequence*) and the spectral sequence

$$E_2^{p,q} = H^p(V/G, \mathcal{H}^q(G, M)) \Rightarrow H^{p+q}(V; G, M), \tag{3}$$

where $\mathcal{H}^q(G, M)$ is the sheaf on V/G associated to the presheaf

$$U \mapsto H^q(G, \pi_* M(U)),$$

by [**Gr57**, Prop. 5.2.2]. It is not hard to check (see [**Gr57**, Th. 5.3.1]) that for $q > 0$ the support of $\mathcal{H}^q(G, M)$ is contained in $\pi(V^G)$, so that $H^p(V/G, \mathcal{H}^q(G, M)) \simeq H^p(V^G, \mathcal{H}^q(G, M))$ for $q > 0$. Hence the morphism of spectral sequences (3) that corresponds to the restriction homomorphism

$$i^*\colon H^n(X(\mathbf{C}); G, M) \to H^n(X(\mathbf{R}); G, M) \tag{4}$$

is an isomorphism on the $E_2^{p,q}$-level for $q>0$.

1.6.1. *Localization.* Since $X(\mathbf{C})/G$ has finite cohomological dimension (see [**Qu71**, Prop. A.11], compare [**Sch94**, Cor. 7.18]), we have that the $E_2^{p,0}$-term $H^p(V/G,\mathcal{H}^0(G,M))$ of spectral sequence (3) is zero for $V=X(\mathbf{C})$ or $X(\mathbf{R})$, and all sufficiently large p. Hence the restriction map (4) is an isomorphism for all sufficiently large n.

In the modern theory of transformation groups this is usually rephrased in terms of *localization* (compare [**Qu71**, §4], [**Hs75**, §III.2]). First we consider the case $M=\mathbf{Z}/2$. Let $H^*(V;G,\mathbf{Z}/2)[\eta^{-1}]$ be the localization of the $H^*(G,\mathbf{Z}/2)$-module $H^*(V;G,\mathbf{Z}/2)$ with respect to the multiplicative subset of $H^*(G,\mathbf{Z}/2)=\mathbf{Z}/2[\eta]$ generated by η. Then the above implies that the restriction map induces an isomorphism of graded $\mathbf{Z}/2[\eta,\eta^{-1}]$-algebras

$$H^*(X(\mathbf{C});G,\mathbf{Z}/2)[\eta^{-1}]\xrightarrow{\sim}H^*(X(\mathbf{R});G,\mathbf{Z}/2)[\eta^{-1}] \tag{5}$$

(compare [**Sch94**, Cor. 7.19]).

In the case $M=\mathbf{Q}/\mathbf{Z}(j)$ for some $j\in\mathbf{Z}$ we localize with respect to the multiplicative subset of $H^*(G,\mathbf{Z})=\mathbf{Z}[\eta^2]/(2\eta^2)$ generated by η^2. Note that $H^*(G,\mathbf{Z})[\eta^{-2}]\simeq\mathbf{Z}/2[\eta^2,\eta^{-2}]$. Hence the isomorphism

$$H^*(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j))[\eta^{-2}]\xrightarrow{\sim}H^*(X(\mathbf{R});G,\mathbf{Q}/\mathbf{Z}(j))[\eta^{-2}] \tag{6}$$

induced by the restriction map is an isomorphism of graded $\mathbf{Z}/2[\eta^2,\eta^{-2}]$-modules.

Later we will also need to know that the natural homomorphisms

$$H^n(X(\mathbf{C});G,\mathbf{Z}/2)\underset{\mathbf{Z}/2}{\otimes}H^*(G,\mathbf{Z}/2)[\eta^{-1}]\to H^*(X(\mathbf{C});G,\mathbf{Z}/2)[\eta^{-1}] \tag{7}$$

and

$$H^n(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j))\underset{\mathbf{Z}}{\otimes}H^*(G,\mathbf{Z})[\eta^{-2}]\to H^{n \pmod 2}(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j))[\eta^{-2}] \tag{8}$$

are isomorphisms for n sufficiently large. We obtain good lower bounds for n by considering the Hochschild–Serre spectral sequence (2). For $M=\mathbf{Z}/2$ we have that on the $E_2^{p,q}$-level the cup product with η coincides with the homomorphism $E_2^{p,q}\to E_2^{p+1,q}$ given by cup product with η in Galois cohomology. Hence the periodicity of the cohomology of G implies that if for some $D\geq 0$ we have that $H^q(X(\mathbf{C}),\mathbf{Z}/2)=0$ for all $q>D$, then cup product with η induces a surjection

$$H^n(X(\mathbf{C});G,\mathbf{Z}/2)\to H^{n+1}(X(\mathbf{C});G,\mathbf{Z}/2)$$

for $n\geq D$ which is an isomorphism if $n>D$. We can always take $D=2\dim(X)$, and even $D=\dim(X)$ if X is affine. We obtain that with such D the homomorphism (7) is a surjection for $n\geq D$ and an isomorphism if $n>D$ (compare [**Sch94**, Cor. 7.19, Cor. 7.20]). By similar reasoning for $M=\mathbf{Q}/\mathbf{Z}(j)$ we obtain the same result for the homomorphism (8).

1.6.2. *Cohomology of the set of real points.* We will now analyse the equivariant cohomology of $X(\mathbf{R})$. For $M=\mathbf{Z}/2$ the $E_2^{p,q}$-terms of the spectral sequence (3) for $V=X(\mathbf{R})$ have the form

$$E_2^{p,q}=H^p(X(\mathbf{R}),\mathbf{Z}/2)$$

for all $p,q\geq 0$. The edge morphisms $E_2^{p,0}=H^p(X(\mathbf{R}),\mathbf{Z}/2)\to H^p(X(\mathbf{R});G,\mathbf{Z}/2)$ for $p\geq 0$, and the cup product induce an isomorphism

$$H^*(X(\mathbf{R}),\mathbf{Z}/2)\underset{\mathbf{Z}/2}{\otimes}H^*(G,\mathbf{Z}/2)\xrightarrow{\sim}H^*(X(\mathbf{R});G,\mathbf{Z}/2) \tag{9}$$

of graded $H^*(G,\mathbf{Z}/2)$-algebras (see [**Gr57**, Th. 4.4.1, Cor. 5.4.1], compare [**Sch94**, Cor. 6.3.2]). In particular, the spectral sequence (3) is trivial for $V = X(\mathbf{R})$ and $M = \mathbf{Z}/2$ (see [**Gr57**, Th. 4.4.1]).

Let $\mathfrak{m} \subset H^*(G,\mathbf{Z}/2)$ be the maximal ideal generated by $(1-\eta)$. Combining the isomorphisms (9) and (5) we get an isomorphism of rings (not of graded rings)

$$H^*(X(\mathbf{C});G,\mathbf{Z}/2)/\mathfrak{m}H^*(X(\mathbf{C});G,\mathbf{Z}/2) \xrightarrow{\sim} H^*(X(\mathbf{R}),\mathbf{Z}/2)$$

(compare [**Sch94**, Cor. 7.19]). The composition of this isomorphism with the quotient map will be denoted by

$$\beta = \beta_{\mathbf{Z}/2}\colon\ H^*(X(\mathbf{C});G,\mathbf{Z}/2) \to H^*(X(\mathbf{R}),\mathbf{Z}/2).$$

Note that our results on the map (7) imply that the map

$$H^n(X(\mathbf{C});G,\mathbf{Z}/2) \to H^*(X(\mathbf{R}),\mathbf{Z}/2)$$

obtained by restricting β to the cohomology in degree n is surjective for $n \geq 2\dim(X)$ (resp. $n \geq \dim(X)$ if X is affine) and bijective for $n > 2\dim(X)$ (resp. $n > \dim(X)$ if X is affine), compare [**Sch94**, Cor. 7.20]. Under the comparison isomorphism (1) this map corresponds for $n > 2\dim(X)$ to the isomorphism given in Section 1.4.

For $M = \mathbf{Q}/\mathbf{Z}(j)$ with $j \in \mathbf{Z}$ we have that the $E_2^{p,q}$-terms of the spectral sequence (3) for $V = X(\mathbf{R})$ have the following form:

$$E_2^{p,q} = \begin{cases} H^p(X(\mathbf{R}),\mathbf{Q}/\mathbf{Z}) & \text{if } q = 0 \text{ and } j \text{ is even,} \\ H^p(X(\mathbf{R}),\mathbf{Z}/2) & \text{if } q \geq 0 \text{ and } j \not\equiv q \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

The coefficient map $\mathbf{Z}/2 \to \mathbf{Q}/\mathbf{Z}(j)$ induces a homomorphism of $E_2^{p,q}$-terms of the spectral sequence (3) which is an isomorphism for all $p \geq 0$, $q \geq 0$ and $j \not\equiv q \pmod 2$. Therefore the spectral sequence (3) is trivial for $M = \mathbf{Q}/\mathbf{Z}(j)$ and $V = X(\mathbf{R})$, and the isomorphism (9) combined with the coefficient map $\mathbf{Z}/2 \to \mathbf{Q}/\mathbf{Z}(j)$ induces for $n > \dim(X)$ an isomorphism

$$\bigoplus_{i \not\equiv n+j \pmod 2} H^i(X(\mathbf{R}),\mathbf{Z}/2) \xrightarrow{\sim} H^n(X(\mathbf{R});G,\mathbf{Q}/\mathbf{Z}(j)).$$

Hence, taking $\mathfrak{m}_{\mathbf{Z}} \subset H^*(G,\mathbf{Z})$ to be the maximal ideal generated by $(1-\eta^2)$, we get an isomorphism of groups

$$H^*(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j))/\mathfrak{m}_{\mathbf{Z}}H^*(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j)) \xrightarrow{\sim} H^*(X(\mathbf{R}),\mathbf{Z}/2).$$

which in this case preserves (resp. reverses) the natural $\mathbf{Z}/2$-gradings when j is odd (resp. even). Note that here the grading modulo 2 on the equivariant cohomology of $X(\mathbf{C})$ descends to the quotient since $\mathfrak{m}_{\mathbf{Z}}$ is generated by the element $1-\eta^2$, which is purely of even degree. The composition of the above isomorphism with the quotient map will be denoted by

$$\beta = \beta_{\mathbf{Q}/\mathbf{Z}(j)}\colon\ H^*(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j)) \to H^*(X(\mathbf{R}),\mathbf{Z}/2).$$

The restriction of β to the cohomology in degree n gives a surjection

$$H^n(X(\mathbf{C});G,\mathbf{Q}/\mathbf{Z}(j)) \to \bigoplus_{i \not\equiv n+j \pmod 2} H^i(X(\mathbf{R}),\mathbf{Z}/2)$$

for $n \geq 2\dim(X)$ (resp. $n \geq \dim(X)$ if X is affine) which is an isomorphism if $n > 2\dim(X)$ (resp. $n > \dim(X)$ if X is affine); compare [**Sch94**, Th. 20.2.11, Th. 20.2.13]).

The notations β, $\beta_{\mathbf{Z}/2}$, and $\beta_{\mathbf{Q}/\mathbf{Z}(j)}$ will also be used for the composition of the comparison isomorphism (1) with the mappings defined above.

REMARK 1.1. In order to give some idea of the nature of the mappings β, let me mention here that for X nonsingular the étale cycle map $\mathrm{cl}\colon \mathcal{Z}^q(X) \to H^{2q}_{\text{ét}}(X, \mathbf{Z}/2)$ composed with $\beta_{\mathbf{Z}/2}$ and the ith projection gives for every pair i, $q \geq 0$ a mapping

$$\mathcal{Z}^q(X) \to H^i(X(\mathbf{R}), \mathbf{Z}/2)$$

which is zero outside the range $q \leq i \leq 2q$, and which coincides for $i = q$ with the cohomological real algebraic cycle map

$$\mathrm{cl}_{\mathbf{R}}\colon \mathcal{Z}^q(X) \to H^q(X(\mathbf{R}), \mathbf{Z}/2)$$

that sends a subvariety $V \subset X$ of codimension q to the Poincaré dual of the class in the 'Borel–Moore' homology group $H_{\dim(X)-q}(X(\mathbf{R}), \mathbf{Z}/2)$ represented by the set of real points $V(\mathbf{R})$, as defined by Borel and Haefliger in [**BH61**]. See [**Kr94**, Th. 0.6], [**Sch95**, Rem. 3.5], or see [**vH96**, §§5.1, 5.2] for a purely topological proof.

