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Microlocal analysis of a seismic linearized inverse problem.

Abstract: The seismic inverse problem is to determine the wavespeed $c(x)$ in the interior of a medium from measurements at the boundary. In this paper we analyze the linearized inverse problem in general acoustic media. The problem is to find a left inverse of the linearized forward map F, or, equivalently, to find the inverse of the normal operator F^*F.

It is well known that in the high frequency approximation the linearized forward map is a Fourier integral operator. If the so called traveltime injectivity condition is satisfied then the normal operator is an invertible pseudodifferential operator. In case this condition is violated the normal operator is the sum of an invertible pseudodifferential operator and a nonlocal part. The normal operator is still asymptotically invertible if the nonlocal part is less singular than the pseudodifferential part. Now there are in general two problems. First the nonlocal part may not be a Fourier integral operator. Second it may be as singular as the pseudodifferential part. We show that both these problems can occur, but that in the generic case they are absent.

Keywords: Migration, microlocal analysis, degenerate Fourier integral operators, seismic inverse problem.

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1 Introduction

In this paper we study an inverse problem for the acoustic wave operator

$$P = \frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

This operator describes the propagation of sound waves in a medium where the speed of sound is given by the function $c(x)$. The waves are generated by a source $f(x,t)$, that is usually assumed to be causal, $f(x,t) = 0$ if $t < 0$. The resulting acoustic velocity field $u(x,t)$ is then given by the solution to the linear partial differential equation

$$Pu = f$$

that satisfies the initial conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$. If the source is a delta function $f(x,t) = \delta(x-s)\delta(t)$, where s is the source position, then the solution is the fundamental solution or Green's function and denoted by $G(x,s,t)$. Solutions for arbitrary f are given by

$$u(x,t) = \int_0^t \int G(x,s,t-t_0) f(s,t_0) \, \mathrm{d}s \, \mathrm{d}t_0.$$

The medium occupies an open part $X \subset \mathbb{R}^n$, with boundary ∂X. Often in a seismic experiment the medium is the subsurface described by $X = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$, with boundary $\partial X = \mathbb{R}^{n-1}$. Sources and receivers are distributed over a source and receiver manifold Σ_s, Σ_r that are assumed to be open parts of the boundary ∂X. One takes measurements during the time interval $I_t = [0, T_{max}]$. The idealized seismic dataset is the set

$$D = \{G(r,s,t) \,|\, r \in \Sigma_r, s \in \Sigma_s, t \in I_t\}.$$

The forward map is the map

$$S : c \mapsto G(r,s,t;c)$$

that maps the model given by c to the data. Essentially one is interested in the inversion of this map, that is to determine the soundspeed c in the medium from the seismic data.

Since the forward map is highly nonlinear the inversion is too difficult in most practical situations. A common approach is to linearize the problem and at the same time apply a high frequency approximation. Thus the full velocity is written as the sum of a smooth (lowfrequency) background velocity c and a perturbation δc, that contains the singularities (highfrequency part). The resulting perturbation term for the Green's function is

$$\delta G(r,s,t) = -\int G(r,x,t-t') \, \delta P \, G(x,s,t') \, \mathrm{d}x \, \mathrm{d}t'. \tag{1}$$

Now it is assumed that, after preprocessing, the data contain only the singly reflected waves. This means that, if $c, \delta c$ are suitably chosen then δG is a highfrequency approximation of the data. Let $Y = \Sigma_r \times \Sigma_s \times I_t$. The operator F that maps the velocity perturbation $\delta c \in E'(X)$ to $\delta G \in D'(Y)$ is called the linearized forward operator. The question is now whether we can invert this operator approximately.

We first say a little bit more about F. Under quite general conditions (see Ten Kroode, Smit and Verdel [6]) the linearized forward operator is a Fourier integral operator (FIO), that is, it has a kernel of the form

$$F(r,s,t,x) = \int A(r,s,t,x,\Theta) \mathrm{e}^{\mathrm{i}\Phi(r,s,t,x,\Theta)}\, \mathrm{d}\Theta.$$

(see Duistermaat [2], Hörmander [3], Treves [14]). This was first proved by Rakesh [12], while in the present setting it was proved by Ten Kroode, Smit and Verdel [6]. Recall that, according to microlocal analysis, the singularities of a function can be localized w.r.t. position x and slowness ξ, that is, in the cotangent bundle with the zero section removed $T^*X\backslash 0$. The positions in $T^*Y\backslash 0$ where F maps the singularities of δc are given by the canonical relation $\Lambda \subset T^*Y\backslash 0 \times T^*X\backslash 0$

$$\begin{aligned}\Lambda = \{(r(x,\alpha), s(x,\beta), \phi_r(x,\alpha) + \phi_s(x,\beta), \omega\rho(x,\alpha), \omega\sigma(x,\beta), -\omega,\\ x, \frac{\omega}{c(x)}(\alpha+\beta)) \,|\, x \in X, \omega \in \mathbb{R}, \alpha, \beta \in S^{n-1}\}.\end{aligned} \tag{2}$$

Here ω is the frequency, (x,α) is the starting point and takeoff direction of the receiver ray, (x,β) is the starting point and takeoff direction of the source ray, $r(x,\alpha), s(x,\beta)$ are the receiver resp. source coordinate, $\omega\rho, \omega\sigma$ are the slownesses associated to r, s, and ϕ_r, ϕ_s are the traveltimes from x to the receiver resp. the source.

The operator F has a left inverse if and only if the normal operator $N = F^*F$ is invertible. In that case a left inverse is given by

$$(F^*F)^{-1}F^*.$$

This left inverse is optimal in the sense of least squares. The normal operator $N = F^*F$ hence plays an important role. Beylkin [1] has shown that if there are no caustics on the rays connecting source and receiver points to the scattering points, then the normal operator is an invertible pseudodifferential operator of order $n-1$. Ten Kroode e.a. [6] have shown that this result still holds when the no-caustics assumption is replaced by the less restrictive *traveltime injectivity condition*. The medium satisfies the traveltime injectivity condition if, given a ray defined by receiver coordinate and slowness and another ray defined by source coordinate and slowness and the travel time, one can uniquely solve for the intersection point if the rays intersect. A pseudodifferential operator of order $n-1$ is a continuous map $H_{(s)} \to H_{(s-(n-1))}$. It can be inverted asymptotically, provided that the amplitude is nonzero, so in this case the operator F has an asymptotic left inverse. See also Nolan and Symes [8], where also the case of lower dimensional acquisition manifolds is discussed.

Ten Kroode e.a. [6] have shown that if the traveltime injectivity condition is violated, but a certain other, "traveltime transversality", condition is satisfied, then the normal operator is the sum of a pseudodifferential operator of order $n-1$, and a more general, Fourier integral operator of order $\frac{n-1}{2}$. This contribution is nonmicrolocal (nonlocal in the cotangent bundle). The condition is that the matrix M in formula (57) of their paper has maximal rank. They also show that if the wave front forms a caustic of generic form then this condition is satisfied. If the nonmicrolocal part of the normal operator is less singular than the pseudodifferential part (i.e. it is continuous $H_{(s)} \to H_{(s-k)}$, with $k < n-1$), then the normal operator is still invertible asymptotically. If this is not the case then in general the normal operator will not be invertible, and artefacts can arise in the image.

It turns out that their analysis is not complete for two reasons. First the normal operator is not always a Fourier integral operator. The case that there are *two* caustics that intersect is not treated. The second point concerns the continuity of the normal operator in Sobolev spaces. The fact that the nonmicrolocal part of the normal operator N_{nonml} is a Fourier integral operator of order $\frac{n-1}{2}$ does *not* necessarily imply that it is continuous from $H_{(s)} \to H_{(s-(n-1)/2)}$. It seems that Theorem 3.2 in [6] therefore is not valid.

In this paper we simplify the traveltime transversality condition and give a geometric interpretation. We show that the condition can be violated in certain special cases, but that it is satisfied generically. So if a medium that violates the condition is perturbed by a small amount, then in general it will satisfy the condition. Mathematically it means that the set of media that satisfies the condition is an open and dense subset of $C^\infty(X)$). A waveguide can be an example a system where the condition is violated.

The question whether N_{nonml} is less singular than the pseudodifferential part $N_{\psi\text{do}}$ is in general quite difficult. It involves the study of so called degenerate Fourier integral operators (see e.g. Hörmander [3], Seeger [13], Phong and Stein [9, 10]). The nonmicrolocal part will be at most as singular as the pseudodifferential part. It is less singular if certain derivatives of the canonical relation of N_{nonml} are nonzero. This is well understood only in the case $n = 2$.

The plan of the paper is as follows. In section 2 we discuss the construction of rays and related quantities using the Hamilton flow. In section 3 we simplify the traveltime transversality condition and we construct the canonical relation of the normal operator. In section 4 the continuity of F in Sobolev spaces is discussed. In section 5 we construct an example where the normal operator is not a FIO and an example where the normal operator is not invertible. In section 6 we discuss how the Hamilton flow (the rays) and its derivatives depend on small perturbations of c. This will be used in section 7, where it is shown that generically F is a FIO, and, in the case $n = 2$, that generically the nonmicrolocal part is less singular than the pseudodifferential part.

2 Preliminaries

In this section the construction of rays, traveltimes and of the derivatives of rays is discussed.

The rays can be found by solving a Hamilton system in T^*X, with parameter t. The Hamiltonian is given by $H(x,\xi) = c(x)\|\xi\|$, the system is

$$\begin{aligned} \frac{\partial x_i}{\partial t} &= \frac{\partial H}{\partial \xi_i}(x,\xi) = c(x)\frac{\xi_i}{\|\xi\|}, \\ \frac{\partial \xi_i}{\partial t} &= -\frac{\partial H}{\partial x_i}(x,\xi) = -\frac{\partial c}{\partial x_i}(x)\|\xi\|. \end{aligned} \tag{3}$$

The mapping $(x_0,\xi_0,t) \mapsto (x(x_0,\xi_0,t),\xi(x_0,\xi_0,t))$, that maps initial values x_0,ξ_0 to the solution of (3) at time t is called the Hamilton flow. The Hamilton flow is homogeneous in $\bar{\xi},\xi$ and it conserves the Hamiltonian. If we let $\alpha_0 = \frac{\xi_0}{\|\xi_0\|}$, $\alpha = \frac{\xi}{\|\xi\|}$, then the Hamilton flow gives a map $(x_0,\alpha_0,t) \mapsto (x(x_0,\alpha_0,t),\alpha(x_0,\alpha_0,t))$.

