A Voronoi poset.

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Abstract

Given a set S of n points in general position, we consider all k-th order Voronoi diagrams on S, for $k = 1, \ldots, n$, simultaneously. We deduce symmetry relations for the number of faces, number of vertices and number of circles of certain orders. These symmetry relations are independent of the position of the sites in S. As a consequence we show that the reduced Euler characteristic of the poset of faces equals zero whenever n odd.

1 Notation.

$\mathcal{A}(S)$	–	the arrangement defined by all bisectors in S.
$B(a, b)$	–	the bisector of a and b.
$CH(S)$	–	the convex hull of S.
δU	–	the boundary of the set U.
$V_k(S)$	–	k-th order Voronoi diagram on S
$\bigodot_{a,b,c}$	–	circle defined by points a, b and c
$\mid \bigodot_{a,b,c} \mid$	–	number of points from S inside the circle.
$[n]$	–	$\{1, \ldots, n\}$
$\Pi(S)$	–	The Voronoi poset on S.
f_i	–	number of regions in $V_i(S)$.
$\bar{f}_i$	–	number of i-dimensional faces.
v_i	–	number of vertices in $V_i(S)$.
e_i	–	number of edges in $V_i(s)$.
c_i	–	number of circles of order i.

2 Introduction.

The dynamics of Voronoi diagrams in the plane is well understood. When $n - 1$ points are fixed and one point is moving continuously somewhere inside the convex hull, combinatorial changes of the Voronoi diagram correspond to changes in the configuration of empty circles. See for example [AGMR]. Changes in the configuration of non-empty circles correspond with combinatorial changes of higher order Voronoi diagrams. Here the k-th order Voronoi diagram associates to each subset of size k of generating sites that area in the plane that consist of points closest to these k sites.

We consider all k-th order Voronoi diagrams simultaneously for k between 1 and n. We do so by introducing the Voronoi poset of a set S of n different sites in the

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plane. The poset consists of all sets of labels that correspond with a subset of sites that defines some non-empty Voronoi region in some k-th order Voronoi diagram.

Higher order Voronoi diagrams have been investigated by numerous people. Many results are published in an article by D.T. Lee, see [Le]. A survey is given in Edelsbrunners book on algorithms in combinatorial geometry, see [Ed]. He ends his paragraph on the complexity of higher order Voronoi diagrams with the following remark.

> Interestingly enough, the number of regions of $V_k(S)$ is thus exactly $kn - k^2 + 1$ if n is odd, if $k = (n+1)/2$ and if no three sites in S are collinear and no four sites are cocircular.

We generalize this result by giving a symmetry relation for the number of regions f_k in the k-th order Voronoi diagram. Assuming general position as in the quote, the following equation holds for any k.

$$f_k + f_{n-k+1} = 2k(n-k+1) + 1 - n$$

See Lemma 5.1. We prove a similar result for the number of vertices, see Lemma 5.2. As mentioned above, circle configurations have a close relation with higher order Voronoi diagrams. Suppose we are given a set S of n points in general positions. Every three points of S define an unique circle, see Figure 1. Let c_i denote the number of circles defined by S that contain exactly i points of S. Then Theorem 5.3 proves

$$c_i + c_{n-i-3} = 2(i+1)(n-i-2)$$

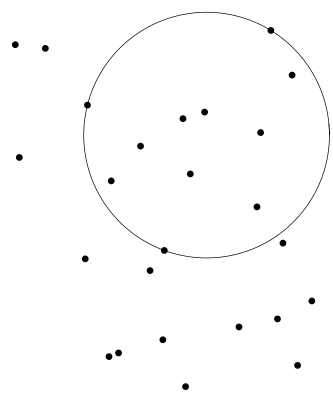

Figure 1: An invariant for circle configurations.

Note that these two symmetry relations are independent of the particular position of the sites in S: although the number of, for example, regions in some k-th order Voronoi diagram will vary, depending on the configuration, the sum of the number of regions in the k-th order diagram and the number of regions in the $(n-k+1)$-th diagram will stay the same.