1.7. The Bloch–Ogus spectral sequence. The mapping from the étale site to the Zariski site induces for any variety X over $\mathbf{R}$ and any étale sheaf $\mathcal{F}$ on X the *local–to–global spectral sequence*

$$E_2^{p,q} = H^p_{\text{Zar}}(X, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathcal{F}), \tag{10}$$

where $\mathcal{H}^q(\mathcal{F})$ is the Zariski sheaf associated to the presheaf

$$U \mapsto H^q_{\text{ét}}(U, \mathcal{F}).$$

For X nonsingular and $\mathcal{F}$ a locally constant torsion sheaf this spectral sequence is often called the *Bloch–Ogus spectral sequence*, since (by [**BO74**, Rem. 6.4]) it coincides with the coniveau spectral sequence studied by Bloch and Ogus in their fundamental paper [**BO74**]. They showed in particular that with the above hypotheses on X and $\mathcal{F}$ the groups $H^p_{\text{Zar}}(X, \mathcal{H}^q(\mathcal{F}))$ vanish for all $p > q$. In the rest of this section we will not need to assume that X is nonsingular.

For X over an algebraically closed field and $\mathcal{F}$ a torsion sheaf we have that $\mathcal{H}^q(\mathcal{F}) = 0$ for $q > \dim(X)$. In view of Section 1.4, this does not hold for X over $\mathbf{R}$ with $X(\mathbf{R})$ nonempty and $\mathcal{F}$ not 2-divisible. However, Colliot-Thélène and Parimala showed in [**CTP90**] for X nonsingular that the sheaves $\mathcal{H}^q(\mathbf{Z}/2)$ do admit a simple description for $q > \dim(X)$. Scheiderer then showed how to derive their result (and the analogue when X is singular) from the cohomological facts presented in Sections 1.5 and 1.6. Moreover, his approach, which we will follow here, immediately gave an easy description of the cohomology groups $H^p(X, \mathcal{H}^q(\mathbf{Z}/2))$ for $q > \dim(X)$, showing that these 'extra' $E_2^{p,q}$-terms of the local–to–global spectral sequence over $\mathbf{R}$ are not so hard to understand.

The homomorphism $\beta_{\mathbf{Z}/2}$ defined in Section 1.6.2 induces for $q > d$ an isomorphism $\tilde{\beta}_{\mathbf{Z}/2}$ from $\mathcal{H}^q(\mathbf{Z}/2)$ to the Zariski-sheaf associated to the presheaf

$$U \mapsto \bigoplus_i H^i(U(\mathbf{R}), \mathbf{Z}/2)$$

and $\beta_{\mathbf{Q}/\mathbf{Z}(j)}$ induces for $q > d$ an isomorphism $\tilde{\beta}_{\mathbf{Q}/\mathbf{Z}(j)}$ from $\mathcal{H}^q(\mathbf{Q}/\mathbf{Z}(j))$ to the Zariski-sheaf associated to the presheaf

$$U \mapsto \bigoplus_{i \not\equiv q+j \pmod 2} H^i(U(\mathbf{R}), \mathbf{Z}/2).$$

It is a remarkable fact that the graded pieces of the above presheaves of degree $i > 0$ vanish after sheafifying for the Zariski topology (Lemma 1.2.i). This observation is due to Scheiderer and it was the key result that allowed him to give a different proof of the result of Colliot-Thélène and Parimala, as well as the easy description referred to above of the higher $E_2^{p,q}$-terms in the local–to–global spectral sequence (see Corollary 1.3).

LEMMA 1.2. *Let X be a variety over $\mathbf{R}$. Let $\varphi\colon X(\mathbf{R}) \to X_{\mathrm{Zar}}$ be the obvious mapping of topological spaces. Let $\mathcal{F}$ be a locally constant sheaf on $X(\mathbf{R})$.*

(i) *For $i > 0$ we have*

$$R^i\varphi_*\mathcal{F} = 0.$$

(ii) *For any $p \geq 0$ we have*

$$H^p_{\mathrm{Zar}}(X, \varphi_*\mathcal{F}) = H^p(X(\mathbf{R}), \mathcal{F}).$$

PROOF. (i) This is the analogue of [**Sch94**, Th. 19.2] for the euclidean topology, so it follows from comparison between the cohomology of $X(\mathbf{R})$ and the cohomology of the real spectrum of X. Scheiderer's earlier approach (in [**Sch90**]) gives a direct proof: it is sufficient to find for every Zariski-neighbourhood U of $P \in X(\mathbf{R})$ a smaller Zariski-open neighbourhood $V \subset U$ of P, such that the restriction

$$H^i(U(\mathbf{R}), \mathcal{F}) \to H^i(V(\mathbf{R}), \mathcal{F})$$

is zero. Such a V is easily constructed from the data that give a semi-algebraic triangulation of $U(\mathbf{R})$ (see [**BCR87**, Th. 9.2.1]).

(ii) This follows from the first statement by general homological algebra. □

COROLLARY 1.3. *Let X be a variety over $\mathbf{R}$ of dimension d. Let $q > d$.*

(i) *The homomorphism $\beta_{\mathbf{Z}/2}$ induces an isomorphism*

$$\tilde{\beta}_{\mathbf{Z}/2}\colon \mathcal{H}^q(\mathbf{Z}/2) \xrightarrow{\sim} \varphi_*\mathbf{Z}/2,$$

hence for any $p \geq 0$ an isomorphism

$$H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Z}/2)) \xrightarrow{\sim} H^p(X(\mathbf{R}), \mathbf{Z}/2).$$

(ii) *For $j \in \mathbf{Z}$ the homomorphism $\beta_{\mathbf{Q}/\mathbf{Z}(j)}$ induces an isomorphism*

$$\tilde{\beta}_{\mathbf{Q}/\mathbf{Z}(j)}\colon \mathcal{H}^q(\mathbf{Q}/\mathbf{Z}(j)) \xrightarrow{\sim} \begin{cases} 0 & \text{if } q \equiv j \pmod 2, \\ \varphi_*\mathbf{Z}/2 & \text{if } q \not\equiv j \pmod 2, \end{cases}$$

hence for every $p \geq 0$ an isomorphism

$$H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Q}/\mathbf{Z}(j))) \xrightarrow{\sim} \begin{cases} 0 & \text{if } q \equiv j \pmod 2, \\ H^p(X(\mathbf{R}), \mathbf{Z}/2) & \text{if } q \not\equiv j \pmod 2. \end{cases}$$

(iii) *For $q \not\equiv j \pmod 2$ the isomorphisms $\tilde{\beta}_{\mathbf{Z}/2}$ and $\tilde{\beta}_{\mathbf{Q}/\mathbf{Z}(j)}$ are compatible with the homomorphism $\mathcal{H}^q(\mathbf{Z}/2) \to \mathcal{H}^q(\mathbf{Q}/\mathbf{Z}(j))$ induced by the coefficient map $\mathbf{Z}/2 \hookrightarrow \mathbf{Q}/\mathbf{Z}(j)$.*

PROOF. Immediate from Lemma 1.2, the above descriptions of $\mathcal{H}^q(\mathbf{Z}/2)$ and $\mathcal{H}^q(\mathbf{Q}/\mathbf{Z}(j))$, and the definition of $\beta_{\mathbf{Z}/2}$ and $\beta_{\mathbf{Q}/\mathbf{Z}(j)}$. □

The fact that $\mathcal{H}^q(\mathbf{Z}/2)$ is isomorphic to $\varphi_*\mathbf{Z}/2$ for $q > d$ is (for X nonsingular) the main theorem of Colliot-Thélène and Parimala (see [**CTP90**, Th. 2.3.1]). The full result is due to Scheiderer (see [**Sch94**, Prop. 19.4, Cor. 19.5.1]).

Of course, the relation between the higher $E_2^{p,q}$-terms of the local–to–global spectral sequence and the cohomology of the set of real points is not only induced by β on the local level, but globally as well. Heuristically, this means that the mapping β maps the $E_2^{p,q}$-terms for $q > \dim(X)$ isomorphically to $H^p(X(\mathbf{R}), \mathbf{Z}/2)$ or 0. In order to make this statement well-defined, we follow Scheiderer's approach.

PROPOSITION 1.4. *Let X be a variety defined over $\mathbf{R}$ of pure dimension d. Let M be a torsion G-module. The comparison isomorphism* (1) *and the inclusion $i\colon X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$ induce a morphism from the local–to–global spectral sequence*

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(M)) \Rightarrow H^{p+q}_{\text{ét}}(X, M),$$

to the spectral sequence (3) *for $V = X(\mathbf{R})$:*

$$E_2^{p,q} = H(X(\mathbf{R}), \mathcal{H}^q(G, M)) \Rightarrow H^{p+q}(X(\mathbf{R}); G, M).$$

On the limit terms this morphism is the comparison isomorphism (1) *followed by the restriction map* (4)*. When $M = \mathbf{Z}/2$ or $\mathbf{Q}/\mathbf{Z}(j)$ for some $j \in \mathbf{Z}$, then on the $E_2^{p,q}$-terms this morphism is for $p \geq 0$, and $q > d$ the isomorphism of Corollary 1.3.*

PROOF. This is the analogue of [**Sch94**, Prop. 19.7]. Let ψ_*^G be the functor sending a G-sheaf $\mathcal{F}$ of abelian groups on $X(\mathbf{C})$ to the sheaf on X_{Zar} associated to the presheaf

$$U \mapsto \mathcal{F}(U)^G.$$

The comparison between étale cohomology and equivariant cohomology gives that the spectral sequence (10) is canonically isomorphic to the spectral sequence associated to the decomposition of derived functors

$$R\Gamma^G_{X(\mathbf{C})} = R\Gamma_{X_{\mathrm{Zar}}} \circ R\psi_*^G, \tag{11}$$

with notations as in [**Gr57**, §5.2]. The inclusion $i\colon X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$ induces an isomorphism of functors

$$\psi_*^G \circ i_* = \varphi_* \circ \mathcal{H}^0(G, -),$$

with φ as in Lemma 1.2 and $\mathcal{H}^0(G,-)$ as in Section 1.6. Hence the canonical morphism $M \to i_* M$ of G-sheaves on $X(\mathbf{C})$ and the above isomorphisms induce a map from the local–to–global spectral sequence with coefficients in M into the spectral sequence associated to the decomposition of derived functors

$$R\Gamma^G_{X(\mathbf{R})} = R\Gamma_{X_{\mathrm{Zar}}} \circ R(\varphi_* \circ \mathcal{H}^0(G, -)) \tag{12}$$

applied to the G-sheaf M on $X(\mathbf{R})$. Since $\Gamma_{X_{\mathrm{Zar}}} \circ \varphi_* = \Gamma_{X(\mathbf{R})}$ and φ_* is exact by Lemma 1.2, the spectral sequence associated to (12) coincides with the spectral sequence (3) for $V = X(\mathbf{R})$, which is the spectral sequence associated to the decomposition

$$R\Gamma^G_{X(\mathbf{R})} = R\Gamma_{X(\mathbf{R})} \circ \mathcal{H}^0(G, -).$$

This proves the first two statements. The last statement follows immediately from the definition of β. □

2. Vanishing of differentials in the Bloch–Ogus spectral sequence

A crucial point in the study of the Bloch–Ogus spectral sequence over $\mathbf{R}$ is the vanishing of the differentials

$$d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r, q-r+1}$$

for large q. See also the historical remarks below. We will prove in this section that for X nonsingular all differentials with source $E_r^{p,q}$ are zero when $q > \dim(X)$ (part (i) of Theorem 2.1). In other words, the 'extra' $q > \dim(X)$ part of the Bloch–Ogus spectral sequence is completely degenerate. For many purposes, and certainly for our purposes here, this means that the Bloch–Ogus spectral sequence for varieties over $\mathbf{R}$ is no more complicated than the Bloch–Ogus spectral sequence over $\mathbf{C}$. Also, we will show that the $q > \dim(X)$ part maps to zero, under the pull-back mapping in étale cohomology associated to base change from $\mathbf{R}$ to $\mathbf{C}$. This is part (ii) of Theorem 2.1. In the rest of the paper we will not use the theorem in its full strength; we will only need Corollary 2.2.