The Jacobi matrix $\frac{\partial(x,\xi)}{\partial(x_0,\xi_0)}(t)$ satisfies an ordinary differential equation along the ray. By differentiating (3) we obtain the Jacobi or neighboring ray equations

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial x_i}{\partial(x_0,\xi_0)} &= \sum_j \left(\frac{\partial^2 H}{\partial \xi_i \partial x_j}\frac{\partial x_j}{\partial(x_0,\xi_0)} + \frac{\partial^2 H}{\partial \xi_i \partial \xi_j}\frac{\partial \xi_j}{\partial(x_0,\xi_0)} \right), \\ \frac{\partial}{\partial t}\frac{\partial \xi_i}{\partial(x_0,\xi_0)} &= \sum_j \left(-\frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial x_j}{\partial(x_0,\xi_0)} - \frac{\partial^2 H}{\partial x_i \partial \xi_j}\frac{\partial \xi_j}{\partial(x_0,\xi_0)} \right). \end{aligned} \tag{4}$$

The Jacobi matrix is symplectic.

We will now investigate the map $(x_0,\alpha_0) \mapsto (r,\rho,\phi_r)$, that maps a pair (x_0,α_0) in the medium to receiver coordinate r, receiver slowness ρ and traveltime ϕ_r. For the map $(x_0,\beta_0) \mapsto (s,\sigma,\phi_s)$ the same results are valid. If $(x_0,\alpha_0,t) \mapsto (x,\alpha)$ denotes the Hamilton flow to a neighborhood of the receiver point, and x_n is the coordinate normal to the surface, then this map is obtained by solving the traveltime to the receiver $\phi_r(x_0,\alpha_0)$ from

$$x_n(x_0,\alpha_0,\phi_r) = 0$$

and then setting

$$\begin{aligned} r(x_0,\alpha_0) &= (x_1(x_0,\alpha_0,\phi_r(x_0,\alpha_0)),\dots,x_{n-1}(x_0,\alpha_0,\phi_r(x_0,\alpha_0))), \\ \rho(x_0,\alpha_0) &= \frac{1}{c(x(x_0,\alpha_0))}(\alpha_1(x_0,\alpha_0,\phi_r(x_0,\alpha_0)),\dots,\alpha_{n-1}(x_0,\alpha_0,\phi_r(x_0,\alpha_0))). \end{aligned}$$

In this way we can define ϕ_r provided there are no rays that come in tangent to the surface (so called grazing rays). The derivatives of this map are

$$
\begin{aligned}
\frac{\partial \phi_r}{\partial(x_0,\alpha_0)} &= \left(\frac{\partial x_n}{\partial t}\right)^{-1} \cdot \frac{\partial x_n}{\partial(x_0,\alpha_0)} \\
\frac{\partial(r,\rho)}{\partial(x_0,\alpha_0)} &= \frac{\partial(x_1,\dots,x_{n-1},\alpha_1,\dots\alpha_{n-1})}{\partial(x_0,\alpha_0)} \\
&\quad + \frac{\partial(x_1,\dots,x_{n-1},\alpha_1,\dots,\alpha_{n-1})}{\partial t} \cdot \left(\frac{\partial x_n}{\partial t}\right)^{-1} \cdot \frac{\partial x_n}{\partial(x_0,\alpha_0)}. \qquad (5)
\end{aligned}
$$

It is not difficult to show that $\frac{\partial(r,\rho,\phi_r)}{\partial(x_0,\alpha_0)}$ is invertible, so that the map $(x_0,\alpha_0) \mapsto (r,\rho,\phi_r)$ is a diffeomorphism. One more property of $\frac{\partial(r,\rho,\phi_r)}{\partial(x_0,\alpha_0)}$ will be needed. The derivative of (r,ρ) with respect to x along the ray vanishes

$$\sum_i \frac{\partial(r,\rho)}{\partial x_0^i}\,\alpha_{0,i} = 0, \qquad (6)$$

while the derivative of the traveltime ϕ_r along the ray satisfies

$$\sum_i \frac{\partial \phi_r}{\partial x_0^i}\,\alpha_{0,i} = -\frac{1}{c(x_0)}. \qquad (7)$$

3 Conditions for the normal operator to be a FIO

The condition under which the normal operator is a Fourier integral operator where derived by Ten Kroode e.a. [6]. It turns out that these conditions can be simplified and have a geometrical interpretation. We also give a characterization of the canonical relation of this Fourier integral operator. This is important since the continuity of the normal operator between Sobolev spaces depends on the properties of the canonical relation.

We recall that the linearized forward operator F is a Fourier integral operator $E'(X) \to D'(Y)$ with canonical relation given by (2). The canonical relation of the adjoint F^* will be denoted by Λ^*. The composition F^*F is a Fourier integral operator if the manifolds $L = \Lambda^* \times \Lambda$ and $M = T^*X \times \Delta_Y^* \times T^*X$ intersect cleanly. Here Δ_Y^* is the diagonal in $T^*Y\backslash 0 \times T^*Y\backslash 0$ and two manifolds L, M are said to intersect cleanly if $L \cap M$ is a manifold and $T(L \cap M) = TL \cap TM$ (see Treves [14], Hörmander [4]). The canonical relation of the compose F^*F is then given by $\Lambda^* \circ \Lambda$ which is defined as

$$\{(\bar{x}, \bar{\xi}, x, \xi) \,|\, \text{there is } (y, \eta) \text{ such that } (y, \eta, x, \xi) \in \Lambda \text{ and } (y, \eta, \bar{x}, \bar{\xi}) \in \Lambda\}.$$

In the present case the condition that L and M intersect is that there are $x, \alpha, \beta, \bar{x}, \bar{\alpha}, \bar{\beta}$, such that

$$\begin{aligned} y(x, \alpha, \beta, \omega) &= y(\bar{x}, \bar{\alpha}, \bar{\beta}, \bar{\omega}) \\ \eta(x, \alpha, \beta, \omega) &= \eta(\bar{x}, \bar{\alpha}, \bar{\beta}, \bar{\omega}). \end{aligned} \tag{8}$$

These equations say that

1. Both $x, \bar{x}$ are on the ray determined by r, ρ and $\alpha, \bar{\alpha}$ are the directions of the ray at x resp. $\bar{x}$.

2. Both $x, \bar{x}$ are on the ray determined by s, σ and $\beta, \bar{\beta}$ are the directions of the ray at x resp. $\bar{x}$.

3. If $x \neq \bar{x}$ then the equality $\phi_r(x, \alpha) + \phi_s(x, \beta) = \phi_r(\bar{x}, \bar{\alpha}) + \phi_s(\bar{x}, \bar{\beta})$ implies that the (r, ρ)-ray hits x first and then $\bar{x}$, while the (s, σ) ray first hits $\bar{x}$ and then x. The traveltimes from x to $\bar{x}$ along the two rays are equal.

4. $\omega = \bar{\omega}$.

In other words this means that there are two rays originating in $\bar{x}$, in the directions $\bar{\alpha}$, resp. $-\bar{\beta}$, that intersect in x at the same traveltime. Therefore (8) is satisfied if and only if there is t such that

$$\Delta x(\bar{x}, \bar{\alpha}, \bar{\beta}, t) := x(\bar{x}, \bar{\alpha}, t) - x(\bar{x}, \bar{\beta}, -t) = 0, \tag{9}$$

and

$$\begin{aligned} x &= x(\bar{x}, \bar{\alpha}, t), \\ \alpha &= \alpha(\bar{x}, \bar{\alpha}, t), \\ \beta &= \beta(\bar{x}, \bar{\beta}, -t). \end{aligned} \tag{10}$$

The relation of the normal operator is then given by

$$\{(\bar{x}, \xi(\bar{x}, \bar{\alpha}, \bar{\beta}, t, \bar{\omega})), (x, \xi(x, \alpha, \beta, t, \omega)) \,|\, \text{equations (9), (10) are satisfied}\}. \tag{11}$$

There are essentially two types of solutions to these equations. First there is the solution where $(x, \alpha, \beta) = (\bar{x}, \bar{\alpha}, \bar{\beta})$, and hence $t = 0$ (we assume there are no periodic rays through source or receiver surface). This will lead to the pseudodifferential part of the normal operator. Secondly there may be solutions such that $(x, \alpha, \beta) \neq (\bar{x}, \bar{\alpha}, \bar{\beta})$. These give a nonlocal contribution. The ray configuration corresponding to a solution of this type is sketched in Figure 1.

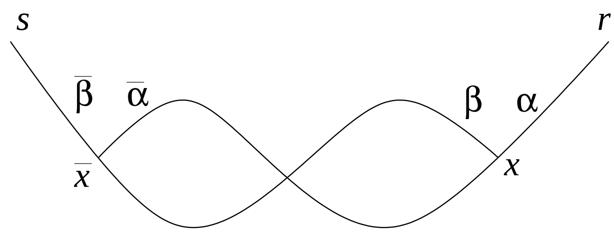

Figure 1: Ray trajectories such that (8) is satisfied and $(x, \alpha, \beta) \neq (\bar{x}, \bar{\alpha}, \bar{\beta})$.

It has been checked in the literature that for solutions of the first type L and M intersect cleanly with excess $e = n - 1$, see e.g. [6], [8]. This is due to the fact that the projection of Λ into T^*Y is an immersion, so that locally the intersection points are precisely the points $(x, \alpha, \beta) = (\bar{x}, \bar{\alpha}, \bar{\beta})$. This leads to a contribution to the normal operator that is a pseudodifferential operator of order $2\,\text{order}\,F + \frac{1}{2}e = n - 1$.

