On the value of the critical point in fractal percolation

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Abstract

We derive a new lower bound $p_c > 0.8107$ for the critical value of Mandelbrot's dyadic fractal percolation model. This is achieved by taking the random fractal set (to be denoted A_∞) and adding to it a countable number of straight line segments, chosen in a certain (non-random) way as to simplify greatly the connectivity structure. We denote the modified model thus obtained by C_∞, and write C_n for the set formed after n steps in its construction. Now it is possible, using an iterative technique, to compute the probability of percolating through C_n for any parameter value p and any finite n. For $p = 0.8107$ and $n = 360$ we obtain a value less than 10^{-5}; using some topological arguments it follows that 0.8107 is subcritical for C_∞ and hence (since C_∞ dominates A_∞) for A_∞.

1 A new lower bound via a new model

The dyadic fractal percolation model [5] can be described informally as follows. Fix $0 \leq p \leq 1$. Divide the unit square $I = [0,1]^2$ into 4 equal smaller squares, and in the natural way retain each of these squares with probability p, or else remove it with probability $1-p$. Iterate this procedure (suitably scaled) of subdivision and random removal on each of the retained squares; in this way we obtain a nested sequence $A_0(\equiv I)$, A_1, $A_2, \ldots$ of random (compact) subsets of I. The intersection of this sequence, which we shall denote A_∞, is a random fractal set. For $0 \leq n \leq \infty$ let $\theta_n = \theta_n(p)$ denote the probability that there is a left-right crossing of A_n in I, that is, that there is a connected component of A_n that intersects both the left side $\{0\} \times [0,1]$ and the right side $\{1\} \times [0,1]$ of the unit square. It is well known (see [1], [3]) that there is a critical value p_c, with $0 < p_c < 1$, such that $\theta_\infty(p)$ is zero if and only if $p < p_c$. In particular, $\theta_\infty(p)$ is discontinuous at p_c.

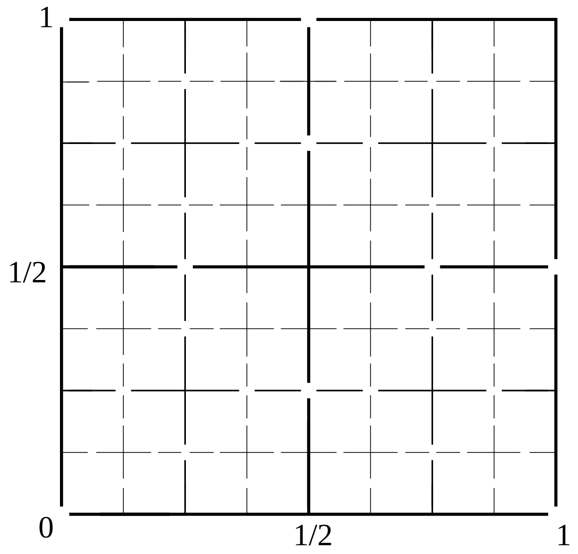

Figure 1: The set F (not in full detail).

It has so far proved intractable to find the exact value of p_c. In this article we introduce a deterministic set $F \subset I$, such that the random fractal process $C_n = A_n \cup F$ has a critical value p_c' that can actually be computed with the help of a fairly simple computer program, giving the value $p_c' = 0.811$ (to three decimal places). As far as rigorous proofs are concerned, we prove (with the aid of the computer) that $p_c' > 0.8107$; moreover, since C_n dominates A_n for all n, it will follow that 0.8107 is a lower bound for the classical critical point p_c. This improves on the previous lower bound of $1/\sqrt{2} \approx 0.7071$ [1].

A precise definition of F is postponed to Section 3, and for now the reader is simply referred to Figure 1. Observe that there are exactly 5 connected components of F that intersect the boundary ∂I of the unit square. (See Section 3 for a justification of this statement.) Since for all $n = 0, \ldots \infty$ $C_n \supset F$, it follows that the number of connected components of C_n that intersect ∂I is less than or equal to five; we shall say that C_n is *wired* if this number is strictly less than five. Thus when C_n is wired, connections have been formed in A_n so that some of the five boundary components of F become linked. For given $0 \leq p \leq 1$ and $n < \infty$, write $\theta_n' = \theta_n'(p)$ for the probability that C_n is wired. Since $A_n \subset C_n$, if there is a left-right crossing in A_n then C_n is certainly wired, so $\theta_n'(p) \geq \theta_n(p)$ for all p and all finite n. Moreover the sequence (C_n) is nested, so $\theta_n'(p)$ is nonincreasing in n and we may define $\theta_\infty'(p) = \lim_{n\to\infty} \theta_n'(p)$.

A note on probability measures: we prefer to think of our underlying probability space in such a way that the underlying probability measure itself depends on the parameter p, so the sequence (A_n) depends on p only through its induced measure (or *law*). The law of (A_n) will be denoted $\mathbf{P}_p$. Thus $\theta'_n(p) = \mathbf{P}_p(C_n \text{ is wired})$ for n finite, and $\theta'_\infty(p) = \lim_{n\to\infty} \mathbf{P}_p(C_n \text{ is wired})$.

We have found it convenient to take what might be called a physicist's approach, in defining θ'_∞ without direct reference to the infinite model C_∞. A more common approach for a mathematician might have been to define $\theta'_\infty(p)$ to be $\mathbf{P}_p(C_\infty \text{ is wired})$, in analogy with the definition of $\theta_\infty(p)$ above. In fact, the two approaches are equivalent, as we see from the following proposition.