The theorem is proved by means of an auxiliary degenerate spectral sequence that maps into the Bloch–Ogus spectral sequence. This will be the local–to–global spectral sequence for equivariant cohomology with supports in $X(\mathbf{R})$. It is defined and studied in Section 2.1. Then in Section 2.2 we study the map into the Bloch–Ogus spectral sequence, and prove the theorem.

It should be said that, using Proposition 1.4, it is actually not very hard to derive Theorem 2.1 from a result of Krasnov in [**Kr94**] concerning the image of the mapping β (Theorem 2.8 in the present paper). This, and further connections with existing results, will be treated in Section 2.3.

THEOREM 2.1. *Let X be a nonsingular variety over $\mathbf{R}$ of dimension d. Let $M = \mathbf{Z}/2$ or $\mathbf{Q}/\mathbf{Z}(j)$ for some $j \in \mathbf{Z}$ and let $n \geq 0$. Consider the Bloch–Ogus spectral sequence with coefficients in M, and let $F^\bullet$ be the associated descending filtration on $H^n_{\text{ét}}(X,M)$, with pth graded piece equal to $E_\infty^{p,n-p}$.*

(i) *For any $p \in \mathbf{Z}$, $q > d$ and $r \geq 2$ the differential $d_r^{p,q}\colon\ E_r^{p,q} \to E_r^{p+r,q-r+1}$ is zero.*

(ii) *The kernel of the canonical mapping*

$$\pi^*\colon\ H^n_{\text{ét}}(X,M) \to H^n_{\text{ét}}(X_{\mathbf{C}},M)$$

maps surjectively onto

$$H^n_{\text{ét}}(X,M)/F^{n-d}H^n_{\text{ét}}(X,M).$$

PROOF. See Section 2.2. □

COROLLARY 2.2. *Let X be a nonsingular variety over $\mathbf{R}$ of dimension d.*

(i) *We have a short exact sequence*

$$0 \to H^{d-1}_{\text{Zar}}(X,\mathcal{H}^d(\mathbf{Q}/\mathbf{Z}(d))) \to H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\beta'} \bigoplus_{i>0} H^{d-2i}(X(\mathbf{R}),\mathbf{Z}/2) \to 0,$$

where β' is the mapping β followed by the projection.

(ii) *The image of the base change map*

$$\pi^*\colon\ H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \to H^{2d-1}_{\text{ét}}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(d))$$

is generated by the image of $H^{d-1}_{\text{Zar}}(X,\mathcal{H}^d(\mathbf{Q}/\mathbf{Z}(d)))$.

PROOF. (i) The exactness follows from part (i) of the theorem and Proposition 1.4.

(ii) Part (ii) of the theorem says in the case of $n = 2d-1$ that the cokernel of the inclusion $H^{d-1}_{\text{Zar}}(X,\mathcal{H}^d(\mathbf{Q}/\mathbf{Z}(d))) \hookrightarrow H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d))$ maps to zero under π^*. □

HISTORICAL REMARKS. The first result on vanishing of differentials of high degree in the Bloch–Ogus spectral sequence over $\mathbf{R}$ is due to Colliot-Thélène and Parimala, who proved the vanishing of all differentials in the Bloch–Ogus spectral sequence for a non-singular geometrically irreducible surface over $\mathbf{R}$ having a compact set of real points (see [**CTP90**, Prop. 3.11]). Inspired by this result, Scheiderer proved for a (possibly singular) variety X of dimension d over an arbitrary real closed field the vanishing of all differentials having target $E_r^{p,q}$ with $r \geq 2$, $p+q \geq 2d$ and $(p,q) \neq (d,d)$ (see [**Sch94**, Prop. 19.8]). Together with Colliot-Thélène he improved the bounds on (p,q) to $p \in \mathbf{Z}$ and $q > d$ (see [**CTS96**, Th. 3.1.b, Rem. 3.1.1]). For X nonsingular they proved in the same paper that the differentials having target $E_r^{d,d}$ vanish as well (see [**CTS96**, Th. 3.2.b]). Actually, all these results are stated for coefficients in $\mathbf{Z}/2$ only, but the case of coefficients in $\mathbf{Q}/\mathbf{Z}(j)$ is an easy consequence.

2.1. Equivariant cohomology with supports. The equivariant cohomology groups of $X(\mathbf{C})$ with supports in $X(\mathbf{R})$ and coefficients in $\mathbf{Z}/2$ needed in the proof of Theorem 2.1 can be defined by

$$H^q_{X(\mathbf{R})}(X(\mathbf{C});G,\mathbf{Z}/2) := R^q\mathrm{Hom}_{G,\mathbf{Z}/2}(i_*\mathbf{Z}/2,\mathbf{Z}/2).$$

Here $i\colon X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$ denotes the inclusion and the subscript appended to $R^q\mathrm{Hom}$ indicates that we are working in the derived category of G-sheaves of $\mathbf{Z}/2$-modules on $X(\mathbf{C})$. This is the obvious equivariant version of ordinary cohomology with supports:

$$H^q_{X(\mathbf{R})}(X(\mathbf{C}),\mathbf{Z}/2) = R^q\mathrm{Hom}_{\mathbf{Z}/2}(i_*\mathbf{Z}/2,\mathbf{Z}/2)$$

(see [**Iv86**, §II.9]). If X is nonsingular of dimension d, then $X(\mathbf{R})$ is a submanifold of $X(\mathbf{C})$ of pure codimension d, so we have for every $n \in \mathbf{Z}$ a canonical isomorphism

$$H^n(X(\mathbf{R}),\mathbf{Z}/2) \xrightarrow{\sim} H^{n+d}_{X(\mathbf{R})}(X(\mathbf{C}),\mathbf{Z}/2),$$

known as the *Thom isomorphism* (see [**Iv86**, Th. VIII.2.3]). The situation is analogous in the equivariant situation:

PROPOSITION 2.3 (Thom isomorphism). *Let X be a nonsingular variety over $\mathbf{R}$ of pure dimension d. We have for any $n \in \mathbf{Z}$ a canonical isomorphism*

$$\tau\colon\ H^n(X(\mathbf{R});G,\mathbf{Z}/2) \xrightarrow{\sim} H^{n+d}_{X(\mathbf{R})}(X(\mathbf{C});G,\mathbf{Z}/2).$$

PROOF. The map τ is the isomorphism $H^n(X(\mathbf{R}),\mathbf{Z}/2) = R^n\mathrm{Hom}_{G,\mathbf{Z}/2}(\mathbf{Z}/2,i_*\mathbf{Z}/2) = R^{n+d}\mathrm{Hom}_{G,\mathbf{Z}/2}(\mathbf{Z}/2,i_*\mathbf{Z}/2[-d]) \xrightarrow[\text{[Iv86, VIII.1]}]{\sim} R^{n+d}\mathrm{Hom}_{G,\mathbf{Z}/2}(\mathbf{Z}/2,R\mathcal{H}om_{\mathbf{Z}/2}(i_*\mathbf{Z}/2,\mathbf{Z}/2)) = R^{n+d}\mathrm{Hom}_{G,\mathbf{Z}/2}(i_*\mathbf{Z}/2,\mathbf{Z}/2) = H^{n+d}_{X(\mathbf{R})}(X(\mathbf{C});G,\mathbf{Z}/2)$. □

The decomposition of derived functors (11) applied to the complex of G-sheaves $R\mathcal{H}om_{\mathbf{Z}/2}(i_*\mathbf{Z}/2,\mathbf{Z}/2)$ gives the local–to–global spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X,\mathcal{H}^q_{X(\mathbf{R})}) \Rightarrow H^{p+q}_{X(\mathbf{R})}(X(\mathbf{C});G,\mathbf{Z}/2), \tag{13}$$

where $\mathcal{H}^q_{X(\mathbf{R})}$ is the Zariski-sheaf associated to the presheaf

$$U \mapsto H^q_{U(\mathbf{R})}(U(\mathbf{C});G,\mathbf{Z}/2).$$

This is the auxiliary spectral sequence mentioned in the introduction of this section.

LEMMA 2.4. *Let X be a nonsingular variety over $\mathbf{R}$ of pure dimension d. Consider the spectral sequence* (3) *for $V = X(\mathbf{R})$ and $M = \mathbf{Z}/2$ shifted vertically by d positions:*

$$E_2^{p,q} = H^p(X(\mathbf{R}),\mathcal{H}^{q-d}(G,\mathbf{Z}/2)) \Rightarrow H^{p+q-d}(X(\mathbf{R});G,\mathbf{Z}/2).$$

There is an isomorphism from this spectral sequence into the spectral sequence (13)*, corresponding to the Thom isomorphism τ on limit terms.*

PROOF. We saw in the proof of Proposition 1.4 that the spectral sequence (3) for $V = X(\mathbf{R})$ and $M = \mathbf{Z}/2$ corresponds to the decomposition of derived functors

$$R\Gamma_{X(\mathbf{R})} = R\Gamma_{X_{\mathrm{Zar}}} \circ (R\psi^G_* \circ i_*).$$

Therefore the required isomorphism is induced by the quasi-isomorphism

$$i_*\mathbf{Z}/2[d] \xrightarrow{\sim} R\mathcal{H}om_{\mathbf{Z}/2}(i_*\mathbf{Z}/2,\mathbf{Z}/2)$$

which was used in the construction of the Thom isomorphism. □

COROLLARY 2.5. *Let X be a nonsingular variety over $\mathbf{R}$ of pure dimension d. Consider the spectral sequence* (13) .

(i) *The differentials $d_r^{p,q}$ are zero for every $p,q \in \mathbf{Z}$ and $r \geq 2$.*
(ii) *We have for every $p \geq 0$ and every $r \geq 2$:*

$$E_r^{p,q} \simeq \begin{cases} 0 & \text{for } q < d, \\ H^p(X(\mathbf{R}), \mathbf{Z}/2) & \text{for } q \geq d \end{cases}$$

(iii) *Cup product with the nontrivial element $\eta \in H^1(G, \mathbf{Z}/2)$ induces for every $q \geq d$, every $p \in \mathbf{Z}$ and every $r \geq 2$ an isomorphism $E_r^{p,q} \xrightarrow{\sim} E_r^{p,q+1}$.*

PROOF. This follows immediately from Lemma 2.4 and the corresponding properties of the spectral sequence (3) for $V = X(\mathbf{R})$ and $M = \mathbf{Z}/2$. $\square$

2.2. Proof of Theorem 2.1. First we will study the canonical mapping

$$\psi\colon\ H^n_{X(\mathbf{R})}(X(\mathbf{C}); G, \mathbf{Z}/2) \to H^n(X(\mathbf{C}); G, \mathbf{Z}/2)$$

obtained by forgetting the supports.

LEMMA 2.6. *Let X be a variety defined over $\mathbf{R}$ of dimension d.*

(i) *The mapping ψ is an isomorphism in degree $n > 2d+1$.*
(ii) *If X is nonsingular, then ψ is injective in every degree $n \geq 0$.*

Note that in the nonequivariant setting part (i) is trivially true, since both groups are zero in degree $n > 2d+1$. However, the nonequivariant analogue of the last statement does not hold (not even for $X = \mathbf{A}^N$ with $N > 0$).

