‘THE MOTHER OF ALL CONTINUED FRACTIONS’

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Dedicated to the memory of Anzelm Iwanik

Abstract. In this paper we give the relationship between the regular continued fraction and the Lehner fractions, using a procedure known as *insertion*. Starting from the regular continued fraction expansion of any real irrational x, when the maximal number of insertions is applied one obtains the Lehner fraction of x. Insertions (and singularizations) show how these (and other) continued fractions expansions are related. We will also investigate the relation between the Lehner fractions and the Farey expansion, and obtain the ergodic system underlying the Farey expansion.

1. Introduction

In 1994, J. Lehner [L] showed that every irrational number $x \in [1,2)$ has a unique continued fraction expansion of the form

$$(1) \qquad b_0 + \cfrac{e_1}{b_1 + \cfrac{e_2}{b_2 + \ddots + \cfrac{e_n}{b_n + \ddots}}} = [\,b_0;\, e_1/b_1,\, e_2/b_2,\, \cdots,\, e_n/b_n,\, \cdots],$$

where (b_i, e_{i+1}) equals either $(1,1)$ or $(2,-1)$. We call these continued fractions *Lehner fractions* or *Lehner expansions.* Each rational number has two finite Lehner expansions. Lehner expansions can be generated dynamically by the map $L : [1,2) \to [1,2)$, given by

$$Lx := \begin{cases} \dfrac{1}{2-x}, & 1 \le x < \frac{3}{2}, \\[2ex] \dfrac{1}{x-1}, & \frac{3}{2} \le x < 2. \end{cases}$$

Notice that in this expansion one has for $x \in [1,2)$ that

$$(b_i, e_{i+1}) = (1,1) \text{ if } L^i(x) \in [\frac{3}{2}, 2),$$

and

$$(b_i, e_{i+1}) = (2,-1) \text{ if } L^i(x) \in [1, \frac{3}{2}).$$

Lehner fractions are examples of the so-called *semi-regular continued fraction expansions.* In general a semi-regular continued fraction expansion (SRCF) is a finite

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or infinite fraction

$$b_0 + \cfrac{e_1}{b_1 + \cfrac{e_2}{b_2 + \ddots + \cfrac{e_n}{b_n + \ddots}}} = [\,b_0;\, e_1/b_1,\, e_2/b_2,\, \cdots,\, e_n/b_n,\, \cdots] \,, \tag{2}$$

with $e_n = \pm 1$; $b_0 \in \mathbb{Z}$; $b_n \in \mathbb{N}$, for $n \geq 1$, subject to the condition

$$e_{n+1} + b_n \geq 1, \text{ for } n \geq 1,$$

and with the restriction that in the infinite case

$$e_{n+1} + b_n \geq 2, \text{ infinitely often.}$$

Moreover we demand that $e_n + b_n \geq 1$ for $n \geq 1$.

Finite truncation in (2) yields the SRCF-convergents

$$A_n/B_n := [\,b_0;\, e_1/b_1,\, e_2/b_2,\, \cdots,\, e_n/b_n] \,,$$

where it is always assumed that $gcd(A_n, B_n) = 1$. We say that (2) is a SRCF-expansion of x in case

$$x = \lim_{n\to\infty} \frac{A_n}{B_n} \,.$$

The best known example of a SRCF is the so-called *regular continued fraction expansion* (RCF); It is well-known that every real irrational number x has a unique RCF-expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = [\,a_0;\, a_1,\, a_2,\, \cdots], \tag{3}$$

where $a_0 \in \mathbb{Z}$ is such, that $x - a_0 \in [0, 1)$, and $a_n \in \mathbb{N}$ for $n \in \mathbb{N}$. Underlying the RCF is the ergodic system

$$([0, 1), \mathcal{B}, \mu, T),$$

where $\mathcal{B}$ is the collection of Borel sets of $[0, 1)$, μ is the Gauss-measure on $[0, 1)$, i.e., the measure with density $(\log 2)^{-1}(1 + x)^{-1}$ on $[0, 1)$, and where $T : [0, 1) \to [0, 1)$ is defined by

$$Tx := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor \,, x \neq 0; T0 := 0.$$

Putting $a_1 = a_1(x) := \lfloor 1/(x - a_0) \rfloor$ and $a_n = a_n(x) := a_1(T^n(x - a_0))$, $n \geq 1$, (3) easily follows from the definition of T. It should be noticed that a rational number has (two) finite RCF-expansions. Denote the RCF-convergents of a real x by $(P_n/Q_n)_{n\geq 1}$, and define the *mediant convergents* of x by

$$\frac{kP_n + P_{n-1}}{kQ_n + Q_{n-1}} \,, \quad 1 \leq k < a_{n+1} \,.$$

In Section 2 we will see that the set of Lehner-convergents of x equals the set of RCF-convergents and mediant convergents of x. Perhaps a more appropriate name for the Lehner fractions would be *the mother of all semi-regular continued fractions*. This becomes transparent with the ideas of *singularization* and *insertion*, discussed in Section 2.