Closely related is the question between which bounds the number of regions of the k-th order diagram can vary. Tight lower bounds for $k - 1 < n/2$ are known, see [Ed]. The symmetry relation gives some bound on the other side.

Another application application of the symmetry relations is the analysis of the reduced Euler characteristic of the Voronoi poset. It turns out that

$$\tilde{\chi}(\Pi(S)) \quad = \quad 0$$

whenever n is odd, see Theorem 6.2.

As there is a tight connection between higher order Voronoi diagrams in the plane and level sets of hyperplane arrangements corresponding with the projection of the Voronoi diagram on the unit paraboloid in $\mathbb{R}^3$, some of these results may help in the investigation of complexity issues with respect to those level sets.

I most certainly want to acknowledge Wilberd van der Kallen and Dirk Siersma for asking me many critical questions guiding me in good directions.

3 Higher order Voronoi diagrams.

Definition of k-th order Voronoi diagram.

Suppose we are given a set $S = \{s_1, \ldots, s_n\}$ of n points in the plane. Essential in this work is the following assumption, the *general position* assumption.

general position.

1. No more than three points from S lie on a common circle.
2. No more than two points from S lie on a common line.

Let $0 \leq k \leq n$. For every point p in the plane we can ask for the k nearest points from S. That is, we look for a subset $A \subset S$, such that

$$|A| = k, \qquad \forall x \in A, \quad \forall y \in S - A : \quad d(p, x) \quad \leq \quad d(p, y)$$

For two points in $\mathbb{R}^2$, we define a halfplane

$$h(x, y) \;:=\; \{p \in \mathbb{R}^2 \mid d(x, p) \leq d(y, p)\}$$

We define the *Voronoi region* of $A \subset S$ of *order* $|A|$ as the intersection of halfplanes

$$V(A) \;:=\; \bigcap_{x \in A,\; y \in S - A} h(x, y)$$

whenever this intersection is not empty. As an intersection of halfplanes, $V(A)$ will be a convex polygon. Assuming general position implies that a Voronoi region cannot degenerate to a line segment or a point. This is proved in Appendix A.

We define the *k-th order Voronoi diagram* as the subdivision of $\mathbb{R}^2$, induced by the set of Voronoi regions of order k. For later purposes, we identify the k-th order Voronoi diagram with the set of non empty k-th order Voronoi regions.

$$V_k(S) := \{V(A) \mid A \subset S,\; |A| = k,\; V(A) \neq \emptyset\}$$

The Voronoi poset.

In the following fix a labelling of the sites in S and identify a set of sites $A \subset S$ that defines a non-empty Voronoi region $V(A)$ with the set of labels $L(A) \subset [n]$ of the sites in A. Thus, a subset L of $[n]$ might or might not correspond to some Voronoi region $V(A_L)$.

Note that for $k = 1$ we get back the ordinary Voronoi diagram, which means that we have the correspondence

$$V_1(S) \;\leftrightarrow\; \{\{1\}, \{2\}, \ldots, \{n\}\}$$

We define that $V_0(S) = \{\emptyset\}$. $V_n(S)$ corresponds to the set $\{\{1, \ldots, n\}\}$. We consider the set of all Voronoi regions that appear for a given set S of points and call the set of corresponding labels the *Voronoi poset* $\Pi(S)$ of S.

$$\Pi(S) \;:=\; \bigcup_k \{\; L(A) \mid V(A) \in V_k(S) \;\}$$

This definition also makes sense when we drop the general position assumption.

Circles and higher order Voronoi diagrams.

We state some elementary properties of higher order Voronoi diagrams.

Every edge in $V_k(S)$ is part of some bisector $B(a, b)$, $a, b \in S$, and the Voronoi vertices are exactly those points that are in the circle centres of three points from S. Therefore, under our general position assumption, every Voronoi vertex has valency three. The following theorem describes the local situation around a Voronoi vertex.