PROOF. (i) This follows from the long exact sequence

$$\cdots \to H^n_{X(\mathbf{R})}(X(\mathbf{C}); G, \mathbf{Z}/2) \to H^n(X(\mathbf{C}); G, \mathbf{Z}/2) \to H^n(X(\mathbf{C}) - X(\mathbf{R}); G, \mathbf{Z}/2) \to \cdots$$

which is obtained by applying $R\mathrm{Hom}_{G,\mathbf{Z}/2}(-, \mathbf{Z}/2)$ to the short exact sequence

$$0 \to i_*\mathbf{Z}/2 \to \mathbf{Z}/2 \to \mathbf{Z}/2_{X(\mathbf{C})-X(\mathbf{R})} \to 0$$

of G-sheaves on $X(\mathbf{C})$ (compare [**Iv86**, §II.9]). Since G acts freely on $X(\mathbf{C}) - X(\mathbf{R})$, we have that $H^n(X(\mathbf{C}) - X(\mathbf{R}); G, \mathbf{Z}/2)$ is canonically isomorphic to the ordinary nth cohomology group of the quotient space, which vanishes for $n > 2d$.

(ii) Consider the following commutative diagram:

$$\begin{tikzcd} H^n_{X(\mathbf{R})}(X(\mathbf{C}); G, \mathbf{Z}/2) \arrow[r, hook, "\cup\eta^N"] \arrow[d, "\psi"] & H^{n+N}_{X(\mathbf{R})}(X(\mathbf{C}); G, \mathbf{Z}/2) \arrow[d, "\psi"] \\ H^n(X(\mathbf{C}); G, \mathbf{Z}/2) \arrow[r, "\cup\eta^N"] & H^{n+N}(X(\mathbf{C}); G, \mathbf{Z}/2) \end{tikzcd}$$

For any $N \geq 0$ the top horizontal arrow is injective by Corollary 2.5.iii. Taking N large enough we get that the right hand vertical arrow is an isomorphism by statement (i); the injectivity of the left hand vertical arrow now is obvious. $\square$

COROLLARY 2.7. *Let X be a nonsingular variety defined over $\mathbf{R}$ of dimension d. The mapping ψ corresponds to a morphism form the local–to–global spectral sequence* (13) *into the Bloch–Ogus spectral sequence* (10) *with coefficients in* $\mathbf{Z}/2$*, that induces on the* $E_2^{p,q}$*-level for any* $p \in \mathbf{Z}$ *and* $q > d$ *an isomorphism*

$$H^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{X(\mathbf{R})}) \xrightarrow{\sim} H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Z}/2)).$$

PROOF. The existence of the morphism of spectral sequences is clear from the definitions. For the last statement, take $q > d$. We will show that ψ induces an isomorphism

$$\tilde{\psi}\colon\ \mathcal{H}^q_{X(\mathbf{R})} \xrightarrow{\sim} \mathcal{H}^q(\mathbf{Z}/2).$$

We see from part (ii) of the above lemma that $\tilde{\psi}$ is injective. Since both $\mathcal{H}^q_{X(\mathbf{R})}$ and $\mathcal{H}^q(\mathbf{Z}/2)$ are isomorphic to $\varphi_*\mathbf{Z}/2$ by Proposition 2.3 and Corollary 1.3.i, this implies that $\tilde{\psi}$ is an isomorphism. □

PROOF OF THEOREM 2.1. Consider the mapping of spectral sequences associated to the injection $\mathbf{Z}/2 \hookrightarrow \mathbf{Q}/\mathbf{Z}(j)$. By Corollary 1.3, this induces a surjection $H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Z}/2)) \to H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Q}/\mathbf{Z}(j)))$ for any $q > d$. Therefore it is sufficient to prove the theorem for $M = \mathbf{Z}/2$.

(i) The first statement follows immediately from Corollary 2.5.i and Corollary 2.7.

(ii) From the Hochschild–Serre spectral sequence for $H^i_{\text{ét}}(X, \mathbf{Z}/2)$ we see that cup product with the nontrivial element $\eta \in H^1(G, \mathbf{Z}/2)$ induces a surjection from $H^{i-1}_{\text{ét}}(X, \mathbf{Z}/2)$ to the kernel of the the pull-back map $H^i_{\text{ét}}(X, \mathbf{Z}/2) \to H^i_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Z}/2)$. Therefore it is sufficient to prove that cup product with η induces for every $q > d$ a surjection on the $E_\infty^{p,q}$-level of the Bloch–Ogus spectral sequence

$$E_\infty^{p,q-1} \xrightarrow{\cup\eta} E_\infty^{p,q} = E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Z}/2)).$$

This follows immediately from Corollary 2.5.iii and Corollary 2.7. More precisely, it follows from the following commutative diagram, where $q > d$ and the vertical arrows are induced by the morphism of spectral sequences of Corollary 2.7:

$$\begin{tikzcd}
H^p_{\mathrm{Zar}}(X, \mathcal{H}^{q-1}_{X(\mathbf{R})}) \arrow[r, "\cup\eta"] \arrow[d] & H^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{X(\mathbf{R})}) \arrow[d] \\
E_\infty^{p,q-1} \arrow[r, "\cup\eta"] & H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(\mathbf{Z}/2))
\end{tikzcd}$$

Here the upper horizontal arrow is an isomorphism by Corollary 2.5.iii and the right hand vertical arrow is an isomorphism by Corollary 2.7. □

2.3. Connections with known results. Consider the following result:

THEOREM 2.8. *Let* X *be a nonsingular variety over* $\mathbf{R}$ *of dimension* d*. For every* $k \geq 0$*, the homomorphism* β *followed by the projection to the* k *lowest degree factors gives a surjection*

$$H^{d+k}_{\text{ét}}(X, \mathbf{Z}/2) \to \bigoplus_{i=0}^{k} H^i(X(\mathbf{R}), \mathbf{Z}/2).$$

For X projective this is due to Krasnov, as a corollary to a more technical, but much stronger result (see [**Kr94**, Th. 3.1, Cor. 3.2]). It also follows (for X complete) from the general result [**AP93**, Prop. 5.3.7] on the relation between β and the Gysin morphism between the cohomology of Poincaré duality spaces, in this case applied to the inclusion $i\colon X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$. Both proofs are purely topological: the important fact is that $X(\mathbf{R})$ is a submanifold of $X(\mathbf{C})$ of codimension $\geq d$ (with equality if X has pure dimension d). In fact, the only reason for the projectiveness condition in Krasnov's result is that in the proof it is used that $X(\mathbf{R})$ is a finite cell complex. As far as I can follow his argument, it seems that this condition can be relaxed in order to allow for arbitrary nonsingular X of pure dimension d. On the other hand, in the approach of Allday and Puppe the compactness of $X(\mathbf{R})$ and $X(\mathbf{C})$ is essential, since Poincaré duality (in the sense of nondegeneracy of the cup product pairing) forms the heart of their methods.

PROPOSITION 2.9. *For any variety X over $\mathbf{R}$ of dimension d the statement of Theorem 2.1 is equivalent to the statement of Theorem 2.8.*

PROOF. As remarked in the proof of Theorem 2.1, we can restrict ourselves to coefficients in $\mathbf{Z}/2$. Then, in view of Proposition 1.4, we have that part (i) of Theorem 2.1 is equivalent to the statement that for $k \geq 0$ the map

$$\beta' \colon H^{d+k+1}_{\text{ét}}(X, \mathbf{Z}/2) \to \bigoplus_{i=0}^{k} H^i(X(\mathbf{R}), \mathbf{Z}/2)$$

given by β followed by the projection is surjective. This is a weaker version of Theorem 2.8. Once we have part (i) of Theorem 2.1, we use the fact that cup product with η induces a surjection from $H^{i-1}_{\text{ét}}(X, \mathbf{Z}/2)$ to the kernel of the map $H^i_{\text{ét}}(X, \mathbf{Z}/2) \to H^i_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Z}/2)$ (as we saw in the proof). Since β respects cup product and sends η to the unit element of $H^*(X(\mathbf{R}), \mathbf{Z}/2)$, we get the following commutative diagram:

$$\begin{tikzcd} H^{d+k}_{\text{ét}}(X, \mathbf{Z}/2) \arrow[rr, "\cup\eta"] \arrow[dr] & & H^{d+k+1}_{\text{ét}}(X, \mathbf{Z}/2) \arrow[dl, two heads, "\beta'"] \\ & \bigoplus_{i=0}^{k} H^i(X(\mathbf{R}), \mathbf{Z}/2) & \end{tikzcd}$$

By Proposition 1.4 we have that the subgroup $F^{k+1}H^{d+k+1}_{\text{ét}}(X, \mathbf{Z}/2)$ maps to zero under the mapping β' defined above, so (since we have part (i) of Theorem 2.1), the diagram gives an immediate equivalence between part (ii) of Theorem 2.1 and the statement of Theorem 2.8. $\square$

It is also possible to give a direct proof of Theorem 2.8 with the methods used here.

SKETCH OF PROOF OF THEOREM 2.8. Without loss of generality we may assume X to be of pure dimension d. Consider the composite mapping

$$\theta \colon H^k(X(\mathbf{R}), \mathbf{Z}/2) \xrightarrow{\tau} H^{d+k}_{X(\mathbf{R})}(X(\mathbf{C}); G, \mathbf{Z}/2) \xrightarrow{\psi} H^{d+k}(X(\mathbf{C}); G, \mathbf{Z}/2) \xrightarrow{i^*} H^{d+k}(X(\mathbf{R}); G, \mathbf{Z}/2).$$

This map corresponds to a morphism θ^{**}_* from the shifted spectral sequence of Lemma 2.4 to the spectral sequence (3) for $V = X(\mathbf{R})$, $M = \mathbf{Z}/2$. In order to prove the theorem, it is sufficient to prove that θ^{**}_* is an isomorphism on the $E^{p,q}_\infty$-level for $q \geq d$.

Since both spectral sequences are trivial, it follows from Lemma 2.4, Corollary 2.7 and Proposition 1.4 that $\theta^{p,q}_r$ is an isomorphism for every $r \geq 2$, $p \in \mathbf{Z}$ and $q > d$; the case $q = d$ follows from Corollary 2.5.iii and the analogue for the spectral sequence (3). $\square$

REMARK 2.10. The use of Corollary 2.7 and Proposition 1.4, hence of the Zariski topology, in the above proof is a purely matter of convenience. With a little more work (using arguments similar to the ones used in the proof of Lemma 2.6 and Corollary 2.7) we could have given a direct proof of the fact that $\theta_r^{p,q}$ is an isomorphism for every $r \geq 2$, $p \in \mathbf{Z}$ and $q \geq d$.

3. The Abel–Jacobi maps for torsion cycle classes

In this section we will see that Bloch's construction of a cohomological Abel–Jacobi map for torsion cycles actually works over the real numbers. Combined with the results of Section 2, this gives us in Section 3.2 precise information on the Abel–Jacobi mapping α_{tors} in terms of étale cohomology. The cohomological Abel–Jacobi map also induces a filtration on $CH_0(X)_{\mathrm{tors}}$, which will be treated in Section 3.3.

3.1. Bloch's cohomological Abel–Jacobi map. We will analyse Bloch's construction of a cohomological Abel–Jacobi map, as presented in [**CT93**, §§3, 4]. Let X be a nonsingular variety over a field k of characteristic zero. Let $\mathcal{K}_i$ be the Zariski-sheaf associated to the presheaf $U \mapsto K_i(\mathcal{O}(U))$ of algebraic K-groups. The Gersten conjecture and the Merkurjev–Suslin theorem on the K_2 of fields give for every $i \geq 0$ a diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^{i-1}_{\mathrm{Zar}}(X,\mathcal{K}_i)\otimes \mathbf{Q}/\mathbf{Z} & \xrightarrow{\varphi_i} & H^{i-1}_{\mathrm{Zar}}(X,\mathcal{H}^i(\mathbf{Q}/\mathbf{Z}(i))) & \xrightarrow{\psi_i} & CH^i(X)_{\mathrm{tors}} & \longrightarrow 0 \\ & & & \Big\downarrow{\scriptstyle \gamma_i} & & & \\ & & & H^{2i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(i)) & & & \end{array} \tag{14}$$

with exact top row (see [**CT93**, §3.2]).