The map L was implicitly given by J. Lehner [L], and is isomorphic to the map $I:[0,1)\to[0,1)$, given by

$$Ix := \begin{cases} \dfrac{x}{1-x}\,, & 0\le x<\frac{1}{2},\\[2ex] \dfrac{1-x}{x}\,, & \frac{1}{2}\le x<1.\end{cases}$$

This map was used by Shunji Ito [I] to generate for every $x\in[0,1)$ the mediant and RCF-convergents of x. However, no semi-regular expansion was associated with this transformation. Ito studied the ergodic properties of this transformation, and showed that[1]

$$([0,1),\mathcal{B},\nu,I)$$

forms an ergodic system, where ν is a infinite σ-finite invariant measure for I with density x^{-1} on $[0,1)$. Due to this, one immediately finds that

$$([1,2),\mathcal{B},\rho,L)$$

forms an ergodic system, where ρ is a infinite σ-finite invariant measure for L with density $(x-1)^{-1}$ on $[1,2)$. Ito's map is also closely related to *the additive continued fraction*, and the *Farey shift map*. For more details on this, see [Rich]. The additive continued fraction yields, like the Lehner fraction, all the RCF- and mediant convergents of any x. In [G], J. Goldman showed that for any $x>0$ related to the additive continued fraction is a so-called *unitary* continued fraction expansion of x of the form (2), with $(b_i,e_i)\in\{(1,1),\,(2,-1)\}$ and $b_0=1$. Notice that this continued fraction expansion (which from now on we will call the *Farey-expansion* of x) is **not** a SRCF-expansion of x. Hitherto no map F was known to generate these Farey-expansions, but due to the close relation with the Lehner fractions we were able to find F, show that it is ergodic, and has a σ-finite infinite invariant measure with density $(x-1)^{-1}-(x+2)^{-1}$ on $[-1,\infty)$, see also Section 3.

Ito also obtained the natural extension

$$([0,1)\times[0,1],\mathcal{B},\bar{\nu},\mathcal{I})$$

of $([0,1),\mathcal{B},\nu,I)$. This system was used by Ito to study the distribution of the sequences of the first and 'last' mediants, and by G. Brown and Q. Yin [BY] to study any sequence of mediant convergents of x of a given order. This was also done by W. Bosma [B], by using the regular system T. Brown and Yin needed the Ratio Ergodic Theorem to circumvent the fact that the invariant measure $\bar{\nu}$ has infinite mass, but Bosma only needed the ergodicity of T. Applying $\phi:[0,1)\times[0,1]\to[0,1)\times[0,1]$ defined by $\phi(x,y)=(x+1,y+1)$ to Ito's natural extension $([0,1)\times[0,1],\mathcal{B},\bar{\nu},\mathcal{I})$ yields $([1,2)\times[1,2],\mathcal{B},\hat{\rho},\mathbf{L})$ as a version of the natural extension of the ergodic system $([0,1),\mathcal{B},\rho,L)$ underlying the Lehner fraction. Here $\mathbf{L}$ is defined by

$$\mathbf{L}(x,y) = \begin{cases} \left(\dfrac{1}{2-x},\,2-\dfrac{1}{y}\right), & 1\le x<\frac{3}{2},\\[2ex] \left(\dfrac{1}{x-1},\,1+\dfrac{1}{y}\right), & \frac{3}{2}\le x<2,\end{cases}$$

[1] All σ-algebras considered are the 1-dimensional or 2-dimensional Lebesgue σ-algebra on the appropriate space. We will always use the notation $\mathcal{B}$ to denote these various σ-algebras, unless it causes confusion.

and $\hat{\rho}$ has density $(2x+2y-xy-3)^{-2}$ on $[1,2)\times[1,2]$. In Section 3 we will study another natural extension of $([0,1),\mathcal{B},\rho,L)$, which will bring out the relation between the Lehner fractions and the Farey-expansion in an easy and clear-cut way. In Section 4 we extend some classical results of the RCF to the Lehner fractions (and the Farey-expansion). These results can not be obtained from its own underlying ergodic system $([0,1),\mathcal{B},\rho,L)$, but follow easily from that of the regular.

2. Insertions

There are two operations on the sequence of digits of a SRCF-expansion (2) of any real x, which transforms this SRCF-expansion into another one: *singularization* and *insertion*. In this section we will deal only with insertions, singularizations will be discussed in Section 5. See also [K], which is a general reference for all statements with respect to singularizations and insertions.