Theorem 3.1 [De] *Let x be the centre of $\bigodot_{a,b,c}$, for $a, b, c \in S$. Let*

$$H \;=\; \{\; z \in S \mid d(x, z) < d(x, a) \;\}$$

and let $k = |H|$. Then x is a Voronoi vertex of $V_{k+1}(S)$ and $V_{k+2}(S)$. The Voronoi edges and regions that contain x are given in Figure 2.

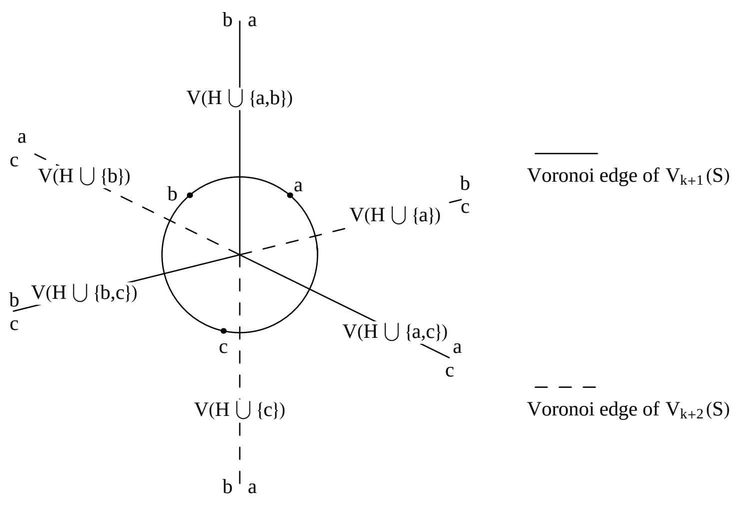

Figure 2: The Voronoi diagram around x.

In fact, all vertices, edges and regions can be described as in Theorem 3.1, see [De] for more details. Let a, b, c and H be as in Theorem 3.1. We will define the *order* of a circle $\bigodot_{a,b,c}$ as $|H|$. Notation: $|\bigodot_{a,b,c}| \;:=\; |H|$. Thus the order of a Voronoi circle $\odot_{a,b,c}$ equals the number of points of $S - \{a, b, c\}$ it contains. Denote the number of circles of order k by c_k and the number of vertices in a k-th order Voronoi diagram by v_k. As a consequence of Theorem 3.1 we get

$$v_k = c_{k-1} + c_{k-2} \tag{3.1}$$

Counting vertices, edges and regions.

The following theorem shows that the total number of vertices, edges and Voronoi regions does not depend on the positions of the points in S, assuming general position.

Theorem 3.2 *Let v_k, e_k and f_k denote the number of vertices, edges and regions in $V_k(S)$ for some set S of size n in general position. Then the total number of vertices, edges and regions in the Voronoi diagram of all orders are as follows.*

(i). $\sum_{k=1}^{n} v_k = \frac{1}{3}n(n-1)(n-2)$

(ii). $\sum_{k=1}^{n} e_k = \frac{1}{2}n(n-1)^2$

(iii). $\sum_{k=1}^{n} f_k = \frac{1}{6}n(n^2+5)$

Proof.
(i). Every circle centre defined by three different sites from S is a Voronoi vertex in some k-th and $(k+1)$-th order Voronoi diagram. As there are $\binom{n}{3}$ different circles, the first claim follows.
(ii). Consider the arrangement of bisectors $\mathcal{A}(S)$. Fix one bisector $B(a, b)$. Without loss of generality we can assume that bisector $B(a, b)$ is divided into $n-1$ line segments by the Voronoi circle centres

$$abx_3, abx_4, \ldots, abx_n$$

where $S = \{a, b, x_3, \ldots, x_n\}$. Every line segment is an edge in some k-th order Voronoi diagram. As there are $\binom{n}{2}$ different bisectors, claim *(ii)* follows.
(iii). The Euler formula holds for every order.

$$v_k - e_k + f_k = 1$$

Thus

$$\sum_{k=1}^{n} f_k = n + \sum_{k=1}^{n} e_k - \sum_{k=1}^{n} v_k$$

This completes the theorem. ■

The number of vertices, edges and regions in $V_k(S)$ is not independent of the configuration of S as already the ordinary Voronoi diagram shows. But the following theorem gives expressions for those numbers, depending on n, k and numbers of unbounded regions. Let f_k^∞ denote the number of unbounded regions in the k-th order Voronoi diagram. By definition $f_0^\infty := 0$.