When k is algebraically closed and X is nonsingular and projective, it can be shown that $\gamma_i \circ \varphi_i = 0$ (see [**CT93**, proof of Th. 4.3]) and ψ_i and γ_i then induce Bloch's cohomological Abel–Jacobi map

$$\lambda_i\colon\ CH^i(X)_{\mathrm{tors}} \to H^{2i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(i)).$$

If k is not algebraically closed, then $\gamma_i \circ \varphi_i$ need not be zero; for example, in the case $i = 1$ the diagram simplifies to the short exact sequence

$$0 \to k^* \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{\gamma_i \circ \varphi_i} H^1_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(1)) \to CH^1(X)_{\mathrm{tors}} \to 0,$$

but $k^* \otimes \mathbf{Q}/\mathbf{Z} \neq 0$ when k is a number field or a p-adic field. When $k = \mathbf{R}$, we do have $k^* \otimes \mathbf{Q}/\mathbf{Z} = 0$, hence $\gamma_1 \circ \varphi_1 = 0$. By the following theorem this generalizes to higher codimensions.

THEOREM 3.1. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$ of dimension d. For any $i \geq 0$ the mapping*

$$\gamma_i \circ \varphi_i\colon\ H^{i-1}_{\mathrm{Zar}}(X,\mathcal{K}_i)\otimes \mathbf{Q}/\mathbf{Z} \to H^{2i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(i))$$

is zero. Hence the cohomological Abel–Jacobi mapping

$$\lambda_i\colon\ CH^i(X)_{\mathrm{tors}} \to H^{2i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(i))$$

is defined. The image of λ_i coincides with the image of the map

$$\gamma_i\colon\ H^{i-1}_{\mathrm{Zar}}(X,\mathcal{H}^i(\mathbf{Q}/\mathbf{Z}(i))) \to H^{2i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(i)).$$

PROOF. In view of diagram (14) it is sufficient to prove that $\gamma_i \circ \varphi_i = 0$. Consider the commutative diagram associated to the base change $\pi\colon X_{\mathbf{C}} \to X$:

$$\begin{tikzcd} H^{i-1}_{\mathrm{Zar}}(X,\mathcal{K}_i)\otimes \mathbf{Q}/\mathbf{Z} \arrow[r, "\gamma_i\circ\varphi_i"] \arrow[d] & H^{2i-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(i)) \arrow[d, "\pi^*"] \\ H^{i-1}_{\mathrm{Zar}}(X_{\mathbf{C}},\mathcal{K}_i)\otimes \mathbf{Q}/\mathbf{Z} \arrow[r, "\gamma_i^{\mathbf{C}}\circ\varphi_i^{\mathbf{C}}"] & H^{2i-1}_{\text{ét}}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(i)) \end{tikzcd}$$

Since $\gamma_i^{\mathbf{C}} \circ \varphi_i^{\mathbf{C}} = 0$ by [**CT93**, proof Th. 4.3] and $H^{i-1}_{\mathrm{Zar}}(X,\mathcal{K}_i)\otimes \mathbf{Q}/\mathbf{Z}$ is divisible, it is sufficient to show that the kernel of π^* does not contain any nonzero divisible subgroup. Since π is étale of degree 2, we have that $\pi_* \circ \pi^*$ is multiplication by 2, hence the kernel of π^* is purely 2-torsion. ☐

Note that if γ_i is injective, then λ_i maps $CH^i(X)_{\mathrm{tors}}$ isomorphically onto the image of γ_i. For X over any field k, the mappings γ_1 and γ_2 are injective for trivial reasons: there is no differential in the Bloch–Ogus spectral sequence that could kill $H^{i-1}_{\mathrm{Zar}}(X,\mathcal{H}^i(\mathbf{Q}/\mathbf{Z}(i)))$ when $i = 1,2$. For $i = d = \dim(X)$ the situation is different in general, since we have for every j smaller or equal to the cohomological dimension of k and satisfying $0 < j \leq d-2$ a potentially nontrivial differential

$$d_{j+1}^{d-2-j,d+j}\colon E_{j+1}^{d-2-j,d+j} \to E_{j+1}^{d-1,d}.$$

However, we proved in Section 2 that these differentials vanish when $k = \mathbf{R}$. We also determined the cokernel of γ_d, so we obtain the following result, which could be considered as a cohomological alternative that works over $\mathbf{R}$ for Roitman's theorem on torsion zero-cycles over algebraically closed fields.

THEOREM 3.2. *Let X be a nonsingular, projective, geometrically irreducible variety over $\mathbf{R}$ of dimension d. We have a short exact sequence*

$$0 \to CH_0(X)_{\mathrm{tors}} \xrightarrow{\lambda_d} H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\beta'} \bigoplus_{i>0} H^{d-2i}(X(\mathbf{R}),\mathbf{Z}/2) \to 0,$$

where β' is the mapping β followed by the projection.

PROOF. By Theorem 3.1 this is an immediate consequence of Corollary 2.2.i. ☐

3.2. Kernel and image of the Abel–Jacobi map. Consider the Abel–Jacobi map α restricted to the torsion subgroups:

$$\alpha_{\mathrm{tors}}\colon A_0(X)_{\mathrm{tors}} \to \mathrm{Alb}(X)(\mathbf{R})_{\mathrm{tors}}$$

and the restriction $\alpha_{\mathrm{tors,div}}$ to the maximal divisible torsion subgroups. We define:

$$\begin{aligned} T(X)_{\mathrm{tors}} &:= \mathrm{Ker}(\alpha_{\mathrm{tors}}), \\ T(X)^0 &:= \mathrm{Ker}(\alpha_{\mathrm{tors,div}}), \\ T(X)^{\mathrm{top}} &:= T(X)_{\mathrm{tors}}/T(X)^0, \\ A_0(X)^{\mathrm{top}} &:= A_0(X)_{\mathrm{tors}}/A_0(X)_{\mathrm{tors,div}} (= A_0(X)/A_0(X)_{\mathrm{div}}), \\ \mathrm{Im}(\alpha)^{\mathrm{top}} &:= \mathrm{Im}(\alpha_{\mathrm{tors}})/\mathrm{Im}(\alpha_{\mathrm{tors,div}}) (= \mathrm{Im}(\alpha)/\mathrm{Im}(\alpha)_{\mathrm{div}}). \end{aligned}$$

The equalities between parentheses follow from the fact that $A_0(X)/A_0(X)_{\mathrm{tors}}$ is uniquely divisible (see Section 1.3). By definition we have short exact sequences

$$0 \to T(X)^{\mathrm{top}} \to A_0(X)^{\mathrm{top}} \to \mathrm{Im}(\alpha)^{\mathrm{top}} \to 0 \tag{15}$$

and

$$0 \to T(X)^0 \to T(X)_{\mathrm{tors}} \to T(X)^{\mathrm{top}} \to 0. \tag{16}$$

These are short exact sequences of finite dimensional $\mathbf{Z}/2$-vector spaces, since $T(X)_{\mathrm{tors}}$ is a finite dimensional $\mathbf{Z}/2$-vector space (by [**CTS96**, Th. 1.6.a] and a trace argument) and

$$A_0(X)^{\mathrm{top}} \simeq \begin{cases} 0 & \text{if } s = 0, \\ (\mathbf{Z}/2)^{s-1} & \text{otherwise,} \end{cases}$$

where s is the number of connected components of X (see Section 1.3). We will now give cohomological descriptions of the above groups.

Over $\mathbf{C}$ we have a commutative diagram

$$\begin{tikzcd} & A_0(X_{\mathbf{C}})_{\mathrm{tors}} \arrow[dl, "\alpha^{\mathbf{C}}_{\mathrm{tors}}"'] \arrow[dr, "\lambda^{\mathbf{C}}_d"] & \\ \mathrm{Alb}(X)(\mathbf{C})_{\mathrm{tors}} \arrow[rr, "\sim"] & & H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d)) \end{tikzcd}$$

in which all arrows are isomorphisms by Roitman's theorem (cf. [**CT93**, Th. 4.2]). Over $\mathbf{R}$ we have the diagram

$$\begin{tikzcd} A_0(X)_{\mathrm{tors}} \arrow[r, hook, "\lambda_d"] \arrow[d, "\alpha_{\mathrm{tors}}"] & H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d)) \arrow[d, "\pi^*"] \\ \mathrm{Alb}(X)(\mathbf{R})_{\mathrm{tors}} \arrow[r, "\sim"] & H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d))^G \end{tikzcd} \tag{17}$$

in which the lower horizontal arrow is an isomorphism.

THEOREM 3.3. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$ of dimension d.*

(i) *The isomorphism*

$$\mathrm{Alb}(X)(\mathbf{R})_{\mathrm{tors}} \simeq H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d))^G$$

sends the image of α_{tors} isomorphically onto the image of

$$\pi^* \colon H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d)) \to H^{2d-1}_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d))^G.$$

(ii) *The injection*

$$\lambda_d \colon A_0(X)_{\mathrm{tors}} \hookrightarrow H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d)$$

sends the group $T(X)^0$ isomorphically onto the kernel of π^* restricted to the maximal divisible subgroup of $H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d))$.

(iii) *If $X(\mathbf{R}) = \emptyset$, then $T(X)_{\mathrm{tors}} \simeq T(X)^0$. Otherwise, we have an isomorphism*

$$T(X)_{\mathrm{tors}} \simeq (\mathbf{Z}/2)^{c+s-1-t},$$

where s is the number of connected components of $X(\mathbf{R})$, t is the $\mathbf{Z}/2$-dimension of $\mathrm{Im}(\alpha)^{\mathrm{top}}$, and c is the $\mathbf{Z}/2$-dimension of $T(X)^0$.

PROOF. (i) Immediate from diagram (17), Theorem 3.1 and Corollary 2.2.ii.

(ii) Immediate from diagram (17) and the fact that λ_d induces an isomorphism

$$A_0(X)_{\mathrm{tors,div}} \simeq H^{2d-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(d))_{\mathrm{div}}$$

by Theorem 3.2.

(iii) Immediate from (i) and (ii) and the short exact sequences (15) and (16). □

COROLLARY 3.4. *Let X_1, X_2 be nonsingular projective geometrically irreducible varieties over $\mathbf{R}$, such that $X_1(\mathbf{C})$ and $X_2(\mathbf{C})$ are equivariantly homeomorphic. The groups $A_0(X_i)_{\mathrm{tors}}$, $\mathrm{Im}(\alpha_{\mathrm{tors}})$, $T(X_i)_{\mathrm{tors}}$, $T(X_i)^0$, $T(X_i)^{\mathrm{top}}$, $A_0(X)^{\mathrm{top}}$, and $\mathrm{Im}(\alpha_{\mathrm{tors}})^{\mathrm{top}}$ are isomorphic for $i = 1, 2$.*

PROOF. Immediate from Theorem 3.3 and the isomorphism (1). □

REMARK 3.5. The two easy examples of varieties with $T(X)_{\mathrm{tors}} \neq 0$ given in the introduction have in fact $T(X)_{\mathrm{tors}} = T(X)^{\mathrm{top}}$. Indeed, the property $T(X)^{\mathrm{top}} \neq 0$ is so common, that it can hardly be considered to be a 'bad' case. On the other hand, we will see in Example 5.2 that the property $T(X)^0 \neq 0$ does not occur for varieties like products of curves or abelian varieties (nor does it occur, trivially, for complete intersections). It corresponds to certain nontrivial differentials in the Hochschild–Serre spectral sequence, which could be considered to be 'bad'.