We will concentrate on the case that there are $(x, \alpha, \beta) \neq (\bar{x}, \bar{\alpha}, \bar{\beta})$ such that (8) holds. We will assume that in that case the intersection is transversal. This is the case precisely when the following matrix has maximal rank (here the part corresponding to $\omega, \bar{\omega}$ is omitted, since it will not cause any problems)

$$M = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial \alpha} & 0 & \frac{\partial r}{\partial \bar{x}} & \frac{\partial r}{\partial \bar{\alpha}} & 0 \\ \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial \alpha} & 0 & \frac{\partial \rho}{\partial \bar{x}} & \frac{\partial \rho}{\partial \bar{\alpha}} & 0 \\ \frac{\partial s}{\partial x} & 0 & \frac{\partial s}{\partial \beta} & \frac{\partial s}{\partial \bar{x}} & 0 & \frac{\partial s}{\partial \bar{\beta}} \\ \frac{\partial \sigma}{\partial x} & 0 & \frac{\partial \sigma}{\partial \beta} & \frac{\partial \sigma}{\partial \bar{x}} & 0 & \frac{\partial \sigma}{\partial \bar{\beta}} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial \alpha} & \frac{\partial \phi}{\partial \beta} & \frac{\partial \phi}{\partial \bar{x}} & \frac{\partial \phi}{\partial \bar{\alpha}} & \frac{\partial \phi}{\partial \bar{\beta}} \end{pmatrix}.$$

We prove in the following lemma that this condition is equivalent to the condition that the rank of the Jacobi matrix $C = \frac{\partial \, \Delta x}{\partial(\bar{x},\bar{\alpha},\bar{\beta},t)}$ corresponding to (9) is maximal. The matrix C is given by

$$C = \begin{pmatrix} \frac{\partial x(\bar{x},\bar{\alpha},t)}{\partial \bar{x}} - \frac{\partial x(\bar{x},\bar{\beta},-t)}{\partial \bar{x}} & \frac{\partial x(\bar{x},\bar{\alpha},t)}{\partial \bar{\alpha}} & -\frac{\partial x(\bar{x},\bar{\beta},-t)}{\partial \bar{\beta}} & \frac{\partial x(\bar{x},\bar{\alpha},t)}{\partial t} + \frac{\partial x}{\partial t}(\bar{x},\bar{\beta},-t) \end{pmatrix}.$$

Note that this parallels the simplification of the system (8) to (9). It implies that the relation of the normal operator (11) is a manifold, as of course it should be.

Lemma 3.1 *Suppose $\bar{x}, \bar{\alpha}, \bar{\beta}, x, \alpha, \beta, t$ are such that equations (9) and (10) are satisfied. Then the rank of the matrix M satisfies*

$$\operatorname{rank} M = 3n - 3 + \operatorname{rank} C. \tag{12}$$

In particular $\operatorname{rank} M$ *is maximal if and only if* $\operatorname{rank} C$ *is maximal.*

Proof Let the assumption of the lemma be satisfied. We write the matrix M as a product of two factors, where one factor contains the derivatives $\frac{\partial(r,\rho,\phi_r)}{\partial(x,\alpha)}, \frac{\partial(s,\sigma,\phi_s)}{\partial(x,\beta)}$ and the other factor relates (x, α, β) to $\bar{x}, \bar{\alpha}, \bar{\beta})$. Here we use that the map

$$(\bar{x}, \bar{\alpha}) \mapsto (r(\bar{x}, \bar{\alpha}), \rho(\bar{x}, \bar{\alpha}), \phi_r(\bar{x}, \bar{\alpha}))$$

equals the composition of the maps

$$\begin{aligned} &(\bar{x}, \bar{\alpha}) \mapsto (x(\bar{x}, \bar{\alpha}, t), \alpha(\bar{x}, \bar{\alpha}, t)), \\ &(x, \alpha) \mapsto (r(x, \alpha), \rho(x, \alpha), \phi_r(x, \alpha) + t). \end{aligned}$$

A similar statement holds for the map $(\bar{x}, \bar{\beta}) \mapsto (s(, \bar{x}, \bar{\beta}), \sigma(\bar{x}, \bar{\beta}), \phi_s(\bar{x}, \bar{\beta}))$, where the role of the variables $\alpha, \bar{\alpha}, r, \rho$ and $\beta, \bar{\beta}, s, \sigma$ is interchanged and $t \leftrightarrow -t$. It follows that

$$\begin{aligned} \frac{\partial(r, \rho, \phi_r)}{\partial(\bar{x}, \bar{\alpha})}(\bar{x}, \bar{\alpha}) &= \frac{\partial(r, \rho, \phi_r)}{\partial(x, \alpha)}(x, \alpha) \cdot \frac{\partial(x, \alpha)}{\partial(\bar{x}, \bar{\alpha})}(\bar{x}, \bar{\alpha}, t), \\ \frac{\partial(s, \sigma, \phi_s)}{\partial(\bar{x}, \bar{\beta})}(\bar{x}, \bar{\beta}) &= \frac{\partial(s, \sigma, \phi_s)}{\partial(x, \beta)}(x, \beta) \cdot \frac{\partial(x, \beta)}{\partial(\bar{x}, \bar{\beta})}(\bar{x}, \bar{\beta}, -t). \end{aligned}$$

This can also be seen from expression (5) above. Thus the matrix M can be decomposed as follows

$$M = A \cdot M',$$

where A and M' are $(4n-3)\times(4n-2)$ resp. $(4n-2)\times(6n-4)$ matrices given by

$$A = \begin{pmatrix} \frac{\partial(r,\rho)}{\partial(x,\alpha)}(x,\alpha) & \\ & \frac{\partial(s,\sigma)}{\partial(x,\beta)}(x,\beta) \\ \frac{\partial\phi_r}{\partial(x,\alpha)}(x,\alpha) & \frac{\partial\phi_s}{\partial(x,\beta)}(x,\beta) \end{pmatrix},$$

$$M' = \begin{pmatrix} I_n & & & \frac{\partial x}{\partial\bar{x}}(\bar{x},\bar{\alpha},t) & \frac{\partial x}{\partial\bar{\alpha}}(\bar{x},\bar{\alpha},t) & \\ & I_{n-1} & & \frac{\partial\alpha}{\partial\bar{x}}(\bar{x},\bar{\alpha},t) & \frac{\partial\alpha}{\partial\bar{\alpha}}(\bar{x},\bar{\alpha},t) & \\ I_n & & & \frac{\partial x}{\partial\bar{x}}(\bar{x},\bar{\beta},-t) & & \frac{\partial x}{\partial\bar{\beta}}(\bar{x},\bar{\beta},-t) \\ & & I_{n-1} & \frac{\partial\beta}{\partial\bar{x}}(\bar{x},\bar{\beta},-t) & & \frac{\partial\beta}{\partial\bar{\beta}}(\bar{x},\bar{\beta},-t) \end{pmatrix}. \tag{13}$$

The matrix A has maximal rank since the matrices

$$\frac{\partial(r,\rho,\phi_r)}{\partial(x,\alpha)}(x,\alpha), \qquad \frac{\partial(s,\sigma,\phi_s)}{\partial(x,\beta)}(x,\beta)$$

are invertible. But it has a nonzero kernel. To find this kernel we note that

$$\frac{\partial(r,\rho)}{\partial(x,\alpha)}(x,\alpha)\cdot\begin{pmatrix} c(x)\,\alpha \\ 0 \end{pmatrix} = 0, \qquad \frac{\partial\phi_r}{\partial(x,\alpha)}(x,\alpha)\cdot\begin{pmatrix} c(x)\,\alpha \\ 0 \end{pmatrix} = -1.$$

The same is true with r,ρ,α and s,σ,β interchanged. Hence

$$\ker A = \operatorname{span}(c(x)\,\alpha, 0, -c(x)\,\beta, 0).$$

Now basic linear algebra says that

$$\operatorname{rank} M = \operatorname{rank} M' - \dim(\operatorname{range} M' \cap \ker A). \tag{14}$$

By trying to solve w in the system $v = M'\cdot w$ one finds that $v = (v_{x,1}, v_\alpha, v_{x,2}, v_\beta) \in \operatorname{range} M'$ if and only if $v_{x,1} - v_{x,2} \in \operatorname{range} C'$, where

$$C' = \begin{pmatrix} \frac{\partial x(\bar{x},\bar{\alpha},t)}{\partial\bar{x}} - \frac{\partial x(\bar{x},\bar{\beta},-t)}{\partial\bar{x}} & \frac{\partial x(\bar{x},\bar{\alpha},t)}{\partial\bar{\alpha}} & -\frac{\partial x(\bar{x},\bar{\beta},-t)}{\partial\bar{\beta}} \end{pmatrix}.$$

Hence $\operatorname{rank} M' = 3n-2+\operatorname{rank} C'$ and

$$\dim(\operatorname{range} M' \cap \ker A) = \dim(\operatorname{range} C' \cap \operatorname{span}(c(x)\alpha + c(x)\beta)).$$

Using (14) and the fact that

$$\operatorname{rank} C = \operatorname{rank} C' + 1 - \dim(\operatorname{range} C' \cap \operatorname{span}(c(x)\alpha + c(x)\beta))$$

the result (12) follows. $\square$

We recapitulate the results in the following theorem.

Theorem 3.2 *Assume the following. If $\bar{x}, \bar{\alpha}, \bar{\beta}, x, \alpha, \beta$ are such that (9), (10) are hold and $\|\bar{\alpha}+\bar{\beta}\| \geq \epsilon$, $\|\alpha+\beta\| > \epsilon$ (no scattering over π) and the $\bar{x}, \bar{\alpha}$-ray ($\bar{x}, \bar{\beta}$-ray) hits the receiver (source) surface, then the matrix C has maximal rank. Under this assumption the normal operator is the sum of a pseudodifferential operator $N_{\psi\mathrm{do}}$ of order $n-1$ and a nonlocal Fourier integral operator N_{nonlocal} of order $\frac{n-1}{2}$ with canonical relation given by (11).*

The new aspect is the simple characterisation using the matrix C. Note that the fact that the nonlocal part has lower order does not imply that it is less singular as an operator between Sobolev spaces. This will be the subject of the next section.