Theorem 3.3 [Ed, Le] *Let S be in general position. Then the number of vertices, edges and regions in the k-th order Voronoi diagram can be expressed as follows.*

(i). $v_k = 2(f_k - 1) - f_k^\infty$

(ii). $e_k = 3(f_k - 1) - f_k^\infty$

(iii). $f_k = (2k-1)n - (k^2-1) - \sum_{i=1}^{k} f_{i-1}^\infty$

Substituting $k = n$ in the formula for f_k in Theorem 3.3 gives a formula for the total number of unbounded regions. Note that $f_n = 1$.

$$\sum_{i=1}^{n} f_{i-1}^\infty = n(n-1) \tag{3.2}$$

The unbounded regions in the k-th order Voronoi diagram can be characterized as follows. Let $\bar{l}_{pq}$ denote all points on the line defined by the points p and q that are in between p and q.

Property 3.4 [OBS] *A region $V(A)$ of the k-th order Voronoi diagram $V_k(S)$ is unbounded if and only if one of the following two conditions holds.*

(i). There exists a line l that separates A from $S - A$.

(ii). There exist two consecutive points p and q, with $p, q \in S - A$, on $\delta CH(S-A)$ such that the points in $A - \overline{pq}$ are in the open half plane defined by l_{pq} opposite to $CH(S-A)$.

As we assume general position, we only have to consider condition *(i)* in Property 3.4. It is clear that in this case the following symmetry holds.

$$f_k^\infty = f_{n-k}^\infty \tag{3.3}$$

4 The Voronoi poset.

Consider the Voronoi poset $\Pi(S)$ introduced above. We can order the faces in the poset by set inclusion of the sets $L(A)$. This gives us indeed a partially ordered set. For more on partially ordered sets consult [Zi]. The poset is bounded as we have the empty set as $\hat{0}$, the unique minimal element and the set $[n]$ as $\hat{1}$, the unique maximal element. In general, a poset is called *graded* if it is bounded and if every maximal chain has equal length. We show that $\Pi(S)$ is graded. Below we give an example that shows that $\Pi(S)$ is in general not a lattice.

Property 4.1 *$\Pi(S)$ is graded.*

Proof. We show that $r(L(A)) = |L(A)|$ is a *rank function* for $\Pi(S)$. A rank function maps an element x from a poset to a unique level in such a way that the level corresponds with the length of any maximal chain from x to $\hat{0}$. Let $L(A) \in \Pi(S)$, with $|L(A)| = k$. Then every point $x \in V(A)$ has the k points from A as its k nearest neighbors. Now order those points with respect to their distance to x. As we assumed general position it is always possible to change the choice of x in such a way that this order is strict. By removing at each step the furthest point still available, we get a chain of length k that descends to $\hat{0}$. ■

We analyse the two smallest cases, assuming general position.

n=3.

For $n = 3$ we only have one poset, the full poset on [3]. That is,

$$\Pi_3(S) = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$$

n=4.

for $n = 4$ we have two essentially different posets. We can see this by looking at the circles defined by the four points. As $n = 4$, Voronoi circles will have order one or two. Because of Eulers formula, we cannot have four circles of order 1. We cannot have less than two circles either as this would give us not enough cells in the first order diagram.

Figure 3: The two different first order Voronoi diagrams.

Two empty Voronoi circles.

Look at the left picture in Figure 3. The only subset of [4] that is missing, is clearly 23, so we are left with

$$\Pi_4(S_1) = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 24, 34, 123, 124, 134, 234, 1234\}$$

This example shows that the Voronoi poset is in general not a lattice. For being a lattice it is required that every two elements of the poset have a unique minimal upper bound. But the elements 2 and 3 have two minimal upper bounds, 123 and 234.