Proposition 1 *We have $\{C_\infty \text{ is wired}\} = \bigcap_{n\geq 0}\{C_n \text{ is wired}\}$. In particular, for all $0 \leq p \leq 1$, $\mathbf{P}_p(C_n \text{ is wired})$ converges to $\mathbf{P}_p(C_\infty \text{ is wired})$ as $n \to \infty$.*

The corresponding result for the A_n is well-known, being a simple consequence of the A_n being nested compact sets [6], and indeed yields $\theta_\infty(p) = \lim_{n\to\infty} \theta_n(p)$. The C_n are however not in general compact, and the proof of Proposition 1 involves a considerable amount of topological work. As the proposition will in any case not be used in this paper we omit the proof (but see [7]).

On the other hand, the next proposition (proved in Section 5) will be crucial to what follows.

Proposition 2 *Suppose $0 < p < 1$ with $\theta'_\infty(p) > 0$. Then*

$$\theta'_\infty(p) \geq \frac{p^{-n} - 3}{2^{2n+3}}$$

for all $n \geq 1$.

An important consequence of Proposition 2 is

Proposition 3 *There is a first order phase transition in θ'_∞, that is, there exists p'_c with $0 < p'_c < 1$ such that $\theta'_\infty(p)$ is zero if and only if $p < p_c$.*

Proof By a simple coupling argument, $\theta'_n(p)$ is nondecreasing in p and nonincreasing in n; moreover, θ'_n is continuous when n is finite, since C_n depends on only finitely many Bernoulli random variables. It therefore follows that θ'_∞ is nondecreasing and right-continuous.

Set $p'_c = \inf\{p : \theta'_\infty(p) > 0\} = \sup\{p : \theta'_\infty(p) = 0\}$. Since

$$\theta'_\infty(p) = \lim_{n\to\infty} \theta'_n(p) \geq \lim_{n\to\infty} \theta_n(p) = \theta_\infty(p),$$

we have $p'_c \leq p_c < 1$. On the other hand, by elementary branching process theory $A_\infty = \emptyset$ ($\mathbf{P}_p$-a.s.) for $p \leq 1/4$, so it follows that $p'_c \geq 1/4 > 0$. Finally, $\theta'_\infty(p'_c) > 0$ since from Proposition 2 and right-continuity we have $\theta'_\infty(p'_c) > 2^{-2n-3}(p'^{-n}_c - 3)$ for all n. □

To make calculations about (C_n) we shall want to be able to express θ'_{n+1} as a function of θ'_n. This cannot be done directly: first we need to break down the event 'C_n is wired' into various sub-events, in other words, to be more specific about the different ways the event 'C_n is wired' can occur. Label the five connected components of $F \cap \partial I$ by $S_1, \ldots S_5$ as in Figure 2. For $0 \leq n \leq \infty$ we define the (random) equivalence relation $\overset{n}{\sim}$ on the numbers 1 to 5 by saying that $i \overset{n}{\sim} j$ if and only if S_i and S_j are contained in the same connected component of C_n. Thus C_n is wired when there exist $i \neq j$ such that $i \overset{n}{\sim} j$; that is, $\{C_n \text{ is wired}\} = \{\overset{n}{\sim} \neq \text{equality}\}$. Hence,

$$\theta'_n(p) = 1 - \mathbf{P}_p(\overset{n}{\sim} = \text{ equality}). \tag{1}$$

Note that not all equivalence relations are geometrically realisable: for example, there is no realisation of C_n for which $\overset{n}{\sim}$ is the relation with equivalence classes $\{1,2,4\}$, $\{3,5\}$ (see Figure 2).

Let E be the set of all possible equivalence relations on the numbers 1 to 5. For $n \geq 0$ and $0 \leq p \leq 1$, let $\pi_n(p)$ be the probability vector indexed by E with $\pi^e_n(p) = \mathbf{P}_p(\overset{n}{\sim} = e)$. In Section 4 we give a constructive proof of the following:

Proposition 4 *There exists a function $f_p : [0,1]^E \to [0,1]^E$ with the property that $\pi_n(p) = f_p(\pi_{n-1}(p))$ for all finite $n \geq 1$. This function can be written down explicitly.*

The explicit formulation of f_p is extremely complicated so for reasons of space will not be given here.

For different values of p, we used Mathematica to iterate f_p a large number (360) of times starting from $\pi_0(p)$. (Since $A_0 \equiv I$, $\pi_0(p)$ is the vector with unit mass on the total equivalence relation, that is, on the relation with one equivalence class $\{1, \ldots 5\}$.) Using (1) we were thus able to compute $\theta'_{360}(p)$. Of particular interest are the values $\theta'_{360}(0.8107) = 2.179 \times 10^{-6}$ and $\theta'_{360}(0.8115) = 0.9659$. These results strongly suggest that $p'_c = 0.811$ (to three decimal places), however we have yet found no way of *proving* that $\theta'_n(0.8115)$ remains bounded away from zero for all n.

To prove that $\theta'_n(0.8107)$ does converge to zero is straightforward: applying Proposition 2 with $p = 0.8107$ and $n = 6$ we see that $p'_c < 0.8107$ implies $\theta'_\infty(0.8107) > 1.595 \times 10^{-5}$; but this would be in contradiction with the computed value of $\theta'_{360}(0.8107)$.

Since (from the proof of Proposition 3) $p'_c \le p_c$, we have therefore proved

Theorem 5 *0.8107 is a lower bound for* p_c *and* p'_c.

The computational results above have been double-checked by Kenery Oron, a student at the University of Utrecht, who independently implemented the algorithms described here in JAVA, without reference to the author's original Mathematica programs. Listings of the Mathematica programs, together with complete details of the function f_p, are available in Notebook form from the author on request.