If there are no solutions to equations (9) such that the rays hit the receiver resp. source manifold, then the normal operator is purely pseudodifferential. This case has been discussed by Ten Kroode e.a. [6] and by Nolan and Symes [8].

4 Sobolev estimates for the nonmicrolocal part and invertibility

In the previous section it is shown that under certain conditions the normal operator is the sum of a pseudodifferential operator $N_{\psi\mathrm{do}}$ or order $n-1$ and a nonmicrolocal Fourier integral operator N_{nonml} of order $\frac{n-1}{2}$. The pseudodifferential part is an invertible operator $H_{(s)} \to H_{(s-n+1)}$. In this section we discuss for what values of m the nonmicrolocal part is continuous $H_{(s)} \to H_{(s-m)}$. If this is the case with $m < n-1$ then the nonmicrolocal part is less singular then the pseudodifferential part and their sum is asymptotically invertible. The infimum of the set

$$\{m \mid N_{\mathrm{nonml}} \text{ is continuous } H_{(s)} \text{ to } H_{(s-m)}\}$$

will be called the Sobelev order of N_{nonml}.

We first show that the nonmicrolocal part is at most as singular as the pseudodifferential part.

Lemma 4.1 *The operator F is continuous $H_{(s)} \to H_{(s-(n-1)/2)}$, and hence N is continuous $H_{(s)} \to H_{(s-n+1)}$.*

Proof We can write F as a finite sum $F = \sum_k F_k$, where the F_k have canonical relation $\Lambda_k \subset \Lambda$ such that $\Lambda_k^* \circ \Lambda_k$ is contained in the diagonal of $(T^*X\backslash 0) \times (T^*X\backslash 0)$. Then $F_k^* F_k$ is a pseudodifferential operator of order $n-1$ and hence continuous $H_{(s)} \to H_{(s-n+1)}$. Therefore the F_k and hence F are continuous $H_{(s)} \to H_{(s-(n-1)/2)}$. □

Some basic facts about the calculation of the Sobolev order for general Fourier integral operators $E'(X_2) \to D'(X_1)$ are described in Hörmander [3], section 25.3. It is well known that if the canonical relation C is the graph of a bijective canonical map $T^*X_1\backslash 0 \to T^*X_2\backslash 0$, then the Sobolev order equals the order of the Fourier integral operator. In that case the projections of C on $T^*X_1\backslash 0, T^*X_2\backslash 0$ are both bijective. If A is such a Fourier integral operator of order m then A^*A is a pseudodifferential operator of order $2m$ and hence A is a continuous map $H_{(s)} \to H_{(s-m)}$. If the principal symbol of such an operator is invertible it is called *elliptic*.

If the canonical relation is not the graph of a bijective canonical map, then the projections π_1, π_2 of C on $T^*X_1\backslash 0$, resp. $T^*X_2\backslash 0$ are not bijective. Denote by λ_1, λ_2 the corank of these projections at some point in C. The number $\lambda_1 + \lambda_2$ plays an important role in estimates of the Sobolev order.

In fact Theorems 25.3.8, 25.3.9 in Hörmander [3] give the estimate

$$\frac{\max_C(\lambda_1+\lambda_2)}{12} \leq \text{Sobolev order} - \text{order} \leq \frac{\max_C(\lambda_1+\lambda_2)}{4}. \qquad (15)$$

If $\lambda_1+\lambda_2$ is constant over C the right hand equality is valid. If the projection is singular only in a lower dimensional subset of C then in general the Sobolev order is better. For instance the left hand equality is valid when both the projections π_1, π_2 have a singularity of fold type.

The next step is the calculation of $\lambda_1+\lambda_2$ in the case at hand. Now X_1 and X_2 are open subsets of X. It follows from Hörmander [3] lemma 25.3.6 that $\lambda_1=\lambda_2$. Now if p is a set of coordinates for C_N then

$$\lambda_1(p) = \operatorname{corank} \frac{\partial(x,\xi)}{\partial p}, \qquad \lambda_2(p) = \operatorname{corank} \frac{\partial(\bar{x},\bar{\xi})}{\partial p}.$$

It is convenient to use new coordinates ν, ψ instead of α, β, such that ν denotes the direction of ξ, $\nu = \frac{\alpha+\beta}{\|\alpha+\beta\|}$, and ψ denotes the remaining $n-1$ coordinates (for instance so called GRT coordinates). Let $\lambda(\bar{x},\bar{\nu},\bar{\psi},\omega)$ be the length of $\bar{\xi}$, $\lambda(\bar{x},\bar{\nu},\bar{\psi},\omega) = \frac{\omega}{c(\bar{x})}\|\alpha(\bar{\nu},\bar{\psi})+\beta(\bar{\nu},\bar{\psi})\|$. Recall how the canonical relation C_N of N_{nonml} is defined. Let $\Delta x(\bar{x},\bar{\nu},\bar{\psi},t) = x(\bar{x},\alpha(\bar{\nu},\bar{\psi}),t) - x(\bar{x},\beta(\bar{\nu},\bar{\psi}),-t)$ and assume that $\frac{\partial \Delta x}{\partial(\bar{x},\bar{\nu},\bar{\psi},t)}$ has maximal rank. Let

$$V = \{(\bar{x},\bar{\nu},\bar{\psi},t) \,|\, \Delta x(\bar{x},\bar{\nu},\bar{\psi},t) = 0\},$$

then C_N can be parametrized by $(\bar{x},\bar{\nu},\bar{\psi},t) \in V$ and $\omega \in \mathbb{R}$

$$\begin{aligned} C_N = \{&(x(\bar{x},\bar{\nu},\bar{\psi},t),\xi(\bar{x},\bar{\nu},\bar{\psi},t,\omega);\bar{x},\lambda(\bar{x},\bar{\nu},\bar{\psi},\omega)\bar{\nu}) \,| \\ &(\bar{x},\bar{\nu},\bar{\psi},t) \in V, \omega \in \mathbb{R}\}. \end{aligned} \qquad (16)$$

For the calculation of λ_1, λ_2 we can omit the conic coordinate, and consider $\frac{\partial(\bar{x},\bar{\nu})}{\partial p}$, since the projection is not singular in the conic direction.

Lemma 4.2 *C_N can locally be parametrized by parameters (p,ω) where $p = (\bar{x}',\bar{\nu}',\bar{\psi}')$, a subset of coordinates of $(\bar{x},\bar{\nu},\bar{\psi},t)$ and ω parametrizes the conic direction. There are invertible matrices $M_1(p), M_2(p)$ such that*

$$\frac{\partial(\bar{x},\bar{\nu})}{\partial p} = M_1(p) \cdot \frac{\partial \Delta x}{\partial(\bar{\psi},t)}(\bar{x}(p),\bar{\nu}(p),\bar{\psi}(p),t(p)) \cdot M_2(p). \qquad (17)$$

It follows that $\lambda_1(p) = \lambda_2(p) = \operatorname{corank} \frac{\partial \Delta x}{\partial(\psi,t)}(\bar{x}(p),\bar{\nu}(p),\bar{\psi}(p),t(p))$.

Proof Because V is a smooth manifold and $\frac{\partial \Delta x}{\partial t} \neq 0$ we can split the set of coordinates $(\bar{x}, \bar{\nu}, \bar{\psi}, t)$ in two disjoint sets $(\bar{x}', \bar{\nu}', \bar{\psi}')$ and $(\bar{x}'', \bar{\nu}'', \bar{\psi}'', t)$ such that locally V can be given as $(\bar{x}'', \bar{\nu}'', \bar{\psi}'', t) = (\bar{x}''(\bar{x}', \bar{\nu}', \bar{\psi}'), \bar{\nu}''(\bar{x}', \bar{\nu}', \bar{\psi}'),$ $\bar{\psi}''(\bar{x}', \bar{\nu}', \bar{\psi}'), t(\bar{x}', \bar{\nu}', \bar{\psi}'))$. We have

$$\frac{\partial(\bar{x}, \bar{\nu})}{\partial p} = \frac{\partial(\bar{x}', \bar{\nu}', \bar{x}'', \bar{\nu}'')}{\partial(\bar{x}', \bar{\nu}', \bar{\psi}')} = \begin{pmatrix} I_k & 0 \\ \frac{\partial(\bar{x}'', \nu'')}{\partial(\bar{x}', \bar{\nu}')} & \frac{\partial(\bar{x}'', \bar{\nu}'')}{\partial(\bar{\psi}')} \end{pmatrix}.$$

On the other hand, by the implicit function theorem we have

$$\left(\frac{\partial \Delta x}{\partial(\bar{x}'', \bar{\nu}'', \bar{\psi}'', t)}\right)^{-1} \cdot \frac{\partial \Delta x}{\partial(\bar{\psi}', \bar{\psi}'', t)} = \begin{pmatrix} \frac{\partial(\bar{x}'', \bar{\nu}'')}{\partial(\bar{\psi}')} & 0 \\ \frac{\partial(\bar{\psi}'', t)}{\partial(\bar{\psi}')} & I_k \end{pmatrix}.$$

Combining these two equations it follows that there are invertible matrices M_1, M_2 so that (17) holds. $\square$

It is desirable to improve the very crude estimate (15). However, in the literature there are only few results on Sobolev estimates for general Fourier integral operators. There are results for the case $n = 2$. Phong and Stein [10, 9] give very precise estimates for certain Fourier integral operators with analytic phase functions that also possess a certain translation symmetry. Seeger [13] obtains somewhat weaker results in the case that the phase function is C^∞ and does not have this translation symmetry. For the case $n \geq 3$ we are not aware of any relevant results.

We will use Seeger's results to obtain an estimate for the nonmicrolocal part of the normal operator in the case $n = 2$. Seeger gives estimates for Fourier integral operators with relation C that is the conormal bundle of a codimension one submanifold M of $\mathbb{R}^2 \times \mathbb{R}^2$. The manifold M projects submersively on both factors $\mathbb{R}^2$. This is in general not the case, but by lemma 25.3.7 in Hörmander [3] one can always apply a symplectic coordinate transformation on X_1 and X_2 such that this condition is satisfied (a coordinate transformation that mixes x and ξ coordinates). Since a symplectic coordinate transformation corresponds to application of an elliptic Fourier integral operator of order 0 the Sobolev estimates are unchanged.