3.3. A cohomological filtration on torsion cycle classes. The filtration on étale cohomology associated to the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(i))) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(i))$$

induces for every i a descending filtration

$$0 = F^{2i} \subset \cdots \subset F^1 \subset F^0 = H^{2i-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(i))$$

with the pth graded piece given by $E_\infty^{p,2i-1-p}$. Pulling back to $CH^i(X)_{\mathrm{tors}}$ via the cohomological Abel–Jacobi map

$$\lambda_i\colon\ CH^i(X)_{\mathrm{tors}} \to H^{2i-1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(i))$$

gives a descending filtration on $CH^i(X)_{\mathrm{tors}}$, which is nondegenerate (i.e., $F^N = 0$ for $N \gg 0$) if and only if λ_i is injective. Note that for $i = \dim(X)$ we get

$$F^1\, CH_0(X)_{\mathrm{tors}} = T(X)_{\mathrm{tors}}.$$

We will see in Section 4.2 that the induced filtration on the maximal divisible subgroup $CH_0(X)_{\mathrm{tors,div}}$ can be determined from the Hochschild–Serre spectral sequence. In Example 5.2.1 we will see that for abelian varieties this filtration coincides with the filtration coming from the Pontryagin product.

REMARK 3.6. A priori the filtration on $CH_0(X)_{\mathrm{tors}}$ can go as deep as F^{2d-1}, but I have no examples of an X of dimension d with $F^{d+1}\, CH_0(X)_{\mathrm{tors}} \neq 0$. For the induced filtration on $A_0(X)^{\mathrm{top}} = A_0(X)_{\mathrm{tors}}/A_0(X)_{\mathrm{tors,div}}$ it can be shown, using Theorem 2.8 and Poincaré duality (compare the discussion in [**MvH98**, §2]), that $F^{d+1}A_0(X)^{\mathrm{top}} = 0$ for any nonsingular projective geometrically irreducible X of dimension d.

4. Methods of calculation

In Section 3.2 we proved that the image of

$$\alpha_{\mathrm{tors}}\colon\ A_0(X)_{\mathrm{tors}} \to \mathrm{Alb}(X)(\mathbf{R})_{\mathrm{tors}}$$

coincides with the image of the pull-back

$$H^{2d-1}(X, \mathbf{Q}/\mathbf{Z}(d)) \to H^{2d-1}(X_{\mathbf{C}}, \mathbf{Q}/\mathbf{Z}(d))^G,$$

which can be computed explicitly from the Hochschild–Serre spectral sequence. In this section we will see in Proposition 4.1 that the kernel $T(X)^0$ of the restriction of α_{tors} to the maximal divisible subgroup of $CH_0(X)_{\text{tors}}$ admits a similar description. As a corollary we get that if the Hochschild–Serre spectral sequence with $\mathbf{Q}/\mathbf{Z}$-coefficients is trivial, then α_{tors} is surjective and $T(X)^0$ is trivial.

By Poincaré duality (Lemma 4.3) the same conclusion holds when the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Z}$ is trivial (in other words, when X is a $\mathbf{Z}$-*GM-variety*). This will be used in Example 5.2. The fact that α_{tors} (or, in fact, α) is surjective for a $\mathbf{Z}$-*GM*-variety was originally proved by V.A. Krasnov in [**Kr84**] without using a cohomological Abel–Jacobi map. For the calculations in Example 5.1 it will be useful that Poincaré duality allows us to transform calculations involving high degree cohomology with coefficients in $\mathbf{Q}/\mathbf{Z}$ into calculations involving low degree cohomology with coefficients in $\mathbf{Z}$ (see Corollary 4.6).

In Section 4.2 we use the methods of Proposition 4.1 for calculating the cohomological filtration on $T(X)^0$ in terms of the Hochschild–Serre spectral sequence.

4.1. Calculating the kernel of the Abel-Jacobi mapping.

PROPOSITION 4.1. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$ of dimension d. Let $T(X)^0$ be the kernel of the Abel–Jacobi mapping α restricted to the maximal divisible subgroup of $A_0(X)_{\text{tors}}$. Then $T(X)^0$ is isomorphic to the cokernel of the base change map*

$$\pi^*\colon H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1)) \to H^{2d-1}_{\text{ét}}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(d+1))^G.$$

PROOF. Let $F^\bullet$ be the descending filtration on étale cohomology associated to the Hochschild–Serre spectral sequence. Since $F^1H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d))$ is the kernel of the base change map π^* for coefficients in $\mathbf{Q}/\mathbf{Z}(d)$, the intersection $F^1H^{2d-1}_{\text{div}}$ with the maximal divisible subgroup of $H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d))$ is isomorphic to $T(X)^0$ by Theorem 3.3.

I claim that $F^1H^{2d-1}_{\text{div}}$ is precisely the kernel of the map

$$F^1H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\cup\eta} F^2H^{2d}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1)) \tag{18}$$

induced by cup-product with the nontrivial element $\eta \in H^1(G,\mathbf{Z}(1))$. Since on the E_2-level of the Hochschild–Serre spectral sequence this gives an isomorphism

$$H^p(G,H^q_{\text{ét}}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(d))) \simeq H^{p+1}(G,H^q_{\text{ét}}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(d+1)))$$

for any $q \geq 0$ and any $p > 0$, we see that the kernel of (18) can be identified with the elements that are killed by differentials from $H^0(G,H^{2d}_{\text{ét}}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(d+1)))$. More precisely, we have an exact sequence

$$\begin{aligned} H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1)) \xrightarrow{\pi^*} H^{2d-1}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}(d+1))^G \longrightarrow \\ F^1H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\cup\eta} F^2H^{2d}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1)), \end{aligned}$$

which gives an isomorphism between the kernel of (18) and the cokernel of π^*.

In order to prove that the kernel of (18) coincides with $F^1H^{2d-1}_{\text{div}}$, hence with $T(X)^0$, it is sufficient to prove that the kernel of the homomorphism

$$H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\cup\eta} H^{2d}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1)) \tag{19}$$

is the maximal divisible subgroup. Since the image of (19) is contained in $F^1H^{2d}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1))$, which is purely 2-torsion (compare the end of the proof of Theorem 3.1), it is sufficient to prove that the kernel of the homomorphism (19) is divisible.

This we will deduce from the long exact sequence

$$(20)\quad \cdots \to H^{2d-1}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z}) \longrightarrow H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\ \delta\ } H^{2d}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1)) \to \cdots$$

obtained from the short exact sequence $0 \to \mathbf{Q}/\mathbf{Z}(d+1) \to \pi_*\mathbf{Q}/\mathbf{Z}_{X_{\mathbf{C}}} \to \mathbf{Q}/\mathbf{Z}(d) \to 0$ of étale sheaves on X (compare [**Sch94**, (7.8.1)]). I claim that the boundary map δ coincides with the homomorphism (19) (compare [**Sch94**, Lemma 7.10.1]). Assuming this, the proposition follows, since the group $H^{2d-1}(X_{\mathbf{C}},\mathbf{Q}/\mathbf{Z})$ is divisible.

In order to prove the last claim, observe that the above short exact sequence is the pullback of a short exact sequence of G-modules. Moreover, the boundary map δ can be considered as the Yoneda product with an element of $[\delta] \in \operatorname{Ext}^1_G(\mathbf{Q}/\mathbf{Z}(d),\mathbf{Q}/\mathbf{Z}(d+1)) \simeq \mathbf{Z}/2$ (this is Ext in the category of G-modules). Since tensor product with $\mathbf{Q}/\mathbf{Z}(d)$ induces a surjection $H^1(G,\mathbf{Z}(1)) = \operatorname{Ext}^1_G(\mathbf{Z},\mathbf{Z}(1)) \twoheadrightarrow \operatorname{Ext}^1_G(\mathbf{Q}/\mathbf{Z}(d),\mathbf{Q}/\mathbf{Z}(d+1))$, we get that δ is either cup product with η or trivial. We rule out this last possibility by considering the long exact sequence (20) for $\operatorname{Spec}\mathbf{R}$ instead of X (without changing d). Since $H^i(\operatorname{Spec}\mathbf{C},\mathbf{Q}/\mathbf{Z}) = 0$ for all $i > 0$ and $H^{d+1}(\operatorname{Spec}\mathbf{R},\mathbf{Q}/\mathbf{Z}(d)) = \mathbf{Z}/2 = H^{d+2}(\operatorname{Spec}\mathbf{R},\mathbf{Q}/\mathbf{Z}(d+1))$, we see that δ is nontrivial. □

COROLLARY 4.2. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$. If the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}$ is trivial, then the mapping*

$$\alpha_{\text{tors}}\colon A_0(X)_{\text{tors}} \to \operatorname{Alb}(X)(\mathbf{R})_{\text{tors}}$$

is surjective and the mapping

$$\alpha_{\text{tors,div}}\colon A_0(X)_{\text{tors,div}} \to \operatorname{Alb}(X)(\mathbf{R})_{\text{tors,div}}$$

is an isomorphism.

PROOF. If the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}$ is trivial, then the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}(1)$ is trivial as well, as we see using cup product with the nontrivial element $\eta \in H^1(G,\mathbf{Z}(1))$ and the periodicity of the cohomology of G. The result now follows immediately from Proposition 4.1, Theorem 3.3.i and the surjectivity of $\alpha_{\text{tors,div}}$. □

LEMMA 4.3. *Let M be a compact oriented manifold of dimension m with an action of $G = \mathbf{Z}/2$. Let $r \geq 2$, let $p \geq 0$, let $j \in \mathbf{Z}$, let $p' \geq r$ with $p' \not\equiv p \pmod 2$ and let $j' \in \mathbf{Z}$ with $j' \equiv j \pmod 2$ if the G-action preserves the orientation and $j' \not\equiv j \pmod 2$ if the G-action reverses the orientation. The image of the differential*

$$d(\mathbf{Q}/\mathbf{Z}(j))\colon E_r^{p,q}(\mathbf{Q}/\mathbf{Z}(j)) \to E_r^{p+r,q-r+1}(\mathbf{Q}/\mathbf{Z}(j))$$

in the Hochschild–Serre spectral sequence of M with coefficients in $\mathbf{Q}/\mathbf{Z}(j)$ is isomorphic to the image of the differential

$$d(\mathbf{Z}(j'))\colon E_r^{p'-r,m-q+r-1}(\mathbf{Z}(j')) \to E_r^{p',m-q}(\mathbf{Z}(j'))$$

in the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Z}(j')$.

PROOF. This is a formal consequence of Poincaré duality for M and G: the cup product pairing induces for any $k \in \mathbf{Z}$ a natural isomorphism of G-modules

$$H^k(X(\mathbf{C}),\mathbf{Q}/\mathbf{Z}(j)) \simeq \operatorname{Hom}(H^{m-k}(X(\mathbf{C}),\mathbf{Z}(j')),\mathbf{Q}/\mathbf{Z}),$$

and for any G-module A, any $i > 0$, and any odd $N > i$, the cup product pairing

$$H^i(G,\operatorname{Hom}(A,\mathbf{Q}/\mathbf{Z})) \otimes H^{N-i}(G,A) \to H^N(G,\mathbf{Q}/\mathbf{Z}) \simeq \mathbf{Z}/2 \hookrightarrow \mathbf{Q}/\mathbf{Z}$$

induces an isomorphism

$$H^i(G,\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})) \simeq \mathrm{Hom}(H^{N-i}(G,\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})),\mathbf{Q}/\mathbf{Z}).$$

Combining these isomorphism, we get for i, k, N as above an isomorphism

$$H^i(G,H^k(M,\mathbf{Q}/\mathbf{Z}(j))) \simeq \mathrm{Hom}(H^{N-i}(G,H^{m-k}(M,\mathbf{Z}(j'))),\mathbf{Q}/\mathbf{Z}). \tag{21}$$

The first two Poincaré duality morphisms given above can be obtained from natural mappings of complexes. This enables us to construct a map Φ from the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}(j)$ into the (shifted) Pontryagin dual

$$\hat{E}_2^{i,k}[N] = \mathrm{Hom}(H^{N-i}(G,H^{m-k}(M,\mathbf{Z}(j'))),\mathbf{Q}/\mathbf{Z}) \Rightarrow \mathrm{Hom}(H^{N+m-i-k}(M;G,\mathbf{Z}(j')),\mathbf{Q}/\mathbf{Z}) \tag{22}$$

of the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Z}(j')$, with the property that Φ induces the isomorphism (21) on the $E_2^{i,k}$-level (still for $i > 0$ and $N > i$ odd). Unfortunately, I do not know any references to the literature for this type of result. Let me just say here that on the one hand this mapping of spectral sequences can be constructed explicitly by writing down a map between well-chosen double complexes. On the other hand, it can be obtained from the following transformation of derived functors (with Γ^G as in [**Gr57**, Ch. V]):

$$F_1 := R\Gamma^G(R\mathrm{Hom}(-,\mathbf{Q}/\mathbf{Z})) \to R\mathrm{Hom}(R\Gamma^G(-),\mathbf{Q}/\mathbf{Z}[-N]) =: F_2,$$

since the composite derived functor $F_1 \circ \Gamma_M(-)$ applied to the sheaf $\mathbf{Z}(j')[m]$ induces the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}(j)$, and $F_2 \circ \Gamma_M(-)$ induces the spectral sequence (22), when applied to $\mathbf{Z}(j')[m]$.