An INSERTION is based upon the identity

$$A+\frac{1}{B+\xi}=A+1+\cfrac{-1}{1+\cfrac{1}{B-1+\xi}}$$

Let (2) be a SRCF-expansion of x, and suppose that for some $n\geq 0$ one has

$$b_{n+1}>1;\ e_{n+1}=1.$$

An *insertion* is the transformation τ_n which changes the continued fraction

$$[\,b_0;\ e_1/b_1,\ldots,\ e_n/b_n,\ 1/b_{n+1},\ldots] \tag{4}$$

into

$$[\,b_0;\ e_1/b_1,\ldots,e_n/(b_n+1),\ -1/1,\ 1/(b_{n+1}-1),\ldots],$$

which is again a SRCF-expansion of x, with convergents, say, $(p_k/q_k)_{k\geq -1}$. Let $(r_k/s_k)_{k\geq -1}$ be the sequence of convergents connected with (4). Using some matrix-identities one easily shows that the sequence of vectors $\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1}$ is obtained from $\begin{pmatrix} r_k \\ s_k \end{pmatrix}_{k\geq -1}$ by *inserting* the term $\begin{pmatrix} r_n+r_{n-1} \\ s_n+s_{n-1} \end{pmatrix}$ before the n^{th} term of the latter sequence, i.e.,

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1}\equiv\begin{pmatrix} r_{-1} \\ s_{-1} \end{pmatrix},\begin{pmatrix} r_0 \\ s_0 \end{pmatrix},\ldots,\begin{pmatrix} r_{n-1} \\ s_{n-1} \end{pmatrix},\begin{pmatrix} r_n+r_{n-1} \\ s_n+s_{n-1} \end{pmatrix},\begin{pmatrix} r_n \\ s_n \end{pmatrix},\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix},\ldots.$$

We leave the proof of the following Proposition to the reader.

Proposition 1. *Let $x\in[1,2)$, with RCF-expansion (3), i.e., $a_0=1$. Then the following algorithm yields the Lehner expansion (1) of x.*

(I) *Let $n\geq 0$ be the first index for which $a_{n+1}>1$. In case $n=0$ (i.e., $a_1>1$) we replace $[\,a_0;\ a_1,\ldots]$ by*

$$[\,2;\ \underbrace{-1/2,\ \ldots,\ -1/2}_{(a_1-2)-times},\ -1/1,\ 1/1,\ 1/a_2,\ \ldots]\,.$$

In case $n\geq 1$ we replace

$$[\,a_0;\ 1,\ \ldots,\ 1,\ a_{n+1},\ \ldots]$$

by

$$\tau_{n+a_{n+1}-1}(\ \ldots\ (\tau_{n+1}(\tau_n([\,a_0;\ 1;\ \ldots,\ 1,\ a_{n+1},\ \ldots]))\ \ldots)$$

$$= [\,a_0;\,1/1,\,\ldots\,,1/2,\,\underbrace{-1/2,\,\ldots\,,\,-1/2}_{(a_{n+1}-2)-times},\,-1/1,\,1/1,\,1/a_{n+2},\,\ldots]\,.$$

Denote this new SRCF-expansion of x by $[\,b_0;\,e_1/b_1,\,e_2/b_2,\,\cdots\,,\,e_n/b_n,\,\cdots]$.

(II) *Let $m \geq n+1$ be the first index in this new SRCF-expansion of x for which $e_{m+1} = 1$ and $b_{m+1} > 1$. Repeat the procedure from* (I) *to this new SCRF-expansion for this value of m.*

Due to the insertion-mechanism it follows that every RCF-convergent or mediant convergent of x is a Lehner-convergent of x.

It is well-known that every quadratic irrational x has a RCF-expansion which is (eventually, i.e., from some point on) periodic. Due to the above algorithm the following corollary is immediate.

Corollary 1. *Let x be a quadratic irrational number, then the Lehner expansion of x is (eventually) periodic.*