We will translate the situation in Seeger's paper to our case to produce a useful estimate. In his article it is assumed that M is given by $\Phi(x, \bar{x}) = 0$. The projections π_1, π_2 are singular if and only if the Monge-Ampere determinant defined by

$$I(x, \bar{x}) := \det \begin{pmatrix} \frac{\partial^2 \Phi}{\partial x \partial \bar{x}} & \left(\frac{\partial \Phi}{\partial x}\right)^t \\ \frac{\partial \Phi}{\partial \bar{x}} & 0 \end{pmatrix}.$$

vanishes. Let $(\delta x, 0)$ be a tangent vector to M. Then M is said to satisfy a right finite type condition of order m if

$$\left((\delta x, 0) \cdot \frac{\partial}{\partial(x, \bar{x})}\right)^k I(x, \bar{x}) \neq 0$$

for some $k \in \{1, \dots m-2\}$, for all $(x, \bar{x}) \in M$ such that $I(x, \bar{x}) = 0$. Similarly, if $(0, \delta\bar{x})$ is a tangent vector to M, then M satisfies a left finite type condition of order m if

$$\left((0, \delta\bar{x}) \cdot \frac{\partial}{\partial(x, \bar{x})}\right)^k I(x, \bar{x}) \neq 0$$

for some $k \in \{1, \dots m-2\}$, for all $(x, \bar{x}) \in M$, such that $I(x, \bar{x}) = 0$.

We give a formulation of these conditions for general canonical relations. In the case just described the canonical relation C is parametrized by $(x, \bar{x}) \in M$, and a parameter for the conic direction. If the projection is singular at some point $(x, \xi, \bar{x}, \bar{\xi}) \in C$, a tangent vector $(\delta x, 0)$ to M must correspond to a tangent vector $(\delta x, \delta\xi, 0, 0) \in T_{(x,\xi,\bar{x},\bar{\xi})}C$, that gives the direction where the projection on $(\bar{x}, \bar{\xi})$ is singular. Similarly a tangent vector $(0, \delta\bar{x})$ in $\mathbb{R}^2 \times \mathbb{R}^2$ corresponds to a tangent vector $(0, 0, \delta\bar{x}, \delta\bar{\xi}) \in T_{(x,\xi,\bar{x},\bar{\xi})}C$. Now suppose we have an arbitrary canonical relation, parametrized by p and suppose that the projections π_1 and π_2 are singular at some value of p. Then there are δp_{R} and δp_{L} that correspond to $(\delta x, \delta\xi, 0, 0) \in T_{(x,\xi,\bar{x},\bar{\xi})}C$, resp. $(0, 0, \delta\bar{x}, \delta\bar{\xi}) \in T_{(x,\xi,\bar{x},\bar{\xi})}C$. So C satisfies a right resp. left finite type condition of order m if

$$\left(\delta p_{\mathrm{R}} \cdot \frac{\partial}{\partial p}\right)^k \det \frac{\partial(\bar{x}, \bar{\xi})}{\partial p} \neq 0, \text{ resp.} \qquad \left(\delta p_{\mathrm{L}} \cdot \frac{\partial}{\partial p}\right)^k \det \frac{\partial(\bar{x}, \bar{\xi})}{\partial p} \neq 0$$

for some $k \in \{1, \dots, m-2\}$, for all p where the projection is singular.

It follows from Lemma 4.2 that in our case $\det \frac{\partial(\bar{x},\bar{\xi})}{\partial p}$ may be replaced by $\det \frac{\partial \Delta x}{\partial(\bar{\psi},t)}$. Now if $\det \frac{\partial \Delta x}{\partial(\bar{\psi},t)} = 0$ and $(\delta\bar{\psi}, \delta t) \in \ker \frac{\partial \Delta x}{\partial(\bar{\psi},t)}$, then $(0, 0, \delta\bar{\psi}, \delta t)$ is a tangent vector to V that corresponds to a vector of the form $(\delta x, \delta\xi, 0, 0) \in T_{(x,\xi,\bar{x},\bar{\xi})}C_N$. So C_N satisfies a right finite type condition of order m if

$$\left((\delta\bar{\psi}, \delta t) \cdot \frac{\partial}{\partial(\bar{\psi}, t)}\right)^k \det \frac{\partial \Delta x}{\partial(\bar{\psi}, t)} \neq 0 \tag{18}$$

for some $k \in 1, \dots m-2$. Suppose the projection is singular and let $(\delta\bar{x}, \delta\bar{\nu}, \delta\bar{\psi}, \delta t)$ correspond to a vector $(0, 0, \delta\bar{x}, \delta\bar{\xi}) \in T_{(x,\xi,\bar{x},\bar{\xi})}C_N$. Then C_N satisfies a left finite type condition of order m if

$$\left((\delta\bar{x}, \delta\bar{\nu}, \delta\bar{\psi}, \delta t) \cdot \frac{\partial}{\partial(\bar{x}, \bar{\nu}, \bar{\psi}, t)}\right)^k \det \frac{\partial \Delta x}{\partial(\bar{\psi}, t)} \neq 0 \tag{19}$$

for some $k \in 1, \ldots m-2$. By interchanging the role of $x, \bar{x}$ the left finite type condition may also be formulated as in (18).

From Theorem 1.1 in Seeger by interpolation the following for us relevant result follows.

Theorem 4.3 *Suppose $m \geq 3$ and suppose that C satisfies both a left and a right finite type condition of order m. Then a Fourier integral operator N of order m_N is continuous $H_{(s)} \to H_{(s-m_N-1/2+1/m-\epsilon)}$.*

The results of Phong and Stein are that, in the case that they consider, all the pairs of integers k, l with

$$\left(\delta p_{\mathrm{R}} \cdot \frac{\partial}{\partial p}\right)^k \left(\delta p_{\mathrm{L}} \cdot \frac{\partial}{\partial p}\right)^l \det \frac{\partial(\bar{x}, \bar{\xi})}{\partial p} \neq 0,$$

are important, including the pairs where both $k, l \neq 0$.

5 The translation invariant case: Maximal degeneracy

We have seen above that there are in general two problems with the normal operator. First it may not be a Fourier integral operator. Second the non-microlocal part may be as singular as the pseudodifferential part and then it is not clear whether the normal operator, and hence the linearized forward operator, is still invertible. We construct examples where these problems occur, for all dimensions n. This can be the case if in some part of X the soundspeed has a translation symmetry, such that it depends only on one space direction, and if there is a waveguide situation. It turns out that in that case the operator F is not fully invertible. In practice the medium often has such a translation symmetry, at least locally.

Assume the soundspeed depends only on the x_n direction, $c(x) = c(x_n)$ (since only the properties in the neighborhood of the two scattering points matter it doesn't need to be the vertical direction). Suppose that this function $c(x_n)$ has a minimum somewhere. Such a configuration acts as a waveguide, rays that are shot not too far from the minimum under small angles with the plane $x_n =$ constant will be deflected back towards the minimum of c. Suppose that a certain ray, shot from $x_0 = (0, \dots, 0, h)$ with an angle α, hits the plane $x_n = h$ again at time $T = T(h, \alpha)$. It follows from equations (4) that the quantities $c(x)\|\xi\|$ and $\xi_1, \dots, \xi_{n-1}$ are conserved so then $\xi(T) = \xi(0)$. Hence the ray is periodic in the sence that $(x(t + kT), \xi(x + kT)) = (x(t) + k(x(T) - x(0)), \xi(t))$. The curve obtained by reflecting the ray in the vertical line is also a solution to the ray equations. Let $\beta = R{\cdot}\alpha$, α reflected in the vertical line. Then $x(x_0, \beta, -T) = x(x_0, \alpha, T)$.

Now it may be the case that the assumptions of theorem 3.2 are violated, and that the nonmicrolocal part of the normal operator is not a Fourier integral operator.

Theorem 5.1 *For $n = 2$, and hence for any $n \geq 2$, there is $c(x) \in C^\infty(X)$ and $\bar{x}, \bar{\alpha}, \bar{\beta}, x, \alpha, \beta, t$, such that equations (9) are satisfied and* $\operatorname{rank} C = n - 1$.

Proof Denote the horizontal coordinate by x_1 and the vertical coordinate by x_2. Suppose we are in the situation described above, that the soundspeed depends only on x_2, $c(x) = c(x_2)$, and that it has a minimum somewhere, say at $x_2 = 0$. An example of a few rays in such a situation is shown in Figure 2. Let $\bar{x} = (0, h)$. It is possible that the wavefront leaving $\bar{x}$ forms caustics. In general the caustic points are on a line that is tangent to the rays. It is possible that there is a caustic somewhere at the line $x_2 = h$, say for an angle $\bar{\alpha}$ and time t. This is also shown in the figure. By definition we have at the caustic points

$$\frac{\partial x}{\partial \bar{\alpha}}(\bar{x}, \bar{\alpha}, t) = 0.$$

Define $\bar{\beta}$ by reflecting $\bar{\alpha}$ in the x_2 axis. Then the direction $\bar{\beta}$ gives a caustic at the same point on the x_1 axis for time coordinate $-t$. So

$$\frac{\partial x}{\partial \bar{\beta}}(\bar{x}, \bar{\beta}, -t) = 0.$$

Due to the translation symmetry we have

$$\frac{\partial x}{\partial \bar{x}}(\bar{x}, \bar{\alpha}, t) = I_2 = \frac{\partial x}{\partial \bar{x}}(\bar{x}, \bar{\beta}, -t).$$

So in this case only the last column of C is nonzero and $\operatorname{rank} C = 1$. $\square$