The rest of this paper is concerned with covering the three points left undone in Section 1: a precise definition of the set F, and proofs of Propositions 2 and 4. In Section 2 we first discuss a toy model that incorporates in a fairly elementary way most of the important ideas we shall need later on, at the same time introducing some useful notation. These ideas will then be built on in Section 3, where we give a precise definition of the set F and discuss some of its consequences for self-similarity properties in the new model (C_n). Having clearly defined F we are then in a position to prove Propositions 2 and 4: Proposition 4 is proved in Section 4 and Proposition 2 in Section 5.

2 A simple example

Before defining the model (C_n) itself, we briefly discuss a toy model that incorporates in a fairly elementary way most of the important ideas concerning (C_n).

Afficianados of fractal percolation will probably be well acquainted with the 'primitive' fractal percolation model called Diamond Percolation [2, 4]. This is usually defined as follows. Begin with a graph consisting of four bonds, as in Figure 3. We call this graph a *diamond*. Now perform independent bond percolation with parameter p on the diamond, and let $\theta_1^\diamond(p) = \mathbf{P}_p(\text{there is an open path in the diamond from point } L \text{ to point } R) = 2p^2 - p^4$. Next, replace each open bond with a new diamond, and iterate the whole process (see Figure 4). In diamond percolation we are interested in the probability $\theta_n^\diamond(p)$ that there is an open path from L to R after n iterations of the process. It is not difficult to see that these probabilities satisfy the recursion

$$\theta_n^\diamond(p) = 2(p\,\theta_{n-1}^\diamond(p))^2 - (p\,\theta_{n-1}^\diamond(p))^4. \tag{2}$$

This enables us easily to compute the value of $\theta_n^\diamond(p)$ for any given p and $n < \infty$. Using elementary methods (see for example Lemma 3.4 of [3]) we can even arrive at an expression

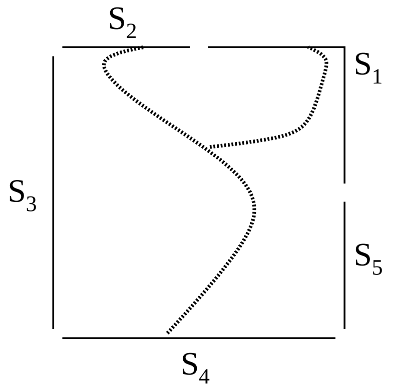

Figure 2: The five components $S_1, \ldots S_5$ of $F \cap \partial I$. In this diagrammatic representation of a realisation of C_n (for some n) we see that S_1, S_2 and S_4 all lie in the same connected component of C_n and thus $1 \overset{n}{\sim} 2 \overset{n}{\sim} 4$. Note that it would not be possible for $\overset{n}{\sim}$ to have equivalence classes $\{1, 2, 4\}$ and $\{3, 5\}$: if A, A' are connected subsets of C_n with $A \supset S_1 \cup S_2 \cup S_4$ and $A' \supset S_3 \cup S_5$, then $A \cap A'$ is nonempty.

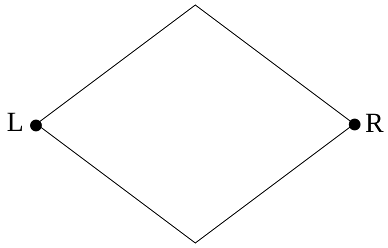

Figure 3: A *diamond* is a simple graph with two vertices distinguished as shown.

for the critical point of the model

$$p_c^{\diamond} = \sup\{p : \theta_n^{\diamond}(p) \to 0 \text{ as } n \to \infty\} = \frac{3}{4}\sqrt{\frac{3}{2}} \approx 0.9186.$$

It is possible to formulate the diamond percolation model in terms of Mandelbrot's fractal percolation process (A_n), as we shall now explain. First, a little notation. For $0 \leq n < \infty$, we say $Q \subset I$ is a *level-n square* if Q is of the form $[a2^{-n}, (a+1)2^{-n}] \times [b2^{-n}, (b+1)2^{-n}]$ for some nonnegative integers $a, b < 2^n$. Thus A_n is always a union of some level-n squares. Q is a *dyadic square* if Q is a level-n square for some n. For $-1 \leq i < \infty$, set $\mathbf{D}_i = \{x \in [0,1] : 2^i x \in \mathbf{Z}\}$, and let $\mathbf{D} = \bigcup_{-1}^{\infty} \mathbf{D}_i$ be the set of all dyadic numbers in $[0,1]$. We define $\phi : \mathbf{D} \to \mathbf{Z}$ by $\phi(x) = \min\{n \geq -1 : x \in \mathbf{D}_n\}$. Thus for example $\phi(0) = -1$, $\phi(1) = 0$, $\phi(5/8) = 3$.