Taking $i = p$, $N = p' + p$ and $k = q$, we get for every $r \geq 2$ a commutative diagram

$$\begin{tikzcd}
E_r^{p,q}(\mathbf{Q}/\mathbf{Z}(j)) \arrow[r, "d(\mathbf{Q}/\mathbf{Z}(j))"] \arrow[d, "\Phi"] & E_r^{p+r,q-r+1}(\mathbf{Q}/\mathbf{Z}(j)) \arrow[d, "\Phi"] \\
\hat{E}_r^{p,q}[N] \arrow[r, "\hat{d}"] \arrow[d, equal] & \hat{E}_r^{p+r,q-r+1}[N] \arrow[d, equal] \\
\mathrm{Hom}(E_r^{p',m-q}(\mathbf{Z}(j')),\mathbf{Q}/\mathbf{Z}) \arrow[r, "d(\mathbf{Z}(j'))^\vee"] & \mathrm{Hom}(E_r^{p'-r,m-q+r+1}(\mathbf{Z}(j')),\mathbf{Q}/\mathbf{Z})
\end{tikzcd}$$

where $d(\mathbf{Z}(j'))^\vee$ denotes the Pontryagin dual of $d(\mathbf{Z}(j'))$. For $p > 0$ and $p' > r$ the vertical arrows are isomorphisms, and all groups in the diagram are finite, so we have proved the statement. The remaining cases $p = 0$ and/or $p' = r$ follow from the periodicity of the cohomology of G. □

For M as above, we deduce that all differentials in the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}$ are trivial if and only if all differentials in the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Z}$ are trivial.

DEFINITION 4.4 ([**Kr83**]). Let X be a nonsingular projective variety over $\mathbf{R}$. We say that X is a $\mathbf{Z}$-*GM-variety* if all differentials are zero in the Hochschild–Serre spectral sequence converging to equivariant cohomology of $X(\mathbf{C})$ with coefficients in $\mathbf{Z}$.

In fact, Krasnov uses *GM*$\mathbf{Z}$ instead of $\mathbf{Z}$-*GM*; the notation $\mathbf{Z}$-*GM* comes from Silhol, and should be read as '$\mathbf{Z}$-Galois-Maximal'.

COROLLARY 4.5. *Let X be a nonsingular projective geometrically irreducible $\mathbf{Z}$-GM-variety over $\mathbf{R}$. Then the mapping α_{tors} is surjective and the mapping $\alpha_{\mathrm{tors,div}}$ is an isomorphism.*

PROOF. Immediate from Corollary 4.2, Lemma 4.3. □

The part of the above corollary concerning the surjectivity of α_{tors} is equivalent to Theorem 3.2 in [**Kr84**].

COROLLARY 4.6. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$.*

(i) *If $X(\mathbf{R}) \neq \emptyset$, then the group $T(X)^0$ is isomorphic to the kernel of the composite mapping*

$$H^1(G, H^1(X(\mathbf{C}), \mathbf{Z}(1))) \to H^2(X(\mathbf{C}); G, \mathbf{Z}(1)) \to H^1(X(\mathbf{R}), \mathbf{Z}/2),$$

where the first map is induced by the Hochschild–Serre spectral sequence, and the second map is induced by β.

(ii) *The group $T(X)^0$ is isomorphic to the kernel of the mapping*

$$\mathrm{Pic}^0(X)/\mathrm{Pic}^0(X)_{\mathrm{div}} \to H^1(X(\mathbf{R}), \mathbf{Z}/2)$$

induced by the cycle map. Here $\mathrm{Pic}^0(X)$ is the kernel of the composite map

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbf{C}}) \to H^2(X(\mathbf{C}), \mathbf{Z}(1)).$$

PROOF. (i) Since $X(\mathbf{R}) \neq \emptyset$, the differential $E_2^{2d+1,1}(\mathbf{Z}(1)) \to E_2^{2d+3,0}(\mathbf{Z}(1)) = H^{2d+3}(G, H^0(X(\mathbf{C}), \mathbf{Z}(1)))$ in the Hochschild–Serre spectral sequence is zero by [**Kr83**, Lemma 2.2]. It then follows from Lemma 4.3 (with $p = 0$, $q = 2d-1$, $j = d+1$, $p' = 2d+1$, $j' = 1$, and $2 \leq r \leq 2d$) that the cokernel of the mapping π^* of Proposition 4.1 is isomorphic to the kernel of the quotient map $E_2^{2d+1,1}(\mathbf{Z}(1)) \to E_\infty^{2d+1,1}(\mathbf{Z}(1))$. Since β maps $E_\infty^{2d+1,1}(\mathbf{Z}(1))$ injectively into $H^1(X(\mathbf{R}), \mathbf{Z}/2)$, the statement follows from the periodicity isomorphism $H^1(G, H^1(X(\mathbf{C}), \mathbf{Z}(1))) = E_2^{1,1}(\mathbf{Z}(1)) \simeq E_2^{2d+1,1}(\mathbf{Z}(1))$.

(ii) This follows from (i) if $X(\mathbf{R}) \neq \emptyset$, since then the equivariant cycle map $\mathrm{Pic}(X) \to H^2(X(\mathbf{C}); G, \mathbf{Z}(1))$ (see [**Kr94**, §4]) induces an isomorphism

$$\mathrm{Pic}^0(X)/\mathrm{Pic}^0(X)_{\mathrm{div}} \simeq H^1(G, H^1(X(\mathbf{C}), \mathbf{Z}(1)))$$

which is compatible with the mappings of statements (i) and (ii). For $X(\mathbf{R}) = \emptyset$, we have that the cycle map induces an isomorphism $\mathrm{Pic}^0(X)/\mathrm{Pic}^0(X)_{\mathrm{div}} \simeq E_3^{1,1}(\mathbf{Z}(1))$ (see [**vH98**, Prop. 3.3]) and we adapt the argument of (i). □

REMARKS 4.7. (i) There is an analogue of Corollary 4.6 for the cokernel of α_{tors}, but it is more cumbersome to state, and we will not need it in the examples, since there the image of α is quite easy to determine.

(ii) I have no method to determine the 'topological quotient' $T(X)^{\mathrm{top}}$ of the kernel of α_{tors} from the Hochschild–Serre spectral sequence without using a priori information about the topology of $X(\mathbf{R})$; it seems unlikely that such a method should exist in general.

4.2. Calculating the cohomological filtration on the divisible subgroup. Let $F^\bullet$ be the filtration defined on $CH_0(X)_{\mathrm{tors}}$ in Section 3.3. After restricting the filtration to the maximal divisible subgroup $CH_0(X)_{\mathrm{tors,div}}$ we can determine the graded pieces from the Hochschild–Serre spectral sequence.

PROPOSITION 4.8. *Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$ of dimension d. We have for any $p \geq 0$ that*

$$\mathrm{Gr}^p F^\bullet CH_0(X)_{\mathrm{tors,div}} \simeq \mathrm{Ker}\{E_\infty^{p,2d-1-p}(\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\cup\eta} E_\infty^{p+1,2d-1-p}(\mathbf{Q}/\mathbf{Z}(d+1))\},$$

where $E_\infty^{p,q}(\mathbf{Q}/\mathbf{Z}(j))$ denotes the $E_\infty^{p,q}$-term of the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Q}/\mathbf{Z}(j)$ and $\eta \in H^1(G,\mathbf{Z}(1))$ is the nontrivial element.

PROOF. This follows immediately from the fact that the cohomological Abel–Jacobi mapping induces an isomorphism between $CH_0(X)_{\mathrm{div}}$ and the kernel of the mapping $H^{2d-1}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d)) \xrightarrow{\cup\eta} H^{2d}_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(d+1))$, as we saw in the proof of Proposition 4.1. □

REMARK 4.9. I do not know whether there are varieties X with $F^p CH_0(X)_{\mathrm{tors,div}}$ nonzero for $p > \dim(X)$. In other words, with nonzero differentials

$$E_r^{0,2d-1}(\mathbf{Q}/\mathbf{Z}(d+1)) \to E_r^{r,2d-r}(\mathbf{Q}/\mathbf{Z}(d+1))$$

for $r > \dim(X)+1$ in the Hochschild–Serre spectral sequence (compare [**vH98**, Remark 3.5.ii]).

5. Examples

Stock examples in the study of zero-cycles are products of curves, abelian varieties and conic bundles. Here we will consider them over the real numbers. We will compute the groups defined in Section 3.2 and in some cases the filtration defined in 3.3 as well. Since the Abel–Jacobi map for torsion zero-cycles is completely determined by the equivariant topology of the complex points, we can in fact replace the geometric conditions by topological conditions. Thus Example 5.1 is an immediate generalization of the example of a conic bundle over a curve of which the real part does not map surjectively onto the real part of the curve (it was already shown in [**Si89**, § V.4] that a smooth conic bundle with that property is not a $\mathbf{Z}$-*GM*-variety). Similarly, the results for products of elliptic curves in Example 5.2 immediately generalize to abelian varieties in Example 5.2.1.

5.1. Fibrations over curves. Let C be a nonsingular projective geometrically irreducible curve over $\mathbf{R}$ of genus g such that $C(\mathbf{R})$ has $c > 0$ connected components. Recall that $c \leq g+1$ (cf. Section 1.2). Let X be a nonsingular projective geometrically irreducible variety over $\mathbf{R}$ such that $X(\mathbf{R})$ has $s > 0$ connected components and let $\varphi\colon X \to C$ be a dominant morphism satisfying the following two conditions:

(a) φ induces an isomorphism $\varphi^*\colon \mathrm{Pic}^0(C) \xrightarrow{\sim} \mathrm{Pic}^0(X)$.

(b) The map $X(\mathbf{R}) \to C(\mathbf{R})$ induced by φ is not surjective.

Let $a > 0$ be the number of connected components V of $C(\mathbf{R})$ such that $\varphi^{-1}(x)$ has real points for some $x \in V$. We denote by h the $\mathbf{Z}/2$-dimension of the kernel of the pull-back map $\varphi^*\colon H^1(C(\mathbf{R}),\mathbf{Z}/2) \to H^1(X(\mathbf{R}),\mathbf{Z}/2)$. Hypothesis (b) implies that $h \geq 1$. Note that it is very easy to construct examples with $a < s$, $a < c$, and/or $h > 1$.

It follows from hypothesis (a) that φ induces an isomorphism $\mathrm{Alb}(X) \xrightarrow{\sim} \mathrm{Jac}(C)$. This isomorphism is compatible with the push-forward of cycles and the Abel–Jacobi map. We see from Sections 1.2 and 1.3 that the image of the composite map

$$A_0(X) \to A_0(C) \to \mathrm{Jac}(C)(\mathbf{R})/\mathrm{Jac}(C)(\mathbf{R})_{\mathrm{div}} \simeq (\mathbf{Z}/2)^{c-1}$$

is generated by zero-cycles of the form $[\varphi(P)]-[\varphi(Q)]$ with $P,Q \in X(\mathbf{R})$, and that for the Albanese map $\alpha\colon A_0(X) \to \mathrm{Alb}(X)$ we have:

(i) $\mathrm{Coker}\,\alpha \simeq \mathbf{Z}/2^{c-a}$, and

(ii) $\mathrm{Im}(\alpha)^{\mathrm{top}} \simeq (\mathbf{Z}/2)^{a-1}$.