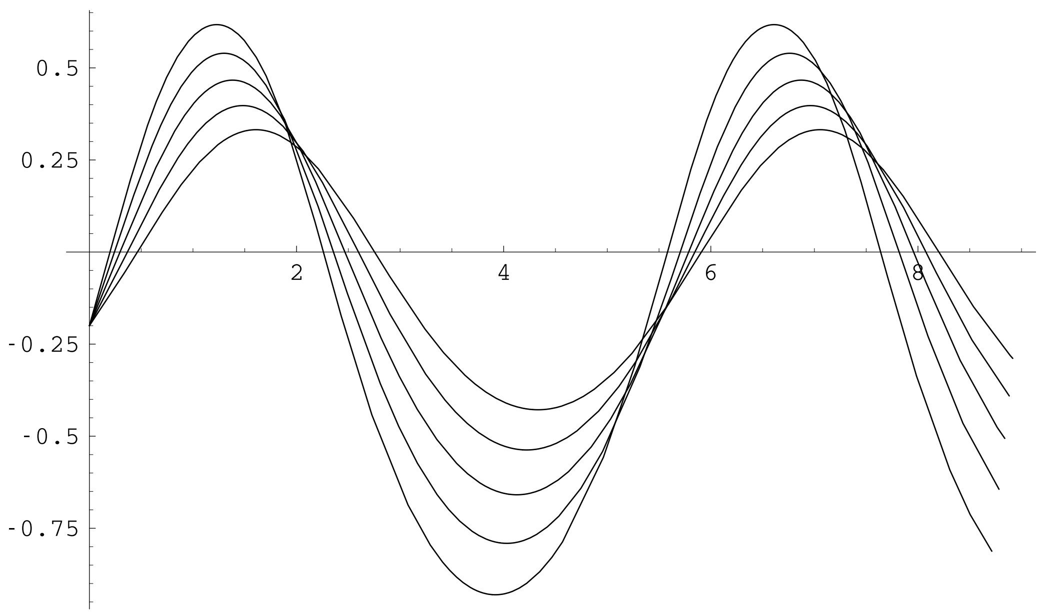

Figure 2: Ray trajectories in $\mathbb{R}^2$ when the wavespeed is given by $c(x) = 1 + .7x_2^2 + .5x_2^3 + .3x_2^4$. Caustics are formed that intersect the line $x_2 = 0.2$ (the vertical take off coordinate).

We proceed to show that the nonmicrolocal part is of the same order as the pseudodifferential part. The manifold V defined by (9) is given by

$$\bar{\beta} = R \cdot \bar{\alpha}, \qquad t = T(\bar{x}_n, \bar{\alpha}).$$

The images of the projections of C on the (x, ξ) and on the $(\bar{x}, \bar{\xi})$ variables are given by open subsets of $X \times \{(0, \dots, 0, \xi_n) \,|\, \xi_n \in \mathbb{R}\}$. So the projection is singular of order $n-1$, it is maximally singular. It follows that the non-microlocal part has Sobolev order $n-1$. According to Hörmander [3] it is continuous $H_{(s)} \to H_{(s-(n-1))}$). It is as singular as the local part.

It follows that if δc is such that $\mathrm{WF}(\delta c) \cap X \times \{(0, \dots, 0, \xi_n) \,|\, \xi_n \in \mathbb{R}\} = \varnothing$, then it is possible to recover δc microlocally. We now show that there are δc, with $\mathrm{WF}(\delta c) \subset X \times \{(0, \dots, 0, \xi_n) \,|\, \xi_n \in \mathbb{R}\}$ so that it is not possible to reconstruct them microlocally.

Theorem 5.2 *Let the medium $c(x)$ be as described above. There is some open set in $U \subset T^*Y$ and two contributions δc_1, δc_2, such that* $\mathrm{WF}(F \cdot \delta c_1) \cap U = \mathrm{WF}(F \cdot \delta c_2) \cap U \neq \varnothing$, *and* $\mathrm{WF}(F \cdot (\delta c_1 - \delta c_2)) \cap U = \varnothing$.

Proof Assume the source and receiver surface are located inside the medium where it is still translationally symmetric, at x_n is h. This is not a restriction. Suppose we have $s_0, \sigma_0, r_0, \rho_0, \bar{x}_0, \bar{\xi}_0, x_0, \xi_0$ in the situation described above (9). Let δc_1 be supported on a neighborhood of $\bar{x}$, and only depend on x_n around $\bar{x}$. Let δc_2 be δc_1 translated to a neighborhood of x. By symmetry

$$F \cdot \delta c_1 = F \cdot \delta c_2 \tag{20}$$

close to s_0, r_0. To see this consider a reflection such that $r \leftrightarrow s$.

$$\begin{aligned}(F \cdot \delta c)(r, s, t) &= (F \cdot \delta c_{\mathrm{refl}})(r_{\mathrm{refl}}, s_{\mathrm{refl}}, t) \\ &= (F \cdot \delta c_{\mathrm{refl}})(s, r, t) \\ &= (F \cdot \delta c_{\mathrm{refl}})(r, s, t).\end{aligned}$$

Now only the microlocal properties of δc around the scattering point matter. Clearly for s, r in small neighborhoods of s_0, r_0 we have $\delta c_{1,\mathrm{refl}} = \delta c_2$ close to the scattering point x. So (20) follows. This proves the theorem. $\square$

6 Generic properties of rays

In the next section it will be necessary to know how the rays and its derivatives are changed when the medium is perturbed. This will be discussed here. A discussion similar to ours can be found in Klingenberg [5], section 3.3. We use some Riemannian geometry, so in this section we will use upper indices for the coordinates, and lower indices for the slownesses.

The rays are the geodesics corresponding to the metric $g^{ij} = c(x)^2\delta^{ij}$. This is a conformal metric (proportional to δ^{ij}). The rays can be calculated by solving the Hamilton system associated to the Hamiltonian $H(x,\xi) = \sqrt{\sum_{i,j} g^{ij}\xi_i\xi_j} = c(x)\|\xi\|$, see (3). The square root is taken in the definition of H so that the parameter t is the traveltime (the arclength). The solution with initial values x_0, ξ_0 and parameter t will be denoted by $(x(x_0,\xi_0,t), \xi(x_0,\xi_0,t))$.

The derivative $\frac{\partial^m(x,\xi)}{\partial(x_0,\xi_0)^m}(x_0,\xi_0,t)$ can be calculated by solving the ordinary differential system

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial^m(x,\xi)}{\partial(x_0,\xi_0)^m}(x_0,\xi_0,t) = \frac{\partial^m}{\partial(x_0,\xi_0)^m}\Bigg(&\frac{\partial H}{\partial \xi}(x(x_0,\xi_0,t),\xi(x_0,\xi_0,t)),\\ &-\frac{\partial H}{\partial x}(x(x_0,\xi_0,t),\xi(x_0,\xi_0,t))\Bigg). \qquad (21)\end{aligned}$$

By applying the derivatives we see that this is a linear system with homogeneous term similar to the ordinary Jacobi equation

$$\begin{pmatrix} \frac{\partial^2 H}{\partial\xi\partial x} & \frac{\partial^2 H}{\partial \xi^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial \xi}\end{pmatrix} \cdot \frac{\partial^m(x,\xi)}{\partial(x_0,\xi_0)^m}.$$

The inhomogeneous term consists of products of lower derivatives $\frac{\partial^k(x,\xi)}{\partial(x_0,\xi_0)^k}$ with higher derivatives of H. The Jacobi equations (4) hence play a special role, and will discussed first, before we look at the the effect of perturbations of c.

The Jacobi equations can be simplified considerably by using coordinates that are centred around the ray, so called Fermi coordinates[1]. Fermi coordinates are described for instance in Klingenberg [5]. The new coordinates consist of a coordinate s that denotes the time along the ray and coordinates $x_{\mathrm{F}}^1, \ldots, x_{\mathrm{F}}^{n-1}$ that denote the distance from the ray, in units of time. To define them let $E_0, \ldots, E_{n-1}$ be a set of orthonormal vectors (with respect

[1]In the geophysical literature one can find so called ray centred coordinates that have similar properties, see e.g. Popov and Pšenčík [11].

to g^{ij}) in $T_{x_0}X$, such that $E_0 = \frac{\partial x}{\partial t}(x_0, \xi_0, 0)$. Denote by $E_i(t)$ the E_i parallel transported along the ray. Consider now the map

$$\Psi : (s, x_{\rm F}^1, \dots, x_{\rm F}^{n-1}) \mapsto \exp_{x(x_0,\xi_0,s)}(\sum_{i=1}^{n-1} E_i(s) x_{\rm F}^i).$$

When the $x_{\rm F}^i$ are sufficiently close to 0 this map defines a set of coordinates around the ray. The transformation matrix can be written

$$\frac{\partial x^i}{\partial s}(s,0) = E_0^i(s), \qquad \frac{\partial x}{\partial x_{\rm F}^j}(s,0) = E_j^i(s),$$

where E_j^i denotes the i-th component of E_j. The slownesses will be denoted by $\sigma, \xi_i^{\rm F}$. Note that if ξ is close to ξ_0 then σ is approximately equal to the length of ξ, while while $\frac{\xi_i^{\rm F}}{\sigma}$ parametrizes the angle with ξ_0. From the definition it follows that for the new metric $g_{ij}^{\rm F}$

$$\begin{aligned} g_{ij}^{\rm F}(s,0) = \delta_{ij}, \qquad \frac{\partial g_{ij}^{\rm F}}{\partial x_{\rm F}^k}(s,0) = 0, \\ \frac{\partial^2 g_{00}^{\rm F}}{\partial x_{\rm F}^i x_{\rm F}^j}(s,0) = -2R_{0i0j}^{\rm F}(s,0), \end{aligned} \tag{22}$$

where the index 0 corresponds to the s coordinate and R_{ijkl} is the Riemann curvature tensor.