Now consider the 'grid'

$$G := (\mathbf{D} \times [0,1]) \cup ([0,1] \times \mathbf{D}).$$

This set exhibits a lot of self-similarity, indeed, for any dyadic square $I_\star$, $G \cap I_\star$ is similar to G. It would however be of little interest to consider the random set $A_n \cup G$ as a percolation model, since this is connected for every realisation of A_n. Instead, we take the following approach. Partition G into a set made up of vertical line segments

$$V = (\mathbf{D} \times [0,1]) \setminus \{(x,y) \in \mathbf{D}^2 : \phi(x) \geq \phi(y)\}$$

and a set made up of horizontal line segments

$$H = ([0,1] \times \mathbf{D}) \setminus \{(x,y) \in \mathbf{D}^2 : \phi(x) < \phi(y)\}$$

(see Figure 5). Define now a new random set

$$\Delta_n = (A_n \cup V) \setminus H$$

for $0 \leq n \leq \infty$, and for $n < \infty$ define

$$\theta_n^{\Delta}(p) = \mathbf{P}_p(\text{there is a left-right crossing of } \Delta_n \text{ in } I).$$

Δ_n has a useful property which enables us to express $\theta_n^{\Delta}(p)$ in terms of simple events defined on the level-1 squares $I_1, \dots I_4$ (as given in Figure 6). For all n,

$$(0,1) \times \{1/2\} \subset H \subset \Delta_n^c \tag{3}$$

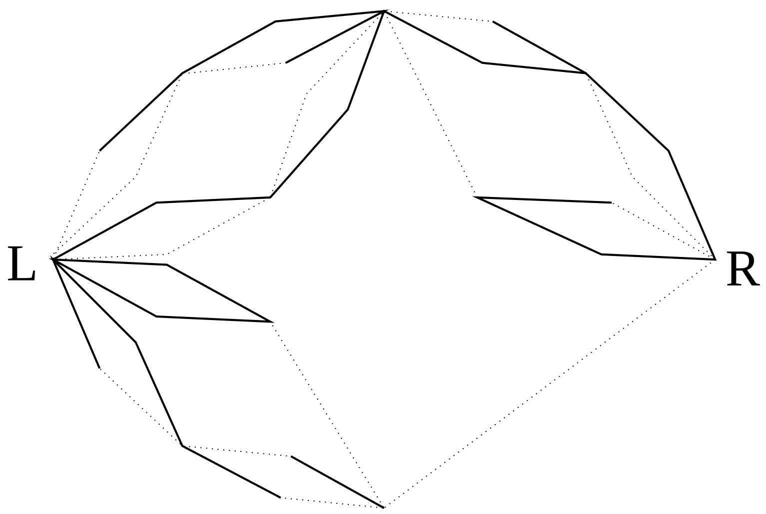

Figure 4: A possible realisation of the diamond percolation process after three iterations. The solid lines represent open bonds and the broken lines closed bonds.

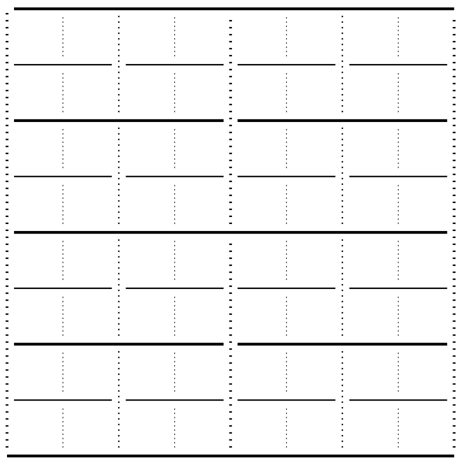

Figure 5: Part of the sets H (in solid lines) and V (in broken lines).

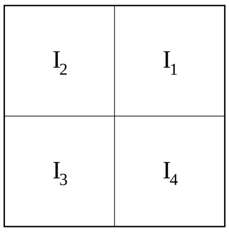

Figure 6: The four level-1 squares $I_1, \dots I_4$.

and

$$\{1/2\} \times ((0,1/2) \cup (1/2,1)) \subset V \subset \Delta_n. \tag{4}$$

This implies that

$$\theta_1^{\Delta}(p) = \mathbf{P}_p(I_1 \cup I_2 \subset A_1 \text{ or } I_3 \cup I_4 \subset A_1) = 2p^2 - p^4$$

and in general that there is a left-right (L-R) crossing of Δ_n in I if and only if either (i) there is a L-R crossing of $\Delta_n \cap I_1$ in I_1 and of $\Delta_n \cap I_2$ in I_2 or (ii) there is a L-R crossing of $\Delta_n \cap I_3$ in I_3 and of $\Delta_n \cap I_4$ in I_4. (Note that it would not be possible to write down a similar characterisation in the classical model (A_n) – see Figure 7.) By self-similarity in Δ_n we see that $\mathbf{P}_p(\text{L-R crossing of } \Delta_n \cap I_i \text{ in } I_i) = p\theta_{n-1}^{\Delta}$ for all $n \geq 1$ and $i = 1, \ldots 4$, hence, θ_n^{Δ} satisfies the recursion

$$\theta_n^{\Delta}(p) = 2(p\,\theta_{n-1}^{\Delta}(p))^2 - (p\,\theta_{n-1}^{\Delta}(p))^4. \tag{5}$$

Of course this comes down to the same calculation as in the first paragraph of this section. Thus $\theta_n^{\Delta} \equiv \theta_n^{\diamond}$ for all n. In particular, the critical point for our model (Δ_n) is known explicitly, having the value $p_c^{\Delta} = \frac{3}{4}\sqrt{\frac{3}{2}}$.

3 The model

The model (C_n) is defined using the dyadic grid G and the function ϕ, in a way reminiscent of the definition of Δ_n in the previous section. Let

$$K = \{(0,0)\} \cup \{(x,y) \in \mathbf{D}^2 : |\phi(x) - \phi(y)| = 1\}$$

and set $F = G \setminus K$. (See Figure 1.) We now define $C_n = A_n \cup F$ for $0 \leq n \leq \infty$.

Before proceeding, a brief remark on connected components of F: notice that the lines $y = 2x$ and $y = x/2$ do not intersect F, so F is certainly disconnected. Indeed, the set Y of Figure 8, made up of ten straight line segments of slope $2^{\pm 1}$, satisfies $Y \cap F = \emptyset$. By inspection, it follows that there are exactly five connected components of F that intersect ∂I, as was stated in Section 1.