3. Farey-expansions

If we define the map $\mathcal{L} : [1,2) \times [-1,\infty) \to [1,2) \times [-1,\infty)$ by

$$\mathcal{L}(x,y) := \left(\frac{e(x)}{x-b(x)},\, \frac{e(x)}{b(x)+y} \right), \quad 1 \leq x < 2\,,\, y \geq -1, \tag{5}$$

where

$$(b(x), e(x)) := \begin{cases} (2,-1) & 1 \leq x < \frac{3}{2}\,, \\ (1,1)\,, & \frac{3}{2} \leq x < 2\,, \end{cases}$$

then $\mathcal{L}$ is bijective, apart from a set of Lebesgue measure zero. The first coordinate map of $\mathcal{L}$ is the Lehner map L, while the second coordinate map yields the 'past' of any $x \in [1,2)$ as a Farey expansion. By this we mean the following. Let $x \in [1,2)\backslash\mathbb{Q}$, with Lehner fraction (1), and let

$$(T_n, V_n) := \mathcal{L}^n(x,0) \qquad \text{for } n \geq 0,$$

then $V_n = [\,0; e_n/b_{n-1},\,\cdots\,,\,e_1/b_0]$ for $n \geq 1$, and $V_0 = 0$. Thus we see that the second coordinate of $\mathcal{L}$ is the inverted Farey map F, and 're-inverting' yields that $F : [-1,\infty) \to [-1,\infty)$ is given by

$$F(x) = \begin{cases} -\frac{1}{x} - 2\,, & -1 \leq x < 0, \\ 0\,, & x = 0, \\ \dfrac{1}{x} - 1\,, & x > 0, \end{cases}$$

i.e.,

$$F(x) = \frac{f(x)}{x} - d(x)$$

where

$$(d(x), f(x)) := \begin{cases} (2,-1) & -1 \leq x < 0, \\ (1,1)\,, & x \geq 0. \end{cases}$$

We have the following theorem.

Theorem 1. *The system*

$$([1,2) \times [-1,\infty)), \bar{\mathcal{B}}, \bar{\rho}, \mathcal{L})$$

forms an ergodic system, which is the natural extension of $([1,2), \mathcal{B}, \rho, L)$. Here $\bar{\rho}$ is a σ-finite, infinite measure, which is invariant under $\mathcal{L}$, with density $(x+y)^{-2}$ on $[1,2) \times [-1,\infty)$.

Proof. Let $\pi : [1,2) \times [-1,\infty) \to [1,2)$ be the natural projection on the first coordinate. Then it is easily seen that $\pi\mathcal{L} = L\pi$, and that for any measurable set A in $[1,2)$,

$$\bar{\rho}(\pi^{-1}A) = \bar{\rho}(A \times [-1,\infty) = \rho(A).$$

Further, $\bar{\rho}$ is an invariant measure; for this it suffices to show that $\bar{\rho}((a,b)\times(c,d)) = \bar{\rho}(\mathcal{L}((a,b)\times(c,d)))$ for any $1 < a < b < 2$ and $-1 < c < d < \infty$. We consider two cases.

(I) For $1 < a < b \leq \frac{3}{2}$, one has

$$\mathcal{L}((a,b)\times(c,d)) = \left(\frac{-1}{a-2}, \frac{-1}{b-2}\right) \times \left(\frac{-1}{c+2}, \frac{-1}{d+2}\right).$$

An easy calculation shows that

$$\begin{aligned}\bar{\rho}\left(\frac{-1}{a-2}, \frac{-1}{b-2}\right) \times \left(\frac{-1}{c+2}, \frac{-1}{d+2}\right) &= \log\left(\frac{(a+d)(b+c)}{(a+c)(b+d)}\right)\\ &= \bar{\rho}((a,b)\times(c,d)).\end{aligned}$$

(II) For $\frac{3}{2} < a < b < 2$, one has

$$\mathcal{L}((a,b)\times(c,d)) = \left(\frac{1}{b-1}, \frac{1}{a-1}\right) \times \left(\frac{1}{d+1}, \frac{1}{c+1}\right).$$

Again a straight forward calculation shows that

$$\bar{\rho}(\mathcal{L}((a,b)\times(c,d))) = \log\left(\frac{(a+d)(b+c)}{(a+c)(b+d)}\right).$$

Finally, we show that $\bar{\mathcal{B}} = \bigvee_{n\geq 0} \mathcal{L}^n\pi^{-1}\mathcal{B}$. To see this, we define for $m,\ n \geq 1$ for the Farey map F and the Lehner map L cylinders $C_m = C_n(f_1/d_1, f_2/d_2, \dots, f_m/d_m)$ resp. $D_n = D_n(b_0; e_1/b_2, \dots, e_n/)$ by

$$C_m(f_1/d_1, \dots, f_m/d_m) := \{x \in [-1,\infty);\, x = [\,0;\, f_1/d_1, \cdots, f_m/d_m, \underbrace{\cdots}_{\text{'free'}}\,]\}$$

and

$$D_n(b_0; e_1/b_2, \dots, e_n/) := \{x \in [1,2);\, x = [\,b_0;\, e_1/b_2, e_2/b_3, \cdots, e_n/ \underbrace{, \cdots}_{\text{'free'}}\,]\},$$

where $(d_i, f_i) \in \{(1,1), (2,-1)\}$ for $i = 1, \dots, m$ and $(b_{i-1}, e_i) \in \{(1,1), (2,-1)\}$ for $i = 1, \dots, n$.