Three empty Voronoi circles.

This situation corresponds with the picture on the right. The region 123 cannot appear in the third order diagram, but all other subsets of [4] do appear, thus

$$\Pi_4(S_2) = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 124, 134, 234, 1234\}$$

5 Symmetry relations.

Given a set S of sites, we can count for every order k the number of vertices, v_k, the number of edges, e_k and the number of non empty Voronoi regions, f_k. By the *f-vector* of $\Pi(S)$ we will mean the vector $\{f_1, f_2, \ldots, f_n\}$. The c- and e-vector are defined analogously.

Symmetry in the number of regions.

It turns out that there exists a symmetry in the f-vectors.

Lemma 5.1 *Consider the f-vector of $\Pi(S)$, where $|S| = n$. Then $f_k + f_{n-k+1}$ is a constant that's independent of the position of the points in S. More precisely,*

$$f_k + f_{n-k+1} \quad = \quad 2k(n-k+1) + 1 - n \tag{5.1}$$

Proof. We apply Theorem 3.3 to f_k and f_{n-k+1}.

$$\begin{aligned} f_k + f_{n-k+1} &= (2k-1)n - k^2 + 1 - \sum_{i=1}^{k} f_{i-1}^{\infty} \\ &+ \quad (2(n-k+1)-1)n - (n-k+1)^2 + 1 - \sum_{i=1}^{n-k+1} f_{i-1}^{\infty} \\ &= 2kn - 2k^2 + 2k + 1 - n + n(n-1) - \left(\sum_{i=1}^{k} f_{i-1}^{\infty} + \sum_{i=1}^{n-k+1} f_{i-1}^{\infty} \right) \end{aligned}$$

We join the two sums by applying the Symmetry Equation 3.3. and evaluate the result by using Equation 3.2.

$$\sum_{i=1}^{k} f_{i-1}^{\infty} + \sum_{i=1}^{n-k+1} f_{i-1}^{\infty} \quad = \quad \sum_{i=1}^{n} f_{i-1}^{\infty} \quad = \quad n(n-1)$$

The lemma follows from combining the two equations. ∎

Symmetry in the number of vertices.

A similar equation holds for the number of vertices of a collection of Voronoi diagrams $V_k(S)$, for $k = 1, \ldots, n-1$.

Lemma 5.2 *Let S be a set of points in general position, $|S| = n$. Let v_k denote the number of vertices in the k-th order Voronoi diagram. Then*

$$v_k + v_{n-k} \quad = \quad 4k(n-k) - 2n \tag{5.2}$$

Proof. Using Theorem 3.3 we can write $v_k + v_{n-k}$ in terms of numbers of faces.

$$v_k + v_{n-k} = 2(f_k - 1) - f_k^{\infty} + 2(f_{n-k} - 1) - f_{n-k}^{\infty}$$

Regroup and apply symmetry, Equation 3.3.

$$= 2(f_k + f_{n-k} - 2 - f_k^{\infty})$$

Apply Theorem 3.3.

$$= 2(n^2 - 2n + 2kn - 2k^2 - (\sum_{i=1}^{k} f_{i-1}^{\infty} + \sum_{i=1}^{n-k} f_{i-1}^{\infty} + f_k^{\infty}))$$

Combine using symmetry again,

$$= 2(n^2 - 2n + 2kn - 2k^2 - \sum_{i=1}^{n} f_{i-1}^{\infty})$$

Use $\sum_{i=1}^{n} f_{i-1}^{\infty} \ = \ n(n-1)$

$$\begin{aligned} &= -2n + 4kn - 4k^2 \\ &= 4k(n-k) - 2n \end{aligned}$$

∎

Symmetry in the number of Voronoi circles.