We shall see that C_n and Δ_n have many similarities, but note one crucial difference. Since Δ_n is obtained from A_n by adding some lines and removing others, in general we have neither $\Delta_n \subset A_n$ nor $A_n \subset \Delta_n$, and so no direct comparison is possible between p_c^{Δ} and p_c. On the other hand, as we saw in Section 1, A_n is a subset of C_n and therefore the critical point for the C_n, p_c', forms a lower bound for p_c.

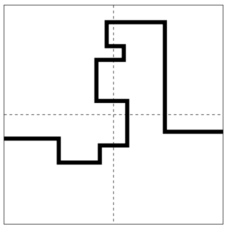

Figure 7: In this realisation of A_n (for a given finite n), the existence of a left-right crossing is guaranteed by complicated interconnections across the boundaries of the level-1 squares. To write down all possible ways such interconnections can occur is practically speaking impossible.

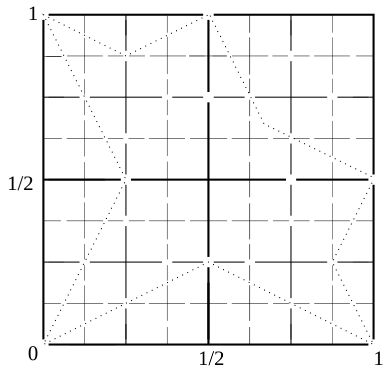

Figure 8: The set Y (in broken lines) does not intersect F (solid lines).

An important similarity between C_n and Δ_n is the way connection probabilities are expressable in terms of fairly simple events defined on the level-1 squares $I_1, \dots I_4$. We have seen already how this property in Δ_n can be used to obtain a simple recursion equation for θ_n^Δ: the same idea will be used (in a more sophisticated way) to prove Proposition 4 in Section 4.

In this section however we concentrate on another point in common between C_n and Δ_n: what might be called stochastic self-similarity. This was rather skirted over in our treatment of Δ_n, but for C_n a more detailed discussion will be necessary. Stochastic self-similarity in Δ_n resulted from the simple stochastic self-similarity of the classical process A_n together with geometric self-similarities in the deterministic sets H and V; for C_n we will again need the properties of A_n, now combined with self-similarities in the set F.

The situation for A_n is quite simple: it is easy to see that for any positive m, $n < \infty$ and any level-m square $I_\star$, the law of $A_{n+m} \cap I_\star$ conditioned on $\{I_\star \subset A_m\}$ (i.e. on '$I_\star$ is retained') is identical to the law of A_n (after scaling). Furthermore this law is invariant under rotation by any multiple of $\pi/2$.

Turn now to F. Looking at Figure 1 we see straightaway that the pattern formed by F within any level-1 square is remarkably similar to the set F itself. To formalise this somewhat, define $J = \{(1,1), (0,1/2), (1/2,0)\}$ and observe that $(F \setminus J) \cap I_i$ is an exact copy of F for $i = 1, \dots 4$, after the application of a similarity mapping ν_i consisting of a scaling and a rotation through a multiple of $\pi/2$. (We omit a proof of this observation but it is not difficult, using the definitions at the beginning of this section.) By induction, we deduce that for any m with $1 \le m < \infty$ and any level-m square $I_\star$, there is a $J_\star \subset F \cap \eth I_\star$ and a similarity mapping $\nu_\star : I_\star \to I$ such that $\nu_\star((F \setminus J_\star) \cap I_\star) = F$. (Here $\eth I_\star$ denotes the set consisting of the four corners of $I_\star$.)

We can now combine these two paragraphs to obtain a stochastic self-similarity property for C_n. If J_0 is any finite subset of F, for $n < \infty$ we define $C_n^{J_0}$ to be the random set $A_n \cup (F \setminus J_0)$, thus $C_n^\emptyset \equiv C_n$. For a level-1 square I_i ($i = 1, \dots 4$) we then see that the law of $\nu_i(C_{n+1}^J \cap I_i)$ conditioned on $\{I_i \subset A_1\}$ is the same as that of C_n. In the general set-up above of a level-m square $I_\star$ we have that $\nu_\star(C_{n+m}^{J_\star} \cap I_\star)$ on $\{I_\star \subset A_m\}$ has the same law as C_n. Let us say that $C_n^{J_0}$ is *wired in* $I_\star$ if there is a connected component of $C_n^{J_0} \cap I_\star$ that contains at least two distinct components of $(F \setminus J_0) \cap \partial I_\star$. Then

$$\mathbf{P}_p(C_{n+m}^{J_\star} \text{ is wired in } I_\star | I_\star \subset A_m) = \theta_n'(p). \tag{6}$$

This result will be important in the proof of Proposition 2 in Section 5.

4 The iteration function

We now prove Proposition 4, that is, we show how to find a function $f_p : [0,1]^E \to [0,1]^E$ such that $\pi_n(p) = f_p(\pi_{n-1}(p))$ for all finite $n \geq 1$.