To obtain the ray and its derivatives in the new coordinates we can just set $H(s, x_{\rm F}, \sigma, \xi^{\rm F}) = \sqrt{\sum_{i=0}^{n-1} g_{\rm F}^{ij} \xi_i^{\rm F} \xi_j^{\rm F}}$. Obviously the ray is given by $s(t) = t, \sigma(t) = \sigma_0, x_{\rm F}(t) = 0, \xi^{\rm F}(t) = 0$. We set $\sigma_0 = 1$. When we now do the calculation for the Jacobi equations we find that the nonzero parts of the Jacobi matrix are $\frac{\partial s}{\partial s_0} = 1$, $\frac{\partial \sigma}{\partial \sigma_0} = 1$ and $\frac{\partial(x_{\rm F}, \xi^{\rm F})}{\partial(x_{{\rm F},0}, \xi^{{\rm F},0})}$ that satisfies the following ODE along the ray

$$\frac{\partial}{\partial t} \frac{\partial(x_{\rm F}, \xi^{\rm F})}{\partial(x_{{\rm F},0}, \xi^{{\rm F},0})}(t) = \begin{pmatrix} 0 & I_{n-1} \\ A(t) & 0 \end{pmatrix} \frac{\partial(x_{\rm F}, \xi^{\rm F})}{\partial(x_{{\rm F},0}, \xi^{{\rm F},0})}(t), \tag{23}$$

where

$$A_{ij}(s) = \tfrac{1}{2} \frac{\partial^2 g_{00}^{\rm F}}{\partial x_{\rm F}^i \partial x_{\rm F}^j} = -R_{i0j0}^{\rm F}(s,0).$$

To obtain an expression for the matrix A_{ij} in terms of the original quantity $c(x)$ one first computes the Riemann curvature in the original coordinates.

Then the transformation rules are used to obtain the curvature in the new coordinates. This leads to the following result

$$A_{ij} = -\delta_{ij} c^{-1} \frac{\partial^2 c}{\partial s^2} - c^{-1} \frac{\partial^2 c}{\partial x_{\mathrm{F}}^i \partial x_{\mathrm{F}}^j} + \text{lower derivatives of } c.$$

So we now have a precise description of the matrix $\frac{\partial(s, x_{\mathrm{F}})}{\partial(s_0, x_{\mathrm{F},0})}$.

Now assume that the soundspeed is perturbed by an amount $\delta c(x)$. We investigate the corresponding first order perturbations of $\delta s(t), \delta x_{\mathrm{F}}(t), \delta\xi^{\mathrm{F}}(t)$. To simplify the notation we define $z_\kappa = (x_{\mathrm{F}}^1, \ldots, x_{\mathrm{F}}^{n-1}, \xi_1^{\mathrm{F}}, \ldots, \xi_{n-1}^{\mathrm{F}})$, $J_{\kappa\lambda} = \begin{pmatrix} 0 & I_{n-1} \\ -I_{n-1} & 0 \end{pmatrix}$.

Lemma 6.1 *A perturbation that is nonzero on the ray results in a nonzero perturbation $\delta s(t)$. One can obtain arbitrary $\delta s(t)$ by choosing δc suitably. A perturbation that vanishes of order $m+1$ on the ray ($m = 0, 1, \ldots$) results in a perturbation $\delta\frac{\partial^m z}{\partial z_0^m}$, while $\delta s(t) = 0$ and $\delta\frac{\partial^k z}{\partial z_0^k} = 0$ for $k < m$. Let $v_1, \ldots, v_m \in \mathbb{R}^{2n-2}$ and let $(v_1, \ldots, v_m)\cdot\delta\frac{\partial^m z}{\partial z_0^m} \in \mathbb{R}^{2n-2}$ be some contraction of $\delta\frac{\partial^m z}{\partial z_0^m}$ with the vectors $v_1, \ldots, v_m$. One can obtain arbitrary values of $(v_1, \ldots, v_m)\cdot\delta\frac{\partial^m z}{\partial z_0^m}(t) \in \mathbb{R}^{2n-2}$ by choosing δc suitably.*

Proof If we have a perturbation $\delta c(x)$ then

$$\delta g^{ij} = 2c\,\delta c\,\delta^{ij}, \qquad \delta g_{\mathrm{F}}^{ij} = \sum_{k,l} \frac{\partial x_{\mathrm{F}}^i}{\partial x^k} \frac{\partial x_{\mathrm{F}}^j}{\partial x^l}\, 2c\,\delta c\,\delta^{kl},$$

and

$$\delta H = \frac{1}{2\sqrt{\sum_{i,j} g_{\mathrm{F}}^{ij} \xi_i^{\mathrm{F}} \xi_j^{\mathrm{F}}}} \sum_{k,l} \xi_k^{\mathrm{F}} \xi_l^{\mathrm{F}}\, \delta g_{\mathrm{F}}^{kl}.$$

Now suppose we set

$$\delta c(s, x_{\mathrm{F}}) = c(s, x_{\mathrm{F}})\, \delta A(s),$$

close to the ray, going smoothly to zero away from the ray. In that case we find $\frac{\partial}{\partial t}\delta s(t) = \frac{\delta c(s)}{c(s)}$, and hence

$$\delta s(t) = \int_0^t \delta A(t')\, \mathrm{d}t'.$$

So by this choice of δc it is possible to obtain an arbitrary perturbation in the direction along the ray.

Next suppose δc is of the form

$$\delta c(s, x_{\mathrm{F}}) = c(s, x_{\mathrm{F}}) \sum_{k_1,\dots,k_{m+1}} \delta A_{k_1\dots k_{m+1}}(s) x_{\mathrm{F}}^{k_1} \dots x_{\mathrm{F}}^{k_{m+1}}, \tag{24}$$

close to the ray, again going smoothly to zero away from the ray. For the perturbation $\delta\frac{\partial^m z}{\partial {z_0}^m}$ we find

$$\begin{aligned}\frac{\partial}{\partial t}\delta\frac{\partial^m z_\kappa}{\partial z_{0,\lambda_1}\dots\partial z_{0,\lambda_m}} = \sum_{\mu,\nu_1,\dots,\nu_m} J_{\kappa\mu}\frac{\partial^{m+1}\delta H}{\partial z_\mu \partial z_{\nu_1}\dots\partial z_{\nu_m}}\frac{\partial z_{\nu_1}}{\partial z_{0,\lambda_1}}\cdots\frac{\partial z_{\nu_m}}{\partial z_{0,\lambda_m}} + \\ + \sum_{\mu,\nu} J_{\kappa\mu}\frac{\partial^2 H}{\partial z_\mu \partial z_\nu}\delta\frac{\partial^m z_\nu}{\partial z_{0,\lambda_1}\dots\partial z_{0,\lambda_m}}.\end{aligned} \tag{25}$$

There are no other nonzero contributions since $\delta\frac{\partial^k z}{\partial {z_0}^k}$ vanishes for $k < m$ and $\frac{\partial^k \delta H}{\partial z^k}$ vanishes for $k < m+1$. The only nonzero part of $\frac{\partial^{m+1}\delta H}{\partial z^{m+1}}$ is when all the derivatives are with respect to x_{F} and they all act on δc, so

$$\frac{\partial^{m+1}\delta H}{\partial z_{\mu_1}\dots\partial z_{\mu_{m+1}}} = \delta A_{\mu_1\dots\mu_{m+1}},$$

where we define $\delta A_{\mu_1\dots\mu_{m+1}}$ to be 0 if any of the indices $\mu_i > n-1$. Let $\Phi_{\kappa\lambda}(t,t')$ be the fundamental solution to (23). The solution of the differential equation is

$$\delta\frac{\partial^m z_\kappa}{\partial z_{0,\lambda_1}\dots\partial z_{0,\lambda_m}}(t) = \int_0^t \sum_{\mu,\nu_1\dots\nu_m,\rho} \Phi_{\kappa\mu}(t,t') J_{\mu\rho}\delta A_{\rho\nu_1\dots\nu_m}\prod_{i=1}^m \Phi_{\nu_i\lambda_i}(t',0)\,\mathrm{d}t'. \tag{26}$$

By choosing $\delta A_{k_1\dots k_{m+1}}$ in different ways a large set of values $\delta\frac{\partial^m z}{\partial {z_0}^m}$ can be obtained. To see this rewrite the solution as

$$\begin{aligned}\frac{\partial^m z_\kappa}{\partial z_{0,\lambda_1}\dots\partial z_{0,\lambda_m}}(t) &= \sum_{\sigma_1,\nu_1,\dots,\nu_m} \delta Z_{\kappa\nu_1\dots\nu_m}\prod_{i=1}^m \Phi_{\nu_i\lambda_i}(t,0), \\ \delta Z_{\kappa\lambda_1\dots\lambda_m} &= \int_0^t \sum_{\mu,\rho,\nu_1,\dots,\nu_m} \Phi_{\kappa\mu}(t,t')J_{\mu\rho}\delta A_{\rho\nu_1\dots\nu_m}(t')\prod_{i=1}^m \Phi_{\nu_i\lambda_i}(t',0)\,\mathrm{d}t'.\end{aligned} \tag{27}$$

If $t - t'$ is small then

$$\Phi(t,t') = \begin{pmatrix} I_{n-1} & (t-t')I_{n-1} \\ (t-t')A(t) & I_{n-1}\end{pmatrix}.$$

up to higher order terms (by this we mean that for each subblock there may be higher order terms of different order). Now suppose that δA is supported in a small interval (t_1, t). Then, to highest order, we can give an explicit expression for δZ. In the following we write $\kappa = k+\alpha(n-1)$, $\lambda_i = l_i+\beta_i(n-1)$, $\alpha, \beta_i \in \{0,1\}$, $k, l_i \in \{1, \ldots, n-1\}$ to indicate whether κ, λ_i refer to x or to ξ coordinates. With this notation

$$\delta Z_{\kappa\lambda_1\ldots\lambda_m} = -\int_{t_1}^{t} (-1)^{1+\beta_1+\ldots+\beta_m}(t-t')^{1-\alpha+\beta_1+\ldots+\beta_m}\delta A_{kl_1\ldots l_m}(t')\,\mathrm{d}t'.$$

The tensor δZ is far from the general tensor, but it follows from this equation that the contraction of δZ with m vectors $\in \mathbb{R}^{2n-2}$ is the general element of $\mathbb{R}^{2n-2}$. It follows that the same is true for $\delta\frac{\partial^m z}{\partial z_0^m}(t)$. This proves the lemma. $\Box$