Recall that the order of a Voronoi circle equals the number of points of S it contains in its interior. Thus we define the *c-vector* of S as the vector

$$c(S) \quad = \quad \{c_0, c_1, \ldots, c_{n-3}\}$$

where c_i denotes the number of circles of order i. The following theorem states that given n arbitrary points in general position, the number of circles that contain exactly i points on their inside plus the number of circles that contain exactly i points on their outside is a constant. We can prove this by applying the above results.

Theorem 5.3 *Consider the c-vector of* $\Pi(S)$, *where* $|S| = n$. *Then* $c_i + c_{n-i-3}$ *is a constant that's independent of the position of the points in* S. *More precisely,*

$$\begin{aligned} c_i + c_{n-i-3} &= 2(i+1)(n-2-i) \\ &= 2i(n-i-3) + 2(n-2) \end{aligned} \tag{5.3}$$

Proof. We prove the theorem by induction.

[i=0]

We use *inversion*. Inversion changes the point-inside-circle relation in 2-dimensional space in a point-below-plane relation in 3-dimensional space. See [BKOS], page 177 for more details and further references. The inversion map ϕ is defined by

$$\begin{aligned} \phi : \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \\ \phi : (x, y) &\mapsto (x, y, x^2 + y^2) \end{aligned}$$

It lifts points in the plane to the unit paraboloid in three-space. As every circle defined by S in the plane contains only three points from S, every hyperplane defined by $\phi(S)$ contains only three points from $\phi(S)$ as well. c_0, the number of empty circles of S in 2D equals the number of faces of the lower hull of $\phi(S)$ in $3D$ and c_{n-3}, the number of circles that contain all other points of S equals the number of faces of the upper hull of $\phi(S)$. Note that all images of points in S under ϕ are part of the convex hull of $\phi(S)$. As the convex hull of a point set consisting of n points consists of $2n-4$ faces, if every face is a triangle, see [BKOS], Theorem 11.1, the claim follows.

[induction step]

$$c_k + c_{n-k-3} = c_{k-1} + c_k + c_{n-k-3} + c_{n-k-2} - (c_{k-1} + c_{n-k-2})$$

Apply Equation 3.1.

$$= v_{k+1} + v_{n-(k+1)} - (c_{k-1} + c_{n-k-2})$$

Use Lemma 5.2 and the induction hypothesis,

$$\begin{aligned} &= 2(2(k+1)-1)(n-(k+1)) - 2(k+1) \\ &\quad - (2(k-1+1)(n-2-(k-1))) \\ &= 2(k+1)(n-2-k) \end{aligned}$$

■

Let $\tilde{f}_k := f_k + f_{n-k+1}$ and $\tilde{c}_i := c_i + c_{n-i-3}$. By the *reduced f-vector*, denoted $\tilde{f}$, we mean the vector of $\tilde{f}_k$'s for all different k. That is

$$\tilde{f} \quad := \quad \{\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{\lfloor \frac{n-1}{2} \rfloor}\}$$

$\tilde{c}$ is defined similarly. As a consequence of Lemma 5.1 and Theorem 5.3, $\tilde{f}$ and $\tilde{c}$ are only dependent on n.

Example.

As an example we present the reduced f- and c-vectors for $n \in \{3, \ldots, 12\}$.

n	$\tilde{f}$	$\tilde{c}$
3	(4, 6)	(2)
4	(5, 9)	(4)
5	(6, 12, 14)	(6, 8)
6	(7, 15, 19)	(8, 12)
7	(8, 18, 24, 26)	(10, 16, 18)
8	(9, 21, 29, 33)	(12, 20, 24)
9	(10, 24, 34, 40, 42)	(14, 24, 30, 32)
10	(11, 27, 39, 47, 51)	(16, 28, 36, 40)
11	(12, 30, 44, 54, 60, 62)	(18, 32, 42, 48, 50)
12	(13, 33, 49, 61, 69, 73)	(20, 36, 48, 56, 60)

Remark.

Computer calculations did not suggest any similar symmetry relation for the number of edges.

Relations between regions and circles.