Then

$$D_n \times C_m = \mathcal{L}^m(D_{m+n}(d_m; f_m/d_{m-1}, \cdots, f_2/d_1, f_1/b_0, e_1/b_1, \cdots, e_n/) \times [-1,\infty)).$$

Since the set of all possible cylinders of the form $D_n \times C_m$ generates $\bar{\mathcal{B}}$, this gives the desired result. $\square$

¿From Theorem 1 the following corollary is immediate.

Corollary 2. *The system*

$$([-1,\infty), \mathcal{B}, \tau, F),$$

which is the dynamical system underlying the Farey-expansion, forms an ergodic system. Here τ is a σ-finite, infinite measure, which is invariant under F, with density $(x+1)^{-1} - (x+2)^{-1}$ on $[-1,\infty)$.

Since natural extensions of a system are isomorphic, the fact that $([1,2), \mathcal{B}, \rho, L)$ has two natural extensions (i.e., one with the (inverted) Lehner map L as its second coordinate map, and the other one with the (inverted) Farey map as its second coordinate), 'shows' that L and T must be isomorphic. We have the following theorem.

Theorem 2. *Let $x \in [-1, \infty)$, with Farey-expansion $x = [\,0;\, f_1/d_1,\, f_2/d_2,\, \cdots]$. Then the map $\xi : [-1,\infty) \to [-1,2)$ defined by*

$$\xi(x) \;=\; [\,d_1;\, f_1/d_2,\, f_2/d_3,\, \cdots]\,,$$

is an isomorphism from $([-1,\infty), \mathcal{B}, \tau, F)$ to $([1,2), \mathcal{B}, \rho, L)$.

Proof. Clearly $\xi : [-1,\infty) \setminus \mathbb{Q} \to [-1,\infty) \setminus \mathbb{Q}$ is a bijection, and since

$$\begin{aligned} \xi(F(x)) \;&=\; \xi([\,0;\, f_2/d_2,\, f_3/d_3,\, \cdots]) \quad &&=\; [\,d_2;\, f_2/d_3,\, f_2/d_3,\, \cdots] \\ &=\; L([\,d_1;\, f_1/d_2,\, f_2/d_3,\, \cdots]) \;&&=\; L(\xi(x)), \end{aligned}$$

we only need to show that ξ is measurable and measure preserving.

For each Farey cylinder $C_n = C_n(f_1/d_1, f_2/d_2, \ldots, f_n/d_n)$ as defined above, one has that $\xi(C_n)$ equals the Lehner cylinder $D_n = D_n(d_1; f_1/d_2, f_2/d_3, \ldots, f_n/\,)$, so that ξ is clearly measurable. It remains to show that

$$\tau(C_n) \;=\; \rho(D_n))\,.$$

For $D_n = D_n(b_0; e_1/b_2, e_2/b_3, \ldots, e_n/\,)$ let D_n^* be defined by

$$D_n^* \;:=\; D_n(b_{n-1}; e_n/b_{n-2}, e_{n-1}/b_{n-3}, \ldots, e_1/\,)\,.$$

From the fact that $\mathcal{L}$ is measure preserving with respect to $\bar{\rho}$ one has

$$\begin{aligned} \tau(C_n) \;&=\; \bar{\rho}([1,2) \times C_n) \quad &&=\; \bar{\rho}(\mathcal{L}^{-n}([1,2) \times C_n)) \\ &=\; \bar{\rho}(D_n^* \times [-1,\infty))) \;&&=\; \rho(D_n^*)\,. \end{aligned}$$

Furthermore, since $\mathbf{L}$ is $\hat{\rho}$-preserving, one has

$$\begin{aligned} \rho(D_n^*) \;&=\; \hat{\rho}(D_n^* \times [1,2)) \;&&=\; \hat{\rho}(\mathbf{L}^n(D_n^* \times [1,2))) \\ &=\; \hat{\rho}([1,2) \times D_n) \;&&=\; \rho(D_n)\,, \end{aligned}$$

and the result follows. $\square$

4. Some classical theorems for Lehner fractions and Farey expansions

In 1935, A.Ya. Khintchine [Kh] obtained the following, classical results on the means of the RCF-digits of almost all x. His proofs are based on the Theorem of Gauss-Kusmin, but easier proofs can be obtained via the Ergodic Theorem.