Using the terminology of Section 3, let $\overset{n}{\approx}$ be the random equivalence relation on $1, \dots 5$ whereby $i \overset{n}{\approx} j$ if and only if $\nu_1^{-1}(S_i)$ and $\nu_1^{-1}(S_j)$ lie in the same connected component of $C_n^J \cap I_1$. Then by self-similarity,

$$\mathbf{P}_p(\overset{n}{\approx} = e \,|I_1 \subset A_1) = \mathbf{P}_p(\overset{n-1}{\sim} = e) = \pi_{n-1}^e(p)$$

for all finite $n \geq 1$ and $e \in E$. Since $\{I_1 \not\subset A_1\} \subset \{\overset{n}{\approx} = \text{equality}\}$, we therefore have

$$\mathbf{P}_p(\overset{n}{\approx} = e) = p\pi_{n-1}^e(p) + (1-p)\mathbf{1}(e = \text{ equality}). \tag{7}$$

Consider now the set $C_n \cap I_1$, and let $\sim_1^n \in E$ be defined by $i \sim_1^n j$ if and only if $\nu_1^{-1}(S_i)$ and $\nu_1^{-1}(S_j)$ lie in the same connected component of $C_n \cap I_1$. We note that $(1,1) \in F$ and that $\nu_1^{-1}(S_3) \cup \{(1,1)\} \cup \nu_1^{-1}(S_4)$ is a connected set; so the sets $\nu_1^{-1}(S_3)$ and $\nu_1^{-1}(S_4)$ are always linked through $C_n \cap I_1$, that is, $3 \sim_1^n 4$ surely. Indeed, $C_n \cap I_1 = (C_n^J \cap I_1) \cup \{(1,1)\}$, so the connected components of $C_n \cap I_1$ are the same as for $C_n^J \cap I_1$ except in the case when a join is made through the point (1,1). That is, the equivalence classes of $\sim_1^n$ consist of any classes of $\overset{n}{\approx}$ that contain neither 3 or 4, together with the class $\{i : i \overset{n}{\approx} 3 \text{ or } i \overset{n}{\approx} 4\}$. This defines a function $\eta : E \to E$ such that $\sim_1^n = \eta(\overset{n}{\approx})$, and we have

$$\mathbf{P}_p(\sim_1^n = e) = \sum_{e_0 : \eta(e_0) = e} \mathbf{P}_p(\overset{n}{\approx} = e). \tag{8}$$

The above discussion can also be applied with I_2, I_3 or I_4 replacing I_1, to obtain random equivalence relations $\sim_i^n$ that describe the connectivity properties of $C_n \cap I_i$ for $i = 2,3,4$ (see Figure 9). With equations similar to (7) and (8), we can express the law of $\sim_i^n$ as a function of the probability vector π_{n-1}. Note moreover that $\sim_i^n$ depends only on the Cantor set construction within the subsquare I_i, therefore $\sim_1^n, \sim_2^n, \sim_3^n$ and $\sim_4^n$ are independent.

We now claim that it is possible to write $\overset{n}{\sim}$ as a function ζ of $\sim_1^n, \sim_2^n, \sim_3^n$ and $\sim_4^n$. It follows that

$$\pi_n^e(p) = \sum_{v \in E^4 : \zeta(v) = e} \mathbf{P}_p(\sim_1^n = v_1, \sim_2^n = v_2, \sim_3^n = v_3, \sim_4^n = v_4)\,;$$

since the $\sim_i^n$ are independent, and we can express each term $\mathbf{P}_p(\sim_i^n = v_i)$ as a function of π_{n-1}, this gives rise to the desired function f satisfying $\pi_n = f(\pi_{n-1})$.

To justify the above claim, let us consider the set $L = \partial(I_1) \cup \ldots \cup \partial(I_4)$ (see Figure 10). This set is crucial in determining how events on the level-1 squares relate to each other, as any interconnection across the boundaries of the level-1 squares must pass through L. For example: in the toy model of Section 2 (formulae 3, 4) we saw that L is contained in $H \cup V$, where V is (surely) a subset of Δ_n for all n, and $H \subset \Delta_n^c$. Thus $L \cap \Delta_n$ is a deterministic set; we have seen that this enables us to set up a recursion (5) for θ_n^{Δ}. On the other hand, we saw in Figure 7 that such a simple idea could not be applied in the classical model A_n, as here interconnections across the boundaries of the level-1 squares can occur in very complicated ways: for large n, $L \cap A_n$ is a complicated random set.

Here, the existence of our function ζ relies on the fact that the relationship between C_n and L is fairly simple. Namely, we can write $L \subset F \cup (K \cap L)$, knowing that F is contained in C_n (surely for all n) and $K \cap L$ is a simple finite set (as given in figure 10). Let the eight connected components of $F \cap L$ be labelled $U_1, \ldots U_8$ as in Figure 9. These components are always contained in C_n; because together they cover the crucial set L (except for a few special points), we see that connections between them in C_n are completely determined by connection events on the four level-1 squares. Formally, we define the random equivalence relation Z on $1, \ldots 8$ with $i\, Z\, j$ if and only if U_i and U_j are contained in the same connected component of C_n: it is possible to write Z as a function of $\sim_1^n, \sim_2^n, \sim_3^n$ and $\sim_4^n$. This completes our argument, as $\overset{n}{\sim}$ is simply the restriction of Z to $\{1, \ldots 5\}$. (See Figure 9.)

To write down f_p we would need explicit expressions for ζ, η, and also for the versions of η corresponding to I_2, I_3 and I_4. In each case, the procedure is a straightfoward matter of processing through the combinatorial structure implied by F. Although the size of E is too great for us realistically to do all this by hand (particularly ζ is infeasible), it is quite straightforward to translate the above argument into a computer algorithm that can be implemented in Mathematica.

5 A bound on θ'_∞

It remains only to prove Proposition 2. To begin with: three preliminary lemmas.

We define a *path* to be a continuous injective map from $[0, 1]$ to I. If J_0 is a finite subset of F and $I_\star$ is a dyadic square, we shall say that $C_n^{J_0} = A_n \cup (F \setminus J_0)$ is *path-wired in* $I_\star$ if there is a path in $C_n^{J_0} \cap I_\star$ linking at least two distinct components of $(F \setminus J_0) \cap \partial I_\star$.