Corollary 5.4

$$\tilde{f}_i \quad = \quad \tilde{f}_0 + \tilde{c}_{i-1} \quad = \quad \tilde{c}_{i-1} + n + 1 \tag{5.4}$$

Proof. This follows directly from Lemma 5.1 and Theorem 5.3. ∎

Property 5.5 *Let f_i^∞ denote the number of unbounded faces in the i-th order diagram and let c_i denote the number of circles of order i.*

$$f_i^\infty + (c_{i-1} - c_{i-2}) \quad = \quad 2(n-i) \tag{5.5}$$

Proof. We prove the property by induction.

[i=1] c_{-1} is zero by definition. The number of vertices v_1 in the first order Voronoi diagram equals the number of circles of order zero, c_0. Thus,

$$f_1^\infty + (c_0 - c_{-1}) = f_1^\infty + v_1$$

Apply Theorem 3.3.

$$\begin{aligned} &= f_1^\infty + 2(f_1 - 1) - f_1^\infty \\ &= 2(n-1) \end{aligned}$$

[induction step] Assume we have proved that

$$f_i^\infty + (c_{i-1} - c_{i-2}) = 2(n-i)$$

We can rewrite this, by using induction again, as

$$c_{i-1} = 2ni - i(i+1) - \sum_{k=1}^{i+1} f_{k-1} \tag{5.6}$$

Now we evaluate $c_i - c_{i-1}$.

$$\begin{aligned} c_i - c_{i-1} &= (c_i + c_{i-1}) - 2c_{i-1} \\ &= v_{i+1} - 2c_{i-1} \\ &= 2(f_{i+1} - 1) - f_{i+1}^\infty - 2c_{i-1} \end{aligned} \tag{5.7}$$

We fill in this expression for $c_i - c_{i-1}$ and apply Theorem 3.3 and Equation 5.6.

$$\begin{aligned} f_{i+1}^\infty + (c_i - c_{i-1}) &= 2(f_{i+1} - 1 - c_{i-1}) \\ &= 2(n-i-1) \end{aligned}$$

This proves the claim. ■

Corollary 5.6 *The c-vector totally determines the f-vector. The correspondence is given by*

$$f_k = n - k + 1 + c_{k-2}$$

Proof. Applying Equation 5.5 we get

$$\sum_{i=1}^{k} f_{i-1}^\infty = (k-1)(2n-k) - c_{k-2}$$

The claim now follows from evaluating Theorem 3.3 using this expression. ■

6 Euler characteristic.

As an application of the symmetry relations we will investigate the reduced Euler characteristic of the Voronoi poset $\Pi(S)$.

By the *reduced Euler characteristic* we mean the quantity

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{n-1} (-1)^i \bar{f}_i \tag{6.1}$$

where $\bar{f}_i$ denotes the number of i-dimensional faces of the complex or poset Δ. For a polytope P, for example, the famous Euler-Poincaré formula states that $\tilde{\chi}(P) = 0$.

Using the symmetry relation 5.1 we can analyse the Euler characteristic of the Voronoi poset. Mind the difference in notation. $\bar{f}_i$ equals the number of faces of *dimension* i, where f_i stands for the number of faces in the *i-th Voronoi diagram*. Thus

$$\bar{f}_i = f_{i+1}$$

Theorem 6.1 *Let S be a set of points in general position, with $|S| = n \geq 3$. Assume n is odd. Then the reduced Euler characteristic of $\Pi(S)$ equals zero.*

$$\tilde{\chi}(\Pi(S)) = 0 \tag{6.2}$$

Proof. Write $\tilde{f}_i = f_{i+1} + f_{n-i} = \bar{f}_i + \bar{f}_{n-i-1}$. Then we get from the definition that

$$\tilde{\chi} = -\bar{f}_{-1} + \tilde{f}_0 + \frac{1}{2}\tilde{f}_{\frac{n-1}{2}} + t_n$$

where

$$t_n := \sum_{i=1}^{\frac{n-3}{2}} (-1)^i \tilde{f}_i$$

$\bar{f}_{-1} = 1$, as $\bar{f}_{-1}$ counts the empty set. $\tilde{f}_0$ is the number of zero dimensional faces plus the number of $n-1$ dimensional faces, so $\tilde{f}_0 = n+1$. Applying Equation 5.1 gives