Theorem 3. (A.Ya. Khintchine) *Let x be a real irrational number, with RCF-expansion* (3). *Then for almost all*[2] *x one has*

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}} &= 1.277\cdots, \\ \lim_{n\to\infty} \sqrt[n]{a_1\cdot a_2\cdots a_n} &= \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}} = 2.6\cdots, \\ \lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} &= \infty\,. \end{aligned}$$

Notice that Khintchine's Theorem 3 plus the concept of insertion provide a heuristic argument why the Lehner system $([0,1),\mathcal{B},\rho,L)$ should be ergodic, with an infinite, σ-finite invariant measure ρ. After all an insertion before the digit $a>1$ is simply building a tower over the RCF-cylinder corresponding to this digit. Since the Lehner expansion of a number x is obtained by using insertion as many times as possible in order to 'shrink away' any (regular digit) $a>1$, it follows that the thus obtained system must be ergodic (it contains the RCF-system $([0,1),\mathcal{B},\mu,T)$ as an induced system), but due to the third statement in Khintchine's Theorem 3 it must also have infinite mass.

With respect to Khintchine's result the situation is quite different for the Lehner expansion; there does not exist a Gauss-Kusmin Theorem for these continued fraction expansions, and we can not apply the Ergodic Theorem directly using the Lehner map L (or the Ito map I), since the underlying dynamical system has an invariant measure which is infinite. In spite of this we will show that for almost all x these *means* do exist, and equal 2. By insertion we know that each RCF-digit corresponds to a certain block of digits of the Lehner fractions, as given in Proposition 1. We have the following theorem.

Theorem 4. *Let x be a real irrational number, with with RCF-expansion* (3) *and Lehner expansion* $[\,b_0;\, e_1/b_1,\, e_2/b_2,\, \cdots,\, e_n/b_n,\, \cdots]$. *Then for almost all x one has*

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{\frac{1}{b_1}+\cdots+\frac{1}{b_n}} &= 2 \\ \lim_{n\to\infty} \sqrt[n]{b_1\cdot b_2\cdots b_n} &= 2\,, \\ \lim_{n\to\infty} \frac{b_1+b_2+\cdots+b_n}{n} &= 2. \end{aligned}$$

Proof. Let $N\in\mathbb{N}$ be sufficiently large, there there exist unique integers k and j, such that

$$N = a_1+\cdots+a_k+j\,, \quad \text{where } 0\le j<a_{k+1}\,.$$

In Proposition 1 we saw that each RCF-digit a_i is replaced by a block of Lehner-digits of length a_i, of the form consisting of a_i-1 2's followed by a digit 1. Then

$$\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_N} = k+\frac{1}{2}\sum_{i=1}^{k}(a_i-1)+\frac{j}{2} = \frac{N+k}{2}.$$

[2] All almost all statements are with respect to Lebesgue measure.

This implies that

$$\frac{N}{\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_N}} = \frac{1}{\frac{1}{2}(1+\frac{k}{N})}.$$

Since $0 \le j < a_{k+1}$ it follows that

$$\frac{k}{N} \le \frac{1}{\frac{1}{k}\sum_{i=1}^{k} a_i},$$

which tends to zero almost surely due to Khintchine's Theorem 1, we find that

$$\lim_{N\to\infty} \frac{N}{\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_N}} = 2.$$

Since all digits in the Lehner fraction of any x are either 1 or 2, and one always has that

$$\frac{N}{\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_N}} \le \sqrt[N]{b_1\cdot b_2\cdots b_N} \le \frac{b_1+b_2+\cdots+b_N}{N} \le 2,$$

see also p. 375-377 in [C], the result follows. □

¿From the above theorem and Theorem 2 the following corollary follows.

Corollary 3. *Let x be a real irrational number, with with RCF-expansion* (3) *and Farey expansion* $[\,0;\, f_1/d_1,\, f_2/d_2,\, \cdots,\, f_n/d_n,\, \cdots]$. *Then for almost all x one has*

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{\frac{1}{d_1}+\cdots+\frac{1}{d_n}} &= 2\\
\lim_{n\to\infty} \sqrt[n]{d_1\cdot d_2\cdots b_n} &= 2\,,\\
\lim_{n\to\infty} \frac{d_1+d_2+\cdots+d_n}{n} &= 2.
\end{aligned}$$

Proof. We give only a proof of the second statement; the other two are obtained in exactly the same way.