7 Generically the normal operator is invertible

Let S be the set of media $c \in C^\infty(X)$ with the property that the normal operator is a Fourier integral operator and that it is invertible. In this section we show that the set S is contains "almost all" of $C^\infty(X)$. Such a property is called generic. Because $C^\infty(X)$ is only a topological space and there is no measure on $C^\infty(X)$, that means that S contains a countable intersection of open dense sets (see e.g. Klingenberg [5], p. 108). The argument

To obtain this result we show that genericly the assumptions of Theorems 3.2 and 4.3 are satisfied. First we discuss the question whether N is a Fourier integral operator. Let

$$\begin{aligned} &u = (\bar{x}, \bar{\alpha}, \bar{\beta}, t, v) \in U, \\ &U = X \times \{(\alpha, \beta) \in S^{n-1} \times S^{n-1} \mid \alpha + \beta \neq 0\} \times I_t \times S^{n-1}. \end{aligned} \tag{28}$$

and let

$$w(u) = (x(\bar{x}, \bar{\alpha}, t) - x(\bar{x}, \bar{\beta}, -t), v \cdot C) \in \mathbb{R}^{4n-1}.$$

We assume there is no scattering at points very close to the boundary of X, so that $\bar{x}$ can be taken in a compact subset of X. We also assume there is no scattering over angles very close to π, so that $\|\alpha + \beta\| \geq \epsilon$ for some $\epsilon > 0$. Under these assumptions it follows from Theorem 3.2 that there is a compact subset $\tilde{U}$ of U such that the normal operator is a Fourier integral operator if $w(u; c) \neq 0$ for all $u \in \tilde{U}$. The following theorem states that this property is generic.

Theorem 7.1 *If $\tilde{U}$ is a compact subset of U (defined in (28)), then the set of media $c \in C^\infty(X)$ such that $w(u; c) \neq 0$ for all $u \in \tilde{U}$ is open and dense. Hence genericly the normal operator is a Fourier integral operator.*

The equation $w(u; c) = 0$ consists of $4n - 1$ equations in $4n - 2$ unknowns, so naively one could argue that the set of media that violate this property is of "codimension 1" and hence the set of allowed media contains "almost all" of $C^\infty(X)$. This argument can be made rigorous by the following lemma of Mather ([7], Lemma 3.2).

Lemma 7.2 *Let F be a topological space. Let U, W be manifolds, and V a submanifold of W. Let for each $f \in F$ there be a mapping $U \to W : u \mapsto w(u; f)$. Suppose for each $f \in F$ there exists an integer k and a continuous*

k-parameter family $\mathbb{R}^k \ni p \mapsto f_p$ in F, $f_{p_0} = f$, such that on a neighborhood of p_0 the map $(u,p) \mapsto w(u; f_p)$ is C^∞ and transversal to V. Then

$$\{f \in F \,|\, u \mapsto w(u,f) \text{ is transversal to } V\}$$

is dense in F.

In the present case we take $F = C^\infty(X)$ and $V = \{0\}$. Because $\dim U + \dim V = 4n-2 < \dim W = 4n-1$ transversal intersection of the map $u \mapsto w(u;c)$ with V means that $w(u;c) \neq 0$ for $u \in U$. We show that in this case the assumption of Lemma 7.2 is satisfied.

Lemma 7.3 *If $w(u_0;c) = 0$ then for suitably chosen small perturbations δc_i of c the perturbations $\delta w_i(u_0;c)$ together with the $\frac{\partial w}{\partial u_j}(u_0;c)$ span $\mathbb{R}^{4n-1}$.*

Proof We first show the following. There is a finite set of variations $\delta c_1, \ldots, \delta c_n$ such that $\delta(x(\bar{x},\bar{\alpha},t) - x(\bar{x},\bar{\beta},-t))$ can be any element of $\mathbb{R}^n$. To prove this consider variations δc supported around the α ray, away from the β ray. The statement now follows from Lemma 6.1.

Now we show that if $v \cdot C = 0$ then there is a finite set $\delta c_1(x), \ldots, \delta c_k(x)$ of perturbations of $c(x)$, that leave the $(\bar{x},\bar{\alpha})$ and $(\bar{x},\bar{\beta})$ rays unchanged, while

$$v \cdot \delta C'$$

can be any element of $\mathbb{R}^{3n-2}$. Here C' is defined as the matrix C with the last column omitted. The last column $C''$of C is always nonzero, so we can choose δv such that $\delta v \cdot C''$ is nonzero. To prove that $v \cdot \delta C'$ can be any element of $\mathbb{R}^{3n-2}$ note that the inner product of v with the last column of C is $v \cdot (\alpha + \beta) = 0$. It follows that v has a nonzero component orthogonal to α, and also a nonzero component orthogonal to β. Denote these, in Fermi coordinates by $v^{\mathrm{F},\alpha}, v^{\mathrm{F},\beta}$. By Lemma 6.1 we can find δc around the α and δc around the β ray such that $v^{\mathrm{F},\alpha} \cdot \delta\left(\frac{\partial(x_{\mathrm{F}},\alpha)}{\partial \bar{x}_{\mathrm{F}}}\right), v^{\mathrm{F},\beta} \cdot \delta\left(\frac{\partial(x_{\mathrm{F}},\beta)}{\partial \bar{x}_{\mathrm{F}}}\right)$ can take any values in $\mathbb{R}^{2n-2}$. We transform back to the original coordinates using the E_i, $i = 1, \ldots, n-1$. Since the E_i corresponding to the α ray together with the E_i corresponding to the β ray span $\mathbb{R}^n$ it follows that $v \cdot \delta C'$ can be any vector in $\mathbb{R}^{3n-2}$. $\square$

Proof of Theorem 7.1 Let $u_0 \in U$. It follows from Lemma 7.3 that there is an open neighborhood U_0 of u_0 and parameters p such that the map $(u,p) \mapsto w(u;c_p)$ intersects $\{0\}$ transversally for $u \in U_0, p \in P$. By Lemma 7.2 this implies that the set S_{U_0} of $c \in C^\infty(X)$ that satisfy $w(u;c) \neq 0$

on U_0 is dense. Since the map $(c,u) \mapsto w(u;c)$ is continuous it is also open. The set $S_{\tilde{U}}$ of $c(x)$ that satisfies $w(u) \neq 0$ on $\tilde{U}$ is a finite intersection of such S_{U_0}. Therefore it is also open and dense. $\square$

Next for the case $n=2$ we apply the same procedure to the question whether the nonmicrolocal part of the normal operator has Sobolev order less than $n-1=1$. We show that if $m \in \mathbb{N}$ is sufficiently large then generically C_N will satisfy both a left and a right finite type condition of order m. Let

$$\begin{aligned} u &= (\bar{x}, \bar{\nu}, \bar{\psi}, t, v) \in U, \\ U &= X \times S^1 \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times I_t \times S^1. \end{aligned} \tag{29}$$

Here $\bar{\nu}, \bar{\psi}$ are coordinates as in section 4,

$$\bar{\nu} = \frac{\bar{\alpha} + \bar{\beta}}{\|\bar{\alpha} + \bar{\beta}\|}, \qquad \bar{\psi} = \angle(\bar{\alpha}, \bar{\nu}).$$

Let $m \in \mathbb{N}$ be the order of the finite type condition and let

$$\begin{aligned} w(u;c) = \Bigg(& \Delta x(\bar{x}, \bar{\nu}, \bar{\psi}, t), \det \frac{\partial\, \Delta x}{\partial(\bar{\psi}, t)}, L(v) \det \frac{\partial\, \Delta x}{\partial(\bar{\psi}, t)}, \ldots, \\ & L(v)^{m-2} \det \frac{\partial\, \Delta x}{\partial(\bar{\psi}, t)}, \frac{\partial\, \Delta x}{\partial(\bar{\psi}, t)} \cdot (\delta\bar{\psi}, \delta t) \Bigg). \end{aligned} \tag{30}$$

where $L(v) = [v \cdot \frac{\partial}{\partial(\bar{\psi},t)}]$. Now according to section 4 and in view of the assumptions made above there is a compact subset $\tilde{U}$ of U such that the canonical relation of N_{nonml} satisfies a right and (by interchanging the role of the $x, \bar{x}$ variables) a left finite type condition of order m if $w(u;c) \neq 0$ for all $u \in \tilde{U}$. This property is generic for $m=6$.

Theorem 7.4 *Let $m=6$. If $\tilde{U}$ is some compact subset of U (defined in (29)) then the set of media $c \in C^\infty(X)$ such that $w(u;c) \neq 0$ for $u \in \tilde{U}$ is open and dense. This implies that generically the normal operator is asymptotically invertible.*

The proof of the theorem parallels that of Theorem 7.1. We will omit the proof and only give the lemma that replaces Lemma 7.3.

Lemma 7.5 *For suitably chosen perturbations δc_i of c the perturbations $\delta w_i(u_0;c)$, together with the $\frac{\partial w}{\partial u_j}(u_0;c)$ span $\mathbb{R}^{m+2}$.*

Proof Let $k \in \{0, \dots, m-1\}$. By suitable perturbations that are supported around the α-ray or the β-ray, and vanish of order $k+1$ on the ray (see (24)) we can obtain arbitrary value of

$$\delta \frac{\partial^k \Delta x}{\partial \bar{\psi}^k},$$

while perturbations of lower derivatives vanish. In the case $k = 0$, we can have arbitrary $\Delta x(\bar{x}, \bar{\nu}, \bar{\psi}, t)$. For $k > 0$ we see that

$$\delta \left[v \cdot \frac{\partial}{\partial(\bar{\psi}, t)} \right]^{k-1} \det \frac{\partial \Delta x}{\partial(\bar{\psi}, t)} = v_1^{k-1} \det \begin{pmatrix} \delta \frac{\partial^k \Delta x}{\partial \bar{\psi}} & \frac{\partial \Delta x}{\partial t} \end{pmatrix},$$

(where v_1 is the first component of v). So by successively choosing perturbations vanishing of order $k = 0, \dots, m-1$ we can have all components of δw arbitrary. $\square$

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