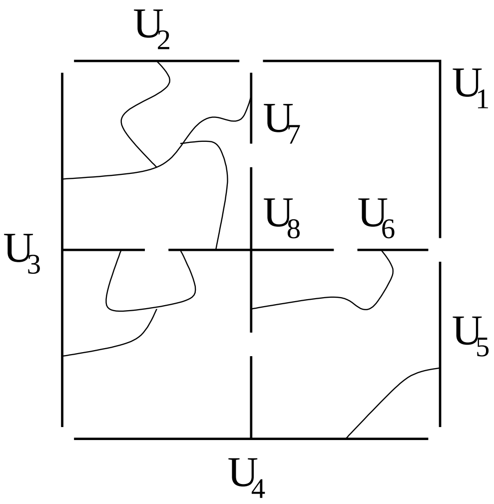

Figure 9: The eight connected components of $F \cap L$ are labelled $U_1, \ldots U_8$. Shown is a representation of a possible realisation of C_n whereby the equivalence classes of the $\sim_n^i$ are given by $\sim_1^n$: {1}, {2}, {3, 4}, {5}; $\sim_2^n$: {1,2,3,4,5}; $\sim_3^n$: {1,2,3}, {4,5}; $\sim_4^n$: {1,5}, {2,3,4}. From this we can deduce that the equivalence classes of $\overset{n}{\sim}$ are {1}, {2,3}, {4,5}.

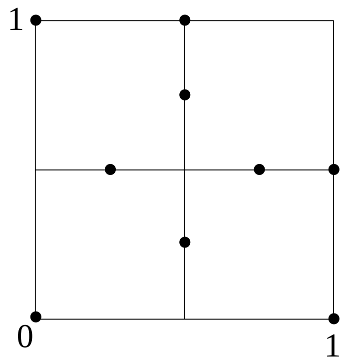

Figure 10: The straight line segments together form the set L; $L \cap K$ consists of the nine points marked.

Lemma 6 *For any finite $J_0 \subset F$ and any dyadic square $I_\star$, we have*

$$\mathbf{P}_p(C_n \textit{ is wired in } I_\star) = \mathbf{P}_p(C_n \textit{ is path-wired in } I_\star)$$

for all $0 \leq n < \infty$ and $0 \leq p \leq 1$.

Proof First fix $I_\star = I$ and let $\theta_n^\gamma(p) = \mathbf{P}_p(C_n$ is path-wired in $I)$; we want to show that $\theta'_n \equiv \theta_n^\gamma$. Now $C_0 = I$ surely, so we know $\theta'_0 \equiv \theta_0^\gamma \equiv 1$. To express θ_n^γ in terms of θ_{n-1}^γ we can proceed in the same way as in Section 4: in fact the combinatorial arguments for θ_n^γ will be exactly the same as they were for θ'_n and therefore we will obtain exactly the same iteration function f; hence it follows that $\theta'_n \equiv \theta_n^\gamma$ for all finite n.

The lemma for general $I_\star$ can be proved by applying the argument of section 4 to $C_n \cap I_\star$ and the four sub-squares into which $I_\star$ divides. $\square$

Lemma 7 *Fix $n \geq 1$ and a realisation C_n, and suppose Q is some level-n square. Then for every connected component Σ of $C_n \cap Q$, $\Sigma \cap \partial Q$ is path-connected.*

Proof If $Q \subset A_n$ then the result is trivial (the only connected component of $C_n \cap Q$ is Q itself) so suppose $Q \not\subset A_n$. It follows that $C_n \cap \mathrm{int}(Q) \subset F$ (where $\mathrm{int}(Q)$ denotes the interior of Q). But from the discussion at the beginning of Section 3, the following is certainly clear: for every connected component Σ of F, $\Sigma \cap \partial I$ either is empty or equals S_i for some $i = 1, \ldots 5$, and therefore is path-connected. Self-similarity of F now completes the proof. $\square$

Lemma 8 *Given positive finite integers m and n and any level m-square $I_\star$,*

$$\mathbf{P}_p(C_{n+m} \textit{ is wired in } I_\star | I_\star \subset A_m) \leq \theta_n(p).$$

Proof In the light of the discussion at the end of Section 3, the statement of the lemma is equivalent to

$$\mathbf{P}_p(C_{n+m} \text{ wired in } I_\star | I_\star \subset A_m) \leq \mathbf{P}_p(C_{n+m}^{J_\star} \text{ wired in } I_\star | I_\star \subset A_m). \tag{9}$$

But on $\{I_\star \not\subset A_m\}$, neither C_{n+m} nor $C_{n+m}^{J_\star}$ is wired in $I_\star$, so (9) is the same as

$$\mathbf{P}_p(C_{n+m} \text{ wired in } I_\star) \leq \mathbf{P}_p(C_{n+m}^{J_\star} \text{ wired in } I_\star).$$

By Lemma 6 this is equivalent to

$$\mathbf{P}_p(C_{n+m} \text{ path-wired in } I_\star) \leq \mathbf{P}_p(C_{n+m}^{J_\star} \text{ wired in } I_\star).$$

So suppose C_{n+m} is path-wired in $I_\star$, that is, there is a path γ in $C_{n+m} \cap I_\star$ linking two distinct components of $F \cap \partial I_\star$. It will be sufficient for us to show that this implies the existence of a path γ' satisfying the same conditions as on γ and in addition with $\gamma' \cap J_\star = \emptyset$.