Let $\mathcal{K}$ be the set of those $x \in [1,2)$ for which

$$\lim_{n\to\infty} \sqrt[n]{b_0\cdot b_1\cdots b_{n-1}} = 2\,,$$

where $[\,b_0;\, e_1/b_1,\, e_2/b_2,\, \cdots,\, e_n/b_n,\, \cdots]$ is the Lehner expansion of x. Due to Theorem 4 we have that $\mathcal{K}^c$ is a null-set, i.e., a set of measure zero. But then one has, that $\xi^{-1}(\mathcal{K}^c)$ is also of measure zero. Now let $y \in \xi^{-1}(\mathcal{K})$, with Farey expansion $[\,0;\, f_1/d_1,\, f_2/d_2,\, \cdots,\, f_n/d_n,\, \cdots]$. Then for each $n \ge 1$ one has that there are as many 2's among the first n digits of y as there are 2's among the first n digits of $x = \xi(y)$, i.e.,

$$\lim_{n\to\infty} \sqrt[n]{d_1\cdot d_2\cdots d_n} = \lim_{n\to\infty} \sqrt[n]{b_0\cdot b_1\cdots b_{n-1}} = 2\,.$$

□

5. Singularizations

In this section we will discuss the concept of the *singularization* of a partial quotient. A singularization is based upon the following identity

$$A + \cfrac{e}{1 + \cfrac{1}{B+\xi}} = A + 1 + \frac{-e}{B+1+\xi}\,.$$

To see the effect of a singularization, let

$$x = [\,b_0;\, e_1/b_1,\, e_2/b_2, \ldots],\ b_n \in \mathbb{N},\ n>0;\ e_i \in \{\pm 1\},\ i \geq 1,$$

be a SRCF-expansion of x. Finite truncation yields the sequence of convergents $(r_k/s_k)_{k\geq -1}$. Suppose that for some $n \geq 0$ one has

$$b_{n+1} = 1;\ e_{n+2} = 1\,,$$

i.e.,

$$[\,b_0;\, e_1/b_1, \ldots] = [\,b_0; e_1/a_1, \ldots,\, e_n/b_n,\, e_{n+1}/\mathbf{1},\, \mathbf{1}/b_{n+2}, \ldots].$$

The transformation σ_n which changes this continued fraction into the continued fraction

$$[\,b_0;\, e_1/b_1, \ldots, e_n/(b_n + e_{n+1}), -e_{n+1}/(b_{n+2}+1), \ldots], \tag{6}$$

which is again a continued fraction expansion of x, with convergents, say $(p_k/q_k)_{k\geq -1}$, is called a SINGULARIZATION. One easily shows that the sequence of vectors $\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1}$ is obtained from $\begin{pmatrix} r_k \\ s_k \end{pmatrix}_{k\geq -1}$ by removing the term $\begin{pmatrix} r_n \\ s_n \end{pmatrix}$ from the latter. Singularizations, and the underlying ergodic theory of a new class of continued fractions, have extensively been studied in [K].

By combining the operatons of singularization and insertion one can obtain any semi-regular continued fraction expansion. In [K] a whole class of semi-regular continued fractions was introduced via singularizations only (some of these SRCF's were new, some classical - like the continued fraction to the nearer integer), and their ergodic theory studied (the main idea in [K] is that the operation of singularization is equivalent to having a induced map on the natural extension of the RCF). As an example of combining the operatons of singularization and insertion we discuss here the *backward continued fraction.*

Each irrational number x in the interval [0,1) has a unique continued fraction expansion of the form

$$1 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \ddots}} = [\,1;\, -1/c_1,\, -1/c_2,\, \cdots], \tag{7}$$

where the c_i's are all integers greater than one. As with the RCF, there is a naturally defined transformation $B : [0,1) \to [0,1)$ which acts as the shift on the continued fraction (7), and which is given by

$$B(x) := \frac{1}{1-x} - \lfloor \frac{1}{1-x} \rfloor,\ x \neq 0;\ B(0) := 0.$$

The graph of B can be obtained from that of the RCF-map T by reflecting the latter in the line $x = \frac{1}{2}$. It is for this reason that the continue fraction (7) has been called 'backward'. It was shown by A. Rényi [R] that B is ergodic, and has a σ-finite, infinite invariant measure with density $1/x$, see also the paper by R.L. Adler and L. Flatto [AF].
As in the case of Proposition 1, we leave the proof of the following proposition to the reader.

Proposition 2. *Let $x \in [0,1)$, with RCF-expansion (3), i.e., $a_0 = 0$. Then the following algorithm yields the backward expansion (7) of x.*

(I) *If $a_1 = 1$, singularize a_1 to arrive at*

$$[\,1;\, -1/(a_2+1),\, 1/a_3,\, \ldots]$$

as a new SRCF-expansion of x.
If $a_1 > 1$, insert $-1/1$ $(a_1 - 1)$-times before a_1 to arrive at

$$[\,1; \underbrace{-1/2,\, \ldots\,,\, -1/2}_{(a_1-2)-times},\, -1/1,\, 1/1,\, 1/a_2,\, \ldots]$$

as a new SRCF-expansion of x. Now singularize $1/1$ appearing at the a_1th position of this new continued fraction expansion of x.