The exact sequence (15) then implies that

(iii) $T(X)^{\mathrm{top}} \simeq (\mathbf{Z}/2)^{s-a}$.

We will determine the group $T(X)^0$ using Corollary 4.6.ii. For this, consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}^0(X) & \xrightarrow{\mathrm{cl}_{X(\mathbf{R})}} & H^1(X(\mathbf{R}),\mathbf{Z}/2) \\ \wr \big\uparrow \varphi^* & & \big\uparrow \varphi^* \\ \mathrm{Pic}^0(C) & \xrightarrow{\mathrm{cl}_{C(\mathbf{R})}} & H^1(C(\mathbf{R}),\mathbf{Z}/2) \end{array}$$

Here the horizontal arrows are the cycle maps (cf. Remark 1.1) and the left hand vertical arrow is an isomorphism by hypothesis (a). It follows from the discussion in Section 1.2 that the kernel of $\mathrm{cl}_{C(\mathbf{R})}$ is divisible, and that the kernel of $\varphi^* \circ \mathrm{cl}_{C(\mathbf{R})}$ modulo its maximal divisible subgroup is isomorphic to $(\mathbf{Z}/2)^{h-1}$ (since $h > 0$). Hence the commutativity of the diagram and Corollary 4.6.ii imply

(iv) $T(X)^0 \simeq (\mathbf{Z}/2)^{h-1}$.

Adding up we obtain

(v) $T(X)_{\mathrm{tors}} \simeq (\mathbf{Z}/2)^{s-a+h-1}$.

The cohomological filtration on $T(X)_{\mathrm{tors}}$ not only depends on the invariants defined so far, but also on the topology of the fibres of φ.

5.1.1. *Smooth conic bundles.* Let C be as above and let $\varphi\colon X \to C$ be a conic bundle, then hypothesis (a) is satisfied. If, moreover, we assume φ to be smooth, then for every connected component $V \subset C(\mathbf{R})$ we have that $\varphi^{-1}(V)(\mathbf{R}) \subset X(\mathbf{R})$ is either empty or an S^1-bundle over V. So $a = s$ and $h = c - s$, which implies that if hypothesis (b) is satisfied, then

$$T(X)_{\mathrm{tors}} \simeq T(X)^0 \simeq (\mathbf{Z}/2)^{c-s-1}.$$

Moreover, Proposition 4.8, Lemma 4.3 and a closer examination of the Hochschild–Serre spectral sequence with coefficients in $\mathbf{Z}$ (which will not be carried out here) yield that

$$\mathrm{Gr}^j F^\bullet CH_0(X)_{\mathrm{tors}} \simeq \begin{cases} (\mathbf{Q}/\mathbf{Z})^g \times (\mathbf{Z}/2)^{s-1} & \text{if } j = 0, \\ 0 & \text{if } j = 1, \\ (\mathbf{Z}/2)^{c-s-1} & \text{if } j = 2, \\ 0 & \text{if } j > 2. \end{cases}$$

Note that for every choice of g, c, s with $0 < s < c \leq g+1$ it is easy to construct a smooth conic bundle as above (compare [**Si89**, § V.4]).

5.2. Products of curves and abelian varieties. For $i = 1, \dots, d$, let C_i be a nonsingular projective geometrically irreducible curve over $\mathbf{R}$ of genus g_i such that $C_i(\mathbf{R})$ has $s_i > 0$ connected components. Take $X = C_1 \times \cdots \times C_d$. We have $\mathrm{Alb}(X) \simeq \mathrm{Alb}(C_1) \times \cdots \times \mathrm{Alb}(C_d)$, so

$$\mathrm{Alb}(X)(\mathbf{R}) \simeq (\mathbf{R}/\mathbf{Z})^{\sum_i g_i} \times (\mathbf{Z}/2)^{\sum_i (s_i - 1)}.$$

On the other hand, $X(\mathbf{R})$ has $s = \prod_i s_i$ connected components, so

$$A_0(X)_{\mathrm{tors}} \simeq (\mathbf{Q}/\mathbf{Z})^{\sum_i g_i} \times (\mathbf{Z}/2)^{s-1},$$

which implies that the kernel $T(X)_{\text{tors}}$ of α_{tors} is nontrivial if $s-1 = (\prod_i s_i) - 1 > \sum_i (s_i - 1)$; in other words, if $s_i > 1$ for more than one index i. We will see that the converse holds as well.

By [**Kr83**, Props 3.6 and 5.6] we have that X is a $\mathbf{Z}$-GM-variety (see Definition 4.4), so it follows from Corollary 4.5 that

(i) α_{tors} *is surjective.*

(ii) $T(X)^0 = 0$.

From the exact sequences (15) and (16) we then see:

(iii) $T(X)_{\text{tors}} = T(X)^{\text{top}} \simeq (\mathbf{Z}/2)^{\prod_i s_i - \sum_i (s_i - 1) - 1}$.

(iv) $T(X)_{\text{tors}}$ *is nontrivial if and only if* $C_i(\mathbf{R})$ *is not connected for more than one index i.*

It will be some more work to determine the cohomological filtration $F^\bullet$ on $CH_0(X)_{\text{tors}}$. Since the total cohomology $H^*(X(\mathbf{C}), \mathbf{Z})$ is torsion free, the boundary map

$$H^{2d-1}(X(\mathbf{C}); G, \mathbf{Q}/\mathbf{Z}(d)) \to H^{2d}(X(\mathbf{C}); G, \mathbf{Z}(d))$$

obtained from the short exact sequence $0 \to \mathbf{Z}(d) \to \mathbf{Q}(d) \to \mathbf{Q}/\mathbf{Z}(d) \to 0$ induces for $p > 0$ an isomorphism

$$H^p(G, H^{2d-1-p}(X(\mathbf{C}), \mathbf{Q}/\mathbf{Z}(d))) \simeq H^{p+1}(G, H^{2d-1-p}(X(\mathbf{C}), \mathbf{Z}(d)))$$

on the E_2-level of the Hochschild–Serre spectral sequences. Since X is a $\mathbf{Z}$-GM-variety, this is still an isomorphism on the E_∞-level. It follows that up to a shift by 1 in the indices the cohomological filtration $F^\bullet$ on $T(X)_{\text{tors}}$ is the same filtration as the one induced by the descending filtration $F^\bullet_{\mathbf{Z}}$ of $H^{2d}(X(\mathbf{C}); G, \mathbf{Z}(d))$ associated to the Hochschild–Serre spectral sequence. In other words,

$$F^j T(X)_{\text{tors}} \simeq F^{j+1}_{\mathbf{Z}} H^d(X(\mathbf{R}), \mathbf{Z}/2)$$

for any $j > 0$, where $F^j_{\mathbf{Z}} H^d(X(\mathbf{R}), \mathbf{Z}/2)$ is the image of $F^j_{\mathbf{Z}} H^{2d}(X(\mathbf{C}); G, \mathbf{Z}(d))$ under β followed by the projection onto $H^d(X(\mathbf{R}), \mathbf{Z}/2)$.

We have for any $j \geq 0$ a commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{\substack{0 \leq j_1, \ldots, j_d \leq 2 \\ \sum_i j_i = j}} \bigotimes\limits_i F^{j_i}_{\mathbf{Z}} H^2(C_i(\mathbf{C}); G, \mathbf{Z}(1)) \arrow[r, "\beta"] \arrow[d] & \bigotimes\limits_i H^*(C_i(\mathbf{R}), \mathbf{Z}/2) \arrow[d] \\
F^j_{\mathbf{Z}} H^{2d}(X(\mathbf{C}); G, \mathbf{Z}(d)) \arrow[r, "\beta"] & H^*(X(\mathbf{R}), \mathbf{Z}/2)
\end{tikzcd} \tag{23}$$

in which, by the Künneth formula, the vertical arrow on the right is an isomorphism and the vertical arrow on the left is a surjection (compare [**Kr83**, § 5.6]).

Since $F^2_{\mathbf{Z}} H^1(C_i(\mathbf{R}), \mathbf{Z}/2) = 0$ for every i, this implies:

(v) $F^j T(X)_{\text{tors}} \simeq \bigoplus\limits_{\substack{0 \leq j_1, \ldots, j_d \leq 1 \\ \sum_i j_i = j+1}} \bigotimes\limits_i F^{j_i}_{\mathbf{Z}} H^1(C_i(\mathbf{R}), \mathbf{Z}/2)$ *for* $j > 0$.

Observe that

$$F^{j_i}_{\mathbf{Z}} H^1(C_i(\mathbf{R}), \mathbf{Z}/2) = \begin{cases} H^1(C_i(\mathbf{R}), \mathbf{Z}/2) & \text{if } j_i = 0, \\ H^1(C_i(\mathbf{R}), \mathbf{Z}/2)^0 & \text{if } j_i = 1, \end{cases}$$

where $H^1(C_i(\mathbf{R}), \mathbf{Z}/2)^0$ is the kernel of the trace map $H^1(C_i(\mathbf{R}), \mathbf{Z}/2) \twoheadrightarrow \mathbf{Z}/2$. Hence $F^1_{\mathbf{Z}} H^1(C_i(\mathbf{R}), \mathbf{Z}/2) = 0$ if and only if $C_i(\mathbf{R})$ is connected. We deduce for $j > 0$:

(vi) $F^j T(X)_{\text{tors}} = 0$ *if and only if* $j > \#\{i : s_i > 1\} - 1$.

If $s_i \in \{1,2\}$ for all i (e.g., if X is a product of elliptic curves), then $X(\mathbf{R})$ has 2^a connected components, where a is the number of indices for which $C_i(\mathbf{R})$ is not connected, and we get the following result.

(vii) *With X and a as above, we have that* $\dim_{\mathbf{Z}/2} T(X)_{\mathrm{tors}} = 2^a - a - 1$ *and* $\dim_{\mathbf{Z}/2}(\mathrm{Gr}^j F^\bullet CH_0(X)_{\mathrm{tors}}) = \binom{a}{j+1}$ *for $j > 0$. In particular, we have for $j > 0$ that $F^j T(X)_{\mathrm{tors}} = 0$ if and only if $j > a-1$.*

5.2.1. *Abelian varieties.* When X is an abelian variety of dimension d defined over $\mathbf{R}$, then $X(\mathbf{C})$ is equivariantly homeomorphic to the set of complex points of a product of d elliptic curves over $\mathbf{R}$. If $X(\mathbf{R})$ has 2^a connected components, we can take a product of a copies of an elliptic curve of which the real part has 2 connected components and $d-a$ copies of an elliptic curve of which the real part has 1 connected component. Therefore by Corollary 3.4 the statements (i), (ii), and (vii) above hold for X, so we have a complete picture of $T(X)_{\mathrm{tors}}$ in this case.

We can also rephrase the results on the filtration in terms of the *Pontryagin product*

$$*\colon\ CH_0(X) \times CH_0(X) \to CH_0(X)$$

defined by $\gamma * \tau = \mu_*(\gamma \times \tau)$, where $\mu\colon X \times X \to X$ is the multiplication (cf. [**Bl76**]). It can be checked that if $\gamma \in F^j CH_0(X)_{\mathrm{tors}}$ and $\tau \in F^{j'} CH_0(X)_{\mathrm{tors}}$, then $\gamma * \tau \in F^{j+j'+1} CH_0(X)_{\mathrm{tors}}$. In other words, the j-fold Pontryagin power $CH_0(X)^{*j}_{\mathrm{tors}} \subset CH_0(X)_{\mathrm{tors}}$, generated by elements of the form $\gamma_1 * \cdots * \gamma_j$, is contained in $F^{j-1} CH_0(X)_{\mathrm{tors}}$. From the representation (v) of $F^\bullet CH_0(X)_{\mathrm{tors}}$ it is a matter of combinatorics to deduce that

(viii) $F^j CH_0(X)_{\mathrm{tors}} = CH_0(X)^{*(j+1)}_{\mathrm{tors}}$ *for any $j \geq 0$.*

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