By considering a sub-path of γ if necessary, we can assume γ intersects $J_\star$ only at its end-points γ_0 and γ_1, if at all. (Recall that $J_\star \subset \boldsymbol{\eth} I_\star$, that is, $J_\star$ contains only corners of $I_\star$; so by construction of F, distinct points of $J_\star$ are in distinct components of $F \cap \partial I_\star$.)

Suppose first that $\gamma_0 \in J_\star$ and $\gamma_1 \notin J_\star$. Then $\gamma_0 \in \boldsymbol{\eth} I_\star$, so there is a unique level-$(n+m)$ square Q with $\gamma_0 \in Q \subset I_\star$; moreover there is a point γ_Q on γ, with $\gamma_Q \in Q \setminus \{\gamma_0\}$, such that the sub-path from γ_0 as far as γ_Q is contained in Q; by continuity of γ we may assume $\gamma_Q \in \partial Q$. Now γ_0 and γ_Q are contained in the same connected component of $C_{n+m} \cap Q$, so by Lemma 7 there is a path $\gamma^\star$ in $C_{n+m} \cap \partial Q$ from γ_0 to γ_Q. Since $\gamma_0 \in \boldsymbol{\eth} I_\star$ we can find an x_0 on $\gamma^\star$ such that $x_0 \in \partial I_\star \setminus J_\star$; let γ' be the path from x_0 to γ_1 obtained by concatening the sub-path of $\gamma^\star$ from x_0 to γ_Q with the sub-path of γ from γ_Q to γ_1.

This completes the case $\gamma_0 \in J_\star$, $\gamma_1 \notin J_\star$. The three other cases are now simple to handle. If neither end-point of γ is in $J_\star$, we take $\gamma' = \gamma$; if $\gamma_0 \notin J_\star$ and $\gamma_1 \in J_\star$ we apply the previous argument to γ_1 instead of γ_0; if both end-points are in $J_\star$ we be apply the previous argument to γ_0 and γ_1. $\square$

We now proceed to prove Proposition 2, which for convenience is re-stated here.

Proposition 2 *Suppose* $0 < p \leq 1$ *with* $\theta'_\infty(p) > 0$. *Then*

$$\theta'_\infty(p) \geq \frac{p^{-n} - 3}{2^{2n+3}}$$

for all $n \geq 1$.

Proof Fix m, n with $1 \leq m, n < \infty$. Consider a realisation of the process for which C_{m+n} is wired. Then by inspection of Figure 8, it follows that there exists some level-n square $I_\star$ with $\mathrm{int}(I_\star) \cap Y \neq \emptyset$, such that C_{m+n} is wired in $I_\star$. Indeed, there exist at least two such squares, $I_\star$ and $I'_\star$, unless $I_\star$ lies at one of the three corners (0,1), (1,0) or (0,0) of I. Hence, for all $0 \leq p \leq 1$,

$$\begin{aligned}
\theta'_{m+n}(p) &= \mathbf{P}_p(C_{m+n} \text{ is wired}) \\
&\leq \sum \mathbf{P}_p(C_{m+n} \text{ is wired in } I_\star) \\
&\quad + \sum \mathbf{P}_p(C_{m+n} \text{ is wired in } I_\star \text{ and in } I'_\star), \qquad (10)
\end{aligned}$$

where the first sum is over the three level-n squares $I_\star$ containing either (0,1), (1,0) or (0,0), and the second sum is over all distinct pairs of level-n squares $I_\star$, $I'_\star$ whose interiors have nonempty intersection with Y.

Consider the case where $I_\star$ and $I'_\star$ are distinct. Conditioned on $\{I_\star \cup I'_\star \subset A_n\}$, the event $\{C_{m+n}$ is wired in $I_\star\}$ depends only on the Cantor set construction within $I_\star$ (and similarly within $I'_\star$), therefore

$$\begin{aligned}
&\mathbf{P}_p(C_{m+n} \text{ is wired in } I_\star \text{ and in } I_\star \,|\, I_\star \cup I'_\star \subset A_n) \\
&= \mathbf{P}_p(C_{m+n} \text{ is wired in } I_\star \,|\, I_\star \subset A_n)\, \mathbf{P}_p(C_{m+n} \text{ is wired in } I'_\star \,|\, I'_\star \subset A_n) \\
&\leq (\theta'_m(p))^2,
\end{aligned}$$

the inequality being an application of Lemma 8. Hence,

$$\begin{aligned}
&\mathbf{P}_p(C_{m+n} \text{ is wired in } I_\star \text{ and in } I'_\star) \\
&= \mathbf{P}_p(I_\star \cup I'_\star \subset A_n)\, \mathbf{P}_p(C_{m+n} \text{ is wired in } I_\star \text{ and in } I'_\star \,|\, I_\star \cup I'_\star \subset A_n) \\
&\leq p^n(\theta'_m(p))^2.
\end{aligned}$$

Applying Lemma 8 also to the first summation in (10) we obtain

$$\begin{aligned}
\theta'_{m+n}(p) &\leq 3p^n\theta'_m(p) + \begin{pmatrix} 2^{n+2} \\ 2 \end{pmatrix} p^n(\theta'_m(p))^2 \\
&\leq 3p^n\theta'_m(p) + 2^{2n+3}p^n(\theta'_m(p))^2.
\end{aligned}$$

Taking the limit $m \to \infty$ we have

$$\theta'_\infty(p) \leq 3p^n\theta'_\infty(p) + 2^{2n+3}p^n(\theta'_\infty(p))^2$$

which is equivalent to the statement of the lemma when p and $\theta_\infty(p)$ are nonzero. □

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