In either case we find as SRCF-expansion of x

$$[\,1;\, (-1/2)^{a_1-1},\, -1/(a_2+1),\, 1/a_3,\, \ldots]\,, \tag{8}$$

here $(-1/2)^{a_1-1}$ is an abbreviation of $\underbrace{-1/2,\, \ldots\,,\, -1/2}_{(a_1-1)-times}$.

(II) *Let $m \geq 1$ be the first index in this new SRCF-expansion of x for which $e_m = 1$. Repeat the procedure from* (I) *to this new SCRF-expansion for this value of m.*

Remark 1. Due to the above insertion/singularization mechanism it follows that x has as backward expansion

$$[\,1;\, (-1/2)^{a_1-1},\, -1/(a_2+2),\, (-1/2)^{a_3-1},\, 1/(a_3+2),\, \ldots]\,, \tag{9}$$

see also Aufgabe 3, p. 131, in [Z]. From (9) it also follows easily that every quadratic irrational number x has an (eventually) periodic backward expansion.

Again, as for the Lehner fractions, it heuristically follows from Khintchine's Theorem 3 and the notion of insertion that the backward continued fraction map B should be ergodic, with an invariant measure of infinite mass. For the Lehner fractions is was also intuively clear that almost surely $\sqrt[n]{b_1 \cdot b_2 \cdot \cdots \cdot b_n} \to 2$ as $n \to \infty$, since there are only digits 1 and 2, and 'there are very few 1's among the 2's' (due to Khintchine's Theorem). For the backward continued fraction clearly such an argument does not exist. We have the following theorem.

Theorem 5. *Let x be a real irrational number, with with RCF-expansion (3) and backward expansion (7). Then for almost all x one has*

$$\lim_{n\to\infty} \sqrt[n]{c_1 \cdot c_2 \cdots c_n} \;=\; 2\,.$$

Proof. Let N be a sufficiently large positive integer, then from (9) we see that there exist unique integers $k \geq 1$ and j, with $0 \leq j < a_{2k+1}$, such that

$$N \;=\; a_1 + a_3 + \cdots + a_{2k-1} + j\,.$$

Then,

$$c_1 \cdot c_2 \cdots c_N \;=\; 2^{\sum_{i=1}^{k}(a_{2i-1}-1)+j-1} \prod_{i=1}^{k}(a_{2i}+2)\,,$$

and therefore

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\log c_i \;&=\; \frac{\log 2}{N}\left(\sum_{i=1}^{k}a_{2i-1} - k + j - 1\right) + \frac{1}{N}\sum_{i=1}^{k}\log(a_{2i}+2) \\
&=\; \log 2\left(1 - \frac{k+1}{\displaystyle\sum_{i=1}^{k}a_{2i-1}+j}\right) + \frac{\displaystyle\sum_{i=1}^{k}\log(a_{2i}+2)}{\displaystyle\sum_{i=1}^{k}a_{2i-1}+j}\,.
\end{aligned}$$

Since

$$\frac{k+1}{\displaystyle\sum_{i=1}^{k}a_{2i-1}+j} \;=\; \frac{1}{\displaystyle\frac{1}{k+1}\sum_{i=1}^{k}a_{2i-1}+\frac{j}{k+1}} \to 0 \quad \text{as } N\to\infty\,,$$

almost surely, and

$$\frac{\displaystyle\sum_{i=1}^{k}\log(a_{2i-1}+2)}{\displaystyle\sum_{i=1}^{k}a_{2i-1}+j} \to 0 \quad \text{as } N\to\infty\,,$$

almost surely, we find that for almost all x

$$\sqrt[N]{c_1 \cdot c_2 \cdot \cdots \cdot c_N} \to 2 \quad \text{as } N\to\infty \quad \text{a.e.}$$

□

Remark 2. Since $c_i \geq 2$ for all $i \geq 1$, it follows that

$$\frac{N}{\dfrac{1}{c_1}+\ldots+\dfrac{1}{c_N}} \;\leq\; 2\,,$$

and therefore we have by Cauchy's result [C] that

$$2 \;\leq\; \lim_{N\to\infty}\frac{N}{\dfrac{1}{c_1}+\ldots+\dfrac{1}{c_N}} \;\leq\; \lim_{N\to\infty}\sqrt[N]{c_1\cdot c_2\cdot\cdots\cdot c_N} \;=\; 2,$$

i.e.,

$$\lim_{N\to\infty}\frac{N}{\dfrac{1}{c_1}+\ldots+\dfrac{1}{c_N}} \;=\; 2 \quad \text{a.e.}$$

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