$$\tilde{f}_{\frac{n-1}{2}} = -(-1)^{\frac{n+1}{2}} \frac{n^2+3}{4}$$

Straightforward calculations show that

$$t_n = (-1)^{\frac{n+1}{2}} \frac{n^2+3}{4} - n$$

So it follows that

$$\tilde{\chi} = -1 + n + 1 - (-1)^{\frac{n+1}{2}} \frac{n^2+3}{4} + (-1)^{\frac{n+1}{2}} \frac{n^2+3}{4} - n = 0$$

■

Remark.

Theorem 6.2 doesn't hold when n is even. But using similar techniques one can prove the following.

$$n \equiv 0(4) \Rightarrow \tilde{\chi}(S) \text{ odd}$$
$$n \equiv 2(4) \Rightarrow \tilde{\chi}(S) \text{ even}$$

Note that as $v_k = c_{k-1} + c_{k-2}$ it follows immediately that

$$\sum_{k=1}^{n-1} (-1)^{k+1} v_k = 0$$

for all n, where v_k denotes the number of vertices in the k-th order Voronoi diagram.

Appendix A. A Voronoi region is non-degenerate.

Property A.1 *Assuming general position implies that a Voronoi region cannot be a line segment or a single point.*

Proof. Suppose first that $V(A)$ is a line segement. That means that $V(A)$ is locally an intersection of at least two halfplanes $h(x,y)$ and $h(u,v)$ such that the bisectors $B(x,y)$ and $B(u,v)$ coincide. This implies that x, y, u and v are cocircular. That's a contradiction.

Now assume that $V(A)$ is a single point p. Then there exist at least three halfplanes h_1, h_2 and h_3 such that $h_1 \cap h_2 \cap h_3 = p$. We consider three cases.

(i). $h_1 = h_1(a,b)$, $h_2 = h_2(u,v)$ and $a \neq u, b \neq v$.

Thus $a, u \in A$ and $b, v \in S - A$. As $V(A)$ is a single point, it must hold that

$$(B(a,b) \cap B(u,v)) \quad \subset \quad (h(u,b) \cap h(a,v))$$

As we have

$$\begin{aligned} p \in B(a,b) \quad &\Leftrightarrow \quad d(p,a) = d(p,b) \\ p \in B(u,v) \quad &\Leftrightarrow \quad d(p,u) = d(p,v) \end{aligned}$$

and

$$\begin{aligned} p \in h(u,b) \quad &\Leftrightarrow \quad d(p,u) \leq d(p,b) \\ p \in h(a,v) \quad &\Leftrightarrow \quad d(p,a) \leq d(p,v) \end{aligned}$$

it follows that

$$d(p,a) \;\leq\; d(p,v) \;=\; d(p,u) \;\leq\; d(p,b) \;=\; d(p,a)$$

But this implies again that a, b, u and v are cocircular. Again a contradiction.

(ii). $h_1 = h_1(a,b)$, $h_2 = h_2(a,v)$ and $b \neq v$.

This implies that a, b and v are cocircular on a circle with centre $B(a,b) \cap B(a,v)$. We consider $h_3(x,y)$.

- $x \neq a$ and $y \neq b \neq u$. See case 1.
- $x = a$ and $y \neq b \neq u$ implies that y and a, b, v are cocircular.
- $x \neq a$ and $y = b$ implies that x and a, b, v are cocircular.

(iii). $h_1 = h_1(a,b)$, $h_2 = h_2(u,b)$ and $a \neq u$.

This implies that a, b and u are cocircular on a circle with centre $B(a,b) \cap B(b,u)$. We consider $h_3(x,y)$.

- $x \neq a \neq u$ and $y \neq b$. See case 1.
- $x = a$ and $y \neq b$. See case 1.
- $x \neq a \neq u$ and $y = b$ implies that x and a, b, u are cocircular.

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